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On the Gronwall Inequality via Discrete Fractional Calculus

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Thesis Approval

On the Gronwall Inequality via Discrete Fractional Calculus

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This thesis was defended successfully on and approved by

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Declaration

My name is Dana Emad Atiya, and I am a student of the AAUP university number 201712810. I confirm that I worked on this Master's thesis myself. And I declare that I have complied with all regulations, instructions, Arab American University standards of Academic codes of conduct. I also adhere to the Dean's Council regulations in withdrawing their confirmation of this degree in case of any violations.

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Dedication

First and foremost, I am grateful to Almighty Allah who helped me in reaching my aim,

To my dear deceased brother Mahmoud,

To my dear parents,

To my husband Mohammad,

To my children Amal and Khaled,

To all of my family.

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Abstract

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In this work, Gronwall's inequality for h-discrete calculus with the nabla operator on $h\mathbb{Z}$ has been proved. We illustrate our results in the context of initial value problem (IVP). Moreover, Gronwall's inequality for discrete calculus on $\mathbb{Z}$ has been verified. For this purpose the equivalence of an initial value problem for a discrete fractional equation and a discrete fractional sum equation on $\mathbb{Z}$ and on $h\mathbb{Z}$ have been considered, and the h-discrete Mittag-Leffler function in term of ${}_xE_h\psi$ and ${}_xF_h\psi$ have been defined. Then an explicit solution have been given for linear discrete fractional sum equation.

Keywords: $Nabla\ \ h-discrete\ \ fractional\ \ sums, Mittag-Leffler\ \ ML, Gronwall's\ \ inequality.$

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Introduction

Fractional calculus is considered as a generalization of the traditional calculus with zero and positive integer order integration and differentiation, and from the confirmed results which are obtained from applying the fractional operators to problems of model real world the fractional calculus has been one of the fastest growing fields of research.

The results on fractional calculus have been accumulated over centuries in various branches of mathematics.

The Mittag-Leffler (ML) function which is a generalization of the exponential function, and the gamma function which was invented by Euler to expand the factorial of complex and real number arguments are the functions that play the most important role in the area of fractional calculus. The gamma function $\Gamma(z)$ is defined by the integral [37]

$$\Gamma(z) = \int_0^{\infty} x^{z-1} \mathrm{e}^{-x}\, \mathrm{d}x$$

In the right half of the complex plane $\mathrm{Re}(z) > 0$ the above integral converges. And, the one parameter ML function is defined by [2]

$$\mathrm{E}_{\lambda}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\lambda + 1)} \qquad , \lambda > 0, t \in \mathbb{C}$$

The two parameter ML function which was introduced by Agarwal which plays a very important role in the fractional calculus, is defined by the series expansion [2]

$$\mathrm{E}_{\lambda,\zeta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\lambda+\zeta)} \qquad ,\lambda,\zeta > 0$$

In 1695, the theory of derivatives of non-integer order in Leibniz's note which goes back to L'Hospital [64] is meaning the derivative of order one half. In one exchange, L'Hospital inquired," what would be the one-half derivative of x?" to which Leibniz responded that the answer "leads to an apparent paradox, from which one day useful consequences will be drawn" [45]. In 1730, Euler interested in the subject of fractional calculus by his invention of gamma function. In 1812, a fractional derivative is defined by P.S.Laplace by means of an integral, and a derivative of arbitrary order having the first mention in 1819 which appears in a text [64]. Also, In 1819, the formula for the n^{th} integer order derivative for polynomials is founded by Lacroix and then he replacing factorials with the gamma function to extend the formula for arbitrary order ζ [37]

$$D^{\zeta}x^{q} = \frac{\Gamma(q+1)}{\Gamma(q-\zeta+1)}x^{q-\zeta},$$

and he used this formula to calculate the half derivative correctly. In 1823 [64], Abel provided the first use of the fractional calculus in the solution of an integral equation which originates in the formulation of the tautochrone (isochrones) problem - the problem of determining the shape of the curve such that the time of landing of an object sliding on the curve under uniform gravity is independent of the starting point of the object. The topic of fractional calculus has been dormant for nearly a decade until the works of Joseph Liouville came out, who conducted the first major study of fractional calculus [43]. In 1832, Joseph Liouville published three large notes and many publications in quick succession [64]. He has been successful in applying his definitions to potential theoretical problems. He found the first formula for a fractional derivative which generalizes a derivative of arbitrary order where the order is any number - rational, irrational or complex. Also, the fractional derivative of x^{-p}.

Riemann studied and developed his theory of fractional integration in fractional calculus when he was student and his work has not been published until his death [43]. A definition for the fractional one which is developed by Riemann is very similar to the modern Riemann-Liouville(RL) fractional integral.

K. B. Oldham and J. Spanier' book which contains applications in physics, chemistry and engineering, played a prominent role in the development of the fractional calculus which can be called applied fractional calculus [57]. Moreover, it was the first book which was fully dedicated to a systematic presentation of the ideas, methods, and applications of the fractional calculus [62].

Many problems in science, engineering and media can be formulated using continuous and discrete fractional calculus [49]-podlubnv1999fractional. In [5]-[9], there are many examples which are applications of fractional calculus. In [69], the nabla fractional sums and differences of order $0 < \eta < 1$ on $h\mathbb{Z}$ where $0 < h \leqslant 1$ are formulated.

The use of discrete fractional calculus (DFC) in analysing non-local behaviour of models has gained great importance in recent years [7]. The study of discrete forward fractional calculus has a little work. Miller and Ross [56] started the study and the authors [14, 16, 17] recently developed discrete forward fractional calculus. The study of discrete backward fractional calculus has more work [31, 40, 41, 46] emerging applications in time series analysis spurred development in discrete backward fractional calculus.

[45] Properties of the generalized falling function, the commutivity of fractional sums and a corresponding power rule for fractional delta-operators have been discussed by Atici and Eloe who interested in discrete fractional calculus. They introduce in more rules to compose fractional sums and differences but leave many important cases unresolved. Inaddition, function domains and lower limits of summation and differentiation which are vital details for a rigorous and correct treatment of the power rule and the fractional composition rules having little attention from Atici and Eloe. Some of the work in discrete fractional calculus has used the backward or forward difference. We refer the reader to [14, 17, 56], for example, and more recently, [20, 24, 25, 38, 39, 44].
We talk about one of the most important inequalities in the theory of differential equations which plays an important role in the qualitative analysis of the solutions to differential and integral equations which was published in 1919 in the work of Gronwall [42]. Since then the literature [58] has a part of many generalizations and extensions of this inequality. It seems that the discrete Gronwall inequality appeared first in 1935 in Mikeladze's work, and now it is used for example in proving convergence of the discrete variable methods for ordinary, partial as well as integral equations [4].
Gronwall's inequality is now the generic name of the device which resolves such questions, after T.H. Gronwall [42] who developed an idea of Peano [59]. Also, Gronwall's inequality has undergone and is still subject to substantial generalization [63], [52].
Gronwall's inequality is a fundamental estimate for (non negative) functions on one real variable satisfying a certain differential or integral inequality by the solutions of the corresponding differential or integral equation [33, 71]. Now, this inequality known as Gronwall-Bellman-Raid inequality which is provided explicit limits on solutions to a class of linear integral inequalities. The Gronwall-

Bellman-Raid inequality has been expanded and used in different cases [73]. The solution of fractional differential equation depending on the initial condition and the order where this dependence is studied by Gronwall's inequality. The thesis organized as follows: in chapter one, we recall some of the definitions and some of the known results that we will use throughout the thesis. In chapter two, we have continuous and discrete fractional calculus and some of their properties especially on discrete fractional sum equations. In chapter three, the Gronwall inequality in its traditional form is defined. In the fourth chapter, we defined the Gronwall inequality for discrete calculus with the nabla operator on scale $h = 1$ and on scale $h\mathbb{Z}$. We illustrate our results with an application of the Gronwall's inequality in discrete fractional calculus to gain sufficient conditions which gives continuous dependence of solutions of IVPs on initial conditions.

Chapter One

Definitions and Preliminary Results

In this chapter, we start by introducing some important basic definitions on discrete fractional calculus. We then move to present some known results.

Definition 1.0.1 *[69] The definitions of the backward and forward difference operator on $h\mathbb{Z}$ are respectively in the form*

$$\nabla_h f(x) = \frac{f(x) - f(x-h)}{h}$$

$$\Delta_h\, f(x) = \frac{f(x+h) - f(x)}{h}$$

Definition 1.0.2 *[69] The forward jump operator on $h\mathbb{Z}$ is defined by*

$$\sigma_h(x) = x + h$$

, and the backward jump operator is defined by

$$\rho_h(x) = x - h.$$

For $a, b \in \mathbb{R}$ with $b > a$, $\dfrac{b-a}{h} \in \mathbb{N}$ and $0 < h \leqslant 1$, the notations

$\mathbb{N}_{a,h} = \{a, a+h, a+2h, \ldots\}$ *and* ${}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \ldots\}$ *will be used.*

Definition 1.0.3 *[69] The nabla h-factorial of x for $\alpha \in \mathbb{R}$ and $0 < h \leqslant 1$ is defined by :*

$$x_h^{\overline{\alpha}} = h^{\alpha} \frac{\Gamma\left(\frac{x}{h} + \alpha\right)}{\Gamma\left(\frac{x}{h}\right)},$$

such that $x \in \mathbb{R} - \{\ldots, -2h, -h, 0\}$, $0_h^{\overline{\alpha}} = 0$ and dividing by poles leads to zero.

Definition 1.0.4 *[69] (Nabla fractional sums)*

For a function $f : \mathbb{N}_a = \{a, a+1, a+2, \ldots\} \longrightarrow \mathbb{R}$, we know that the nabla left fractional sum of order $\alpha > 0$ is

$$\begin{aligned}
\left({}_a\nabla^{-\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x - \rho(s))^{\overline{\alpha - 1}} f(s)\, \nabla s \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=a+1}^{x} (x - \rho(k))^{\overline{\alpha - 1}} f(k), \qquad x \in \mathbb{N}_a.
\end{aligned}$$

For a function $f : {}_b\mathbb{N} = \{b, b-1, b-2, \ldots\} \longrightarrow \mathbb{R}$, the nabla right fractional sum of order $\alpha > 0$ is defined by

$$\begin{aligned}
\left(\nabla_b^{-\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (s - \rho(x))^{\overline{\alpha - 1}} f(s)\, \Delta s \\
&= \frac{1}{\Gamma(\alpha)} \sum_{k=x}^{b-1} (k - \rho(x))^{\overline{\alpha - 1}} f(k), \qquad x \in {}_b\mathbb{N}.
\end{aligned}$$

Definition 1.0.5 *[69] (Nabla h-fractional sums)*

For a fuction $f : \mathbb{N}_{a,h} = \{a, a+h, a+2h, \ldots\} \longrightarrow \mathbb{R}$, we knowledge before that the nabla *left h-fractional sum of order $\alpha > 0$ is*

$$\begin{aligned} \left({}_a\nabla_h^{-\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-\rho_h(s))_h^{\overline{\alpha-1}} f(s)\, \nabla_h s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=a/h+1}^{x/h} (x-\rho_h(kh))_h^{\overline{\alpha-1}} f(kh) h, \qquad x \in \mathbb{N}_{a,h}. \end{aligned}$$

For a function $f:\ {}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \ldots\} \longrightarrow \mathbb{R}$, we knowledge before that the nabla right h-fractional sum of order $\alpha > 0$ is

$$\begin{aligned} \left({}_h\nabla_b^{-\alpha} f\right)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (s-\rho_h(x))_h^{\overline{\alpha-1}} f(s)\, \Delta_h s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k=x/h}^{b/h-1} (kh-\rho_h(x))_h^{\overline{\alpha-1}} f(kh) h, \qquad x \in\ {}_{b,h}\mathbb{N}. \end{aligned}$$

Here we will write the definition of Nabla discrete Mittag-Leffler.

Definition 1.0.6 *[1] (Nabla discrete Mittag-Leffler)*
For $\gamma \in \mathbb{R}$ such that $|\gamma| < 1$, $\gamma = \dfrac{-\alpha}{1-\alpha}$, and $\alpha, \beta, \tau \in \mathbb{C}$ with $Re(\alpha) > 0$, the nabla discrete ML function is defined by

$$E_{\overline{\alpha,\beta}}(\gamma, \tau) = \sum_{k=0}^{\infty} \gamma^k \frac{\tau^{\overline{k\alpha+\beta-1}}}{\Gamma(k\alpha+\beta)}, \qquad |\gamma| < 1$$

For $\beta = 1$ we have,

$$E_{\overline{\alpha}}(\gamma, \tau) \triangleq E_{\overline{\alpha,1}}(\gamma, \tau) = \sum_{k=0}^{\infty} \gamma^k \frac{\tau^{\overline{k\alpha}}}{\Gamma(k\alpha+1)}, \qquad |\gamma| < 1$$

Definition 1.0.7 *[1] (Nabla h-discrete Mittag-Leffler)*

For $\gamma \in \mathbb{R}$ such that $|\gamma h^{\alpha}| < 1$, $\gamma = \frac{-\alpha}{1-\alpha}$, and $\alpha, \beta, \tau \in \mathbb{C}$ with $Re(\alpha) > 0$, the nabla h-discrete ML function is defined by

$$_{h}E_{\overline{\alpha,\beta}}(\gamma,\tau) = \sum_{k=0}^{\infty} \gamma^{k} \frac{\tau_{h}^{\overline{k\alpha+\beta-1}}}{\Gamma(k\alpha+\beta)}, \quad |\gamma h^{\alpha}| < 1$$

For $\beta = 1$ we have,

$$_{h}E_{\overline{\alpha}}(\gamma,\tau) \triangleq \ _{h}E_{\overline{\alpha,1}}(\gamma,\tau) = \sum_{k=0}^{\infty} \gamma^{k} \frac{\tau_{h}^{\overline{k\alpha}}}{\Gamma(k\alpha+1)}, \quad |\gamma h^{\alpha}| < 1$$

Definition 1.0.8 *[70] (Lipschitz condition)*

Let $D \subset \mathbb{R}^2$ and a function $f : D \longrightarrow \mathbb{R}^2$. If there exists a constant $L \geq 0$ such that $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$, for all $(t,y_1), (t,y_2) \in D$, then we say f satisfies a lipschitz condition on a set D with Lipschitz constant$L$.

Lemma 1.0.1 *[19] If $\alpha > 0, \mu > -1$, and $t \in \mathbb{N}_a$, then*

$$\nabla_{a}^{-\alpha}(t-a+1)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a+1)^{\overline{\alpha+\mu}}$$

Proof:

$$\begin{aligned} \nabla_{a}^{-\alpha}(t-a+1)^{\overline{\mu}} &= \sum_{s=a}^{t} \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}(s-a+1)^{\overline{\mu}} \\ &= \sum_{s=1}^{t-a+1} \frac{(t-a+1-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)} s^{\overline{\mu}} \end{aligned}$$

Hence from [15], we have

$$\sum_{s=1}^{t-a+1} \frac{(t-a+1-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)} s^{\overline{\mu}} = (\nabla_1^{-\alpha} f)(t-a+1)$$
$$= \nabla_1^{-\alpha}(t-a+1)^{\overline{\mu}}$$
$$= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a+1)^{\overline{\alpha+\mu}},$$

where $f(t) = t^{\overline{\mu}}$. □

Lemma 1.0.2 *[69] If* $\alpha > 0, \mu > -1, h > 0$*, and* $t \in \mathbb{N}_{a,h}$*, then*

$${}_a\nabla_h^{-\alpha}(t-a)_h^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)_h^{\overline{\alpha+\mu}}$$

Theorem 1.0.3 *[19] For a function* $f : \mathbb{N}_a \to \mathbb{R}$*, and for any* $0 < \alpha < 1$*, and , the following equality holds:*

$$\nabla_{a+1}^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)$$

Proof: Using the product rule of the nabla operator:

$$\nabla_s[(t-\rho(s))^{\overline{\alpha-1}} f(s)] = (t-\rho(s))^{\overline{\alpha-1}} \nabla_s f(s) - (\alpha-1)(t+1-\rho(s))^{\overline{\alpha-2}} f(\rho(s)).$$

we have

$$
\begin{aligned}
\nabla_{a+1}^{-\alpha}\nabla f(t) &= \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\nabla f(s)\\
&= \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(s)|_a^t+\frac{(\alpha-1)}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t+1-\rho(s))^{\overline{\alpha-2}}f(\rho(s))\\
&= f(t)-\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a)+\frac{1}{\Gamma(\alpha-1)}\sum_{s=a}^{t-1}(t-\rho(s))^{\overline{\alpha-2}}f(s)\\
&= -\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a)+\frac{1}{\Gamma(\alpha-1)}\sum_{s=a}^{t}(t-\rho(s))^{\overline{\alpha-2}}f(s)\\
&= -\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a)+\nabla_a^{-(\alpha-1)}f(t)\\
&= -\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a)+\nabla_a^{1-\alpha}f(t)\\
&= \nabla\nabla_a^{-\alpha}f(t)-\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a).
\end{aligned}
$$

□

The Riemann-Liouville form of the fractional difference is used to obtain the proof,

$$(\nabla_a^{\alpha}f)(t) := \nabla^k(\nabla_a^{\alpha-k}f)(t)$$

where k is a nonnegative integer and $k-1<\alpha\le k$.

Theorem 1.0.4 *[69] Let $\alpha,\mu>0$, and f be a real valued function. Then*

$$\nabla_a^{-\alpha}[\nabla_a^{-\mu}f(t)] = \nabla_a^{-(\mu+\alpha)}f(t) = \nabla_a^{-\mu}[\nabla_a^{-\alpha}f(t)].$$

Chapter Two

Discrete Fractional Calculus

In this chapter, we present the IVP for a fractional difference equation on $h\mathbb{Z}$ and discrete sum equations. In Section 2.1, we introduce the initial value problem when $h = 1$ and on $h\mathbb{Z}$ in general. In Section 2.2, we present discrete fractional equation and very important theorems.

2.1 An Initial Value Problem on Fractional Calculus

For a fractional difference equation, consider the following IVP.

$$\nabla_a^{\alpha} y(t) = f(t, y(t)) \qquad for \;\; t = a+1, a+2, \ldots \tag{2.1}$$

$$\nabla_a^{-(1-\alpha)} y(t)|_{t=a} = y(a) = c, \tag{2.2}$$

where $0 < \alpha \leq 1$ and $a \in \mathbb{R}$. When Equation (2.1)is linear, the IVP can be solved explicitly for y. However, the nonlinear case of the problem is implicit and interested.

To obtain the solution of this IVP, we apply the operator $\nabla_{a+1}^{-\alpha}$ to each side of the equation (2.1) to obtain:

$$\nabla_{a+1}^{-\alpha}\nabla_a^{\alpha}y(t) = \nabla_{a+1}^{-\alpha}f(t, y(t)),$$

when we use Riemann-Liouville form of the fractional difference at $k = 1$, we have the form

$$\nabla_{a+1}^{-\alpha}\nabla\nabla_a^{-(1-\alpha)}y(t) = \nabla_{a+1}^{-\alpha}f(t, y(t)),$$

From Theorem (1.0.3) we have

$$\nabla\nabla_a^{-\alpha}\nabla_a^{-(1-\alpha)}y(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\nabla_a^{-(1-\alpha)}y(t)|_{t=a} = \nabla_{a+1}^{-\alpha}f(t, y(t)).$$

Then,

$$\nabla\nabla_a^{-\alpha}\nabla_a^{-(1-\alpha)}y(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a) + \nabla_{a+1}^{-\alpha}f(t, y(t)).$$

Apply Theorem (1.0.4) to have

$$\begin{aligned}
\nabla\nabla_a^{-\alpha}\nabla_a^{-(1-\alpha)}y(t) &= \nabla\nabla_a^{-(1-\alpha)}(\nabla_a^{-\alpha}y(t)) \\
&= \nabla\nabla_a^{\alpha-1}(\nabla_a^{-\alpha}y(t)) \\
&= \nabla_a^{\alpha}\nabla_a^{-\alpha}y(t) \\
&= \nabla_a^{0}y(t) = y(t).
\end{aligned}$$

So,

$$y(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a) + \nabla_{a+1}^{-\alpha}f(t, y(t)). \tag{2.3}$$

Conversely, assume that y has the representation Equation (2.3), and from applying Lemma (1.0.1)

$$\begin{aligned}
\nabla_a^{\alpha}(t-a+1)^{\overline{\alpha-1}} &= \nabla\nabla_a^{\alpha-1}(t-a+1)^{\overline{\alpha-1}} \\
&= \nabla\nabla_a^{-(1-\alpha)}(t-a+1)^{\overline{\alpha-1}} \\
&= \nabla\frac{\Gamma(\alpha-1+1)}{\Gamma(\alpha-1+1-\alpha+1)}(t-a+1)^{1-\alpha+\alpha-1} = \nabla\Gamma(\alpha) = 0
\end{aligned}$$

moreover,

$$\begin{aligned}
\nabla_{a+1}^{-\alpha}f(t,y(t)) &= \sum_{s=a+1}^{t}\frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(s,y(s)) \\
&= \sum_{s=a}^{t}\frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(s,y(s)) - \frac{(t-\rho(a))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a,y(a)) \\
&= \nabla_a^{-\alpha}f(t,y(t)) - \frac{(t-\rho(a))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a,y(a)) \\
\nabla_a^{\alpha}\nabla_{a+1}^{-\alpha}f(t,y(t)) &= \nabla_a^{\alpha}\nabla_a^{-\alpha}f(t,y(t)) - \nabla_a^{\alpha}\frac{(t-\rho(a))^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a,y(a)) \\
&= \nabla_a^{0}f(t,y(t)) - 0 \\
&= f(t,y(t)). \qquad \square
\end{aligned}$$

Thus, we have proved the following theorem [19].

Theorem 2.1.1 *[19] y is a solution of the IVP, equations((2.1), (2.2)) if and only if y has the form*

$$y(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a) + \nabla_{a+1}^{-\alpha}f(t,y(t)).$$

Lemma 2.1.2 *[69] For a function* $f : \mathbb{N}_{a+h,h} \to \mathbb{R}$*, and any* $0 < \alpha, h \leq 1$*, we have the following equality:*

$$ {}_a\nabla_h^{-\alpha}\nabla_h f(t) = \nabla_h \, {}_a\nabla_h^{-\alpha} f(t) - \frac{(t-a)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a). $$

Lemma 2.1.3 *[69] For a function* $y : \mathbb{N}_{a+h,h} \to \mathbb{R}$*, and any* $0 < \alpha, h \leq 1$*, we have the following equality:*

$$ {}_{a-h}\nabla_h^{\alpha} y(t) = {}_a\nabla_h^{\alpha} y(t) + \frac{(t-a+h)_h^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} y(a)h. $$

Theorem 2.1.4 *[69] For a function* $y : \mathbb{N}_{a+h,h} \to \mathbb{R}$*, and any* $0 < \alpha, h \leq 1$*, we have the following equality:*

$$ {}_a\nabla_h^{-\alpha} \, {}_{a-h}\nabla_h^{\alpha} y(t) = y(t) - \frac{h^{1-\alpha}}{\Gamma(\alpha)} (t-a+h)_h^{\overline{\alpha-1}} y(a). $$

Considering the following initial fractional difference equation:

$$ {}_{a-h}\nabla_h^{\alpha} y(t) = f(t, y(t)) \qquad for \quad t = a+h, a+2h, \dots \tag{2.4} $$

$$ {}_{a-h}\nabla_h^{-(1-\alpha)} y(t)|_{t=a} = h^{1-\alpha} y(a) = c, \tag{2.5} $$

where $0 < \alpha, h < 1$ and $a \in \mathbb{R}$.

Using Theorem (2.1.4), we can state the following theorem:

Theorem 2.1.5 *[69] y is a solution of the IVP, equations ((2.4), (2.5)) if and only if it has the form*

$$y(t) = \frac{(t-a+1)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} c +_a \nabla_h^{-\alpha} f(t, y(t)). \tag{2.6}$$

2.2 Discrete Fractional Sum Equations

Considering the following discrete fractional sum equation

$$y(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a) + \nabla_{a+1}^{-\alpha} f(t, y(t)).$$

If w and v satisfy

$$w(t) \geq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} w(a) + \nabla_{a+1}^{-\alpha} f(t, w(t)) \tag{2.7}$$

and

$$v(t) \leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} v(a) + \nabla_{a+1}^{-\alpha} f(t, v(t)) \tag{2.8}$$

then we replace $f(t, y(t))$ by $g(t)y(t)$ where $-1 < g(t) < 1$ for $t = a, a+1, a+2, \ldots$.

Firstly, the following theorem is valid if $g(t) < 1$, the condition $-1 < g(t) < 1$ applies to the method of solution of Equation (2.3); in particular, it will appear that the method of successive approximations converge if $-1 < g(t) < 1$. Secondly, we consider the inequalities (2.7) and (2.8) can be solved explicitly if $f(t, y(t)) = g(t)y(t)$.

Theorem 2.2.1 *[19] Let w and v satisfy the inequalities (2.7) and (2.8), respectively. If $v(a) \leq w(a)$, then $v(t) \leq w(t)$ for $t = a+1, a+2, \ldots$.*

Proof: Let $u(t) = v(t) - w(t)$.

Using the induction principle we prove that $u(t) \leq 0$ for $t = a, a+1, a+2, \ldots$

when $k = a$ we have from the condition $u(a) = v(a) - w(a) \leq 0$.

If $u(k) \leq 0$ for $k = a, a+1, \ldots, t-1$, then we have

$$\begin{aligned} u(t) &= v(t) - w(t) \leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} v(a) + \nabla_{a+1}^{-\alpha} b(t)v(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} w(a) - \nabla_{a+1}^{-\alpha} b(t)w(t) \\ &= \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} (v(a) - w(a)) + \sum_{k=a+1}^{t} \frac{(t-\rho(k))^{\overline{\alpha-1}}}{\Gamma(\alpha)} b(k)v(k) - \sum_{k=a+1}^{t} \frac{(t-\rho(k))^{\overline{\alpha-1}}}{\Gamma(\alpha)} b(k)w(k) \\ &\leq \sum_{k=a+1}^{t} \frac{(t-\rho(k))^{\overline{\alpha-1}}}{\Gamma(\alpha)} b(k)\left(v(k) - w(k)\right) \end{aligned}$$

since, $u(k) \leq 0$ for $s = a, a+1, \ldots, t-1$, then

$$\begin{aligned} u(t) &\leq \frac{(t-\rho(t))^{\overline{\alpha-1}}}{\Gamma(\alpha)} b(t)\left(v(t) - w(t)\right) \\ &= \frac{(t-t+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} b(t)\left(v(t) - w(t)\right) \\ &= b(t)\left(v(t) - w(t)\right) \\ v(t) - w(t) &\leq b(t)\left(v(t) - w(t)\right) \end{aligned}$$

hence we have

$$(1 - b(t))\left(v(t) - w(t)\right) \leq 0$$

then we obtain $v(t) - w(t) \leq 0$ Since $-1 < g(t) < 1$.

This completes the proof. □

Define $E_x\phi = \nabla_{a+1}^{-\alpha}x(t)\phi(t)$ and $F_x\phi = \nabla_a^{-\alpha}x(t)\phi(t)$.The discrete ML function can be written in term of $E_x\phi$ or $F_x\phi$ if ϕ is constant. In the References [?, 18], these discrete functions have been discussed and in [?, ?] they have been discussed for q-fractional case.

Lemma 2.2.2 *[19] The following inequality holds for any λ constant,*

$$|E_\lambda(t-a+1)^{\overline{\alpha-1}}| \leq F_{|\lambda|}(t-a+1)^{\overline{\alpha-1}},$$

where $t = a, a+1, a+2, \ldots.$

Proof: In this proof we use the fact that $\Gamma(y) > 0$ for $y > 0$.
We have

$$\begin{aligned} |E_\lambda(t-a+1)^{\overline{\alpha-1}}| &= |\nabla_{a+1}^{-\alpha}\lambda(t-a+1)^{\overline{\alpha-1}}| \\ &= |\lambda||\nabla_{a+1}^{-\alpha}(t-a+1)^{\overline{\alpha-1}}| \\ &= |\lambda|\nabla_{a+1}^{-\alpha}(t-a+1)^{\overline{\alpha-1}} \\ &= |\lambda| \sum_{s=a+1}^{t} \frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}(s-a+1)^{\overline{\alpha-1}} \end{aligned}$$

$$|E_\lambda(t-a+1)^{\overline{\alpha-1}}| = |\lambda|\left(\sum_{s=a}^{t}\frac{(t-\rho(s))^{\overline{\alpha-1}}}{\Gamma(\alpha)}(s-a+1)^{\overline{\alpha-1}} - \frac{(t-\rho(a))^{\overline{\alpha-1}}}{\Gamma(\alpha)}(1)^{\overline{\alpha-1}}\right)$$

$$= |\lambda|\left(\nabla_a^{-\alpha}(t-a+1)^{\overline{\alpha-1}} - (t-a+1)^{\overline{\alpha-1}}\right)$$

$$\leq |\lambda|\nabla_a^{-\alpha}(t-a+1)^{\overline{\alpha-1}}$$

$$= \nabla_a^{-\alpha}|\lambda|(t-a+1)^{\overline{\alpha-1}}$$

$$= F_{|\lambda|}(t-a+1)^{\overline{\alpha-1}},$$

where $t = a, a+1, a+2, \ldots$. □

Lemma 2.2.3 *[19] The following equality holds for any λ constant,*

$$F_\lambda^n(t-a+1)^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\lambda^n(t-a+1)^{\overline{(n+1)\alpha-1}},$$

where $n \in \mathbb{N}$.

Proof: Using the induction principle and Lemma (1.0.1) we obtain the proof.

for $n = 1$,

$$F_\lambda(t-a+1)^{\overline{\alpha-1}} = \nabla_a^{-\alpha}\lambda(t-a+1)^{\overline{\alpha-1}}$$

from Lemma (1.0.1):

$$F_\lambda(t-a+1)^{\overline{\alpha-1}} = \frac{\lambda\Gamma(\alpha-1+1)}{\Gamma(\alpha-1+\alpha+1)}(t-a+1)^{\overline{\alpha-1+\alpha}}$$

$$= \frac{\lambda\Gamma(\alpha)}{\Gamma(2\alpha)}(t-a+1)^{\overline{2\alpha-1}}$$

Assume it is true for n,

$$F_\lambda^n(t-a+1)^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\lambda^n(t-a+1)^{\overline{(n+1)\alpha-1}},$$

We need to prove it for $n+1$:

$$F_\lambda^{n+1}(t-a+1)^{\overline{\alpha-1}} = F_\lambda F_\lambda^n(t-a+1)^{\overline{\alpha-1}}$$

$$= F_\lambda\frac{\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\lambda^n(t-a+1)^{\overline{(n+1)\alpha-1}}$$

$$= \frac{\lambda^n\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}F_\lambda(t-a+1)^{\overline{(n+1)\alpha-1}}$$

$$= \frac{\lambda^n\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\nabla_a^{-\alpha}\lambda(t-a+1)^{\overline{\alpha-1}}$$

$$= \frac{\lambda^{n+1}\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\frac{\Gamma((n+1)\alpha-1+1)}{\Gamma((n+1)\alpha-1+\alpha+1)}(t-a+1)^{\overline{(n+1)\alpha-1+\alpha}}$$

$$= \frac{\Gamma(\alpha)}{\Gamma((n+1)\alpha+\alpha)}\lambda^{n+1}(t-a+1)^{\overline{(n+2)\alpha-1}}$$

□

Lemma 2.2.4 *[19] If* $|y(t)| \leq \lambda$ *for* $t = a, a+1, a+2, \ldots$ *where* $\lambda > 0$*, then*

$$|E_y^n(t-a+1)^{\overline{\alpha-1}}| \leq F_\lambda^n(t-a+1)^{\overline{\alpha-1}}, \qquad n \in \mathbb{N}.$$

Proof: We prove this lemma by using the induction principle.
for $n = 1$ the inequality is valid from Lemma (2.2.2):

$$|E_y(t-a+1)^{\overline{\alpha-1}}| \leq F_{|y|}(t-a+1)^{\overline{\alpha-1}} \leq F_\lambda(t-a+1)^{\overline{\alpha-1}}$$

Let's assume that $|E_y^s(t-a+1)^{\overline{\alpha-1}}| \leq F_\lambda^s(t-a+1)^{\overline{\alpha-1}}$ is valid for $s = 1, 2, \ldots, n-1$.
we have for $s = n$

$$\begin{aligned}
|E_y^n(t-a+1)^{\overline{\alpha-1}}| &= |E_y E_y^{n-1}(t-a+1)^{\overline{\alpha-1}}| \\
&\leq |E_y F_\lambda^{n-1}(t-a+1)^{\overline{\alpha-1}} \\
&\leq F_\lambda F_\lambda^{n-1}(t-a+1)^{\overline{\alpha-1}} \\
&= F_\lambda^n(t-a+1)^{\overline{\alpha-1}}
\end{aligned}$$

□

Theorem 2.2.5 *[19] If* $|x(t)| < 1$ *for* $t \in \mathbb{N}_a \cap [a, b]$*, then the discrete fractional sum equation*

$$y(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a) + \nabla_{a+1}^{-\alpha} x(t) y(t)$$

for $t \in \mathbb{N}_a \cap [a, b]$*, where* $b \in \mathbb{R}$*, has a solution*

$$y(t) = \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}}.$$

Proof: The method of successive approximations can be used to find the solution.

Set

$$y_0(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a),$$

$$y_n(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a) + \nabla_{a+1}^{-\alpha} x(t) y_{n-1}(t), \quad \text{for } n \geq 1.$$

now

$$
\begin{aligned}
y_1(t) &= y_0(t) + \nabla_{a+1}^{-\alpha} x(t) y_0(t) \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + E_x y_0 \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x (t-a+1)^{\overline{\alpha-1}}.
\end{aligned}
$$

$$
\begin{aligned}
y_2(t) &= y_0(t) + \nabla_{a+1}^{-\alpha} x(t) y_1(t) \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \nabla_{a+1}^{-\alpha} x(t) \left(y_0(t) + \nabla_{a+1}^{-\alpha} x(t) y_0(t) \right) \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \nabla_{a+1}^{-\alpha} x(t) y_0(t) + \nabla_{a+1}^{-\alpha} x(t) \left(\nabla_{a+1}^{-\alpha} x(t) y_0(t) \right) \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x (t-a+1)^{\overline{\alpha-1}} + \nabla_{a+1}^{-\alpha} x(t) \left(\frac{y(a)}{\Gamma(\alpha)} E_x (t-a+1)^{\overline{\alpha-1}} \right) \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x E_x (t-a+1)^{\overline{\alpha-1}} \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x^2 (t-a+1)^{\overline{\alpha-1}}.
\end{aligned}
$$

Using the induction principle we have

$$y_n(t) = \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{n} E_x^k (t-a+1)^{\overline{\alpha-1}} \quad n = 0, 1, 2, \dots.$$

Let's assume that $y_s(t) = \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{s} E_x^k (t-a+1)^{\overline{\alpha-1}}$ is valid for $s = 0, 1, \dots, n-1$.

For $s = n$, we have

$$\begin{aligned}
y_n(t) &= \frac{y(a)}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \nabla_{a+1}^{-\alpha} x(t) y_{n-1}(t) \\
&= \frac{y(a)}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} \nabla_{a+1}^{-\alpha} x(t) \left(\sum_{k=0}^{n-1} E_x^k (t-a+1)^{\overline{\alpha-1}} \right) \\
&= \frac{y(a)}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} E_x \sum_{k=1}^{n} E_x^{k-1} (t-a+1)^{\overline{\alpha-1}} \\
&= \frac{y(a)}{\Gamma(\alpha)} E_x^0 (t-a+1)^{\overline{\alpha-1}} + \frac{y(a)}{\Gamma(\alpha)} \sum_{k=1}^{n} E_x E_x^{k-1} (t-a+1)^{\overline{\alpha-1}} \\
&= \frac{y(a)}{\Gamma(\alpha)} \left(E_x^0 (t-a+1)^{\overline{\alpha-1}} + \sum_{k=1}^{n} E_x^k (t-a+1)^{\overline{\alpha-1}} \right) \\
y_n(t) &= \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{n} E_x^k (t-a+1)^{\overline{\alpha-1}}
\end{aligned}$$

taking the limit of $y_n(t)$ as $n \to \infty$, we get

$$y(t) = \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}}. \tag{2.9}$$

where $y(t)$ is the solution of the discrete fractional sum equation.

The absolute convergence of the series (2.9) can be shown by using Lemmas

(2.2.2) and (2.2.4), the comparison test in addition to the d'Alembert ratio test.

$$
\begin{aligned}
y(t) &= \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_x^k (t-a+1)^{\overline{\alpha-1}} \\
&\leq \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \left| E_x^k (t-a+1)^{\overline{\alpha-1}} \right| \\
&\leq \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} F_{|x|}^k (t-a+1)^{\overline{\alpha-1}} \\
&\leq \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} F_{\lambda}^k (t-a+1)^{\overline{\alpha-1}} \\
&= \frac{y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(k\alpha+\alpha)} \lambda^k (t-a+1)^{\overline{(k+1)\alpha-1}} \\
&= y(a) \sum_{k=0}^{\infty} \lambda^k \frac{(t-a+1)^{\overline{(k+1)\alpha-1}}}{\Gamma(k\alpha+\alpha)}.
\end{aligned}
$$

Indeed, the following series

$$
\sum_{n=0}^{\infty} \lambda^n \frac{(t-a+1)^{\overline{(n+1)\alpha-1}}}{\Gamma(n\alpha+\alpha)}
$$

where $0 \leq \lambda < 1$, is convergent.

Let

$$
a_n = \lambda^n \frac{(t-a+1)^{\overline{(n+1)\alpha-1}}}{\Gamma(n\alpha+\alpha)} = \lambda^n \frac{\Gamma(t-a)}{B(t-a,(n+1)\alpha)} \frac{1}{\Gamma(t-a+1)}
$$

.

Then we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\lambda\frac{B(t-a,(n+1)\alpha)}{B(t-a,(n+2)\alpha)}\right|$$

, where $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. We will use the approximation formula (called a Stiriling approximation formula) for the Beta function

$$B(x,\epsilon) \sim \Gamma(\epsilon)x^{-\epsilon}$$

, where x is large and ϵ is fixed. Now, our gool is to show that

$$\lim_{x\to\infty}\frac{B(x,\epsilon)}{\Gamma(\epsilon)x^{-\epsilon}} = 1$$

In other words we have to show that

$$\lim_{x\to\infty} x^{\epsilon}B(x,\epsilon) = \Gamma(\epsilon)$$

. From the definition of Beta function we have

$$x^{\epsilon}B(x,\epsilon) = \Gamma(\epsilon)x^{\epsilon}\frac{\Gamma(x)}{\Gamma(x+\epsilon)}$$

where the Gamma function is $\Gamma(x) \approx \sqrt{2\pi}\ x^{(x+\frac{1}{2})}e^{-x}$, from that we have

$$\Gamma(x+\epsilon) \approx \sqrt{2\pi}\ (x+\epsilon)^{(x+\epsilon+\frac{1}{2})}e^{-(x+\epsilon)}$$

. Also, the exponential is defined by

$$e^{\epsilon} = \lim_{x\to\infty} \left(1+\frac{\epsilon}{x}\right)^{x}$$

.

$$(x+\epsilon)^{(x+\epsilon+\frac{1}{2})} = \left(x\left(1+\frac{\epsilon}{x}\right)\right)^{(x+\epsilon+\frac{1}{2})} = x^{(x+\epsilon+\frac{1}{2})}\left(1+\frac{\epsilon}{x}\right)^{(x+\epsilon+\frac{1}{2})} = x^{(x+\frac{1}{2})}x^{\epsilon}\left(1+\frac{\epsilon}{x}\right)^{x}\left(1+\frac{\epsilon}{x}\right)^{(\epsilon+\frac{1}{2})}$$

So,

$$\Gamma(x+\epsilon) \approx \sqrt{2\pi}\ x^{(x+\frac{1}{2})}x^{\epsilon}\left(1+\frac{\epsilon}{x}\right)^{x}\left(1+\frac{\epsilon}{x}\right)^{(\epsilon+\frac{1}{2})} e^{-(x+\epsilon)}.$$

Then we have

$$\begin{aligned}
\lim_{x\to\infty}\frac{x^{\epsilon}\Gamma(x)}{\Gamma(x+\epsilon)} &= \lim_{x\to\infty}\frac{x^{\epsilon}\sqrt{2\pi}\ x^{(x+\frac{1}{2})}e^{-x}}{\sqrt{2\pi}\ x^{(x+\frac{1}{2})}x^{\epsilon}\left(1+\frac{\epsilon}{x}\right)^{x}\left(1+\frac{\epsilon}{x}\right)^{(\epsilon+\frac{1}{2})} e^{-(x+\epsilon)}} \\
&= e^{\epsilon}\lim_{x\to\infty}\frac{1}{\left(1+\frac{\epsilon}{x}\right)^{x}\left(1+\frac{\epsilon}{x}\right)^{(\epsilon+\frac{1}{2})}} \\
&= e^{\epsilon}\frac{1}{e^{\epsilon}\times 1} = 1
\end{aligned}$$

Now, we have

$$\lim_{n\to\infty}\frac{B(t-a,(n+1)v)}{\Gamma(t-a)((n+1)\alpha)^{-(t-a)}} = 1,$$

$$\lim_{n\to\infty}\frac{B(t-a,(n+2)v)}{\Gamma(t-a)((n+2)\alpha)^{-(t-a)}} = 1.$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\lambda\frac{\Gamma(t-a)((n+1)\alpha)^{-(t-a)}}{\Gamma(t-a)((n+2)\alpha)^{-(t-a)}}\right| \\
&= \lim_{n\to\infty}\lambda\left(\frac{(n+2)\alpha}{(n+1)\alpha}\right)^{(t-a)} \\
&= \lambda\left(\lim_{n\to\infty}\frac{n+2}{n+1}\right)^{(t-a)} \\
&= \lambda\left(\lim_{n\to\infty}\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)^{(t-a)} = \lambda.1 = \lambda < 1.
\end{aligned}$$

□

2.2.1 Discrete Fractional Sum Equations on $h\mathbb{Z}$

Consider the following h-discrete fractional sum equation on $h\mathbb{Z}$

$$y(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}y(a) +_a \nabla_h^{-\alpha}f(t,y(t)).$$

Assume that w and v satisfy the following inequalities respectively

$$w(t) \geq \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}w(a) +_a \nabla_h^{-\alpha}f(t,w(t)) \tag{2.10}$$

and

$$v(t) \leq \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h^{1-\alpha}v(a) +_a \nabla_h^{-\alpha}f(t,v(t)) \tag{2.11}$$

then we put $f(t,y(t)) = g(t)y(t)$ where $-1 < g(t) < 1$ for $t = a, a+1, a+2, \ldots$.

Theorem 2.2.6 *Let w and v satisfy the inequalities (2.10) and (2.11), respectively. If $v(a) \leq w(a)$, then $v(t) \leq w(t)$ for $t = a+1, a+2, \ldots$.*

Proof: Let $u(t) = v(t) - w(t)$.

Using the induction principle, we prove that $u(t) \leq 0$ for $t = a, a+1, a+2, \ldots$

When $s = a$ we have $u(a) = v(a) - w(a) \leq 0$ from the condition.

If $u(k) \leq 0$ for $k = a, a+1, \ldots, t-1$, then we have

$$\begin{aligned}
u(t) &= v(t) - w(t) \\
&\leq \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} v(a) +_a \nabla_h^{-\alpha} f(t, v(t)) - \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} w(a) -_a \nabla_h^{-\alpha} f(t, w(t)) \\
&= \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} \left(v(a) - w(a)\right) + \sum_{k=a/h+1}^{t/h} \frac{(t-\rho_h(kh))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} hb(kh) \left(v(kh) - w(kh)\right) \\
&\leq \frac{(t-\rho_h(t))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} hb(t) \left(v(t) - w(t)\right)
\end{aligned}$$

$$\begin{aligned}
v(t) - w(t) &= \frac{(h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} hb(t) \left(v(t) - w(t)\right) \\
&= \frac{h^\alpha \Gamma(1+\alpha)}{\Gamma(\alpha)\Gamma(1)} hb(t) \left(v(t) - w(t)\right) \\
&= \alpha h^{\alpha+1} b(t) \left(v(t) - w(t)\right)
\end{aligned}$$

$$v(t) - w(t) \leq \alpha h^{\alpha+1} b(t) \left(v(t) - w(t)\right)$$

Hence we have

$$\left(1-\alpha h^{\alpha+1}b(t)\right)(v(t)-w(t)) \;\leq\; 0$$

We obtain $v(t)-w(t)\leq 0$ since $-1<b(t)<1, 0<\alpha<1$ and $0<h<1$.
This completes the proof. □

Define ${}_xE_h\psi = {}_{a+1}\nabla_h^{-\alpha}x(t)\psi(t)$ and ${}_xF_h\psi = {}_a\nabla_h^{-\alpha}x(t)\psi(t)$.

Lemma 2.2.7 *The following inequality holds for any λ constant,*

$$|{}_\lambda E_h(t-a+1)_h^{\overline{\alpha-1}}| \leq {}_{|\lambda|}F_h(t-a+1)_h^{\overline{\alpha-1}},$$

where $t=a,a+1,a+2,\ldots$.

Proof: Since $\Gamma(y)>0$ for $y>0$, we have:

$$\begin{aligned}
|{}_\lambda E_h(t-a+1)_h^{\overline{\alpha-1}}| &= |{}_{a+1}\nabla_h^{-\alpha}\lambda(t-a+1)_h^{\overline{\alpha-1}}| \\
&= |\lambda||{}_{a+1}\nabla_h^{-\alpha}(t-a+1)_h^{\overline{\alpha-1}}| \\
&= |\lambda|{}_{a+1}\nabla_h^{-\alpha}(t-a+1)_h^{\overline{\alpha-1}} \\
&= |\lambda|\sum_{s=\frac{(a+1)}{h}+1}^{t/h}\frac{(t-\rho_h(sh))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h(sh-a+1)_h^{\overline{\alpha-1}} \\
&= |\lambda|\sum_{s=\frac{a}{h}+1}^{\frac{t-1}{h}}\frac{(t-\rho_h(sh+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}h(sh+1-a+1)_h^{\overline{\alpha-1}}
\end{aligned}$$

$$
\begin{aligned}
|{}_\lambda E_h(t-a+1)_h^{\overline{\alpha-1}}| &= |\lambda|\Big(\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \frac{(t-\rho_h(sh+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(sh-a+2)_h^{\overline{\alpha-1}} \\
&\qquad - \frac{(t-\rho_h(t+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(t-a+2)_h^{\overline{\alpha-1}}\Big) \\
&= |\lambda|\Big(\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \frac{(t-\rho_h(sh+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(sh-a+2)_h^{\overline{\alpha-1}} \\
&\qquad - \frac{(t-t-1+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(t-a+2)_h^{\overline{\alpha-1}}\Big) \\
&= |\lambda|\left(\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \frac{(t-\rho_h(sh+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(sh-a+2)_h^{\overline{\alpha-1}} - h^{\alpha-1}\frac{\Gamma\left(\frac{h-1}{h}+\alpha-1\right)}{\Gamma\left(\frac{h-1}{h}\right)}\right) \\
&= |\lambda|\left(\sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \frac{(t-\rho_h(sh+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(sh-a+2)_h^{\overline{\alpha-1}} - h^{\alpha-1}\frac{\Gamma\left(\alpha-\frac{1}{h}\right)}{\Gamma\left(1-\frac{1}{h}\right)}\right) \\
&\leq |\lambda| \sum_{s=\frac{a}{h}+1}^{\frac{t}{h}} \frac{(t-\rho_h(sh+1))_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h(sh-a+2)_h^{\overline{\alpha-1}}
\end{aligned}
$$

$$
|{}_\lambda E_h(t-a+1)_h^{\overline{\alpha-1}}| \leq {}_{|\lambda|}F_h(t-a+1)_h^{\overline{\alpha-1}}
$$

□

Lemma 2.2.8 *For any λ constant, we have*

$$
{}_\lambda F_h^n(t-a+1)_h^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\lambda^n(t-a+1)_h^{\overline{(n+1)\alpha-1}}, \qquad n \in \mathbb{N}.
$$

Proof: Using the induction principle and Lemma (1.0.2), we obtain the proof. For $n=1$,

$$
{}_\lambda F_h(t-a+1)_h^{\overline{\alpha-1}} = {}_a\nabla_h^{-\alpha}\lambda(t-a+1)_h^{\overline{\alpha-1}}
$$

from Lemma (1.0.2):

$$ {}_{\lambda}F_h(t-a+1)_h^{\overline{\alpha-1}} = \frac{\lambda\Gamma(\alpha-1+1)}{\Gamma(\alpha-1+\alpha+1)}(t-a+1)_h^{\overline{\alpha-1+\alpha}} $$

$$ = \frac{\lambda\Gamma(\alpha)}{\Gamma(2\alpha)}(t-a+1)_h^{\overline{2\alpha-1}} $$

Assume it is true for n,

$$ {}_{\lambda}F_h^n(t-a+1)_h^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\lambda^n(t-a+1)_h^{\overline{(n+1)\alpha-1}} $$

We need to prove it for $n+1$:

$$ {}_{\lambda}F_h^{n+1}(t-a+1)_h^{\overline{\alpha-1}} = {}_{\lambda}F_h\ {}_{\lambda}F_h^n(t-a+1)_h^{\overline{\alpha-1}} $$

$$ = {}_{\lambda}F_h\frac{\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\lambda^n(t-a+1)_h^{\overline{(n+1)\alpha-1}} $$

$$ = \frac{\lambda^n\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\,{}_{\lambda}F_h(t-a+1)_h^{\overline{(n+1)\alpha-1}} $$

$$ = \frac{\lambda^n\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\,{}_{a}\nabla_h^{-\alpha}\lambda(t-a+1)_h^{\overline{\alpha-1}} $$

$$ = \frac{\lambda^{n+1}\Gamma(\alpha)}{\Gamma(n\alpha+\alpha)}\frac{\Gamma((n+1)\alpha-1+1)}{\Gamma((n+1)\alpha-1+\alpha+1)}(t-a+1)_h^{\overline{(n+1)\alpha-1+\alpha}} $$

$$ {}_{\lambda}F_h^{n+1}(t-a+1)_h^{\overline{\alpha-1}} = \frac{\Gamma(\alpha)}{\Gamma((n+1)\alpha+\alpha)}\lambda^{n+1}(t-a+1)_h^{\overline{(n+2)\alpha-1}} $$

□

Theorem 2.2.9 *If $|x(t)| < 1$ for $t \in \mathbb{N}_a \cap [a,b]$, then the h-discrete fractional sum equation*

$$y(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} y(a) + {}_a\nabla_h^{-\alpha} x(t) y(t)$$

for $t \in \mathbb{N}_a \cap [a,b]$, where $b \in \mathbb{R}$, has a solution

$$y(t) = \frac{h^{1-\alpha} y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} {}_x F_h^k (t-a+h)_h^{\overline{\alpha-1}}.$$

Proof: The method of successive approximations can be used to find the solution. Set

$$y_0(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} y(a),$$

$$y_n(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} y(a) + {}_a\nabla_h^{-\alpha} x(t) y_{n-1}(t), \qquad \text{for } n \geq 1.$$

Now,

$$\begin{aligned}
y_1(t) &= y_0(t) + {}_a\nabla_h^{-\alpha} x(t) y_0(t) \\
&= \frac{h^{1-\alpha} y(a)}{\Gamma(\alpha)} {}_x F_h^0 (t-a+h)_h^{\overline{\alpha-1}} + {}_x F_h y_0(t) \\
&= \frac{h^{1-\alpha} y(a)}{\Gamma(\alpha)} {}_x F_h^0 (t-a+h)_h^{\overline{\alpha-1}} + {}_x F_h \frac{h^{1-\alpha} y(a)}{\Gamma(\alpha)} (t-a+h)_h^{\overline{\alpha-1}} \\
&= \frac{h^{1-\alpha} y(a)}{\Gamma(\alpha)} {}_x F_h^0 (t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha} y(a)}{\Gamma(\alpha)} {}_x F_h (t-a+h)_h^{\overline{\alpha-1}}.
\end{aligned}$$

$$
\begin{aligned}
y_2(t) &= y_0(t) + {}_a\nabla_h^{-\alpha} x(t) y_1(t) \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + {}_a\nabla_h^{-\alpha} x(t) \left(y_0(t) + {}_a\nabla_h^{-\alpha} x(t) y_0(t)\right) \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + {}_a\nabla_h^{-\alpha} x(t) y_0(t) + {}_a\nabla_h^{-\alpha} x(t) \left({}_a\nabla_h^{-\alpha} x(t) y_0(t)\right) \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h(t-a+h)_h^{\overline{\alpha-1}} \\
&\quad + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_a\nabla_h^{-\alpha} x(t) \left({}_a\nabla_h^{-\alpha} x(t)(t-a+h)_h^{\overline{\alpha-1}}\right) \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h(t-a+h)_h^{\overline{\alpha-1}} \\
&\quad + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_a\nabla_h^{-\alpha} x(t)\ {}_xF_h(t-a+h)_h^{\overline{\alpha-1}} \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h(t-a+h)_h^{\overline{\alpha-1}} \\
&\quad + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h\ {}_xF_h(t-a+h)_h^{\overline{\alpha-1}} \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^2(t-a+h)_h^{\overline{\alpha-1}}
\end{aligned}
$$

Using the induction principle, we have

$$
y_n(t) = \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)} \sum_{k=0}^{n} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}} \quad n = 0, 1, 2, \ldots.
$$

Assume $y_s(t) = \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)} \sum_{k=0}^{s} {}_xF_h^s(t-a+h)_h^{\overline{\alpha-1}}$ is valid for $s = 0, 1, \ldots, n-1$.

For $s = n$, we have

$$\begin{aligned}
y_n(t) &= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}} + {}_a\nabla_h^{-\alpha}x(t)y_{n-1}(t) \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_a\nabla_h^{-\alpha}x(t)\left(\sum_{k=0}^{n-1} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}}\right) \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h \sum_{k=1}^{n} {}_xF_h^{k-1}(t-a+h)_h^{\overline{\alpha-1}} \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\ {}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)} \sum_{k=1}^{n} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}} \\
&= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\left({}_xF_h^0(t-a+h)_h^{\overline{\alpha-1}} + \sum_{k=1}^{n} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}}\right) \\
y_n(t) &= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)} \sum_{k=0}^{n} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}}
\end{aligned}$$

taking the limit of $y_n(t)$ as $n \to \infty$, we get

$$y(t) = \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}}. \tag{2.12}$$

where $y(t)$ is the solution of the h-discrete fractional sum equation.

The absolute convergence of the series (2.12) can be shown by using Lemmas

(2.2.7) and(2.2.8), the comparison test in addition to the d'Alembert ratio test.

$$\begin{aligned} y(t) &= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\sum_{k=0}^{\infty} {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}} \\ &\leq \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\left| {}_xF_h^k(t-a+h)_h^{\overline{\alpha-1}}\right| \\ &\leq \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\sum_{k=0}^{\infty} {}_{|x|}F_h^k(t-a+h)_h^{\overline{\alpha-1}} \\ &\leq \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\sum_{k=0}^{\infty} {}_{\lambda}F_h^k(t-a+h)_h^{\overline{\alpha-1}} \\ &= \frac{h^{1-\alpha}y(a)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha)}{\Gamma(k\alpha+\alpha)}\lambda^k(t-a+h)_h^{\overline{(k+1)\alpha-1}} \\ &= h^{1-\alpha}y(a)\sum_{k=0}^{\infty}\lambda^k\frac{(t-a+h)_h^{\overline{(k+1)\alpha-1}}}{\Gamma(k\alpha+\alpha)} \end{aligned}$$

Indeed, the following series

$$\sum_{n=0}^{\infty}\lambda^n\frac{(t-a+h)_h^{\overline{(n+1)\alpha-1}}}{\Gamma(n\alpha+\alpha)}$$

where $0 \leq \lambda < 1$, is convergent.

Let

$$b_n = \lambda^n\frac{(t-a+h)_h^{\overline{(n+1)\alpha-1}}}{\Gamma(n\alpha+\alpha)}$$

From Definition (1.0.3) we have

$$\begin{aligned} b_n &= \lambda^n h^{((n+1)\alpha-1)} \frac{\Gamma((\frac{t-a+h}{h}) + (n+1)\alpha - 1)}{\Gamma(\frac{t-a+h}{h})\Gamma((n+1)\alpha)} \\ &= h^{\alpha-1}\lambda^n h^{n\alpha} \frac{\Gamma((\frac{t-a}{h}) + (n+1)\alpha)}{\Gamma(\frac{t-a+h}{h})\Gamma((n+1)\alpha)} \frac{\Gamma(\frac{t-a}{h})}{\Gamma(\frac{t-a}{h})} \\ &= h^{\alpha-1}(\lambda h^\alpha)^n \frac{\Gamma(\frac{t-a}{h})}{B(\frac{t-a}{h}, (n+1)\alpha)} \frac{1}{\Gamma(\frac{t-a+h}{h})} \end{aligned}$$

Then we have

$$\frac{b_{n+1}}{b_n} = \frac{h^{\alpha-1}(\lambda h^\alpha)^{n+1}\Gamma(\frac{t-a}{h})}{B(\frac{t-a}{h}, (n+2)\alpha)\Gamma(\frac{t-a+h}{h})} \frac{B(\frac{t-a}{h}, (n+1)\alpha)\Gamma(\frac{t-a+h}{h})}{h^{\alpha-1}(\lambda h^\alpha)^n \Gamma(\frac{t-a}{h})}$$

So,

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \lambda h^\alpha \frac{B(\frac{t-a}{h}, (n+1)\alpha)}{B(\frac{t-a}{h}, (n+2)\alpha)} \right|$$

,

where $B(x, y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. We use the approximation formula (called a Stiriling approximation formula) for the Beta function

$$B(x, \epsilon) \sim \Gamma(\epsilon) x^{-\epsilon}$$

, where x is large and ϵ is fixed. Now, our goal is to show that

$$\lim_{x\to\infty} \frac{B(x, \epsilon)}{\Gamma(\epsilon) x^{-\epsilon}} = 1$$

Then we have

$$\lim_{n\to\infty} \frac{B(\frac{t-a}{h},(n+1)\alpha)}{\Gamma(\frac{t-a}{h})((n+1)\alpha)^{-(\frac{t-a}{h})}} = 1,$$

$$\lim_{n\to\infty} \frac{B(\frac{t-a}{h},(n+2)\alpha)}{\Gamma(\frac{t-a}{h})((n+2)\alpha)^{-(\frac{t-a}{h})}} = 1.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| &= \lim_{n\to\infty}\left|\lambda h^\alpha \frac{\Gamma(\frac{t-a}{h})((n+1)\alpha)^{-(\frac{t-a}{h})}}{\Gamma(\frac{t-a}{h})((n+2)\alpha)^{-(\frac{t-a}{h})}}\right| \\ &= \lim_{n\to\infty} \lambda h^\alpha \left(\frac{(n+2)\alpha}{(n+1)\alpha}\right)^{(\frac{t-a}{h})}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| &= \lambda h^\alpha \left(\lim_{n\to\infty} \frac{n+2}{n+1}\right)^{(\frac{t-a}{h})} \\ &= \lambda h^\alpha \left(\lim_{n\to\infty} \frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right)^{(\frac{t-a}{h})} = \lambda h^\alpha.1 = \lambda h^\alpha < 1.\end{aligned}$$

□

Chapter Three

Gronwall's Inequality in Discrete Fractional Calculus

In this chapter, we introduce theorem which is parallel to the Gronwall's inequality in discrete fractional calculus and in h-discrete fractional calculus. Also, we obtain the Gronwall's inequality in discrete calculus and in h-discrete calculus. We have an application on the Gronwall's inequality.

Theorem 3.0.1 *[19] If v and y are nonnegative real valued functions where $0 \leq y(t) < 1$ for all $t \in \mathbb{N}_a$ and*

$$v(t) \leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} v(a) + \nabla_{a+1}^{-\alpha} v(t) y(t),$$

then

$$v(t) \leq \frac{v(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} E_y^k (t-a+1)^{\overline{\alpha-1}}.$$

The Gronwall's inequality in discrete calculus will be obtained from Theorem (3.0.1) if $\alpha = 1$. As far as we know, the Gronwall's inequality is not mentioned explicitly elsewhere.

Corollary 3.0.1 *[19] Let* $0 \le y(t) < 1$. *If*

$$v(t) \le v(a) + \sum_{s=a+1}^{t} y(s)v(s),$$

then

$$v(t) \le v(a) \ \hat{e}_y(t,a),$$

where $\hat{e}_y(t,a)$ *is the nabla exponential function for time scale* $\mathbb{T} = \mathbb{Z}$.

3.1 Gronwall's Inequality in h-Discrete Fractional Calculus

Theorem 3.1.1 *If* v *and* y *be nonnegative real valued functions where* $0 \le y(t) < 1$ *for all* $t \in \mathbb{N}_a$ *and*

$$v(t) \le \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} h^{1-\alpha} v(a) + \ {}_a\nabla_h^{-\alpha} v(t) y(t),$$

then

$$v(t) \le \frac{h^{1-\alpha} v(a)}{\Gamma(\alpha)} \sum_{k=0}^{\infty} \ {}_yF_h^k (t-a+h)_h^{\overline{\alpha-1}}.$$

The Gronwall's inequality in h-discrete calculus will be obtained from Theorem 3.1.1 if $\alpha = 1$.

Corollary 3.1.1 *Let* $0 \le y(t) < 1$. *If*

$$v(t) \le v(a) + \sum_{s=a/h+1}^{t/h} h \ v(sh) y(sh)$$

then

$$v(t) \leq v(a) \ {}_h\hat{e}_y(t,a),$$

where ${}_h\hat{e}_y(t,a)$ *is the nabla exponential function for time scale* $\mathbb{T} = \mathbb{Z}$.

Proof: From Theorem (3.1.1) we have

$$v(t) \leq v(a) \sum_{k=0}^{\infty} {}_yF_h^k 1.$$

We claim that $\sum_{k=0}^{\infty} {}_yF_h^k 1 = {}_h\hat{\mathrm{e}}_y(t,a)$. Where ${}_h\hat{\mathrm{e}}_y(t,a)$ is the unique solution of the following initial value problem

$$\nabla_h x(t) = y(t)x(t), \qquad x(a) = 1.$$

Thus, we show that $\sum_{k=0}^{\infty} {}_yF_h^k 1$ satisfies this initial value problem

$$\nabla_h x(t) = y(t)x(t), x(a) = 1.$$

Indeed,

$$\begin{aligned}
\nabla_h \sum_{k=0}^{\infty} {}_yF_h^k 1 &= \sum_{k=0}^{\infty} \nabla_h \ {}_yF_h^k 1 \\
&= \sum_{k=1}^{\infty} \nabla_h \ {}_yF_h(\ {}_yF_h^{k-1} 1) \\
&= \sum_{k=1}^{\infty} \nabla_h \ {}_a\nabla_h^{-1} y(t) \ {}_yF_h^{k-1} 1 \\
&= y(t) \sum_{k=0}^{\infty} {}_yF_h^k 1
\end{aligned}$$

and $\sum_{k=0}^{\infty}\ {}_yF_h^k1(a)=1$. Since the above IVP has a unique solution, we obtain the result. So, we have $\sum_{k=0}^{\infty}\ {}_yF_h^k1=\ {}_h\hat{\mathrm{e}}_y(t,a)$. □

3.2 An Application of the Gronwall's Inequality

In this section, we will state an application of the Gronwall's inequality on discrete calculus and on h-discrete calculus.

The application on $\mathbb{Z}$ is
Suppose that $f(t,y)$ satisfies a Lipschitz condition(1.0.8) with constant $0\leq K<1$ for every t and y.
Let the function $\varphi(t)$ satisfies

$$\nabla_a^{\alpha}\varphi(t)=f(t,\varphi(t)) \quad for \quad t=a+1,a+2,\ldots, \tag{3.1}$$

$$\nabla_a^{-(1-\alpha)}\varphi(t)|_{t=a}=\gamma \tag{3.2}$$

and the function ψ satisfies

$$\nabla_a^{\alpha}\psi(t)=f(t,\psi(t)) \quad for \quad t=a+1,a+2,\ldots, \tag{3.3}$$

$$\nabla_a^{-(1-\alpha)}\psi(t)|_{t=a}=\beta \tag{3.4}$$

We use the equivalent summation equation (2.3) to have

$$\varphi(t)=\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\gamma+\nabla_{a+1}^{-\alpha}f(t,\varphi(t))$$

$$\psi(t)=\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}\beta+\nabla_{a+1}^{-\alpha}f(t,\psi(t))$$

We take the difference of φ and ψ to have

$$\varphi(t) - \psi(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(\gamma - \beta) + \nabla_{a+1}^{-\alpha}(f(t,\varphi(t)) - f(t,\psi(t))).$$

Next, we obtain the following:

$$|\varphi(t) - \psi(t)| = \left| \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}(\gamma - \beta) + \nabla_{a+1}^{-\alpha}(f(t,\varphi(t)) - f(t,\psi(t))) \right|$$

$$\leq \left| \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \right| |\gamma - \beta| + |\nabla_{a+1}^{-\alpha}(f(t,\varphi(t)) - f(t,\psi(t)))|$$

$$\leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} |\gamma - \beta| + K\nabla_{a+1}^{-\alpha}|\varphi(t) - \psi(t)|.$$

Using the Gronwall's inequality and Lemma (2.2.3) we have

$$|\varphi(t) - \psi(t)| \leq \frac{|\gamma - \beta|}{\Gamma(\alpha)} \sum_{i=0}^{\infty} E_K^i (t-a+1)^{\overline{\alpha-1}}$$

$$= |\gamma - \beta| \sum_{i=0}^{\infty} \frac{K^i}{\Gamma(i\alpha + \alpha)}(t-a+1)^{\overline{(i+1)\alpha-1}}$$

$$= |\gamma - \beta|(t-a+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(K(t-a+\alpha)^{\overline{\alpha}}),$$

where $F_{\alpha,\alpha}$ is the discrete ML function in Definition 1.0.6.

Next we replace equations (3.3) and (3.4) by

$$\nabla_a^{\alpha}\psi(t) = f(t, \psi(t)) \quad for \quad t = a+1, a+2, \ldots, \tag{3.5}$$

$$\nabla_a^{-(1-\alpha)}\psi(t)|_{t=a} = \gamma_n, \tag{3.6}$$

where $\gamma_n \longrightarrow \gamma$.

Let ψ_n be the solution of equations (3.5) and (3.6) and consider the fixed interval $[a, a+T] \cap \mathbb{N}_a$. After that we have

$$\begin{aligned}|\varphi(t) - \psi_n(t)| &\leq |\gamma - \gamma_n|(t-a+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(K(t-a+\alpha)^{\overline{\alpha}}) \\ &\leq |\gamma - \gamma_n|(T+1)^{\overline{\alpha-1}} F_{\alpha,\alpha}(K(T+\alpha)^{\overline{\alpha}}).\end{aligned}$$

As $n \longrightarrow \infty$ and $\gamma_n \longrightarrow \gamma$, then $|\varphi(t) - \psi_n(t)| \longrightarrow 0$. We conclude that small changes in the solution will be obtained from the small changes in the initial condition.

Also, the application of the Gronwall's inequality on $h\mathbb{Z}$ is

Let $g(t, y)$ satisfies a Lipschitz condition(1.0.8) with constant $0 \leq M < 1$ for every t and y. Assume that the function $\varphi(t)$ satisfies the following condition

$${}_{a-h}\nabla_h^{\alpha}\varphi(t) = g(t, \varphi(t)) \quad for \quad t = a+h, a+2h, \ldots, \tag{3.7}$$

$${}_{a-h}\nabla_h^{-(1-\alpha)}\varphi(t)|_{t=a} = c \tag{3.8}$$

and the function ψ satisfies

$${}_{a-h}\nabla_h^{\alpha}\psi(t) = g(t, \psi(t)) \quad for \quad t = a+h, a+2h, \ldots, \tag{3.9}$$

$$ {}_{a-h}\nabla_h^{-(1-\alpha)}\psi(t)|_{t=a}= d. \tag{3.10}$$

We use the equivalent summation equation (2.6) to have

$$\varphi(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}c + {}_a\nabla_h^{-\alpha}g(t,\varphi(t))$$

$$\psi(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}d + {}_a\nabla_h^{-\alpha}g(t,\psi(t))$$

We take the difference of φ and ψ to have

$$\varphi(t) - \psi(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}(c-d) + {}_a\nabla_h^{-\alpha}(g(t,\varphi(t)) - g(t,\psi(t))).$$

Next, we obtain the following:

$$|\varphi(t) - \psi(t)| = \left|\frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}(c-d) + {}_a\nabla_h^{-\alpha}(g(t,\varphi(t)) - g(t,\psi(t)))\right|$$

$$\leq \left|\frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}\right||c-d| + |{}_a\nabla_h^{-\alpha}(g(t,\varphi(t)) - g(t,\psi(t)))|$$

$$\leq \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)}|c-d| + M\,{}_a\nabla_h^{-\alpha}|\varphi(t) - \psi(t)|.$$

Using the Gronwall's inequality and Lemma (2.2.8), we have

$$|\varphi(t)-\psi(t)| \le \frac{|c-d|}{\Gamma(\alpha)} \sum_{i=0}^{\infty} {}_{M}F_h^{i}(t-a+h)_h^{\overline{\alpha-1}}$$

$$= |c-d| \sum_{i=0}^{\infty} \frac{M^i}{\Gamma(i\alpha+\alpha)} (t-a+h)_h^{\overline{(i+1)\alpha-1}}$$

$$= |\gamma-\beta|(t-a+1)^{\overline{\alpha-1}}\ {}_{h}E_{\alpha,\alpha}(M(t-a+\alpha)^{\overline{\alpha}}),$$

where ${}_{h}E_{\alpha,\alpha}$ is the h-discrete ML function in Definition 1.0.7.
Next, we replace equations (3.9) and (3.10) by

$$_{a-h}\nabla_h^{\alpha}\psi(t) = g(t,\psi(t)) \quad for \quad t = a+h, a+2h, \ldots, \tag{3.11}$$

$$_{a-h}\nabla_h^{-(1-\alpha)}\psi(t)|_{t=a} = c_n, \tag{3.12}$$

where $c_n \longrightarrow c$. We call the solution of equations (3.11) and (3.12) by ψ_n and consider the fixed interval $[a, a+T] \cap \mathbb{N}_a$. Then we have

$$|\varphi(t)-\psi_n(t)| \le |c-c_n|(t-a+h)_h^{\overline{\alpha-1}}\ {}_{h}E_{\alpha,\alpha}(M(t-a+\alpha)^{\overline{\alpha}})$$
$$\le |c-c_n|(T+h)^{\overline{\alpha-1}}\ {}_{h}E_{\alpha,\alpha}(M(T+\alpha)^{\overline{\alpha}}).$$

As $n \longrightarrow \infty$ and $c_n \longrightarrow c$, then $|\varphi(t)-\psi_n(t)| \longrightarrow 0$. We conclude that small changes in the solution will be obtained from the small changes in the initial condition.

3.3 Conclusions and Future work

Firstly, the definition of Nabla fractional h-sums and Nabla h-discrete Mittag-Leffler function on the time scale $h\mathbb{Z}$ have been presented. Then the explicit solutions of the IVP for a discrete and h-discrete fractional equations have been obtained. Secondly, the Gronwall's inequality on $h\mathbb{Z}$ have been proved (Let $0 \leq y(t) < 1$. If $v(t) \leq v(a) + \sum_{s=a/h+1}^{t/h} h\ v(sh)y(sh)$, then $v(t) \leq v(a)\ \ {}_h\hat{e}_y(t,a)$). Thirdly, as an application using Gronwall's inequality, in solving IVP it has been shown that small changes in the initial condition imply a small changes in the solution(the stability) of the discrete fractional sum equation.

As a future work, we will extend our work for $\alpha > 0$ on scale $h\mathbb{Z}$ where $h > 1$.

Appendix

Theorem 3.3.1 *[54] (Absolute Convergence Test)*
If $\sum_{n=0}^{\infty} |a_n|$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *converges.*

Definition 3.3.1 *[54] (Absolutely Convergent)*
Given a series $\sum_{n=1}^{\infty} a_n$*. If the corresponding series* $\sum_{n=1}^{\infty} |a_n|$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

Theorem 3.3.2 *[54] (Comparison Test for Series)*

Suppose that a_n *and* b_n *are non-negative for all* n *and that* $a_n \leq b_n$ *when* $n \geq N$*, for some* N.

1. If $\sum_{n=0}^{\infty} b_n$ *converges, then* $\sum_{n=0}^{\infty} a_n$ *also converges.*

2. If $\sum_{n=0}^{\infty} a_n$ *diverges, then* $\sum_{n=0}^{\infty} b_n$ *also diverges.*

Theorem 3.3.3 *[26] (d'Alembert's Ratio Test)*

Suppose $\{a_n\}$ *is a sequence of positive numbers.*

1. If $\frac{a_{n+1}}{a_n} < r$ *for all sufficiently large* n*, where* $r < 1$*, then the series* $\sum_{n=0}^{\infty} a_n$ *converges. On the other hand, if* $\frac{a_{n+1}}{a_n} \geq 1$ *for all sufficiently large* n*, then the series* $\sum_{n=0}^{\infty} a_n$ *diverges.*

2. Suppose that $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ *exists. Then the series* $\sum_{n=0}^{\infty} a_n$ *converges if* $l < 1$ *and diverges if* $l > 1$*. No conclusion can be drawn if* $l = 1$.

Definition 3.3.2 *[72] (Beta Function)*

The beta function $B(x, y)$ is the name used by Legendre and Whittaker and Watson (1990) for the beta integral (also called the Eulerian integral of the first kind). It is defined by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

where x and y are positive real numbers.

Definition 3.3.3 *[29] (Stability)*

Stability is a condition in which a slight disturbance in a system does not produce too disrupting an effect on that system.

Let $y(t)$ be a solution of a differential equation, we said $y(t)$ is stable if any other solution of the equation that starts out sufficiently close to it when $x = 0$ remains close to it for succeeding values of x.

$y(t)$ is asymptotically stable if the difference between the solutions approaches zero as x increases.

$y(t)$ is unstable if it does not have either of these properties.

Definition 3.3.4 *[48] (The Method of Successive Approximations)*

One method of solving what appears at first to be very daunting equations, where the steps of this method are

1. Assume an approximate value for the variable that will simplify the equation.

2. Solve for the variable.

3. Use the answer as the second apporximate value and solve the equation again.

4. Repeat this process until a constant value for the variable is obtained.

Definition 3.3.5 *[29] (Principle of Mathematical Induction)*

Mathematical Induction is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n.

Consider a statement $P(n)$, *where* n *is a natural number. Then to determine the validity of* $P(n)$ *for every* n, *use the following principle:*

Step 1: Check whether the given statement is true for $n = 1$.

Step 2: Assume that given statement $P(n)$ *is also true for* $n = k$, *where* k *is any positive integer.*

Step 3: Prove that the result is true for $P(k + 1)$ *for any positive integer* k.

If the above-mentioned conditions are satisfied, then it can be concluded that $P(n)$ *is true for all* $n \in \mathbb{N}$.

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ملخص

حول متباينة جرون وال في نطاق حساب التفاضل و التكامل الكسري المنفصل

دانا عماد محمود عطية

في هذا العمل ، تم إثبات متباينة جرون وال لحساب التفاضل والتكامل المنفصل بالخطوة h مع عامل تشغيل نابلا على $h\mathbb{Z}$. نوضح نتائجنا في سياق مشكلة القيمة الأولية. علاوة على ذلك ، تم التحقق من متباينة جرون وال في حساب التفاضل والتكامل المنفصل على Z . لهذا الغرض ، تم النظر في معادلة مشكلة القيمة الأولية لمعادلة كسرية منفصلة ومعادلة مجموع كسري منفصل على Z و $h\mathbb{Z}$ ، و تم تعريف دالة ميتاج ليفيلار المنفصل بالخطوة h بمصطلح ${}_xE_h\psi$ و ${}_xF_h\psi$. ثم تم إعطاء حل صريح لمعادلة مجموع كسري خطي منفصل.