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# On solution of generalized proportional fractional integral via a new fixed point theorem

Anupam Das[1], Iyad Suwan[2], Bhuban Chandra Deuri[3] and Thabet Abdeljawad[4,5,6]*

*Correspondence:
tabdeljawad@psu.edu.sa
[4]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
[5]Department of Medical Research, China Medical University, Taichung, 40402, Taiwan
Full list of author information is available at the end of the article

**Abstract**

The aim of this paper is the solvability of generalized proportional fractional(GPF) integral equation at Banach space $\mathbb{E}$. Herein, we have established a new fixed point theorem which is then applied to the GPF integral equation in order to establish the existence of solution on the Banach space. At last, we have illustrated a genuine example that verified our theorem and gave a strong support to prove it.

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**Keywords:** Measure of noncompactness (MNC); Fixed point theorem; Generalized proportional fractional integral

## 1 Introduction

In 1930 Kuratowski [1] introduced the notion of a measure of noncompactness. In functional analysis, this idea is particularly important in metric fixed point theory and operator equation theory in Banach spaces. The theory of infinite systems of fractional integral equations (FIEs) plays a pivotal role in different fields, which includes various implications in the scaling system theory, the theory of algorithms, etc. There are many real life problems which can be formulated by infinite systems of integral equations with fractional order in a very effective manner.

In recent times, the fixed point theory (FPT) has applications in various scientific fields. Also, FPT can be applied seeking solutions for FIE.

Different real life situations which are formulated via FIEs can be studied using FPT and measure of noncompactness (MNC) (see [2–24]).

Let a real Banach space $(\mathbb{E}, \|.\|)$ and $B(x,r) = \{y \in \mathbb{E} : \|y - x\| \leq r\}$. If $\Omega (\neq \phi) \subseteq \mathbb{E}$. Also, $\bar{\Omega}$ and Conv$\Omega$ represent the closure and convex closure of $\Omega$. Moreover, let

a. $\mathfrak{M}_{\mathbb{E}}$ = collection of all nonempty and bounded subsets of $\mathbb{E}$,
b. $\mathfrak{N}_{\mathbb{E}}$ = collection of all relatively compact sets,
c. $\mathbb{R}$ = collection of all real numbers,
 and
d. $\mathbb{R}_+$ = collection of all nonnegative real numbers.

The following definition of an MNC is given in [25].

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**Definition 1.1** A function $\Pi : \mathfrak{M}_{\mathbb{E}} \to [0,\infty)$ is called an MNC in $\mathbb{E}$ if it satisfies the following conditions:

(i) The family $\ker \Pi = \{\Omega \in \mathfrak{M}_{\mathbb{E}} : \Pi(\Omega) = 0\}$ is nonempty and $\ker \Pi \subset \mathfrak{N}_{\mathbb{E}}$.
(ii) $\Omega \subseteq \Omega_1 \implies \Pi(\Omega) \leq \Pi(\Omega_1)$.
(iii) $\Pi(\bar{\Omega}) = \Pi(\Omega)$.
(iv) $\Pi(\operatorname{Conv} \Omega) = \Pi(\Omega)$.
(v) $\Pi(\rho\Omega + (1-\rho)P) \leq \rho\Pi(\Omega) + (1-\rho)\Pi(P)$ for $\rho \in [0,1]$.
(vi) If $\Omega_n \in \mathfrak{M}_{\mathbb{E}}$, $\Omega_n = \bar{\Omega}_n$, $\Omega_{n+1} \subset \Omega_n$ for $n = 1,2,3,\ldots$ and $\lim_{n\to\infty} \Pi(\Omega_n) = 0$ then $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n \neq \phi$.

The $\ker \Pi$ family is *kernel of measure* $\Pi$. Note that the intersection set $\Omega_\infty$ from (vi) is a member of the family $\ker \Pi$. In fact, since $\Pi(\Omega_\infty) \leq \Pi(\Omega_n)$ for any $n$, we conclude that $\Pi(\Omega_\infty) = 0$. This gives $\Omega_\infty \in \ker \Pi$.

The fixed point principle and theorem play a key role in the theory of fixed point.

**Theorem 1.2** (Shauder [26]) *Let $\mathbb{V}$ be a nonempty, closed, and convex subset of a Banach space $\mathbb{E}$. Then every compact, continuous map $\Upsilon : \mathbb{V} \to \mathbb{V}$ has at least one fixed point($\mathbb{FP}$) in $\mathbb{V}$.*

**Theorem 1.3** (Darbo [27]) *Let $V$ be a nonempty, bounded, closed, and convex(NBCC) subset of a Banach space $\mathbb{E}$. Let $\Upsilon : V \to V$ be a continuous mapping. Assume that there is a constant $p \in [0,1)$ such that*

$$\eta(\Upsilon\Omega) \leq p\eta(\Omega), \quad \Omega \subseteq V,$$

*where $\eta$ is an arbitrary MNC. Then $\Upsilon$ has an $\mathbb{FP}$ in $V$.*

We introduced the following generalization of the Banach contraction principle, in which we get a variety of contractive inequalities by substituting different functions $g$.

**Theorem 1.4** *Let $(\gamma, d)$ be a complete metric space. Also, let $J : \gamma \mapsto \gamma$ be a continuous self-mapping. Suppose that there exists a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t\to o^+} g(t) = 0$, $g(0) = 0$, and*

$$d(Jx, Jy) \leq g\bigl(d(x,y)\bigr) - g\bigl(d(Jx,Jy)\bigr); \quad \forall x,y \in \gamma.$$

*Then $J$ has a unique $\mathbb{FP}$.*

**Definition 1.5** ([28]) Let $\mathbb{F}$ be the class of all functions $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying:

(1) $\max\{m_1, m_2\} \leq F(m_1, m_2)$ for $m_1, m_2 \geq 0$;
(2) $F$ is continuous;
(3) $F(m_1 + m_2, n_1 + n_2) \leq F(m_1, n_1) + F(m_2, n_2)$;

e.g. $F(m_1, m_2) = m_1 + m_2$.

## 2 Main result

**Theorem 2.1** *Let $\mathbb{V}$ be an NBCC subset of a Banach space $\mathbb{E}$, and let $\Upsilon : \mathbb{V} \to \mathbb{V}$ be a continuous operator such that*

$$F\bigl[\Pi(\Upsilon X), \phi\bigl(\Pi(\Upsilon X)\bigr)\bigr] \leq \Delta\bigl[F\bigl\{\Pi(X), \phi\bigl(\Pi(X)\bigr)\bigr\}\bigr] - \Delta\bigl[F\bigl\{\Pi(\Upsilon X), \phi\bigl(\Pi(\Upsilon X)\bigr)\bigr\}\bigr] \quad (2.1)$$

*for all* $X \subseteq \mathbb{V}$, *where* $\Delta, \phi : \mathbb{R}_+ \to \mathbb{R}_+$ *are nondecreasing continuous functions and* $\Pi$ *is an arbitrary MNC. Then* $\Upsilon$ *has at least one* $\mathbb{FP}$ *in* $\mathbb{V}$.

*Proof* Let $\mathbb{V}_0 = \mathbb{V}$ and construct a sequence $\{\mathbb{V}_n\}$ such that $\mathbb{V}_{n+1} = \operatorname{Conv}(\Upsilon \mathbb{V}_n)$ for all $n \in \mathbb{N}$. If there exists a positive integer $N_0 \in \mathbb{N}$ such that $\Pi(\mathbb{V}_{N_0}) = 0$, so $\mathbb{V}_{N_0}$ is relatively compact. And by Theorem 2.1, we give that $\Upsilon$ has an $\mathbb{FP}$.

If possible, assume that $\Pi(\mathbb{V}_n) > 0$ for all $n$. Also, we have

$$\mathbb{V}_1 \supseteq \mathbb{V}_2 \supseteq \cdots \supseteq \mathbb{V}_n \supseteq \mathbb{V}_{n+1} \supseteq \ldots$$

Since the sequence $\{\Pi(\mathbb{V}_n)\}$ is decreasing. So, $\phi(\Pi(\mathbb{V}_n))$ is decreasing.

Hence, the sequence $F[\Pi(\mathbb{V}_n), \phi(\Pi(\mathbb{V}_n))]$ is decreasing.

Since $\lim_{n\to\infty} F[\Pi(\mathbb{V}_n), \phi(\Pi(\mathbb{V}_n))] = L$.

By using equation (2.1), we have

$$\begin{aligned}
0 &\le F\bigl[\Pi(\mathbb{V}_{n+1}), \phi\bigl(\Pi(\mathbb{V}_{n+1})\bigr)\bigr] \\
&= F\bigl[\Pi(\Upsilon \mathbb{V}_n), \phi\bigl(\Pi(\Upsilon \mathbb{V}_n)\bigr)\bigr] \\
&\le \Delta\bigl[F\bigl\{\Pi(\mathbb{V}_n), \phi\bigl(\Pi(\mathbb{V}_n)\bigr)\bigr\}\bigr] - \Delta\bigl[F\bigl\{\Pi(\Upsilon \mathbb{V}_n), \phi\bigl(\Pi(\Upsilon \mathbb{V}_n)\bigr)\bigr\}\bigr] \\
&= \Delta\bigl[F\bigl\{\Pi(\mathbb{V}_n), \phi\bigl(\Pi(\mathbb{V}_n)\bigr)\bigr\}\bigr] - \Delta\bigl[F\bigl\{\Pi(\mathbb{V}_{n+1}), \phi\bigl(\Pi(\mathbb{V}_{n+1})\bigr)\bigr\}\bigr].
\end{aligned}$$

As $n \to \infty$, we get

$$0 \le L \le \Delta(L) - \Delta(L) = 0,$$

that is, $L = 0$.

Therefore, $\lim_{n\to\infty} \Pi(\mathbb{V}_n) = 0$. According to axiom (vi) of Definition 1.1, we conclude that $\mathbb{V}_\infty = \bigcap_{n=1}^{\infty} \mathbb{V}_n$ is an NBCC set, invariant under the mapping $\Upsilon$ and belongs to $\ker \Pi$. By Theorem 1.2, we have $\Upsilon$ has an $\mathbb{FP}$. □

**Theorem 2.2** *Let* $V$ *be an NBCC subset of a Banach space* $\mathbb{E}$, *and let* $\Upsilon : \mathbb{V} \to \mathbb{V}$ *be a continuous operator such that*

$$2F\bigl[\Pi(\Upsilon X), \phi\bigl(\Pi(\Upsilon X)\bigr)\bigr] \le F\bigl\{\Pi(X), \phi\bigl(\Pi(X)\bigr)\bigr\} \tag{2.2}$$

*for all* $X \subseteq \mathbb{V}$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a nondecreasing continuous function and* $\Pi$ *is an arbitrary MNC. Then* $\Upsilon$ *has at least one* $\mathbb{FP}$ *in* $\mathbb{V}$.

*Proof* Taking $\Delta(t) = t; t \ge 0$ in Theorem 2.1. □

The statement in the next corollary is a result of Theorem 2.1.

**Corollary 2.3** *Let* $V$ *be an NBCC subset of a Banach space* $\mathbb{E}$, *and let* $\Upsilon : \mathbb{V} \to \mathbb{V}$ *be a continuous operator such that*

$$2\Pi(\Upsilon X) + 2\phi\bigl(\Pi(\Upsilon X)\bigr) \le \Pi(X) + \phi\bigl(\Pi(X)\bigr) \tag{2.3}$$

*for all $X \subseteq \mathbb{V}$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function and $\Pi$ is an arbitrary MNC. Then $\Upsilon$ has at least one $\mathbb{FP}$ in $\mathbb{V}$.*

*Proof* Taking $F(m_1, m_2) = m_1 + m_2$ in Theorem 2.2. So, we get the required result. □

**Corollary 2.4** *Let $V$ be an NBCC subset of a Banach space $\mathbb{E}$, and let $\Upsilon : \mathbb{V} \to \mathbb{V}$ be a continuous operator such that*

$$\Pi(\Upsilon X) \leq p\Pi(X) \tag{2.4}$$

*for all $X \subseteq \mathbb{V}$, where $p = \frac{1}{2} \in (0, 1]$ and $\Pi$ is an arbitrary MNC. Then $\Upsilon$ has at least one $\mathbb{FP}$ in $\mathbb{V}$.*

*Proof* Taking $\phi(t) = 0$ in Corollary 2.3, we get the required result. □

**Theorem 2.5** *Let $V$ be an NBCC subset of a Banach space $\mathbb{E}$, and let $\Upsilon : \mathbb{V} \to \mathbb{V}$ be a continuous operator such that*

$$F\bigl[\Pi(\Upsilon X), \phi\bigl(\Pi(\Upsilon X)\bigr)\bigr] \leq \lambda F\bigl\{\Pi(X), \phi\bigl(\Pi(X)\bigr)\bigr\} \tag{2.5}$$

*for all $X \subseteq \mathbb{V}$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing continuous function and $\Pi$ is an arbitrary MNC, where $\lambda = \frac{k}{k+1} \in [0, 1)$. Then $\Upsilon$ has at least one $\mathbb{FP}$ in $\mathbb{V}$.*

*Proof* Taking $\Delta(t) = kt$ where $t \geq 0$, $k \geq 0$ in Theorem 2.1. □

**Corollary 2.6** *Let $V$ be an NBCC subset of a Banach space $\mathbb{E}$, and let $\Upsilon : \mathbb{V} \to \mathbb{V}$ be a continuous operator such that*

$$\Pi(\Upsilon X) \leq \lambda\Pi(X) \tag{2.6}$$

*for all $X \subseteq \mathbb{V}$, where $\lambda \in (0, 1]$ and $\Pi$ is an arbitrary MNC. Then $\Upsilon$ has at least one $\mathbb{FP}$ in $\mathbb{V}$.*

*Proof* Taking $F(m_1, m_2) = m_1 + m_2$ and $\phi(t) \equiv 0$ in Theorem 2.5. So, we get the result which is Darbo's fixed point theorem. □

**Definition 2.7** ([29]) An element $(\mathcal{A}, \mathcal{B}) \in \mathcal{X} \times \mathcal{X}$ is called a coupled fixed point of a mapping $\mathcal{T} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ if $\mathcal{T}(\mathcal{A}, \mathcal{B}) = \mathcal{A}$ and $\mathcal{T}(\mathcal{B}, \mathcal{A}) = \mathcal{B}$.

**Theorem 2.8** ([25]) *Suppose that $\Pi_1, \Pi_2, \ldots, \Pi_n$ is the MNC in $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$ respectively. Moreover, suppose that the function $\mathcal{X} : \mathbb{R}_+^n \to \mathbb{R}_+$ is convex and $\mathcal{F}(y_1, y_2, \ldots, y_n) = 0 \Leftrightarrow y_t = 0$ for $t = 1, 2, \ldots, n$, then $\Pi(\mathcal{X}) = \mathcal{F}(\Pi_1(\mathcal{X}_1), \Pi_2(\mathcal{X}_2), \ldots, \Pi_n(\mathcal{X}_n))$ defines an MNC in $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$, where $\mathcal{X}_t$ denotes the natural projection of $\mathcal{X}$ into $\mathbb{E}_t$ for $t = 1, 2, \ldots, n$.*

*Example* 2.9 ([25]) Let $\Pi$ be an MNC on $\mathbb{E}$. Define $\mathcal{F}(\mathcal{A}, \mathcal{B}) = \mathcal{A} + \mathcal{B}$; $\mathcal{A}, \mathcal{B} \in \mathbb{R}_+$. Then $\mathcal{F}$ has all the properties mentioned in Theorem 2.8. Hence, $\Pi^{cf}(\mathcal{X}) = \Pi_1(\mathcal{X}_1) + \Pi_2(\mathcal{X}_2)$ is an MNC in the space $\mathbb{E} \times \mathbb{E}$, where $\mathcal{X}_t$, $t = 1, 2$, denotes the natural projections of $\mathcal{X}$.

**Definition 2.10** ([30]) Suppose that $G$ is the set of all functions $\mu : \mathbb{R}_+ \to \mathbb{R}$ satisfying the following conditions:

(1) $\mu$ is a continuous strictly increasing function.

(2) $\lim_{n\to\infty} \mu(s_n) = -\infty \Leftrightarrow \lim_{n\to\infty} s_n = 0$ for all $s_n \subseteq \mathbb{R}_+$.

For example,

i. $\mu_1(s) = \ln(s)$,

ii. $\mu_2(s) = 1 - \frac{1}{s^t}$, $t > 0$.

**Theorem 2.11** *Let $\mathbb{V}$ be an NBCC subset of a Banach space $\mathbb{E}$, and let $\Upsilon : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ be a continuous operator such that*

$$\mu\bigl[F\bigl\{\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr), \phi\bigl(\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr)\bigr)\bigr\}\bigr] \le \frac{\Delta}{2}\bigl[\mu\bigl\{\Pi(s_1 \times s_2) + \phi\bigl(\Pi(s_1 \times s_2)\bigr)\bigr\}\bigr] \tag{2.7}$$

*for all $s_1, s_2 \subseteq \mathbb{V}$, where $\Delta$, $F$, and $\phi$ are as in Theorem 2.1 and $\Pi$ is an arbitrary MNC. In addition, we assume $\mu(\mathcal{A} + \mathcal{B}) \le \mu(\mathcal{A}) + \mu(\mathcal{B})$; $\mathcal{A}, \mathcal{B} \ge 0$ and $\phi(\mathcal{A} + \mathcal{B}) \le \phi(\mathcal{A}) + \phi(\mathcal{B})$; $\mathcal{A}, \mathcal{B} \ge 0$. Then $\Upsilon$ has at least a couple of $\mathbb{FP}$ in $\mathbb{V}$.*

*Proof* Consider a mapping $\Upsilon^{cf} : \mathbb{V} \times \mathbb{V} \to \mathbb{V} \times \mathbb{V}$ by $\Upsilon^{cf}(\mathcal{A}, \mathcal{B}) = (\Upsilon(\mathcal{A}, \mathcal{B}), \Upsilon(\mathcal{B}, \mathcal{A}))$; $\mathcal{A}, \mathcal{B} \in \mathbb{V}$. It is trivial that $\Upsilon^{cf}$ is continuous.

Let $s \subseteq \mathbb{V} \times \mathbb{V}$ be nonempty. We have $\Pi^{cf}(s) = \Pi(s_1) + \Pi(s_2)$ is an MNC, where $s_1, s_2$ are the natural projections of $s$ into $\mathbb{E}$.

We get

$$\begin{aligned}
&\mu\bigl[F\bigl\{\Pi^{cf}\bigl(\Upsilon^{cf}(s)\bigr), \phi\bigl(\Pi^{cf}\bigl(\Upsilon^{cf}(s)\bigr)\bigr)\bigr\}\bigr] \\
&\quad \le \mu\bigl[F\bigl\{\Pi^{cf}\bigl(\Upsilon(s_1 \times s_2) \times \Upsilon(s_2 \times s_1)\bigr), \phi\bigl(\Pi^{cf}\bigl(\Upsilon(s_1 \times s_2) \times \Upsilon(s_2 \times s_1)\bigr)\bigr)\bigr\}\bigr] \\
&\quad = \mu\bigl[F\bigl\{\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr) + \Pi\bigl(\Upsilon(s_2 \times s_1)\bigr), \phi\bigl(\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr) + \Pi\bigl(\Upsilon(s_2 \times s_1)\bigr)\bigr)\bigr\}\bigr] \\
&\quad \le \mu\bigl[F\bigl\{\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr) + \Pi\bigl(\Upsilon(s_2 \times s_1)\bigr), \phi\bigl(\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr)\bigr) + \phi\bigl(\Pi\bigl(\Upsilon(s_2 \times s_1)\bigr)\bigr)\bigr\}\bigr] \\
&\quad \le \mu\bigl[F\bigl\{\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr), \phi\bigl(\Pi\bigl(\Upsilon(s_1 \times s_2)\bigr)\bigr)\bigr\}\bigr] \\
&\qquad + \mu\bigl[F\bigl\{\Pi\bigl(\Upsilon(s_2 \times s_1)\bigr), \phi\bigl(\Pi\bigl(\Upsilon(s_2 \times s_1)\bigr)\bigr)\bigr\}\bigr] \\
&\quad \le \Delta\bigl[\mu\bigl\{\Pi(s_1) + \Pi(s_2) + \phi\bigl(\Pi(s_1) + \Pi(s_2)\bigr)\bigr\}\bigr] \\
&\quad = \Delta\bigl[\mu\bigl\{\Pi^{cf}(s) + \phi\bigl(\Pi^{cf}(s)\bigr)\bigr\}\bigr] \\
&\quad = \Delta\bigl[\mu\bigl\{F\bigl(\Pi^{cf}(s), \phi\bigl(\Pi^{cf}(s)\bigr)\bigr)\bigr\}\bigr].
\end{aligned}$$

By Theorem 2.1, we conclude that $\Upsilon^{cf}$ has minimum of one fixed point in $\mathbb{V} \times \mathbb{V}$. That is, $\Upsilon$ has minimum of one coupled fixed point. □

## 3 Measure of noncompactness on $C([0,T])$

Consider the space $\mathbf{E} = C(I)$ which is the set of real continuous functions on $I$, where $I = [0,T]$. Then $\mathbf{E}$ is a Banach space with the norm

$$\|\varrho\| = \sup\bigl\{\bigl|\varrho(\varsigma)\bigr| : \varsigma \in I\bigr\}, \quad \varrho \in \mathbf{E}.$$

Let $\Upsilon(\neq \phi) \subseteq \mathbf{E}$ be bounded. For $\varrho \in \Upsilon$ and $\epsilon > 0$, denote by $\omega(\varrho, \epsilon)$ the modulus of the continuity of $\varrho$, i.e.,

$$\omega(\varrho, \epsilon) = \sup\{|\varrho(\varsigma_1) - \varrho(\varsigma_2)| : \varsigma_1, \varsigma_2 \in I, |\varsigma_1 - \varsigma_1| \leq \epsilon\}.$$

Further, we define

$$\omega(\Upsilon, \epsilon) = \sup\{\omega(\varrho, \epsilon) : \varrho \in \Upsilon\}; \qquad \omega_0(\Upsilon) = \lim_{\epsilon \to 0} \omega(\Upsilon, \epsilon).$$

It is well known that the function $\omega_0$ is an MNC in $\mathbf{E}$ such that the Hausdorff measure of noncompactness $\chi$ is given by $\chi(\Upsilon) = \frac{1}{2}\omega_0(\Upsilon)$ (see [25]).

## 4 Solvability of fractional integral equation

For $\rho \in (0, 1]$ and $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0$, we define the left GPF integral of $f$ defined by [31]

$$\left({}_aI^{\alpha,\rho} f\right)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{(\rho-1)(t-\tau)}{\rho}} (t - \tau)^{\alpha-1} f(\tau)\, d\tau.$$

In this part, we study the following fractional integral equation:

$$\mathcal{Z}(\varsigma) = \Delta\left(\varsigma, \mathcal{L}\left(\varsigma, \mathcal{Z}(\varsigma)\right), \left({}_0I^{\alpha,\rho} \mathcal{Z}\right)(\varsigma)\right), \tag{4.1}$$

where $\alpha > 1, \rho \in (0, 1], \varsigma \in I = [0, T]$.

Let

$$B_{d_0} = \{\mathcal{Z} \in \mathbf{E} : \|\mathcal{Z}\| \leq d_0\}.$$

Assume that

(A) $\Delta : I \times \mathbb{R}^2 \to \mathbb{R}, \mathcal{L} : I \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exist constants $\delta_1, \delta_2, \delta_3 \geq 0$ satisfying

$$\left|\Delta(\varsigma, \mathcal{L}, I_1) - \Delta(\varsigma, \bar{\mathcal{L}}, \bar{I}_1)\right| \leq \delta_1|\mathcal{L} - \bar{\mathcal{L}}| + \delta_2|I_1 - \bar{I}_1|, \quad \varsigma \in I; \mathcal{L}, I_1, \bar{\mathcal{L}}, \bar{I}_1 \in \mathbb{R}$$

and

$$\left|\mathcal{L}(\varsigma, J_1) - \mathcal{L}(\varsigma, J_2)\right| \leq \delta_3|J_1 - J_2|, \quad J_1, J_2 \in \mathbb{R}.$$

(B) There exists $d_0 > 0$ satisfying

$$\bar{\Delta} = \sup\{|\Delta(\varsigma, \mathcal{L}, I_1)| : \varsigma \in I, \mathcal{L} \in [-\hat{\mathcal{L}}, \hat{\mathcal{L}}], I_1 \in [-\hat{\mathcal{I}}, \hat{\mathcal{I}}]\} \leq d_0$$

and

$$\delta_1\delta_3 < 1,$$

where

$$\hat{\mathcal{L}} = \sup\{|\mathcal{L}(\varsigma, \mathcal{Z}(\varsigma))| : \varsigma \in I, \mathcal{Z}(\varsigma) \in [-d_0, d_0]\}$$

and

$$\hat{\mathcal{I}} = \sup\big\{\big|\big({}_0I^{\alpha,\rho}\mathcal{Z}\big)(\varsigma)\big| : \varsigma \in I, \mathcal{Z}(\varsigma) \in [-d_0, d_0]\big\}.$$

(C) $|\Delta(\varsigma,0,0)| = 0, \mathcal{L}(\varsigma,0) = 0$.

(D) There exists a positive solution $d_0$ of the inequality

$$\delta_1\delta_3 r + \frac{\delta_2 r T^\alpha}{\rho^\alpha\Gamma(\alpha+1)}.e^{\frac{(\rho-1)T}{\rho}} \le r.$$

**Theorem 4.1** *If conditions* (*A*)–(*D*) *hold, then Eq.* (4.1) *has a solution in* $\mathbf{E} = C(I)$.

*Proof* Define the operator $\mathcal{T} : \mathbf{E} \to \mathbf{E}$ as follows:

$$(\mathcal{T}\mathcal{Z})(\varsigma) = \Delta\big(\varsigma, \mathcal{L}\big(\varsigma, \mathcal{Z}(\varsigma)\big), \big({}_0I^{\alpha,\rho}\mathcal{Z}\big)(\varsigma)\big).$$

*Step* 1: We prove that the function $\mathcal{Q}$ maps $B_{d_0}$ into $B_{d_0}$. Let $\Upsilon \in B_{d_0}$. We have

$$\begin{aligned}
&\big|(\mathcal{T}\mathcal{Z})(\varsigma)\big| \\
&\quad\le \big|\Delta\big(\varsigma,\mathcal{L}\big(\varsigma,\mathcal{Z}(\varsigma)\big),\big({}_0I^{\alpha,\rho}\mathcal{Z}\big)(\varsigma)\big) - \Delta(\varsigma,0,0)\big| + \big|\Delta(\varsigma,0,0)\big| \\
&\quad\le \delta_1\big|\mathcal{L}\big(\varsigma,\mathcal{Z}(\varsigma)\big) - 0\big| + \delta_2\big|\big({}_0I^{\alpha,\rho}\mathcal{Z}\big)(\varsigma) - 0\big| \\
&\quad\le \delta_1\delta_3\big|\mathcal{Z}(\varsigma)\big| + \delta_2\big|\big({}_0I^{\alpha,\rho}\mathcal{Z}\big)(\varsigma)\big|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\big|\big({}_0I^{\alpha,\rho}\mathcal{Z}\big)(\varsigma)\big| \\
&\quad= \left|\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\varsigma e^{\frac{(\rho-1)(\varsigma-\tau)}{\rho}}(\varsigma-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau\right| \\
&\quad\le \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\varsigma e^{\frac{(\rho-1)(\varsigma-\tau)}{\rho}}(\varsigma-\tau)^{\alpha-1}\big|\mathcal{Z}(\tau)\big|\,d\tau \\
&\quad\le \frac{d_0 e^{\frac{(\rho-1)T}{\rho}}}{\rho^\alpha\Gamma(\alpha)}\int_0^\varsigma (\varsigma-\tau)^{\alpha-1}\,d\tau \\
&\quad\le \frac{d_0 T^\alpha e^{\frac{(\rho-1)T}{\rho}}}{\rho^\alpha\Gamma(\alpha+1)}.
\end{aligned}$$

Hence, $\|\mathcal{T}\| < d_0$ gives

$$\|\mathcal{T}\| \le \delta_1\delta_3 d_0 + \frac{\delta_2 d_0 T^\alpha}{\rho^\alpha\Gamma(\alpha+1)}.e^{\frac{(\rho-1)T}{\rho}} \le d_0.$$

Due to assumption (D), $\mathcal{T}$ maps $B_{d_0}$ into $B_{d_0}$.

*Step* 2: We prove that $\mathcal{T}$ is continuous on $B_{d_0}$. Let $\epsilon > 0$ and $\mathcal{Z}, \bar{\mathcal{Z}} \in B_{r_0}$ such that $\|\mathcal{Z} - \bar{\mathcal{Z}}\| < \epsilon$. We have

$$\big|(\mathcal{T}\mathcal{Z})(\varsigma) - (\mathcal{T}\bar{\mathcal{Z}})(\varsigma)\big|$$

$$\begin{aligned}&\leq \bigl|\Delta\bigl(\varsigma,\mathcal{L}\bigl(\varsigma,\mathcal{Z}(\varsigma)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma)\bigr)-\Delta\bigl(\varsigma,\mathcal{L}\bigl(\varsigma,\bar{\mathcal{Z}}(\varsigma)\bigr),\bigl({}_0I^{\alpha,\rho}\bar{\mathcal{Z}}\bigr)(\varsigma)\bigr)\bigr|\\&\leq \delta_1\bigl|\mathcal{L}\bigl(\varsigma,\mathcal{Z}(\varsigma)\bigr)-\mathcal{L}\bigl(\varsigma,\bar{\mathcal{Z}}(\varsigma)\bigr)\bigr|+\delta_2\bigl|\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma)-\bigl({}_0I^{\alpha,\rho}\bar{\mathcal{Z}}\bigr)(\varsigma)\bigr|.\end{aligned}$$

Also,

$$\begin{aligned}&\bigl|\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma)-\bigl({}_0I^{\alpha,\rho}\bar{\mathcal{Z}}\bigr)(\varsigma)\bigr|\\&\quad=\left|\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\varsigma e^{\frac{(\rho-1)(\varsigma-\tau)}{\rho}}(\varsigma-\tau)^{\alpha-1}\bigl\{\mathcal{Z}(\tau)-\bar{\mathcal{Z}}(\tau)\bigr\}\,d\tau\right|\\&\quad\leq\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^\varsigma e^{\frac{(\rho-1)(\varsigma-\tau)}{\rho}}(\varsigma-\tau)^{\alpha-1}\bigl|\mathcal{Z}(\tau)-\bar{\mathcal{Z}}(\tau)\bigr|\,d\tau\\&\quad<\frac{\epsilon T^\alpha e^{\frac{(\rho-1)T}{\rho}}}{\rho^\alpha\Gamma(\alpha+1)}.\end{aligned}$$

Hence, $\|\mathcal{Z}-\bar{\mathcal{Z}}\|<\epsilon$ gives

$$\bigl|(\mathcal{T}\mathcal{Z})(\varsigma)-(\mathcal{T}\bar{\mathcal{Z}})(\varsigma)\bigr|<\delta_1\delta_3\epsilon+\frac{\epsilon T^\alpha e^{\frac{(\rho-1)T}{\rho}}}{\rho^\alpha\Gamma(\alpha+1)}.$$

As $\epsilon\to0$ we get $|(\mathcal{T}\mathcal{Z})(\varsigma)-(\mathcal{T}\bar{\mathcal{Z}})(\varsigma)|\to0$. This shows that $\mathcal{T}$ is continuous on $B_{d_0}$.

*Step* 3: An estimate of $\mathcal{T}$ with respect to $\omega_0$: Assume that $\Omega(\neq\phi)\subseteq B_{d_0}$. Let $\epsilon>0$ be arbitrary and choose $\mathcal{Z}\in\Omega$ and $\varsigma_1,\varsigma_2\in I$ such that $|\varsigma_2-\varsigma_1|\leq\epsilon$ and $\varsigma_2\geq\varsigma_1$.

Now,

$$\begin{aligned}&\bigl|(\mathcal{T}\mathcal{Z})(\varsigma_2)-(\mathcal{T}\mathcal{Z})(\varsigma_1)\bigr|\\&\quad=\bigl|\Delta\bigl(\varsigma_2,\mathcal{L}\bigl(\varsigma_2,\mathcal{Z}(\varsigma_2)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_2)\bigr)-\Delta\bigl(\varsigma_1,\mathcal{L}\bigl(\varsigma_1,\mathcal{Z}(\varsigma_1)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr)\bigr|\\&\quad\leq\bigl|\Delta\bigl(\varsigma_2,\mathcal{L}\bigl(\varsigma_2,\mathcal{Z}(\varsigma_2)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_2)\bigr)-\Delta\bigl(\varsigma_2,\mathcal{L}\bigl(\varsigma_2,\mathcal{Z}(\varsigma_2)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr)\bigr|\\&\qquad+\bigl|\Delta\bigl(\varsigma_2,\mathcal{L}\bigl(\varsigma_2,\mathcal{Z}(\varsigma_2)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr)-\Delta\bigl(\varsigma_2,\mathcal{L}\bigl(\varsigma_1,\mathcal{Z}(\varsigma_1)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr)\bigr|\\&\quad+\bigl|\Delta\bigl(\varsigma_2,\mathcal{L}\bigl(\varsigma_1,\mathcal{Z}(\varsigma_1)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr)-\Delta\bigl(\varsigma_1,\mathcal{L}\bigl(\varsigma_1,\mathcal{Z}(\varsigma_1)\bigr),\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr)\bigr|\\&\quad\leq\delta_2\bigl|\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_2)-\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr|+\delta_1\bigl|\mathcal{L}\bigl(\varsigma_2,\mathcal{Z}(\varsigma_2)\bigr)-\mathcal{L}\bigl(\varsigma_1,\mathcal{Z}(\varsigma_1)\bigr)\bigr|+\omega_\Delta(I,\epsilon)\\&\quad\leq\delta_2\bigl|\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_2)-\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr|+\delta_1\delta_3\bigl|\mathcal{Z}(\varsigma_2)-\mathcal{Z}(\varsigma_1)\bigr|+\omega_\Delta(I,\epsilon),\end{aligned}$$

where

$$\omega_\Delta(I,\epsilon)=\sup\left\{\begin{matrix}|\Delta(\varsigma_2,\mathcal{L},\mathcal{I}_1)-\Delta(\varsigma_1,\mathcal{L},\mathcal{I}_1)|:|\varsigma_2-\varsigma_1|\leq\epsilon;\varsigma_1,\varsigma_2\in I;\\ \mathcal{L}\in[-\hat{\mathcal{L}},\hat{\mathcal{L}}];\mathcal{I}_1\in[-\hat{\mathcal{I}},\hat{\mathcal{I}}]\end{matrix}\right\}.$$

Also,

$$\begin{aligned}&\bigl|\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_2)-\bigl({}_0I^{\alpha,\rho}\mathcal{Z}\bigr)(\varsigma_1)\bigr|\\&\quad=\left|\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varsigma_2}e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau\right.\\&\qquad\left.-\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varsigma_1}e^{\frac{(\rho-1)(\varsigma_1-\tau)}{\rho}}(\varsigma_1-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau\right|\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{\varsigma_2} e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau - \int_0^{\varsigma_1} e^{\frac{(\rho-1)(\varsigma_1-\tau)}{\rho}}(\varsigma_1-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau\right| \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{\varsigma_2} e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau - \int_0^{\varsigma_1} e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau\right| \\
&\quad + \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_0^{\varsigma_1} e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau - \int_0^{\varsigma_1} e^{\frac{(\rho-1)(\varsigma_1-\tau)}{\rho}}(\varsigma_1-\tau)^{\alpha-1}\mathcal{Z}(\tau)\,d\tau\right| \\
&\leq \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{\varsigma_1}^{\varsigma_2} e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1}\left|\mathcal{Z}(\tau)\right|d\tau \\
&\quad + \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varsigma_1}\left|\left(e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1} - e^{\frac{(\rho-1)(\varsigma_1-\tau)}{\rho}}(\varsigma_1-\tau)^{\alpha-1}\right)\mathcal{Z}(\tau)\right|d\tau \\
&\leq \frac{-e^{\frac{(\rho-1)T}{\rho}}}{\rho^\alpha\Gamma(\alpha+1)}\|\mathcal{Z}\|(\varsigma_2-\varsigma_1)^\alpha \\
&\quad + \frac{\|\mathcal{Z}\|}{\rho^\alpha\Gamma(\alpha)}\int_0^{\varsigma_1}\left|e^{\frac{(\rho-1)(\varsigma_2-\tau)}{\rho}}(\varsigma_2-\tau)^{\alpha-1} - e^{\frac{(\rho-1)(\varsigma_1-\tau)}{\rho}}(\varsigma_1-\tau)^{\alpha-1}\right|d\tau.
\end{aligned}$$

As $\epsilon \to 0$, then $\varsigma_2 \to \varsigma_1$, and so $|({}_0I^{\alpha,\rho}\mathcal{Z})(\varsigma_2) - ({}_0I^{\alpha,\rho}\mathcal{Z})(\varsigma_1)| \to 0$.
Hence,

$$\begin{aligned}
&\left|(\mathcal{T}\mathcal{Z})(\varsigma_2) - (\mathcal{T}\mathcal{Z})(\varsigma_1)\right| \\
&\quad\leq \delta_2\left|\left({}_0I^{\alpha,\rho}\mathcal{Z}\right)(\varsigma_2) - \left({}_0I^{\alpha,\rho}\mathcal{Z}\right)(\varsigma_1)\right| + \delta_1\delta_3\omega(\mathcal{Z},\epsilon) + \omega_\Delta(I,\epsilon)
\end{aligned}$$

gives

$$\omega(\mathcal{T}\mathcal{Z},\epsilon) \leq \delta_2\left|\left({}_0I^{\alpha,\rho}\mathcal{Z}\right)(\varsigma_2) - \left({}_0I^{\alpha,\rho}\mathcal{Z}\right)(\varsigma_1)\right| + \delta_1\delta_3\omega(\mathcal{Z},\epsilon) + \omega_\Delta(I,\epsilon).$$

By the uniform continuity of $\Delta$ on $I \times [-\hat{\mathcal{L}},\hat{\mathcal{L}}] \times [-\hat{\mathcal{I}},\hat{\mathcal{I}}]$, we have $\omega_\Delta(I,\epsilon) \to 0$ as $\epsilon \to 0$. Taking $\sup_{\mathcal{Z}\in\Omega}$ and $\epsilon \to 0$, we get

$$\omega_0(\mathcal{T}\Omega) \leq \delta_1\delta_3\omega_0(\Omega).$$

Thus, by Corollary 2.6, $\mathcal{Q}$ has a fixed point in $\Omega \subseteq B_{d_0}$, i.e., equation (4.1) has a solution in **E**. □

*Example* 4.2 Consider the following equation:

$$\mathcal{Z}(\varsigma) = \frac{\mathcal{Z}(\varsigma)}{7+\varsigma^2} + \frac{({}_0I^{2,\frac{1}{2}}\mathcal{Z})(\varsigma)}{10} \tag{4.2}$$

for $\varsigma \in [0,2] = I$.

We have

$$\left({}_0I^{2,\frac{1}{2}}\mathcal{Z}\right)(\varsigma) = \frac{4}{\Gamma(2)}\int_0^{\varsigma} e^{-(\varsigma-\tau)}(\varsigma-\tau)\mathcal{Z}(\tau)\,d\tau.$$

Also, $\Delta(\varsigma,\mathcal{L},\mathcal{I}_1) = \mathcal{L} + \frac{\mathcal{I}_1}{10}$ and $\mathcal{L}(\varsigma,\mathcal{Z}) = \frac{\mathcal{Z}}{7+\varsigma^2}$. It is trivial that both $\Delta, \mathcal{L}$ are continuous satisfying

$$\left|\mathcal{L}(\varsigma,J_1) - \mathcal{L}(\varsigma,J_2)\right| \leq \frac{|J_1 - J_2|}{8}$$

and

$$\left|\Delta(\varsigma,\mathcal{L},\mathcal{I}_1) - \Delta(\varsigma,\bar{\mathcal{L}},\bar{\mathcal{I}}_1)\right| \leq |\mathcal{U} - \bar{\mathcal{U}}| + \frac{1}{10}|\mathcal{I}_1 - \bar{\mathcal{I}}_1|.$$

Therefore, $\delta_1 = 1, \delta_2 = \frac{1}{10}, \delta_3 = \frac{1}{8}$, and $\delta_1\delta_3 = \frac{1}{8} < 1$.

If $\|\mathcal{Z}\| \leq d_0$, then

$$\hat{\mathcal{L}} = \frac{d_0}{8}$$

and

$$\hat{\mathcal{I}} = \frac{8d_0}{e^2}.$$

Further,

$$\left|\Delta(\varsigma,\mathcal{L},\mathcal{I}_1)\right| \leq \frac{d_0}{8} + \frac{8d_0}{10e^2} \leq d_0.$$

If we choose $d_0 = 2$, then

$$\hat{\mathcal{L}} = \frac{1}{4}, \qquad \hat{\mathcal{I}} = \frac{16}{e^2},$$

which gives

$$\bar{\Delta} \leq 2.$$

On the other hand, assumption (D) is also satisfied for $d_0 = 2$.

We observe that all the assumption from (A)–(D) of Theorem 4.1 are satisfied. By Theorem 4.1, it can be said that equation (4.2) has a solution in $\mathbf{E} = C(I)$.

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**Author details**
[1]Department of Mathematics, Cotton University, Panbazar, Guwahati-781001, Assam, India. [2]Department of Mathematics and Statistics, The Arab American University, P.O. Box 240, 13 Zababdeh, Jenin, Palestine. [3]Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791112, Arunachal Pradesh, India. [4]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia. [5]Department of Medical Research, China Medical University, Taichung, 40402, Taiwan. [6]Department of Computer Science and Information Engineering, Asia University, Taichung, Taiwan.

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