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Fractional h-differences with exponential kernels and their monotonicity properties

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In this work, the nabla fractional differences of order $0 < \mu < 1$ with discrete exponential kernels are formulated on the time scale $h\mathbb{Z}$, where $0 < h \leq 1$. Hence, the earlier results obtained in *Adv. Differ. Equ.*, 2017, (78) (2017) are generalized. The monotonicity properties of the h–Caputo-Fabrizio (CF) fractional difference operator are concluded using its relation with the nabla h–Riemann-Liouville (RL) fractional difference operator. It is shown that the monotonicity coefficient depends on the step h, and this dependency is explicitly derived. As an application, a fractional difference version of the mean value theorem (MVT) on $h\mathbb{Z}$ is proved.

KEYWORDS

nabla h-discrete exponential kernel, nabla h-RL fractional difference, nabla h-CF fractional difference

MSC CLASSIFICATION

39A12; 34A25; 26A33

1 | INTRODUCTION

By the late 19th century, a lot of fruitful results have been obtained in the subject of fractional calculus in the continuous setting. Since then, many problems in science and engineering have been modeled by fractional calculus. For example, we refer the reader to Podlubny.[1] Despite the huge developments in the field of continuous fractional calculus in both theory and applications, notably less is known about discrete fractional calculus. In 1974, Diaz and Osler published their paper introducing a discrete fractional difference operator defined as an infinite series.[2] In 1988, Gray and Zhang developed Leibniz' formula, a limited composition rule and a version of a power rule for differentiation.[3] Since 1989, discrete fractional calculus has been under focus by many researchers. The forward or delta difference has been employed in previous studies[4-12]; other work has been developed for the fractional backward or nabla difference.[13-17] The relation between the forward and the backward fractional operators has been developed in Atici and Eloe,[16] and the fractional derivatives on time scales have been defined.[18,19] Up to now, the researchers have paid little attention to function domains or to lower limits of summation and differentiation, which may lead to wrong claims; however, many applications of discrete fractional calculus in different directions of science are successfully discussed in many papers in the last decade, for example, in previous studies.[20-28]

Not long ago, a new fractional derivative based on nonsingular kernels is introduced.[29] It appears that Caputo-Fabrizio (CF) derivative is very profitable in describing real-word problems, which behaves exponentially. Atangana and Baleanu[30] solved the fractional heat transfer model using new fractional derivatives with exponential kernels. In Caputo and Fabrizio,[31] many applications about the new notations of fractional derivatives with exponential kernels were presented, and the coherence with the thermodynamic laws were proved.

Recently, studying the monotonicity for fractional difference operators with nonsingular discrete kernels is under consideration.[32-45] The monotonicity with discrete exponential kernels were studied in Abdeljawad and Baleanu[46] when the time step $h = 1$. Suwan et al[35] studied the monotonicity results for h-discrete fractional operators, and they formulated a new version of the mean value theorem (MVT) as an application for $0 < h \leq 1$.

In this work, CF fractional differences and their correspondent fractional sums are studied with exponential kernels, and their monotonicity results are obtained for small enough time step $0 < h \leq 1$, and then the results are prettified by formulating a new version of the MVT as an application. As a general conclusion, the time scale $h\mathbb{Z}$ provides an accurate description of the discrete phenomena.

In Section 2 of this article, the needed definitions and preliminaries are presented. In Section 3, we investigate the monotonicity results in Suwan et al[35] for fractional differences with discrete exponential kernel, and hence, we generalize the results in Abdeljawad and Baleanu.[46] In the fourth section, we formulate a new version of the MVT as an application. Finally, the conclusions are provided in Section 5.

2 | DEFINITIONS AND PRELIMINARY RESULTS

For $p, q \in \mathbb{R}$ with $p < q$, $\frac{q-p}{h} \in \mathbb{N}$, and $0 < h \leq 1$, let $\mathbb{N}_{p,h} = \{p, p+h, p+2h, \dots\}$ and ${}_{q,h}\mathbb{N} = \{q, q-h, q-2h, \dots\}$, the h-nabla discrete exponential kernel is expressed as ${}_h\hat{e}_\lambda(t, \rho_h(s)) = \left(\frac{1}{1-h\lambda}\right)^{\frac{t-\rho_h(s)}{h}} = \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-(s-h)}{h}}$, where $\lambda = \frac{-\mu}{1-\mu}$.[37]

In the following, we propose the h-discrete version of the difference operators:

Definition 2.1. (Suwan et al[35]) The backward difference and the forward difference operators on $h\mathbb{Z}$ are defined by

$$\nabla_h \phi(t) = \frac{\phi(t) - \phi(t-h)}{h}$$

and

$$\Delta_h \phi(t) = \frac{\phi(t+h) - \phi(t)}{h}$$

respectively.

Definition 2.2. (Suwan et al[35]) The backward jump and the forward jump operators on the time scale $h\mathbb{Z}$ are defined by $\rho_h(t) = t - h$ and $\sigma_h(t) = t + h$, respectively.

Definition 2.3. Let ϕ be real-valued function defined on $\mathbb{N}_{p,h} = \{p, p+h, p+2h, \dots\}$ in the left case or on ${}_{q,h}\mathbb{N} = \{q, q-h, q-2h, \dots\}$ in right case. Let $L(\mu, h) = B(\mu)\left(\frac{\mu}{h} + 1 - \mu\right)$, where $B(\mu)$ is a normalizing positive constant satisfying $B(0) = B(1) = 1$, then for $\mu \in (0, 1)$ and $0 < h \leq 1$, we define

The left h-nabla CF fractional difference by

$$({}_{p}^{CFC}\nabla_h^\mu \phi)(t) = L(\mu, h) \sum_{m=p/h+1}^{t/h} h(\nabla_h \phi(mh)) \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}, \ t \in \mathbb{N}_{p+h,h}.$$

The right h-nabla CF fractional difference by

$$({}_{h}^{CFC}\nabla_q^\mu \phi)(t) = L(\mu, h) \sum_{m=t/h}^{q/h-1} h(-\Delta_h \phi(mh)) \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}, \ t \in {}_{q-h,h}\mathbb{N}.$$

The left h-nabla RL fractional difference by

$$({}_{p}^{CFR}\nabla_h^\mu \phi)(t) = L(\mu, h) \nabla_h \sum_{m=p/h+1}^{t/h} h\phi(mh) \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}, \ t \in \mathbb{N}_{p+h,h}.$$

The right h-nabla RL fractional difference by

$$({}_{h}^{CFR}\nabla_q^\mu \phi)(t) = L(\mu, h) \left(-\Delta_h \sum_{m=t/h}^{q/h-1} h\phi(mh) \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}\right), \ t \in {}_{q-h,h}\mathbb{N}.$$

Notice that

$$L(0, h) = B(0)\left(\frac{0}{h} + 1 - 0\right) = 1, \ L(1, h) = B(1)\left(\frac{1}{h} + 1 - 1\right) = \frac{1}{h}.$$

Remark 2.1. After noting that $L(0,h)=1$ and $L(1,h)=\frac{1}{h}$, in the limiting case $\mu\to 0$ and $\mu\to 1$, we remark the following:

- $\left({}_{p}^{CFC}\nabla_{h}^{\mu}\phi\right)(t)\to\phi(t)-\phi(p)$ as $\mu\to 0$,

  Proof.
$$\begin{aligned}
\lim_{\mu\to 0}({}_{p}^{CFC}\nabla_{h}^{\mu}\phi)(t) &= \lim_{\mu\to 0}L(\mu,h)\sum_{m=p/h+1}^{t/h}h(\nabla_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&= L(0,h)\sum_{m=p/h+1}^{t/h}h(\nabla_h\phi(mh))\left(\frac{1-0}{1-0+0}\right)^{\frac{t-mh}{h}}\\
&= \sum_{m=p/h+1}^{t/h}h\left(\frac{\phi(mh)-\phi(mh-h)}{h}\right)\\
&= \phi(t)-\phi(t-h)+\phi(t-h)-\phi(t-2h)+\ldots+\phi(p+h)-\phi(p)\\
&= \phi(t)-\phi(p).
\end{aligned}$$
□

- $\left({}_{p}^{CFC}\nabla_{h}^{\mu}\phi\right)(t)\to\nabla_h\phi(t)$ as $\mu\to 1$,

  Proof.
$$\begin{aligned}
\lim_{\mu\to 1}({}_{p}^{CFC}\nabla_{h}^{\mu}\phi)(t) &= \lim_{\mu\to 1}L(\mu,h)\sum_{m=p/h+1}^{t/h}h(\nabla_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&= L(1,h)\lim_{\mu\to 1}\left(h(\nabla_h\phi(t))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-t}{h}}+\sum_{m=p/h+1}^{t/h-1}h(\nabla_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right)\\
&= \frac{1}{h}\lim_{\mu\to 1}(h\nabla_h\phi(t))+\frac{1}{h}\sum_{m=p/h+1}^{t/h-1}h(\nabla_h\phi(mh))\left(\frac{1-1}{1-1+h}\right)^{\frac{t-mh}{h}}\\
&= \nabla_h\phi(t).
\end{aligned}$$
□

- $\left({}_{h}^{CFC}\nabla_{q}^{\mu}\phi\right)(t)\to\phi(t)-\phi(q)$ as $\mu\to 0$,

  Proof.
$$\begin{aligned}
\lim_{\mu\to 0}({}_{h}^{CFC}\nabla_{q}^{\mu}\phi)(t) &= \lim_{\mu\to 0}L(\mu,h)\sum_{m=t/h}^{q/h-1}h(-\Delta_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}\\
&= L(0,h)\sum_{m=t/h}^{q/h-1}h(-\Delta_h\phi(mh))\left(\frac{1-0}{1-0+0}\right)^{\frac{mh-t}{h}}\\
&= \sum_{m=t/h}^{q/h-1}h\left(\frac{\phi(mh)-\phi(mh+h)}{h}\right)\\
&= \phi(t)-\phi(t+h)+\phi(t+h)-\phi(t+2h)+\ldots+\phi(q-h)-\phi(q)\\
&= \phi(t)-\phi(q).
\end{aligned}$$
□

- $\left({}_{h}^{CFC}\nabla_{q}^{\mu}\phi\right)(t)\to-\Delta_h\phi(t)$ as $\mu\to 1$,

  Proof.
$$\begin{aligned}
\lim_{\mu\to 1}(({}_{h}^{CFC}\nabla_{q}^{\mu}\phi)(t) &= \lim_{\mu\to 1}L(\mu,h)\sum_{m=t/h}^{q/h-1}h(-\Delta_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}\\
&= L(1,h)\lim_{\mu\to 1}\left(h(-\Delta_h\phi(t))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-t}{h}}+\sum_{m=t/h+1}^{q/h-1}h(-\Delta_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}\right)\\
&= \frac{1}{h}\lim_{\mu\to 1}(-h\Delta_h\phi(t))+\frac{1}{h}\sum_{m=t/h+1}^{q/h-1}h(-\Delta_h\phi(mh))\left(\frac{1-1}{1-1+h}\right)^{\frac{mh-t}{h}}\\
&= -\Delta_h\phi(t).
\end{aligned}$$
□

Similarly, we can prove the following:

- $\left({}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t) \to \phi(t)$ as $\mu \to 0$,
- $\left({}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t) \to \nabla_h\phi(t)$ as $\mu \to 1$,
- $\left({}_{h}^{CFR}\nabla_{q}^{\mu}\phi\right)(t) \to \phi(t)$ as $\mu \to 0$,
- $\left({}_{h}^{CFC}\nabla_{q}^{\mu}\phi\right)(t) \to -\Delta\phi(t)$ as $\mu \to 1$.

Definition 2.4. Let $u(t)$ be a function such that $u(t) = \left({}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t)$, $0 < \mu < 1$ and $0 < h \leq 1$. The correspondent left discrete fractional integral of $\left({}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t)$ is defined by

$$\left({}_{p}^{CF}\nabla_{h}^{-\mu}u\right)(t) = \frac{1-\mu}{L(\mu,h)(1-\mu+h\mu)}u(t) + \frac{\mu}{L(\mu,h)(1-\mu+h\mu)}\sum_{m=p/h+1}^{t/h} hu(mh), \ t \in \mathbb{N}_{p+h,h}.$$

Also, we define the right fractional integral by

$$\left({}_{h}^{CF}\nabla_{q}^{-\mu}u\right)(t) = \frac{1-\mu}{L(\mu,h)(1-\mu+h\mu)}u(t) + \frac{\mu}{L(\mu,h)(1-\mu+h\mu)}\sum_{m=t/h}^{q/h-1} hu(mh), \ t \in {}_{q-h,h}\mathbb{N}.$$

Note that from the definition we have,

$$\left({}_{p}^{CF}\nabla_{h}^{-\mu}\,{}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t) = \phi(t),$$

and

$$\left({}_{h}^{CF}\nabla_{q}^{-\mu}\,{}_{h}^{CFR}\nabla_{q}^{\mu}\phi\right)(t) = \phi(t).$$

Proposition 2.1. *(The relation between Riemann and Caputo type h-fractional differences with exponential kernels)*
For $0 < \mu < 1$ *and* $0 < h \leq 1$ *we have,*

1. $$\left({}_{p}^{CFC}\nabla_{h}^{\mu}\phi\right)(t) = \left({}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t) - \frac{L(\mu,h)(1-\mu+h\mu)}{1-\mu}\phi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}}.$$
2. $$\left({}_{h}^{CFC}\nabla_{q}^{\mu}\phi\right)(t) = \left({}_{h}^{CFR}\nabla_{q}^{\mu}\phi\right)(t) - \frac{L(\mu,h)(1-\mu+h\mu)}{1-\mu}\phi(q)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{q-t}{h}}.$$

In the following, we provide an alternative proof to the one appeared in Abdeljawad,[37] where the authors used the discrete Laplace transformations.

Proof. In this proof, we start with the definition of the left and right Caputo h-fractional differences with exponential kernels, and we conclude its relation with the left and right *RL* h-fractional differences with exponential kernels, respectively.

1. $$\begin{aligned}
({}_{p}^{CFC}\nabla_{h}^{\mu}\phi)(t) &= L(\mu,h)\sum_{m=p/h+1}^{t/h} h(\nabla_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} \\
&= L(\mu,h)\sum_{m=p/h+1}^{t/h} h\left(\frac{\phi(mh)-\phi(mh-h)}{h}\right)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} \\
&= L(\mu,h)\sum_{m=p/h+1}^{t/h} \phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - L(\mu,h)\sum_{m=p/h+1}^{t/h} \phi(mh-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} \\
&= L(\mu,h)\sum_{m=p/h+1}^{t/h} \phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - L(\mu,h)\sum_{m=p/h}^{t/h-1} \phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh-h}{h}} \\
&= L(\mu,h)\sum_{m=p/h+1}^{t/h} \phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - L(\mu,h)\sum_{m=p/h+1}^{t/h-1} \phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-h-mh}{h}} \\
&\quad - L(\mu,h)\phi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p-h}{h}} \\
&= L(\mu,h)\nabla_h\sum_{m=p/h+1}^{t/h} h\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - L(\mu,h)\phi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p-h}{h}} \\
&= \left({}_{p}^{CFR}\nabla_{h}^{\mu}\phi\right)(t) - \frac{L(\mu,h)(1-\mu+h\mu)}{1-\mu}\phi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}}.
\end{aligned}$$

2. $$\left({}_{h}^{CFC}\nabla_{q}^{\mu}\phi\right)(t) = L(\mu,h)\sum_{m=t/h}^{q/h-1} h(-\Delta_h\phi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}$$
$$= L(\mu,h)\sum_{m=t/h}^{q/h-1}\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}} - L(\mu,h)\sum_{m=t/h}^{q/h-1}\phi(mh+h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}}$$
$$= L(\mu,h)\sum_{m=t/h}^{q/h-1}\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}} - L(\mu,h)\sum_{m=t/h+1}^{q/h}\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-h-t}{h}}$$
$$= L(\mu,h)\sum_{m=t/h}^{q/h-1}\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}} - L(\mu,h)\sum_{m=t/h+1}^{q/h-1}\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-(t+h)}{h}}$$
$$- L(\mu,h)\phi(q)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{q-t-h}{h}}$$
$$= L(\mu,h)(-\Delta_h)\sum_{m=t/h}^{q/h-1} h\phi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{mh-t}{h}} - L(\mu,h)\phi(q)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{q-t-h}{h}}$$
$$= \left({}_{h}^{CFR}\nabla_{q}^{\mu}\phi\right)(t) - \frac{L(\mu,h)(1-\mu+h\mu)}{1-\mu}\phi(q)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{q-t}{h}}. \qquad \square$$

Numerical calculations have been done in order to verify the first equation in Proposition 2.1. The data used are $p = 2.5$, $h = 0.5$, $\mu = 0.3$, $L(\mu,h) = \frac{\mu}{h} + 1 - \mu$, and $\phi(t) = t^2$. The results are illustrated in Figure 1 and Table 1.

In addition, the data are presented in Table 1.

Lemma 2.2.

1. $${}_{p}^{CF}\nabla_{h}^{-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h} = \frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1}.$$

2. $$\nabla_{h,s}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-s)/h} = \frac{\mu}{1-\mu+h\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-s)/h}.$$

3. $$\left({}_{p}^{CF}\nabla_{h}^{-\mu}\ \nabla_h\phi\right)(t) = \left(\nabla_h\ {}_{p}^{CF}\nabla_{h}^{-\mu}\phi\right)(t) - \frac{\mu}{L(\mu,h)(1-\mu+h\mu)}\phi(p).$$

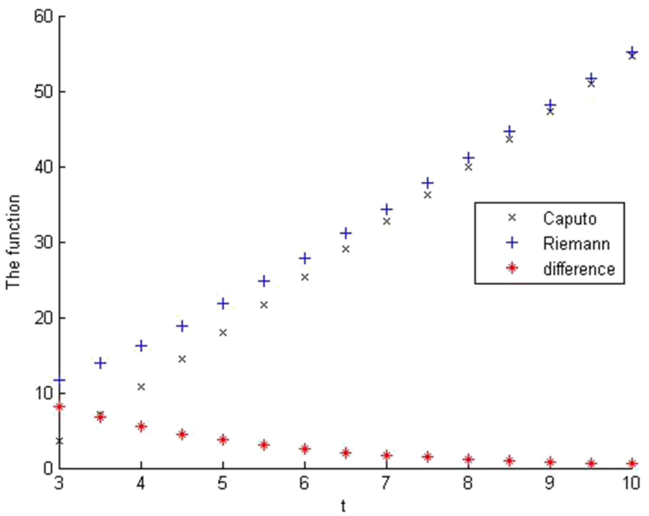

FIGURE 1 The relation between *RL* and *CF* *h*-fractional differences with exponential kernels [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 The relation between *RL* and *CF* h-fractional differences with exponential kernels

t	Caputo	Riemann	Difference
3.0	11.700000	3.575000	8.125000
3.5	13.860294	7.169118	6.691176
4.0	16.289360	10.778979	5.510381
4.5	18.939767	14.401806	4.537961
5.0	21.772455	18.035311	3.737144
5.5	24.755257	21.677609	3.077648
6.0	27.861682	25.327149	2.534534
6.5	31.069915	28.982652	2.087263
7.0	34.361989	32.643066	1.718922
7.5	37.723108	36.307525	1.415583
8.0	41.141089	39.975315	1.165774
8.5	44.605897	43.645848	0.960050
9.0	48.109268	47.318639	0.790629
9.5	51.644397	50.993291	0.651106
10.0	55.205680	54.669475	0.536205

4.
$$\nabla_h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}=\frac{-\mu}{1-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}.$$

5.
$${}_{p}^{CFR}\nabla_h^{\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}=L(\mu,h)\left(1-\frac{\mu}{1-\mu+h\mu}(t-p)\right)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h-1}.$$

6.
$$\left({}_{p}^{CFR}\nabla_h^{\mu}1\right)(t)=L(\mu,h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-p)/h-1}.$$

Proof.

1.
$$\begin{aligned}
{}_{p}^{CF}\nabla_h^{-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h} &= \frac{1-\mu}{L(\mu,h)(1-\mu+h\mu)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}+\frac{\mu}{L(\mu,h)(1-\mu+h\mu)}\sum_{m=p/h+1}^{t/h}h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{mh/h}\\
&=\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h+1}\\
&\quad+\frac{h\mu}{L(\mu,h)(1-\mu+h\mu)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1}\left(\frac{1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-p)/h}}{1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)}\right)\\
&=\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h+1}\\
&\quad+\frac{h\mu}{L(\mu,h)(1-\mu+h\mu)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1}\left(\frac{1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-p)/h}}{\frac{h\mu}{1-\mu+h\mu}}\right)\\
&=\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h+1}+\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1}\left(1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-p)/h}\right)\\
&=\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h+1}+\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1}-\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h+1}\\
&=\frac{1}{L(\mu,h)}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1}.
\end{aligned}$$

□

2.
$$\nabla_{h,s}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-s)/h} = \frac{\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-s)/h} - \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-(s-h))/h}}{h}$$
$$= \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-s)/h}\frac{1}{h}\left(\frac{h\mu}{1-\mu+h\mu}\right)$$
$$= \frac{\mu}{1-\mu+h\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-s)/h}.$$

□

3.
$$\left({}_{p}^{CF}\nabla_{h}^{-\mu}\ \nabla_h\phi\right)(t) = \frac{1-\mu}{L(\mu,h)\,(1-\mu+h\mu)}\nabla_h\phi(t) + \frac{\mu}{L(\mu,h)\,(1-\mu+h\mu)}\sum_{m=p/h+1}^{t/h} h\nabla_h\phi(mh)$$
$$= \frac{1-\mu}{L(\mu,h)\,(1-\mu+h\mu)}\nabla_h\phi(t) + \frac{\mu}{L(\mu,h)\,(1-\mu+h\mu)}\left(\sum_{m=p/h+1}^{t/h}\phi(mh) - \sum_{p/h+1}^{t/h}\phi(mh-h)\right)$$
$$= \frac{1-\mu}{L(\mu,h)\,(1-\mu+h\mu)}\nabla_h\phi(t) + \frac{\mu}{L(\mu,h)\,(1-\mu+h\mu)}\left(\sum_{m=p/h+1}^{t/h}\phi(mh) - \sum_{m=p/h}^{t/h-1}\phi(mh)\right)$$
$$= \frac{1-\mu}{L(\mu,h)\,(1-\mu+h\mu)}\nabla_h\phi(t) + \frac{h\mu}{L(\mu,h)\,(1-\mu+h\mu)}\ \nabla_h\sum_{m=p/h+1}^{t/h}\phi(mh)$$
$$- \frac{\mu}{L(\mu,h)\,(1-\mu+h\mu)}\phi(p)$$
$$= \nabla_h\left(\frac{1-\mu}{L(\mu,h)\,(1-\mu+h\mu)}\phi(t) + \frac{\mu}{L(\mu,h)\,(1-\mu+h\mu)}\sum_{m=p/h+1}^{t/h} h\phi(mh)\right)$$
$$- \frac{\mu}{L(\mu,h)\,(1-\mu+h\mu)}\phi(p)$$
$$= \left(\nabla_h\ {}_{p}^{CF}\nabla_{h}^{-\mu}\phi\right)(t) - \frac{\mu}{L(\mu,h)(1-\mu+h\mu)}\phi(p).$$

□

4.
$$\nabla_h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h} = \frac{\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h} - \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h-1}}{h}$$
$$= \frac{-\mu}{1-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}.$$

□

5.
$${}_{p}^{CFR}\nabla_{h}^{\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h} = L(\mu,h)\nabla_h\sum_{m=p/h+1}^{t/h} h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{mh/h}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-mh)/h}$$
$$= L(\mu,h)\nabla_h\sum_{m=p/h+1}^{t/h} h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}$$
$$= L(\mu,h)h\nabla_h\left(\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}\left(\frac{t-p}{h}\right)\right)$$
$$= L(\mu,h)\left(\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}\left(\frac{t-p}{h}\right) - \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h-1}\left(\frac{t-p}{h}-1\right)\right)$$
$$= L(\mu,h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h-1}\left(1+\frac{t-p}{h}\left(\frac{1-\mu}{1-\mu+h\mu}-1\right)\right)$$
$$= L(\mu,h)\left(1-\frac{\mu}{1-\mu+h\mu}(t-p)\right)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h-1}.$$

6. $$\begin{aligned}&\left({}_{p}^{CFR}\nabla_h^{\mu}1\right)(t)\\ &= L(\mu,h)\nabla_h \sum_{m=p/h+1}^{t/h} h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\ &= L(\mu,h)\frac{1}{h}\left(\sum_{m=p/h+1}^{t/h} h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - \sum_{m=p/h+1}^{t/h-1} h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-h-mh}{h}}\right)\\ &= L(\mu,h)\left(\sum_{m=p/h+1}^{t/h} \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - \sum_{m=p/h+2}^{t/h} \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right)\\ &= L(\mu,h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{(t-p)/h-1}.\end{aligned}$$

□

3 | THE MONOTONICITY RESULTS

In the following, we define the $h\mathbb{Z}$ versions of the monotonic functions that appear in Atıcı and Uyanik.[36]

Definition 3.1. Let $\psi : \mathbb{N}_{p,h} \to \mathbb{R}$ such that $\psi(p) \geq 0$. Then $\psi(t)$ is called μ-increasing function on $\mathbb{N}_{p,h}$ if $\psi(t+h) \geq \mu\, \psi(t)\ \forall t \in \mathbb{N}_{p,h}$.

If $\psi(t)$ is increasing on $\mathbb{N}_{p,h}$ ($\psi(t+h) \geq \psi(t)\ \forall\ t \in \mathbb{N}_{p,h}$), then $\psi(t)$ is μ-increasing on $\mathbb{N}_{p,h}$. For $\mu = 1$, then the increasing and the μ-increasing are the same.

Definition 3.2. Let $\psi : \mathbb{N}_{p,h} \to \mathbb{R}$ such that $\psi(p) \leq 0$. Then $\psi(t)$ is called μ-decreasing on $\mathbb{N}_{p,h}$ if $\psi(t+h) \leq \mu\psi(t)\ \forall t \in \mathbb{N}_{p,h}$.

If $\psi(t)$ is decreasing on $\mathbb{N}_{p,h}$ ($\psi(t+h) \leq \psi(t) \forall t \in \mathbb{N}_{p,h}$), then $\psi(t)$ is μ-decreasing on $\mathbb{N}_{p,h}$. For $\mu = 1$, the decreasing and the μ-decreasing are the same.

Theorem 3.1. *Let $\psi : \mathbb{N}_{p-h,h} \to \mathbb{R}$ such that $\left({}_{p-h}^{CFR}\nabla_h^{\mu}\psi\right)(t) \geq 0$ for $0 < \mu < 1$, and $t \in \mathbb{N}_{p,h}$. Then $\psi(t)$ is $\left(\frac{h\mu}{1-\mu+h\mu}\right)$-increasing.*

Proof. In our proof, we will show that when *CFR* is nonnegative, then μ in Definition 3.1 will be replaced by $\frac{h\mu}{1-\mu+h\mu}$.

First, we recall that

$$\left({}_{p-h}^{CFR}\nabla_h^{\mu}\psi\right)(t) = L(\mu,h)\nabla_h \sum_{m=p/h}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}.$$

Let

$$J(t) = \sum_{m=p/h}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}.$$

Then $\nabla_h J(t) \geq 0$. To show that

$$\begin{aligned}\nabla_h J(t) &= \frac{J(t)-J(t-h)}{h}\\ &= \sum_{m=p/h}^{t/h} \psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - \sum_{m=p/h}^{t/h-1} \psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-h-mh}{h}}\\ &= \psi(t) + \sum_{m=p/h}^{t/h-1} \psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} - \sum_{m=p/h}^{t/h-1} \psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-h-mh}{h}}\\ &= \psi(t) + \sum_{m=p/h}^{t/h-1} \psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\left(1-\frac{1-\mu+h\mu}{1-\mu}\right)\end{aligned}$$

$$= \psi(t) - \frac{h\mu}{1-\mu} \sum_{m=p/h}^{t/h-1} \psi(mh) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{t-mh}{h}} \geq 0.$$

Therefore,

$$\nabla_h J(t) = \psi(t) - \frac{h\mu}{1-\mu} \sum_{m=p/h}^{t/h-1} \psi(mh) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{t-mh}{h}} \geq 0. \tag{1}$$

When $t = p$, we have

$$\nabla_h J(p) = \psi(p) \geq 0, \text{ and hence, } \psi(p) \geq 0.$$

When $t = p + h$, we get

$$\begin{aligned}
\nabla_h J(p+h) &= \psi(p+h) - \frac{h\mu}{1-\mu} \sum_{m=p/h}^{p/h} \psi(mh) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{p+h-mh}{h}} \\
&= \psi(p+h) - \frac{h\mu}{1-\mu}\, \psi(p) \left(\frac{1-\mu}{1-\mu+h\mu} \right) \\
&= \psi(p+h) - \frac{h\mu}{1-\mu+h\mu}\, \psi(p) \geq 0;
\end{aligned}$$

hence, $\psi(p+h) \geq \frac{h\mu}{1-\mu+h\mu}\, \psi(p)$.

Now, we use induction to show that

$$\psi(t+h) \geq \frac{h\mu}{1-\mu+h\mu}\, \psi(t), \ \forall\, t \in \mathbb{N}_{p,h}.$$

Let $\psi(m+h) \geq \frac{h\mu}{1-\mu+h\mu}\, \psi(m) \geq 0,\ \forall\, m < t$ such that $m, t \in \mathbb{N}_{p,h}$.

We should prove that $\psi(t+h) \geq \left(\frac{h\mu}{1-\mu+h\mu} \right) \psi(t)$.

Now, in (1), replacing t by $t+h$ gives

$$\psi(t+h) - \frac{h\mu}{1-\mu} \sum_{m=p/h}^{t/h} \psi(mh) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{t+h-mh}{h}} \geq 0,$$

$$\psi(t+h) - \frac{h\mu}{1-\mu} \left(\psi(p) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{t+h-p}{h}} + \ldots + \psi(t) \left(\frac{1-\mu}{1-\mu+h\mu} \right) \right) \geq 0,$$

$$\begin{aligned}
\psi(t+h) &\geq \frac{h\mu}{1-\mu} \left(\psi(p) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{t+h-p}{h}} + \ldots + \psi(t) \left(\frac{1-\mu}{1-\mu+h\mu} \right) \right) \\
&\geq \frac{h\mu}{1-\mu}\, \psi(t) \left(\frac{1-\mu}{1-\mu+h\mu} \right), \\
&= \left(\frac{h\mu}{1-\mu+h\mu} \right) \psi(t).
\end{aligned}$$

Hence, $\psi(t+h) \geq \left(\frac{h\mu}{1-\mu+h\mu} \right) \psi(t)$. □

By Proposition 2.1 and Theorem 3.1, the following *CF h*-fractional result is concluded.

Corollary 3.2. *Let* $\psi : \mathbb{N}_{p-h,h} \to \mathbb{R}$, *and let*

$$\left({}^{CFC}_{p-h}\nabla^{\mu}_{h}\, \psi \right)(t) \geq \frac{-L(\mu,h)\,(1-\mu+h\mu)}{1-\mu} \psi(p-h) \left(\frac{1-\mu}{1-\mu+h\mu} \right)^{\frac{t-p+h}{h}}, \ t \in \mathbb{N}_{p,h}$$

for $0 < \mu < 1$, *and* $0 < h \leq 1$, *then* $\psi(t)$ *is* $\left(\frac{h\mu}{1-\mu+h\mu} \right)$*-increasing.*

Proof.

$$\left({}^{CFC}_{p-h}\nabla^{\mu}_{h}\psi\right)(t)=\left({}^{CFR}_{p-h}\nabla^{\mu}_{h}\psi\right)(t)-\frac{L(\mu,h)(1-\mu+h\mu)}{1-\mu}\psi(p-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p+h}{h}}.$$

Now,

$$({}^{CFC}_{p-h}\nabla^{\mu}_{h}\ \psi\ )(t)\ \geq\frac{-L(\mu,h)\ (1-\mu+h\mu)}{1-\mu}\psi(p-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p+h}{h}};$$

hence,

$$({}^{CFC}_{p-h}\nabla^{\mu}_{h}\ \psi\ )(t)+\frac{L(\mu,h)\ (1-\mu+h\mu)}{1-\mu}\psi(p-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p+h}{h}}\geq 0,$$

which means that

$$\left({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \psi\ \right)(t)\ \geq 0;$$

hence, by Theorem 3.1, $\psi(t)$ *is* $\left(\frac{h\mu}{1-\mu+h\mu}\right)$-increasing. □

Theorem 3.3. *Let* $\psi\,:\,\mathbb{N}_{p-h,h}\to\mathbb{R}$ *such that* $\psi(p)\geq 0$*, and assume* $0<\mu<1$*. If* ψ *is increasing on* $\mathbb{N}_{p,h}$*, then*

$$({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \psi\ )(t)\ \geq 0\ ,\forall t\in\mathbb{N}_{p,h}.$$

Proof. Since we have

$$({}^{CFR}_{p-h}\nabla^{\mu}_{h}\psi)(t)=L(\mu,h)\nabla_{h}J(t),\ \ t\in\mathbb{N}_{p,h},$$

it is enough to show that

$$J(t)=\sum_{m=p/h}^{t/h}h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}$$

is an increasing function on $\mathbb{N}_{p,h}$. In reference to the proof of Theorem 3.1, when $t=p$, $\nabla_{h}J(p)=\psi(p)\geq 0$.

Now, we will show that $\nabla_{h}J(t)\geq 0$.

Since, from assumption, $\psi(t)$ is increasing, then $\psi(t)\geq\psi(t-h)\geq\psi(p)\geq 0,\ \forall\ t\in\mathbb{N}_{p,h}$.

From (1), we recall that

$$\nabla_{h}J(t)=\psi(t)-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-1}\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}.$$

Then we have

$$\begin{aligned}
\nabla_{h}J(t)&=\psi(t)-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-1}\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&=\psi(t)-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-2}\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}-\frac{h\mu}{1-\mu}\psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)\\
&=\psi(t)-\frac{h\mu}{1-\mu+h\mu}\psi(t-h)-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-2}\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&\quad+\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-2}\psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-2}\psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&=\psi(t)-\frac{h\mu}{1-\mu+h\mu}\psi(t-h)-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-2}\psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&\quad+\frac{h\mu}{1-\mu}\left(\sum_{m=p/h}^{t/h-2}(\psi(t-h)-\psi(mh))\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right)\\
&\geq\psi(t)-\frac{h\mu}{1-\mu+h\mu}\psi(t-h)-\frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-2}\psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}
\end{aligned}$$

$$
\begin{aligned}
&= \psi(t) - \frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-1} \psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} \\
&= \psi(t) - \psi(t-h) + \psi(t-h) - \frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-1} \psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} \\
&\geq \psi(t-h) - \frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-1} \psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}} \\
&= \psi(t-h)\left(1 - \frac{h\mu}{1-\mu}\sum_{m=p/h}^{t/h-1}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right) \\
&= \psi(t-h)\left(1 - \frac{h\mu}{1-\mu}\left(\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}} + \ldots + \left(\frac{1-\mu}{1-\mu+h\mu}\right)\right)\right) \\
&= \psi(t-h)\left(1 - \frac{h\mu}{1-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)\left(\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p-h}{h}} + \ldots + 1\right)\right) \\
&= \psi(t-h)\left(1 - \left(\frac{h\mu}{1-\mu+h\mu}\right)\left(\frac{1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}}}{1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)}\right)\right) \\
&= \psi(t-h)\left(1 - \left(\frac{h\mu}{1-\mu+h\mu}\right)\left(\frac{1-\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}}}{\left(\frac{h\mu}{1-\mu+h\mu}\right)}\right)\right) \\
&= \psi(t-h)\left(1 - \left(1 - \left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}}\right)\right) \\
&= \psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}} \geq 0, \textit{which completes the proof}.
\end{aligned}
$$

□

Theorem 3.4. *Let $\psi : \mathbb{N}_{p-h,h} \to \mathbb{R}$ such that $\psi(p) > 0$ is strictly increasing on $\mathbb{N}_{p,h}$, where $0 < \mu < 1$. Then*

$$
({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \psi\ )(t) > 0,\ \ t \in \mathbb{N}_{p,h}.
$$

Proof. Since we have

$$
({}^{CFR}_{p-h}\nabla^{\mu}_{h}\psi)(t) = L(\mu, h)\nabla_h J(t),\ \ t \in \mathbb{N}_{p,h},
$$

it is enough to show that

$$
J(t) = \sum_{m=p/h}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}
$$

is strictly increasing on $\mathbb{N}_{p,h}$. In reference to the proof of Theorem 3.1, when $t = p$, we have $\nabla_h J(p) = \psi(p) > 0$.

Now, we will show that $\nabla_h J(t) > 0$.

Since $\psi(t)$ is strictly increasing, it follows that $\psi(t) > \psi(t-h) > \psi(p) > 0,\ \forall\ t \in \mathbb{N}_{p,h}$.

In reference to the proof of Theorem 3.3, we have $\nabla_h J(t) \geq \psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}}$; hence,

$$
\begin{aligned}
\nabla_h J(t) &\geq \psi(t-h)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}} \\
&> \psi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}} \\
&> 0, \textit{which completes the proof}.
\end{aligned}
$$

□

Theorem 3.5. *Let* $\psi : \mathbb{N}_{p-h,h} \to \mathbb{R}$ *such that* $\left({}^{CFR}_{p-h}\nabla^{\mu}_{h}\, \psi\right)(t) \leq 0$ *for* $0 < \mu < 1$, *and* $t \in \mathbb{N}_{p,h}$. *Then* $\psi(t)$ *is* $\left(\frac{h\mu}{1-\mu+h\mu}\right)$*-decreasing.*

Proof. Let $\varphi : \mathbb{N}_{p-h,h} \to \mathbb{R}$ such that $\varphi(t) = -\psi(t)$; hence,

$$({}^{CFR}_{p-h}\nabla^{\mu}_{h}\, \varphi)(t) = ({}^{CFR}_{p-h}\nabla^{\mu}_{h}\, (-\psi))(t) = -({}^{CFR}_{p-h}\nabla^{\mu}_{h}\psi)(t) \geq 0.$$

Applying Theorem 3.1 to $\varphi(t)$ finishes the proof. □

Theorem 3.6. *Let* $\psi : \mathbb{N}_{p-h,h} \to \mathbb{R}$ *such that* $\psi(p) \leq 0$ *be decreasing on* $\mathbb{N}_{p,h}$. *Then for* $0 < \mu < 1$,

$$({}^{CFR}_{p-h}\nabla^{\mu}_{h}\, \psi\,)(t) \leq 0,\ \forall\, t \in \mathbb{N}_{p,h}.$$

Proof. Apply Theorem 3.3 to $\varphi(t) = -\psi(t)$. □

4 | APPLICATION: MVT

In this section, the h-fractional difference version of the MVT will be presented. For this purpose, we need the following theorem.

Theorem 4.1. *For* $0 < \mu < 1$, *then*

$$\left({}^{CF}_{p}\nabla^{-\mu}_{h}\, {}^{CFR}_{p-h}\nabla^{\mu}_{h}\psi\right)(t) = \psi(t) - \frac{h\mu}{1-\mu+h\mu}\, \psi(p),\ \ t \in \mathbb{N}_{p,h}.$$

Proof. By definition and Lemma 2.2 we have,

$$\begin{aligned}
\left({}^{CF}_{p}\nabla^{-\mu}_{h}\, {}^{CFR}_{p-h}\nabla^{\mu}_{h}\psi\right)(t) &= \left({}^{CF}_{p}\nabla^{-\mu}_{h}\, L(\mu,h)\nabla_h \sum_{m=p/h}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right)\\
&= L(\mu,h){}^{CF}_{p}\nabla^{-\mu}_{h}\, \nabla_h\left(h\psi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-p}{h}} + \sum_{m=p/h+1}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right)\\
&= L(\mu,h){}^{CF}_{p}\nabla^{-\mu}_{h}\, \nabla_h\left(h\psi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t}{h}}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{-p}{h}}\right)\\
&\quad + L(\mu,h)\, {}^{CF}_{p}\nabla^{-\mu}_{h}\, \nabla_h\left(\sum_{m=p/h+1}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\right)\\
&= L(\mu,h)\; h\psi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{-p/h}_{p} CF\nabla^{-\mu}_{h}\, \nabla_h\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{t/h}\\
&\quad +{}^{CF}_{p}\, \nabla^{-\mu}_{h}\, L(\mu,h)\nabla_h \sum_{m=p/h+1}^{t/h} h\psi(mh)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t-mh}{h}}\\
&= L(\mu,h)\; h\psi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{-p/h}_{p} CF\nabla^{-\mu}_{h}\, \frac{-\mu}{1-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{\frac{t}{h}} +{}^{CF}_{p}\, \nabla^{-\mu}_{h}\, {}^{CFR}_{p}\nabla^{\mu}_{h}\, \psi(t)\\
&= L(\mu,h)\; h\psi(p)\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{-p/h} \frac{1}{L(\mu,h)}\, \frac{-\mu}{1-\mu}\left(\frac{1-\mu}{1-\mu+h\mu}\right)^{p/h+1} + \psi(t)\\
&= \psi(t) - \frac{h\mu}{1-\mu+h\mu}\, \psi(p).
\end{aligned}$$

□

Theorem 4.2. *(The h-fractional difference MVT)*

Let ϕ and φ be defined on $\mathbb{N}_{p,h} \cap {}_{q,h}\mathbb{N}$, where $q = p + nh$ for some $n \in \mathbb{N}$. Assume that φ is strictly increasing, $\varphi(p) > 0$, and $0 < \mu < 1$. Then, there exist $w_1,\ w_2 \in \mathbb{N}_{p,h} \cap {}_{q,h}\mathbb{N}$ such that

$$\frac{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\phi)(w_1)}{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\varphi)(w_1)} \le \frac{\phi(q) - \frac{h\mu}{1-\mu+h\mu}\phi(p)}{\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\varphi(p)} \le \frac{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\phi)(w_2)}{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\varphi)(w_2)}. \tag{2}$$

Proof. First, we need to show that $\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\varphi(p) > 0$. Since φ is strictly increasing, then by Theorem 3.4,

$$({}^{CFR}_{p-h}\nabla^{\mu}_{h}\varphi)(t) > 0\ \forall\ t \in \mathbb{N}_{p,h} \cap {}_{q,h}\mathbb{N}.$$

Applying the fractional sum operator on both sides of the inequality, we get

$${}^{CF}_{p}\nabla^{-\mu}_{h}({}^{CFR}_{p-h}\nabla^{\mu}_{h}\varphi)(t) > {}^{CF}_{p}\nabla^{-\mu}_{h}(0)\ \forall\ t \in \mathbb{N}_{p,h} \cap {}_{q,h}\mathbb{N}.$$

Hence, we have

$$\varphi(t) - \frac{h\mu}{1-\mu+h\mu}\varphi(p) > 0\ \forall\ t \in \mathbb{N}_{p,h} \cap {}_{q,h}\mathbb{N}.$$

For $t = q$, we get

$$\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\ \varphi(p) > 0.$$

The proof by contradiction will be used now. Assume (2) is not true, then either

$$\frac{\phi(q) - \frac{h\mu}{1-\mu+h\mu}\ \phi(p)}{\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\ \varphi(p)} < \frac{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \phi\ )(t)}{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \varphi\ )(t)}\ ,\ \forall\ t \in \mathbb{N}_{p,h}\ \cap {}_{q,h}\mathbb{N}, \tag{3}$$

or

$$\frac{\phi(q) - \frac{h\mu}{1-\mu+h\mu}\ \phi(p)}{\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\ \varphi(p)} > \frac{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \phi\ )(t)}{({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \varphi\ )(t)}\ ,\ \forall\ t \in \mathbb{N}_{p,h}\ \cap {}_{q,h}\mathbb{N}. \tag{4}$$

Since φ is strictly increasing, then by Theorem 3.4,

$$({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \varphi\ )(t) > 0\ \forall\ t \in \mathbb{N}_{p,h}\ \cap {}_{q,h}\mathbb{N}.$$

Hence, (3) becomes

$$\frac{\phi(q) - \frac{h\mu}{1-\mu+h\mu}\ \phi(p)}{\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\ \varphi(p)}\ ({}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \varphi\ )(t) < (\ {}^{CFR}_{p-h}\nabla^{\mu}_{h}\ \phi\ )(t),\ \forall\ t \in \mathbb{N}_{p,h} \cap {}_{q,h}\mathbb{N}.$$

Applying the fractional sum operator at $t = q$, we see that

$$\frac{\phi(q) - \frac{h\mu}{1-\mu+h\mu}\ \phi(p)}{\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\ \varphi(p)} \left(\varphi(q) - \frac{h\mu}{1-\mu+h\mu}\ \varphi(p) \right) < \left(\phi(q) - \frac{h\mu}{1-\mu+h\mu}\ \phi(p) \right),$$

and hence, $\phi(q) < \phi(q)$, which is a contradiction. Similar way can be used for (4). □

Notice that if we let $h = 1$ in Theorem 4.2, the results in Abdeljawad and Baleanu[46] are easily obtained.

5 | CONCLUSIONS

The conclusions of this article are summarized as follows:

1. The h-nabla Riemann-Liouville and CF fractional differences of order $0 < \mu < 1$ with discrete exponential kernels on $h\mathbb{Z}$ is formulated.
2. The correspondent left discrete fractional integral of the h-nabla Riemann-Liuville difference operator is defined.

3. The relation between h-nabla Riemann-Liouville and CF fractional differences is detected.
4. If $({}_{p-h}^{CFR}\nabla_h^{\mu}\psi)(t) > 0$, then $\psi(t)$ is $\frac{h\mu}{1-\mu+h\mu}$-increasing.
5. If $\psi(t)$ is increasing on $\mathbb{N}_{p,h}$ and $\psi(p) > 0$, then $({}_{p-h}^{CFR}\nabla_h^{\mu}\psi)(t) > 0$.
6. A monotonicity results for the h-nabla CF fractional difference operator is proved.
7. The monotonicity factor, which is $\frac{h\mu}{1-\mu+h\mu}$, affected by the discretization step h.
8. The achievements of this paper are generalization of those obtained in Abdeljawad and Baleanu.[46]
9. As an application, an h-fractional difference MVT with exponential kernel on $h\mathbb{Z}$ has been proved.

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