Arab American University
Faculty of Graduate Studies

Monotonicity Analysis of Fractional Proportional h-Differences

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This thesis was submitted in partial fulfillment of the requirements for the Master's degree in Applied Mathematics

Febreuary/2022

Thesis Approval

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Declaration

My name is Eman Hashem Sadi, and I am a student of the AAUP university number 201820268. I confirm that I worked on this Master's thesis myself. And I declare that I have complied with all regulations, instructions, Arab American University standards of Academic codes of conduct. I also adhere to the Dean's Council regulations in withdrawing their confirmation of this degree in case of any violations.

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Dedication

I dedicate this thesis with my great gratitude
To Almighty Allah who helped me to reach my aim,
To my parents,
To my brother Samer,
To my sisters Hanin, Rana , Heba,
To my supervisor Dr. Iyad Suwan, and,
To everyone who encouraged me to complete this work.

Acknowledgements

First, the great thanks to God Almighty, who guided me and inspired me with patience and ability to complete my education satisfactorily.

I would like to express my thanks and appreciation to my supervisor Dr. Iyad Suwan for his continuous support during the period in preparing my thesis.

I sincerely thank the examination committee for their careful reading of my thesis and their valuable suggestions.

All thanks and gratitude to my hard-working parents for their endless support and, I will never forget my whole family for their support and encouragement.

Great thanks to the Department of Mathematics and Statistics at the Arab American University - Palestine.

Abstract

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In this work, the definitions of Nabla fractional proportional h-sums and Reiman-Liouville RL fractional proportional h-differences of order $0 < \alpha \leq 1$, $0 < \rho \leq 1$ have been formulated, some applications on the old monotonicity results of fractional proportional differences and fractional h-differences have been detected.

Moreover, the monotonicity analysis of fractional proportional-h differences for $0 < \alpha \leq 1$, $0 < \rho \leq 1$ have been proved.

Keywords: *Nabla h-fractional proportional sums, Reiman-Liouville RL h-fractional proportional differences.*

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Introduction

Fractional Calculus is one of the subjects that have been under focus by many researchers for many centuries. It can help us give a generalization of ordinary differentiation and integration to non-integer order. It began in 1695, when L'Hopital asked Leibniz for the derivative of order $n = \frac{1}{2}$ [27].

In 1729, Euler invented the gamma function to extend the factorial of complex and real number arguments. The gamma function is defined by [21]

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \ ,$$

This integral converges when $Re(z) > 0$. The 'beauty' of the gamma function can be found in its properties as seen in the following equation: [27]:

$$\Gamma(z+1) = z\Gamma(z), \Gamma(z) = (z-1)! \tag{0.1}$$

where z$\in \mathbb{N}_+$.

In 1819, Lacroix found a formula for the n^{th} integer-order derivative for polynomials of exponent P and then he extended it for arbitrary order α by replacing factorials with the gamma function yielding [21]:

$$D^\alpha X^P = \frac{\Gamma(P+1)}{\Gamma(P-\alpha+1)} X^{P-\alpha} \tag{0.2}$$

and he used this formula to calculate the half derivative correctly. Nevertheless, Lacroix's method did not offer hints for possible applications [18]. In 1822, Fourier also mentioned arbitrary derivatives for sine and cosine functions, but he gave no applications. Neils Abel firstly provided an application for fractional calculus in 1823 with his Tautochrone problem [25]. The Tautochrone problem involves finding a curve in which the time needed to reach the bottom is independent of the starting position.

The first systematic study was made by Joseph Liouville, initially, he defined a derivative of arbitrary order as an infinite series, then he formulated a second definition in which he was able to give a fractional derivative of X^{-P}.After Liouville, Bernhard Reiman formulated a generalization of a Taylor series to derive a formula for integration of arbitrary order [18]. The approaches of Reiman and Liouville can be abridged in a single formula which is nowadays known as Reiman-Liouville fractional integral formula [23]

$$J_c^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_c^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \qquad (0.3)$$

where J^{α} represents the fractional integral operator of order $\alpha \in \mathbb{R}_{+}$, $f(t)$ is a function of time, C is a lower limit of integration and Γ is the Gamma function. If $c = 0$ then it gives the Reiman formula . If $c = -\infty$ then equation (0.3) becomes Liouville formula [18]. The fractional derivative formula of order $\alpha > 0$ is obtained starting from equation (0.3) for $c = 0$ [23]

$$D^{\alpha} f(T) := D^m J^{m-\alpha} f(t) = \frac{d^m}{dt^m} \times [\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau] \qquad (0.4)$$

where $m \in \mathbb{Z}^{+}, m-1 < \alpha \leq m$.

In 1967, Caputo developed the Caputo fractional derivative by reformulating Reiman-Liouville fractional derivative so he could use integer order initial conditions when working with fractional differential equations [25].

Many problems in science and engineering can be formulated and solved using continuous and discrete fractional calculus [2, 19, 28]. Many applications can be found in [8, 10–12]. The theory of fractional sums and fractional differences including their monotonicity properties are extensively studied in [1,3,4,9,13–17,20–22,24,26,33].

Monotonicity results for fractional differences operators with discrete exponential kernels were studied in [5] when time step $h = 1$. In [30], the nabla fractional sums and differences of order $0 < \alpha < 1$ on the time scale $h\mathbb{Z}$ where $0 < h < 1$ are formulated and the monotonicity results for the fractional h-differences were concluded. In [32], the monotonicity analysis of fractional proportional differences were studied.

As a generalization of the previous studies, we study the monotonicity analysis of fractional proportional h-differences,

In the first chapter of the thesis, section one, we present the definitions we will use throughout the thesis. In section two, we state the monotonicity results for fractional proportional differences, and in section three, we present some applications of the monotonicity analysis of the fractional proportional differences.

In chapter two, section one, we conclude the monotonicity analysis for fractional h-differences. In section two, we present some applications of the monotonicity analysis of the fractional h-differences and in section three, we conclude the h-fractional difference version of the mean value theorem and an application on it.

In the third chapter, section one, we conclude the monotonicity analysis of fractional proportional h difference. In section two, we present conclusions and future work.

Chapter One

Fractional Proportional Differences

In this chapter, we start by introducing some basic definitions in discrete fractional calculus, then the monotonicity analysis of fractional proportional differences and some applications.

1.1 Definitions

Definition 1.1.1 *[30] The backward difference operator on $h\mathbb{Z}$ is defined by:*

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h}$$

and the forward difference operator on $h\mathbb{Z}$ is defined by:

$$\nabla_h f(t) = \frac{f(t+h) - f(t)}{h}$$

Definition 1.1.2 *[30] The backward jump operator on $h\mathbb{Z}$ is defined by:*

$$\rho_h(t) = t - h$$

and the forward jump operator on $h\mathbb{Z}$ is defined by:

$$\sigma_h(t) = t + h$$

For a,b $\in \mathbb{R}$ *with a<b,* $\dfrac{b-a}{h} \in \mathbb{N}$*, and 0<h≤1, we use the notations:*

$$\mathbb{N}_{a,h} = \{a, a+h, a+2h, ...\}$$

$$_{b,h}\mathbb{N} = \{b, b-h, b-2h, ...\}$$

Definition 1.1.3 *[32] The discrete proportional difference of order* $0 < \rho \leq 1$*, for the function f is defined by:*

$$\nabla^{\rho} f(t) = (1-\rho) f(t) + \rho \nabla f(t) \tag{1.1}$$

$t \in \mathbb{N}_{c+1} = \{c+1, c+2, c+3, ...\}$ *where c is a positive integer.*

Definition 1.1.4 *Let* $z \in \mathbb{N}_c$, $0 < \rho \leq 1$, *and* $P = \frac{\rho-1}{\rho}$, *then*

$$e_P(z, c) = \rho^{z-c}.$$

Definition 1.1.5 *[31] For p,q*$\in \mathbb{R}$ *with p<q,* $\dfrac{q-p}{h} \in \mathbb{N}$*, and* $0 < h \leq 1$*.*
Let $\mathbb{N}_{p,h} = \{p, p+h, p+2h, ...\}$ *and* $_{q,h}\mathbb{N} = \{q, q-h, q-2h, ...\}$*. The h-Nabla discrete exponential kernal is expressed as:*

$$_h\hat{e}_{\lambda}(t, \rho_h(s)) = \left(\frac{1}{1-h\lambda}\right)^{\dfrac{t-\rho_h(s)}{h}}$$

where $\lambda = \left(\dfrac{-\mu}{1-\mu}\right)$.

Definition 1.1.6 *[30]Let* $\alpha \in \mathbb{R}$ *and 0<h≤1, the nabla h-factorial of t is defined by*

$$t_h^{\bar{\alpha}} = h^{\alpha} \frac{\Gamma(\frac{t}{h} + \alpha)}{\Gamma(\frac{t}{h})}$$

such that t$\in \mathbb{R}$*-*$\{\ldots, -2h, -h, 0\}, 0_h^{\bar{\alpha}} = 0$ *and dividing by poles leads to zero.*

Definition 1.1.7 *[6] Let $s \in \mathbb{R}$, $0 < \alpha < 1$ and $f, g : \mathbb{N}_{a,h} \to \mathbb{R}$ be functions. The nabla h-discrete convolution of f with g is defined by:*

$$(f * g)(t) = \int_a^t g(t - \rho_h(s) + a) f(s) \nabla_h s = h \sum_{k=a\ h+1}^{\frac{t}{h}} g(t - \rho_h(kh) + a) f(kh) \qquad (1.2)$$

Definition 1.1.8 (Nabla fractional proportional h-sums) *for a function $f : \mathbb{N}_{a,h} = \{a, a+h, a+2h, \ldots\} \to \mathbb{R}$, the left fractional proportional h-sum of f of order $\alpha > 0$ is defined by [7]*

$$({}_a\nabla_h^{-\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t \hat{e}_\rho(t - \tau + \alpha h, 0)(t - \rho_h(\tau))_h^{\overline{\alpha-1}} f(\tau) \nabla_h \tau$$

using the convolution equation (1.2), *we get:*

$$({}_a\nabla_h^{-\alpha,\rho} f)(t) = \frac{h}{\rho^\alpha \Gamma(\alpha)} \sum_{k=\frac{a}{h+1}}^{\frac{t}{h}} \hat{e}_\rho(t - kh + \alpha h, 0)(t - \rho_h(kh))_h^{\overline{\alpha-1}} f(kh)$$

for a function $f : {}_{b,h}\mathbb{N} = \{b, b-h, b-2h, \ldots\} \to \mathbb{R}$, the right fractional proportional h-sum of f with order $\alpha > 0$ is defined by [7]

$$({}_h\nabla_b^{-\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_t^b \hat{e}_\rho(\tau - t + \alpha h, 0)(\tau - \rho_h(t))_h^{\overline{\alpha-1}} f(\tau) \triangle_h \tau$$

using the convolution equation (1.2), *we get:*

$$({}_h\nabla_b^{-\alpha,\rho} f)(t) = \frac{h}{\rho^\alpha \Gamma(\alpha)} \sum_{\frac{t}{h}}^{k=\frac{b}{h-1}} \hat{e}_\rho(kh - t + \alpha h, 0)(kh - \rho_h(t))_h^{\overline{\alpha-1}} f(kh)$$

Definition 1.1.9 (RL fractional proportional h-differences) *For $\rho > 0$ and $\alpha \in \mathbb{C}$, Re(α) > 0, we define the left RL proportional fractional h-difference of f by [7]*

$$\begin{aligned}({}_a\nabla_h^{\alpha,\rho} f)(t) &= \nabla_h^{n,\rho}\,{}_a\nabla_h^{-(n-\alpha),\rho} f(t)\\ &= \frac{\nabla_{h,t}^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_a^t \hat{e}_\rho(t-\tau+h(n-\alpha),0)(t-\rho_h(\tau))_h^{\overline{n-\alpha-1}} f(\tau)\nabla_h\tau\end{aligned}$$

using the convolution equation (1.2), *we get:*

$$({}_a\nabla_h^{\alpha,\rho} f)(t) = h\frac{\nabla_{h,t}^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\sum_{k=\frac{a}{h+1}}^{\frac{t}{h}} \hat{e}_\rho(t-(kh)+h(n-\alpha),0)(t-\rho_h(kh))_h^{\overline{n-\alpha-1}} f(kh)$$

The right proportional fractional h-difference ending at b is defined by [7]

$$\begin{aligned}({}_h\nabla_b^{\alpha,\rho} f)(t) &= {}_\ominus\Delta_h^{n,\rho}\,{}_h\nabla_b^{-(n-\alpha),\rho} f(t)\\ &= \frac{{}_\ominus\Delta_{h,t}^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\int_t^b \hat{e}_\rho(\tau-t+h(n-\alpha),0)(\tau-\rho_h(t))_h^{\overline{n-\alpha-1}} f(\tau)\,\Delta_h\,\tau\end{aligned}$$

using the convolution equation (1.2), *we get:*

$$\begin{aligned}({}_h\nabla_b^{\alpha,\rho} f)(t) &= h\frac{{}_\ominus\Delta_{h,t}^{n,\rho}}{\rho^{n-\alpha}\Gamma(n-\alpha)}\sum_{\frac{t}{h}}^{k=\frac{b}{h-1}} \hat{e}_\rho(kh-t+h(n-\alpha),0)(kh-\rho_h(t))_h^{\overline{n-\alpha-1}} f(kh)\\ where\, n &= [Re(\alpha)]+1\; and ({}_\ominus\Delta_h^{n,p} g)(t) = ({}_\ominus\Delta_h^p{}_\ominus\Delta_h^p\cdots{}_\ominus\Delta_h^p g)(t)\end{aligned}$$

1.2 Monotonicity Analysis of Fractional Proportional Differences

We start by introducing some definitions related to the monotonicity analysis of fractional proportional differences, and then Theorems of the monotonicity analysis.

Definition 1.2.1 *[32] Let y:* $\mathbb{N}_a \rightarrow \mathbb{R}$ *be a function satisfying* $y(a) \geq 0,\ 0 < \alpha < 1$. *Then, y(t) is called an* α*-increasing function on* $\mathbb{N}_a$ *if* $y(t+1) \geq \alpha y(t)\ \forall t \in \mathbb{N}_a$.

Definition 1.2.2 *[32] Let y:* $\mathbb{N}_a \rightarrow \mathbb{R}$ *be a function satisfying* $y(a) \leq 0,\ 0 < \alpha < 1$. *Then, y(t) is called an* α*-decreasing function on* $\mathbb{N}_a$ *if* $y(t+1) \leq \alpha y(t)\ \forall t \in \mathbb{N}_a$.

Theorem 1.2.1 *[32]Let* $\chi : \mathbb{N}_c \rightarrow \mathbb{R}$ *be a function, and suppose that* $({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\ )(z) \geq 0$ *for* $0 < \varepsilon < 1$, *and* $0 < \rho \leq 1,\ z \in \mathbb{N}_{c-1}$. *Then* $\chi(z)$ *is* $\varepsilon\rho - increasing$.

Proof can be found in [32].

Proof:

$$\begin{aligned}({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\ )(z) &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z}\widehat{e}_{p}(z-\iota,0)\,(\,z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\\ &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z}\rho^{z-\iota}\,(\,z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\end{aligned}$$

Let

$$S(z) = \sum_{\iota=c}^{z}\rho^{z-\iota}\,(\,z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota).$$

Then,

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\ )(z) = \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}S(z).$$

Hence, from the assumption, we have $\nabla^{\rho}S(z) \geq 0$. That is

$$\begin{aligned}
\nabla^{\rho}S(z) &= (1-\rho)S(z) + \rho\nabla S(z) \\
&= (1-\rho)S(z) + \rho(S(z) - S(z-1)) \\
&= S(z) - \rho S(z) + \rho S(z) - \rho S(z-1) \\
&= S(z) - \rho S(z-1) \\
&= \sum_{\iota=c}^{z} \rho^{z-\iota}\,(\, z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) - \rho \sum_{\iota=c}^{z-1} \rho^{z-1-\iota}\,(\, z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&= (z-\varsigma(z))^{\overline{-\varepsilon}}\chi(z) + \sum_{\iota=c}^{z-1} \rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&- \sum_{\iota=c}^{z-1} \rho^{z-\iota}\,(z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&= (\, z-z+1)^{\overline{-\varepsilon}}\chi(z) \; + \sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\;\left(\,(\, z-\varsigma(\iota))^{\overline{-\varepsilon}} \; - \; (\, z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\;\right) \\
&= (\,1)^{\overline{-\varepsilon}}\chi(z) \; + \sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\;\;\nabla(\, z-\varsigma(\iota))^{\overline{-\varepsilon}} \\
&= \frac{\Gamma(1-\varepsilon)}{\Gamma(1)}\chi(z) \; + \sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\,(-\varepsilon\;\,(\, z-\varsigma(\iota))^{\overline{-\varepsilon-1}}) \\
&= \Gamma(1-\varepsilon)\chi(z) \; - \; \varepsilon\sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\;\,(\, z-\varsigma(\iota))^{\overline{-\varepsilon-1}} \\
&\geq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\,\chi\,)(z) &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\,S(z) \\
&= \frac{1}{\Gamma(1-\varepsilon)}\;\left(\Gamma(1-\varepsilon)\chi(z) \; - \; \varepsilon\sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\;\,(\, z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right) \\
&= \chi(z) \; - \; \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\;\,(\, z-\varsigma(\iota))^{\overline{-\varepsilon-1}} \\
&\geq 0
\end{aligned}$$

Hence,

$$\chi(z) \;\geq\; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota}\chi(\iota)\ \left(z-\varsigma(\iota)\right)^{\overline{-\varepsilon-1}}.$$

Clearly,

$$\chi(c-1)=0.$$

So we can start the induction from the next step.

When $z=c$, we get $\chi(c)\geq 0$;

also, when $z=c+1$ we have:

$$\begin{aligned}
\chi(c+1) \;&\geq\; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{c} \rho^{c+1-\iota}\chi(\iota)\ \left(c+1-\varsigma(\iota)\right)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\, \rho^{c+1-c}\chi(c)\ \left(c+1-\varsigma(c)\right)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\, \rho\chi(c)\ \left(c+1-c+1\right)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon\rho}{\Gamma(1-\varepsilon)}\, \chi(c)\, \frac{\Gamma(1-\varepsilon)}{\Gamma(2)} \\
&= \varepsilon\rho\, \chi(c),
\end{aligned}$$

Now for $z+1$, replace z by $z+1$, we get

$$\begin{aligned}
\chi(z+1) \;&\geq\; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z} \rho^{z+1-\iota}\chi(\iota)\ \left(z+1-\varsigma(\iota)\right)^{\overline{-\varepsilon-1}} \\
&\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)}\, \rho^{z+1-z}\chi(z)\ \left(z+1-\varsigma(z)\right)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\, \rho\chi(z)\ \left(z+1-z+1\right)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon\rho}{\Gamma(1-\varepsilon)}\, \chi(z)\ 2^{\overline{-\varepsilon-1}} \\
&= \varepsilon\rho\, \chi(z),
\end{aligned}$$

Hence $\chi(z)$ is $\varepsilon\rho - \textit{increasing}$.

$\square$

Theorem 1.2.2 *[32] Let* $\chi : \mathbb{N}_{c-1} \to \mathbb{R}$ *be a function satisfies* $\chi(c) \geq 0$, *and suppose that for* $0 < \varepsilon < 1$, *and* $0 < \rho \leq 1$. *If* $\chi(z)$ *is increasing on* $\mathbb{N}_c$, *then we have*

$$\left({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi\right)(z) \geq 0, \ \forall z \in \mathbb{N}_{c-1}.$$

Proof can be found in [32].

Theorem 1.2.3 *[32] Let* $\chi : \mathbb{N}_{c-1} \to \mathbb{R}$ *be a function satisfies* $\chi(c) > 0$ *and be strictly increasing on* $\mathbb{N}_c$, *where* $0 < \varepsilon < 1$, *and* $0 < \rho \leq 1$. *Then*

$$\left({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi\right)(z) > 0, \quad z \in \mathbb{N}_{c-1}.$$

Proof can be found in [32].

Theorem 1.2.4 *[32] Let* $\chi : \mathbb{N}_{c-1} \to \mathbb{R}$ *be a function, and suppose that* $\left({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi\right)(z) \leq 0$ *for* $0 < \varepsilon < 1$, *and* $0 < \rho \leq 1, z \in \mathbb{N}_{c-1}$. *Then* $\chi(z)$ *is* $\varepsilon\rho -$ *decreasing.*

Proof can be found in [32].

Theorem 1.2.5 *[32] Let a function* $\chi : \mathbb{N}_{c-1} \to \mathbb{R}$ *be decreasing on* $\mathbb{N}_c$ *such that* $\chi(c) \leq 0$. *Then for* $0 < \varepsilon < 1$, *and* $0 < \rho \leq 1$, *we have*

$$\left({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi\right)(z) \leq 0, \ \forall z \in \mathbb{N}_{c-1}.$$

Proof can be found in [32].

1.3 Examples on Monotonicity Analysis of Fractional Proportional Differences

Example 1.3.1

This example is an illustration of theorem 1.2.1.

Let $\chi(z) = z^2, \chi(z) : \mathbb{N}_1 \to \mathbb{R}$

Assume that $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\ \chi\ )(z) \geq 0, \forall z \in \mathbb{N}_{c-1}$, we want to show that $\chi(z) is\ \varepsilon\rho - increasing.$

$$
\begin{aligned}
({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\ \chi\ )(z) &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z} \widehat{e}_p(z-\iota,0)\,(\ z-\varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2 \\
&= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z} \rho^{z-\iota}\,(\ z-\varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2
\end{aligned}
$$

Let

$$
S(z) = \sum_{\iota=c}^{z} \rho^{z-\iota}\,(\ z-\varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2.
$$

Then,

$$
({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\ \chi\ )(z) = \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\,S(z).
$$

Hence, from the assumption, we have $\nabla^{\rho}S(z) \geq 0$. That is

$$
\begin{aligned}
\nabla^{\rho}S(z) &= (1-\rho)S(z) + \rho\nabla S(z) \\
&= (1-\rho)S(z) + \rho(S(z) - S(z-1)) \\
&= S(z) - \rho S(z) + \rho S(z) - \rho S(z-1) \\
&= S(z) - \rho S(z-1) \\
&= \sum_{\iota=c}^{z} \rho^{z-\iota}\,(\ z-\varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2 - \rho \sum_{\iota=c}^{z-1} \rho^{z-1-\iota}\,(\ z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2
\end{aligned}
$$

$$\begin{aligned}
&= (z - \varsigma(z))^{\overline{-\varepsilon}}\,(z)^2 + \sum_{\iota=c}^{z-1} \rho^{z-\iota}\,(z - \varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2 \\
&- \sum_{\iota=c}^{z-1} \rho^{z-\iota}\,(z - 1 - \varsigma(\iota))^{\overline{-\varepsilon}}\,(\iota)^2 \\
&= (\, z - z + 1)^{\overline{-\varepsilon}}\,(z)^2 \; + \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \; \left((\, z - \varsigma(\iota))^{\overline{-\varepsilon}} \; - \; (\, z - 1 - \varsigma(\iota))^{\overline{-\varepsilon}} \; \right) \\
&= (\, 1)^{\overline{-\varepsilon}}\,(z)^2 \; + \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \;\; \nabla (\, z - \varsigma(\iota))^{\overline{-\varepsilon}} \\
&= \frac{\Gamma(1-\varepsilon)}{\Gamma(1)}(z)^2 \; + \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \,(-\varepsilon \;(\, z - \varsigma(\iota))^{\overline{-\varepsilon-1}}) \\
&= \Gamma(1-\varepsilon)(z)^2 \; - \; \varepsilon \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \;(\, z - \varsigma(\iota))^{\overline{-\varepsilon-1}} \\
&\geq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\, \chi\,)(z) &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\, S(z) \\
&= \frac{1}{\Gamma(1-\varepsilon)} \left(\Gamma(1-\varepsilon)(z)^2 \; - \; \varepsilon \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \;(\, z - \varsigma(\iota))^{\overline{-\varepsilon-1}} \right) \\
&= (z)^2 \; - \; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \;(\, z - \varsigma(\iota))^{\overline{-\varepsilon-1}} \\
&\geq 0
\end{aligned}$$

Hence,

$$(z)^2 \;\geq\; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota}(\iota)^2 \;(\, z - \varsigma(\iota))^{\overline{-\varepsilon-1}}.$$

Clearly,

$$\chi(c-1) = 0.$$

So we can start the induction from the next step.

When $z = c$, we get $\chi(c) \geq 0$;

also, when $z = c + 1$ we have:

$$\begin{aligned}
(c+1)^2 &\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{c} \rho^{c+1-\iota}(\iota)^2 \ (\, c+1-\varsigma(\iota))^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho^{c+1-c}(c)^2 \ (\, c+1-\varsigma(c))^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho(c)^2 \ (\, c+1-c+1)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon\rho}{\Gamma(1-\varepsilon)} \,(c)^2\, \frac{\Gamma(1-\varepsilon)}{\Gamma(2)} \\
&= \varepsilon\rho\,(c)^2,
\end{aligned}$$

Now for $z+1$, replace z by $z+1$, we get

$$\begin{aligned}
(z+1)^2 &\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z} \rho^{z+1-\iota}(\iota)^2 \ (\, z+1-\varsigma(\iota))^{\overline{-\varepsilon-1}} \\
&\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho^{z+1-z}(z)^2 \ (\, z+1-\varsigma(z))^{\overline{-\varepsilon-1}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho(z)^2 \ (\, z+1-z+1)^{\overline{-\varepsilon-1}} \\
&= \frac{\varepsilon\rho}{\Gamma(1-\varepsilon)} \,(z)^2\, 2^{\overline{-\varepsilon-1}} \\
&= \varepsilon\rho\,(z)^2,
\end{aligned}$$

Hence $\chi(z)$ is $\varepsilon\rho - increasing.$

Example 1.3.2

This example is an illustration of theorem 1.2.2.

Let $\chi(z) = z^2, \chi(z) : \mathbb{N}_0 \to \mathbb{R}$, $\chi(z)$ is increasing on $\mathbb{N}_0$.

We want to show that

$$({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\,\chi\,)(z) \geq 0, \forall z \in \mathbb{N}_{c-1}.$$

Since

$$({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) = (z)^2 \; - \; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(\iota)^2, \;\; \forall z \in \mathbb{N}_{c-1},$$

When $z = c$, $\chi(c) \geq 0$, we have from the assumption $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(c) = (c)^2 \geq 0$.
Clearly, $\chi(c-1) = (c-1)^2 = 0$.
So we can start the induction from the next step.
Assume that $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(i) \geq 0,\ \forall\, i < z$. We shall show that $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0$.
Since, from assumption, $\chi(z)$ is increasing, it follows that

$$\chi(z) \geq \chi(z-1) \geq \chi(c) \geq 0, \forall\; z \in \mathbb{N}_c\,.$$

$$\begin{aligned}
({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) &= (z)^2 \; - \; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(\iota)^2 \\
&= (z)^2 - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho^{z-z+1}\,(z-\varsigma(z-1))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&- \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-2} \rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(\iota)^2 \\
&= (z)^2 \; - \; \varepsilon\rho\,(z-1)^2 \; - \; \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-2} \rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(\iota)^2 \\
&+ \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-2} \rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&- \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-2} \rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2
\end{aligned}$$

$$\begin{aligned}
&= (z)^2 - \varepsilon\rho\,(z-1)^2 \\
&+ \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,((z-1)^2-(\iota)^2) \\
&- \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&\geq (z)^2 - \varepsilon\rho\,(z-1)^2 - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&= (z)^2 - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&= (z)^2 - (z-1)^2 + (z-1)^2 \\
&- \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&\geq (z-1)^2 - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2 \\
&= (z-1)^2\left(1-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right) \\
&\geq 0.
\end{aligned}$$

Example 1.3.3

This example is an illustration of theorem 1.2.3.

Let $\chi(z) = z^2$, $\chi(z) : \mathbb{N}_0 \to \mathbb{R}$ $\chi(z)$ is increasing on $\mathbb{N}_0$, we want to show that

$$({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\,\chi\,)(z) > 0, \forall z \in \mathbb{N}_{c-1}.$$

Since when $z = c$, we have $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(c) = \chi(c) > 0$.

Clearly $\chi(c-1) = 0$, so we can start the induction from the next step.

Assume that $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(i) > 0$, $\forall\, i < z$, we shall show that $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) > 0$.

Since, from assumption, $\chi(z)$ is increasing it follows that

$$\chi(z) > \chi(z-1) > \chi(c) > 0, \ \forall\ z \in \mathbb{N}_c\,.$$

$$\begin{aligned}({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) &= (z)^2 - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(\iota)^2\\ &> (z)^2 - (z-1)^2 + (z-1)^2\\ &- \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2\\ &> (z-1)^2 - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\,(z-1)^2\\ &= (z-1)^2\left(1 - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\,(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right)\\ &> 0.\end{aligned}$$

Example 1.3.4

This example is an illustration of theorem 1.2.4.

Let $\chi(z) = -z^2$, $\chi(z) : \mathbb{N}_0 \to \mathbb{R}$

suppose $({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \leq 0, \forall z \in \mathbb{N}_{c-1}$, where $c = 1$.

We need to show χ is $\varepsilon\rho - \mathit{decreasing}$ on $\mathbb{N}_0$.

Let $\theta :\ \mathbb{N}_{c-1} \to\ \mathbb{R}$ be a function such that $\theta(z) = -\chi(z) = z^2$; hence,

$$(\,{}^{R}_{c-1}\nabla^{\varepsilon,\rho}\,\theta)(z) = (\,{}^{R}_{c-1}\nabla^{\varepsilon,\rho}\,(-\chi))(z) = -({}^{R}_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0.$$

Now, by Theorem 1.2.1, we conclude that $\theta(z)$ is $\varepsilon\rho\ -\ \mathit{increasing}$ on $\mathbb{N}_0$.

Hence,

$$\theta(z+1)\ \geq\ \varepsilon\rho\theta(z),$$

So,

$$(z+1)^2\ \geq\ \varepsilon\rho(z)^2$$

which is

$$-(z+1)^2\ \leq\ \varepsilon\rho - ((z)^2)$$

$$\chi(z+1) \ \leq \ \varepsilon\rho(\chi(z)),$$

that is to say, $\chi(z)$ is $\varepsilon\rho \ - \ decreasing.$

Example 1.3.5

This example is an illustration of theorem 1.2.5.

Let $\chi(z) = -Z^2$, $\chi(z) : \mathbb{N}_0 \to \mathbb{R}$

Suppose that $\chi(z)\ is\ \varepsilon\rho - decreasing$ and $\chi(c) \leq 0$,we want to show that $({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) \leq 0.$

Let $\theta :\ \mathbb{N}_{c-1} \to \ \mathbb{R}$ be a function such that $\theta(z) \ = \ -\chi(z) \ = \ Z^2$; θ is increasing on $\mathbb{N}_0$, $\theta(c) = c^2 \geq 0$, where $c = 1$.

By Theorem 1.2.2

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z) \geq 0, \forall z \in \mathbb{N}_{c-1}.$$

then

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho} \chi)(z) = ({}_{c-1}^{R}\nabla^{\varepsilon,\rho} (-\theta))(z) = -({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z) \leq 0, \forall z \in \mathbb{N}_{c-1}.$$

Chapter Two

Fractional h-Differences

In this chapter, we start by introducing the monotonicity analysis of fractional h-differences, some applications. Finally, we state the h- fractional difference version of the mean value theorem and an application of it.

2.1 Monotonicity Analysis of Fractional h-Differences

Definition 2.1.1 *[30] Let y:* $\mathbb{N}_{a,h} \rightarrow \mathbb{R}$ *be a function satisfying* $y(a) \geq 0$, *and let* $0 \leq h < 1$. *Then, y(t) is called an* α*-increasing function on* $\mathbb{N}_{a,h}$ *if* $y(t+h) \geq \alpha y(t)$, $\forall t \in \mathbb{N}_{a,h}$.

Definition 2.1.2 *[30] Let y:* $\mathbb{N}_{a,h} \rightarrow \mathbb{R}$ *be a function satisfying* $y(a) \leq 0$, *and let* $0 \leq h < 1$. *Then, y(t) is called an* α*-decreasing function on* $\mathbb{N}_{a,h}$ *if* $y(t+h) \leq \alpha y(t)$, $\forall t \in \mathbb{N}_{a,h}$.

Theorem 2.1.1 *[30] Let* $y : \mathbb{N}_{a-h,h} \rightarrow \mathbb{R}$, *and suppose that* $({}_{a-h}\nabla_h^{\alpha} y)(t) \geq 0$ *for* $0 < \alpha \leq 1$, *and* $0 < h \leq 1$, $t \in \mathbb{N}_{a-h,h}$. *Then* $y(t)$ *is* $\alpha - increasing$.

Proof can be found in [30].

Theorem 2.1.2 *[30] Assume that the function* $y : \mathbb{N}_{a-h,h} \rightarrow \mathbb{R}$ *satisfies* $y(a) \geq 0$, *and assume that for* $0 < \alpha \leq 1$, *and* $0 < h \leq 1$. *If y is increasing on* $\mathbb{N}_{a,h}$, *then we have*

$$({}_{a-h}\nabla_h^{\alpha} y)(t) \geq 0, \forall t \in \mathbb{N}_{a-h,h}$$

Proof can be found in [30].

Theorem 2.1.3 *[30] Let a function* $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *satisfies* $y(a) > 0$ *and be strictly increasing on* $\mathbb{N}_{a,h}$ *where* $0 < \alpha \leq 1$ *and* $0 < h \leq 1$*. Then*

$$({}_{a-h}\nabla_h^\alpha y)(t) > 0$$

Proof can be found in [30].

Theorem 2.1.4 *[30] Let* $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$*, and suppose that* $({}_{a-h}\nabla_h^\alpha y)(t) \leq 0$ *for* $0 < \alpha \leq 1$*, and* $0 < h \leq 1$*,* $t \in \mathbb{N}_{a-h,h}$*. Then* $y(t)$ *is* $\alpha -$ *decreasing.*

Proof can be found in [30].

Theorem 2.1.5 *[30] Let a function* $y : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *satisfies* $y(a) \leq 0$ *and be decreasing on* $\mathbb{N}_{a,h}$*.Then, for* $0 < \alpha \leq 1$ *and* $0 < h \leq 1$*, we have*

$$({}_{a-h}\nabla_h^\alpha y)(t) \leq 0, \forall t \in \mathbb{N}_{a-h,h}$$

Proof can be found in [30].

2.2 Examples on Monotonicity Analysis of Fractional h-Differences

Example 2.2.1

This example is an illustration of theorem 2.1.1

Let $y(t) = t^2, y(t) : \mathbb{N}_{1-h,h} \to \mathbb{R}$

Suppose that $({}_{a-h}\nabla_h^\alpha\, y\,)(t) \geq 0$ for $0 < \alpha \leq 1$ and $0 < h \leq 1$, $t \in \mathbb{N}_{a-h,h}$.

We need to show that $y(t)$ is $\alpha - increasing$

It is known that

$$({}_{a-h}\nabla_h^\alpha\, y\,)(t) \;= \frac{\nabla_h}{\Gamma(1-\alpha)} \sum_{k=\frac{a-h}{h+1}}^{\frac{t}{h}} \;(\,t - \rho_h(kh))^{\overline{-\alpha}}\,(kh)^2 h$$

If we let

$$S(z) = \sum_{k=\frac{a-h}{h+1}}^{\frac{t}{h}} \;(\,t - \rho_h(kh))^{\overline{-\alpha}}\,(kh)^2 h$$

then ,

$$({}_{a-h}\nabla_h^\alpha\, y\,)(t) \;= \frac{\nabla_h}{\Gamma(1-\alpha)}\, S(t).$$

Let

$$\nabla_h S(t) \geq 0 \;.$$

$$\begin{aligned}
\nabla_h S(t) \;&=\; \nabla_h \sum_{k=\frac{a}{h}}^{\frac{t}{h}} \;(\,t - \rho_h(kh))^{\overline{-\alpha}}\,(kh)^2 h \\
&=\; \frac{1}{h}\Big(\sum_{k=\frac{a}{h}}^{\frac{t}{h}}(t-\rho_h(kh))_h^{\overline{-\alpha}}(kh)^2 h - \sum_{k=\frac{a}{h}}^{\frac{t-h}{h}}(t-h-\rho_h(kh))_h^{\overline{-\alpha}}(kh)^2 h\Big) \\
&=\; \frac{1}{h}\Big(\sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}(t-\rho_h(kh))_h^{\overline{-\alpha}}(kh)^2 h - \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}(t-h-\rho_h(kh))_h^{\overline{-\alpha}}(kh)^2 h + (t-h-\rho_h(t))_h^{\overline{-\alpha}}(t)^2 h\Big) \\
&=\; (t-t+h)_h^{\overline{-\alpha}}(t)^2 + \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}(kh)^2 h \nabla_h (t-\rho_h(kh))_h^{\overline{-\alpha}} \\
&=\; (h)_h^{\overline{-\alpha}}(t)^2 + \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}(kh)^2 h\Big[h^{-\alpha}\frac{\Gamma(\frac{t}{h}-k+1-\alpha)}{\Gamma(\frac{t}{h}-k+1} - h^{-\alpha}\frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k}\Big]\frac{1}{h}
\end{aligned}$$

$$
\begin{aligned}
&= (h)_h^{\overline{-\alpha}}(t)^2 + \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h \Big[\frac{h^{-\alpha}}{h}\frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k)}\Big[\frac{(\frac{t}{h}-k-\alpha)}{(\frac{t}{h}-k)} - 1\Big] \\
&= (h)_h^{\overline{-\alpha}}(t)^2 + \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h \Big[\frac{h^{-\alpha}}{h}\frac{\Gamma(\frac{t}{h}-k-\alpha)}{\Gamma(\frac{t}{h}-k)}\Big[\frac{(\frac{t}{h}-k-\alpha-\frac{t}{h}+k)}{(\frac{t}{h}-k)}\Big] \\
&= (h)_h^{\overline{-\alpha}}(t)^2 - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h \Big[h^{-\alpha-1}\frac{\Gamma(\frac{t}{h}-k+1-\alpha-1)}{\Gamma(\frac{t}{h}-k+1)} \\
&= (h)_h^{\overline{-\alpha}}(t)^2 - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}} \geq 0
\end{aligned}
$$

Then

$$
\nabla_h S(t) = (h)_h^{\overline{-\alpha}}(t)^2 - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}} \geq 0 \tag{2.1}
$$

When $t = a = 1$, we have

$$
\nabla_h S(a) = (h)_h^{\overline{-\alpha}} a^2 = (h)^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)} a^2 = (h)^{-\alpha}\Gamma(1-\alpha)a^2 \geq 0
$$

Hence

$$
y(a) \geq 0
$$

When $t = a + h$, we get

$$
\nabla_h S(a+h) = (h)_h^{\overline{-\alpha}}(a+h)^2 - \alpha\,(a+h-a+h))^{\overline{-\alpha-1}}\,(a)^2 h
$$

$$
= (h)^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)}(a+h)^2 - \alpha(h)^{-\alpha-1}\frac{\Gamma(2-\alpha-1)}{\Gamma(1)}a^2 h \geq 0
$$

$$
= (h)^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)}(a+h)^2 - \alpha(h)^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)}a^2 \geq 0
$$

Hence

$$
(a+h)^2 \geq \alpha(a)^2
$$

$$y(a+h) \geq \alpha y(a)$$

Now, we follow inductively to show that

$$y(t+h) = (t+h)^2 \geq \alpha (t)^2 = \alpha y(t), \forall t \in \mathbb{N}_{a,h}$$

Assume

$$y(k+h) \geq \alpha y(k), \forall k < t \text{ } such \text{ } that \text{ } k, t \in \mathbb{N}_{a,h}$$

We know that

$$\nabla_h S(t) = (h)_h^{\overline{-\alpha}}(t)^2 - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h\,(\,t - \rho_h(kh)))^{\overline{-\alpha-1}} \geq 0$$

Replace t by $t+h$. Then, we have

$$\begin{aligned}
\nabla_h S(t+h) &= (h)_h^{\overline{-\alpha}}(t+h)^2 - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h}} (kh)^2 h\,(\,t+h - \rho_h(kh)))^{\overline{-\alpha-1}} \\
&= (h)_h^{\overline{-\alpha}}(t+h)^2 - \alpha\big[(t+h-\rho_h(a))_h^{\overline{-\alpha-1}}(a)^2 h \\
&+ (t+h-\rho_h(a+h))_h^{\overline{-\alpha-1}}(a+h)^2 h + \ldots + (t+h-\rho_h(t))_h^{\overline{-\alpha-1}}(t)^2 h\big] \geq 0
\end{aligned}$$

Or

$$\begin{aligned}
&(h)_h^{\overline{-\alpha}}(t+h)^2 \\
&\qquad \geq \alpha\big[(t+h-\rho_h(a))_h^{\overline{-\alpha-1}}(a)^2 h \\
&\qquad + (t+h-\rho_h(a+h))_h^{\overline{-\alpha-1}}(a+h)^2 h + \ldots + (t+h-\rho_h(t))_h^{\overline{-\alpha-1}}(t)^2 h\big] \\
&\qquad \geq \alpha(t+h-t+h)_h^{\overline{-\alpha-1}}(t)^2 h \\
&\qquad = \alpha(2h)_h^{\overline{-\alpha-1}}(t)^2 h
\end{aligned}$$

It follows that

$$(h)^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)}(t+h)^2 \geq \alpha(h)^{-\alpha}\frac{\Gamma(2-\alpha-1)}{\Gamma(2)}t^2$$

$$(h)^{-\alpha}\Gamma(1-\alpha)(t+h)^2 \geq \alpha(h)^{-\alpha}\Gamma(1-\alpha)t^2$$

Hence $(t+h)^2 \geq \alpha(t)^2$

Which means $y(t+h) \geq \alpha y(t)$ then $y(t)$ is $\alpha - increasing$

Example 2.2.2

This example is an illustration of theorem 2.1.2 .

Let $y(t) = t^2, y(t) : \mathbb{N}_{1-h,h} \to \mathbb{R}$, $y(a) = y(1) = 1 \geq 0$. $y(t)$is increasing on $\mathbb{N}_{a,h}$.

We want to show that $({}_{a-h}\nabla_h^\alpha\, y\,)(t) \geq 0$ for $0 < \alpha \leq 1$ and $0 < h \leq 1$, $t \in \mathbb{N}_{a-h,h}$.

It is known that

$$({}_{a-h}\nabla_h^\alpha\, y\,)(t) \;=\frac{\nabla_h}{\Gamma(1-\alpha)}\sum_{k=\frac{a-h}{h+1}}^{\frac{t}{h}} (\,t-\rho_h(kh))^{\overline{-\alpha}}\,(kh)^2h$$

If we let

$$S(t) = \sum_{k=\frac{a-h}{h+1}}^{\frac{t}{h}} (\,t-\rho_h(kh))^{\overline{-\alpha}}\,(kh)^2h$$

then,

$$({}_{a-h}\nabla_h^\alpha\, y\,)(t) \;= \frac{\nabla_h}{\Gamma(1-\alpha)}\,S(t)$$

We need to show

$$\nabla_h S(t) \geq 0$$

From the application of theorem 2.1.1 When $t = a$, we have

$$\nabla_h S(a) = (h)_{h}^{\overline{-\alpha}} a^2 = (h)^{-\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1)}a^2 = (h)^{-\alpha}\Gamma(1-\alpha)a^2 \geq 0$$

hence

$$y(a) \geq 0$$

since

$$(h)_{h}^{\overline{-\alpha}} > 0, a^2 \geq 0 and \Gamma(1-\alpha) > 0$$

.

Assume that $\nabla_h S(\iota) \geq 0, \forall \iota < t$, we shall show that $\nabla_h S(t) \geq 0$

Since $y(t) = t^2$ is increasing , it follows that

$$t^2 \geq (t+h)^2 \geq a^2 \geq 0, \forall t \in \mathbb{N}_{a,h}$$

From equation (2.1)

$$\begin{aligned}
\nabla_h S(t) &= (h)_{h}^{\overline{-\alpha}}(t)^2 - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-1}} (kh)^2 h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}} \\
&= (h)_{h}^{\overline{-\alpha}}(t)^2 - \alpha\,(\,t-\rho_h(t-h))^{\overline{-\alpha-1}}\,(t-h)^2 h - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-2}} (kh)^2 h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}} \\
&= (h)_{h}^{\overline{-\alpha}}(t)^2 - \alpha\,(\,2h)^{\overline{-\alpha-1}}\,(t-h)^2 h - \alpha \sum_{k=\frac{a}{h}}^{\frac{t}{h-2}} (kh)^2 h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}
\end{aligned}$$

$$\begin{aligned}
&= (h)_h^{\overline{-\alpha}}(t)^2 - \alpha\,(\,2h)^{\overline{-\alpha-1}}(t-h)^2h - \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-2}}(kh)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&- \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-2}}(t-h)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&+ \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-2}}(t-h)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&= (h)_h^{\overline{-\alpha}}(t)^2 - \alpha\,(\,2h)^{\overline{-\alpha-1}}(t-h)^2h + \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-2}}((t-h)^2-(kh)^2)h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&- \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-2}}(t-h)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}
\end{aligned}$$

Since $y=t^2$ is increasing , then $((t-h)^2-(kh)^2)\geq 0, \forall k=\frac{a}{h},\frac{a}{h}+1,...,\frac{t}{h}-2$.
From which it follows that

$$\begin{aligned}
\nabla_h S(t) &= (h)_h^{\overline{-\alpha}}(t)^2 - \alpha\,(\,2h)^{\overline{-\alpha-1}}(t-h)^2h - \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-2}}(t-h)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&= (h)_h^{\overline{-\alpha}}(t)^2 - \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}(t-h)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&= (h)_h^{\overline{-\alpha}}(t)^2 - (h)_h^{\overline{-\alpha}}(t-h)^2 + (h)_h^{\overline{-\alpha}}(t-h)^2 - \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}(t-h)^2h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\\
&= (h)_h^{\overline{-\alpha}}\big[(t)^2-(t-h)^2\big] + (t-h)^2\Big[(h)_h^{\overline{-\alpha}} - \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\Big]\\
&\geq (t-h)^2\Big[(h)_h^{\overline{-\alpha}} - \alpha\sum_{k=\frac{a}{h}}^{\frac{t}{h-1}}h\,(\,t-\rho_h(kh))^{\overline{-\alpha-1}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= (t-h)^2\big[(h)_h^{\overline{-\alpha}} - \alpha h\,(\,t-a+h))^{\overline{-\alpha-1}} + (\,t-a-h+h))^{\overline{-\alpha-1}} + ... + (\,2h))^{\overline{-\alpha-1}}\big] \\
&= (t-h)^2(h)_h^{-\alpha}[\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)} - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha-1)}{\Gamma(\frac{t-a}{h})} - ... - \alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)}] \\
&= (t-h)^2(h)_h^{-\alpha}[\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)} - \alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - ... - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}] \\
&= (t-h)^2(h)_h^{-\alpha}[\frac{\Gamma(1-\alpha)(1-\alpha)}{1\Gamma(1)} - \alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - ... - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}] \\
&= (t-h)^2(h)_h^{-\alpha}[\frac{2\Gamma(2-\alpha)}{2\Gamma(2)} - \alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - ... - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}] \\
&= (t-h)^2(h)_h^{-\alpha}[\frac{\Gamma(2-\alpha)(2-\alpha)}{\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - ... - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}] \\
&= (t-h)^2(h)_h^{-\alpha}[\frac{3\Gamma(3-\alpha)}{3\Gamma(3)} - \alpha\frac{\Gamma(3-\alpha)}{\Gamma(4)} - ... - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}]
\end{aligned}
$$

If we continue in this manner, we conclude that

$$
\begin{aligned}
\nabla_h S(t) &\geq (t-h)^2(h)_h^{-\alpha}[\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h})} - \alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)}] \\
&= (t-h)^2(h)_h^{-\alpha}\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h})}[1-\alpha\frac{1}{\frac{t-a}{h}}] \\
&= (t-h)^2(h)_h^{-\alpha}\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h})}(\frac{t-a}{h}-\alpha) \\
&= (t-h)^2(h)_h^{-\alpha}\frac{\Gamma(\frac{t-a+h}{h}-\alpha)}{\Gamma(\frac{t-a+h}{h})} \\
&= (t-h)^2(t-a+h)_h^{\overline{-\alpha}} \geq 0
\end{aligned}
$$

Then

$$({}_{a-h}\nabla_h^\alpha\, y\,)(t)\ = \frac{\nabla_h S(t)}{\Gamma(1-\alpha)} \geq 0$$

Example 2.2.4

This example is an illustration of theorem 2.1.4

Let $y(t) = -t^2$, $y(t) : \mathbb{N}_{a-h,h} \to \mathbb{R}$

Suppose $({}_{a-h}\nabla^\alpha y)(t) \leq 0$ for $0 < \alpha \leq 1$ and $0 < h \leq 1$, $t \in \mathbb{N}_{a-h,h}$, we want to show y is $\alpha - decreasing$

Let $g:\ \mathbb{N}_{c-1} \to\ \mathbb{R}$ be a function such that $g(t) = -y(t) = t^2$; hence,

$$(\,{}_{a-h}\nabla^\alpha\ g)(t) = (\,{}_{a-h}\nabla^\alpha\ (-y))(t) = -({}_{a-h}\nabla^\alpha y)(t) \geq 0.$$

From Theorem 2.1.1, we conclude that $g(t)$ is $\alpha\ -\ increasing.$

hence,

$$g(t+1)\ \geq\ \alpha g(t),$$

so,

$$(t+1)^2\ \geq\ \alpha(t)^2$$

which is

$$-(t+1)^2\ \leq\ \alpha - ((t)^2)$$

$$y(t+1)\ \leq\ \alpha(y(t)),$$

that is to say, $y(t))$ is $\alpha\ -\ decreasing.$

Example 2.2.5

This example is an illustration of theorem 2.1.5.

Let $y(t) = -t^2$, $y(t) : \mathbb{N}_{a-h,h} \to \mathbb{R}$, $y(t)$ is decreasing on $\mathbb{N}^{a,h}$, $y(a=1) \leq 0$.

we want to show $({}_{a-h}\nabla^\alpha y)(t) \leq 0$, for $0 < \alpha \leq 1$ and $0 < h \leq 1$.

Let $g : \mathbb{N}_{c-1} \to \mathbb{R}$ be a function such that $g(t) = -y(t) = t^2$, then its increasing, $g(a) = a^2 \geq 0$

From Theorem 2.1.2, $({}_{a-h}\nabla^{\alpha} g)(t) \geq 0$

then

$$({}_{a-h}\nabla^{\alpha}\, y)(t) = ({}_{a-h}\nabla^{\alpha}\, (-g))(t) = -({}_{a-h}\nabla^{\alpha} g)(t) \leq 0.$$

2.3 The h-Fractional Difference Version of Mean Value Theorem MVT

The following Lemmas and Theorems are needed to proceed with the mean value theorem.

Lemma 2.3.1 *[30] For any $0 < \alpha \leq 1$, $0 < h \leq 1$, and $f : \mathbb{N}_{a+h,h} \to \mathbb{R}$ the following holds :*

$$ {}_{a}\nabla_{h}^{-\alpha} \nabla_{h} f(t) = \nabla_{h\,a}\nabla_{h}^{-\alpha} f(t) - \frac{(t-a)_{h}^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) \tag{2.2}$$

Lemma 2.3.2 *[30] For any $0 < \alpha \leq 1$, $0 < h \leq 1$, and $y : \mathbb{N}_{a+h,h} \to \mathbb{R}$ the following holds :*

$$ {}_{a-h}\nabla_{h}^{\alpha} y(t) = {}_{a}\nabla_{h}^{\alpha} y(t) + \frac{(t-a+h)_{h}^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} y(a)h \tag{2.3}$$

Theorem 2.3.3 *[30] For any $0 < \alpha \leq 1$, $0 < h \leq 1$, and $y : \mathbb{N}_{a+h,h} \to \mathbb{R}$ the following holds :*

$$ {}_{a}\nabla_{h}^{-\alpha}\, {}_{a-h}\nabla_{h}^{\alpha} y(t) = y(t) - \frac{h^{1-\alpha}}{\Gamma(\alpha)} (t-a+h)_{h}^{\overline{\alpha-1}} y(a) \tag{2.4}$$

Consider the following initial fractional difference equations :

$$ {}_{a-h}\nabla_{h}^{\alpha} y(t) = f(t, y(t)) \, for \; t = a+h, a+2h, ..., \tag{2.5}$$

$$ {}_{a-h}\nabla_{h}^{-(1-\alpha)} y(t)|_{t=a} = h^{1-\alpha} y(a) = c \tag{2.6}$$

Where $0 < \alpha, h < 1$ *and* a *is any real number.*

Theorem 2.3.4 *[30]* y *is a solution of the initial value problem* (2.5), (2.6) *if and only if it has been formulated by*

$$y(t) = \frac{(t-a+h)_h^{\overline{\alpha-1}}}{\Gamma(\alpha)} c +_a \nabla_h^{-\alpha} f(t, y(t)) \tag{2.7}$$

depending on Theorem (2.3.3), *we can write :*

$${}_a\nabla_h^{-\alpha}{}_{a-h}\nabla_h^{\alpha} y(t) = y(t) - R_h(\alpha, t, a) y(a) \tag{2.8}$$

where $R_h(\alpha, t, a) = \frac{h^{\overline{1-\alpha}}}{\Gamma(\alpha)}(t-a+h)_h^{\alpha-1}$.

Theorem 2.3.5 ***(The h-Fractional Difference MVT) [30]***

Let f *and* g *be functions defined on* $\mathbb{N}_{a,h} \cap_{b,h} \mathbb{N} = \{a, a+h, a+2h, ...b-2h, b-h, b\}$*where* $b = a + kh$ *for some* $k \in \mathbb{N}$*. Assume that* g *is strictly increasing* $g(a) > 0$, *and* $0 < \alpha \leq 1$, $0 < h \leq 1$. *Then there exist* $s_1, s_2 \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N}$ *such that :*

$$\frac{({}_{a-h}\nabla_h^{\alpha} f)(s_1)}{({}_{a-h}\nabla_h^{\alpha} g)(s_1)} \leq \frac{f(b) - R_h(\alpha, b, a) f(a)}{g(b) - R_h(\alpha, b, a) g(a)} \leq \frac{({}_{a-h}^{R}\nabla_h^{\alpha} f)(s_2)}{({}_{a-h}^{R}\nabla_h^{\alpha} g)(s_2)} \tag{2.9}$$

Application of the h-Fractional Difference MVT

We used the article [30] as a reference.

Let $f(t)$, $g(t)$ be two functions defined on $\mathbb{N}_{a,h} \cap_{b,h} \mathbb{N} = \{a, a+h, a+2h, ...b-2h, b-h, b\}$ where $a = 1$ and $b = a + kh$ for some $k \in \mathbb{N}$.

$$f(t) = t$$

$$g(t) = t^2$$

g is strictly increasing, $g(a) > 0$, and $0 < \alpha \leq 1, 0 < h \leq 1$. we want to show that there exist $s_1, s_2 \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N}$ such that :

$$\frac{({}_{a-h}\nabla_h^\alpha)(s_1)}{({}_{a-h}\nabla_h^\alpha)(s_1)^2} \leq \frac{b - R_h(\alpha, b, a)a}{b^2 - R_h(\alpha, b, a)a^2} \leq \frac{({}^R_{a-h}\nabla_h^\alpha)(s_2)}{({}^R_{a-h}\nabla_h^\alpha)(s_2)^2} \tag{2.10}$$

we need to show

$$b^2 - R_h(\alpha, b, a)a^2 > 0$$

since g is strictly increasing, then by Theorem (2.1.3)

$$({}_{a-h}\nabla_h^\alpha g)(t) > 0 \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N}$$

Applying the fractional sum operator on both sides of the inequality

$${}_a\nabla_h^{-\alpha}({}_{a-h}\nabla_h^\alpha g)(t) >_a \nabla_h^{-\alpha}(0) \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N}.$$

or using equation (2.8) we get

$$g(t) - R_h(\alpha, t, a)g(a) > 0$$

for $t = b$, we have

$$g(b) - R_h(\alpha, b, a)g(a) > 0$$

hence,

$$b^2 - R_h(\alpha, b, a)a^2 > 0$$

to prove equation (2.10), we use the method of contradiction. Assume that equation (2.10) is not true, then either

$$\frac{({}_{a-h}\nabla_h^\alpha)(s_1)}{({}_{a-h}\nabla_h^\alpha)(s_1)^2} > \frac{b - R_h(\alpha, b, a)a}{b^2 - R_h(\alpha, b, a)a^2}, \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N} \tag{2.11}$$

or

$$\frac{({}_{a-h}\nabla_h^\alpha)(s_1)}{({}_{a-h}\nabla_h^\alpha)(s_1)^2} < \frac{b - R_h(\alpha, b, a)a}{b^2 - R_h(\alpha, b, a)a^2}, \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N} \tag{2.12}$$

again, since g is strictly increasing, then by Theorem (2.1.3) we conclude that

$$({}_{a-h}\nabla_h^\alpha g)(t) > 0 \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N}$$

hence, equation (2.11) becomes

$$({}_{a-h}\nabla_h^\alpha)(s_1) > ({}_{a-h}\nabla_h^\alpha)(s_1)^2 \frac{b - R_h(\alpha, b, a)a}{b^2 - R_h(\alpha, b, a)a^2}, \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N} \tag{2.13}$$

Applying the fractional sum operator on both sides of the inequality at $t = b$ and using equation (2.8), we see that

$$(b - R_h(\alpha, b, a)a) > (b^2 - R_h(\alpha, b, a)a^2) \frac{b - R_h(\alpha, b, a)a}{b^2 - R_h(\alpha, b, a)a^2}, \forall t \in \mathbb{N}_{a,h} \cap_{b,h} \mathbb{N} \tag{2.14}$$

and hence $f(b) < f(b)$, which is a contradiction. In a similar way equation (2.12) will lead to a contradiction.

Chapter Three

Fractional Proportional h-Differences

In the previous chapters, we introduced the monotonicity analysis of fractional proportional differences and the monotonicity analysis of fractional h-differences, now we will introduce the monotonicity analysis of fractional proportional h-differences.

3.1 Monotonicity Analysis

In this section, we will introduce the monotonicity analysis of fractional proportional h-differences.

Definition 3.1.1 *Let* $f\colon \mathbb{N}_{a,h} \to \mathbb{R}$ *be a function satisfying* $f(a) \geq 0$,*and let* $0 \leq h < 1$, $0 \leq \rho < 1$ *Then,* $f(t)$ *is called an* $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)$ *−increasing function on* $\mathbb{N}_{a,h}$ *if* $y(t+h) \geq \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) y(t)\ \forall t \in \mathbb{N}_{a,h}$.

Definition 3.1.2 *Let* $f\colon \mathbb{N}_{a,h} \to \mathbb{R}$ *be a function satisfying* $f(a) \leq 0$,*and let* $0 \leq h < 1$, $0 \leq \rho < 1$ *Then,* $f(t)$ *is called an* $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)$ *−decreasing function on* $\mathbb{N}_{a,h}$ *if* $y(t+h) \leq \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) y(t)\ \forall t \in \mathbb{N}_{a,h}$.

Theorem 3.1.1 *Let* $f : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *be a function, suppose that* $(\ {}_{a-h}\nabla_h^{\alpha,\rho} f\ )(t) \geq 0\ \forall t \in \mathbb{N}_{a-h,h}$*for* $0 < \alpha \leq 1$, $0 < h \leq 1$ *and* $0 < \rho \leq 1$, $t \in \mathbb{N}_{a-h,h}$. *Then* $f(t)$ *is* $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)-$ *increasing.*

Proof: Using definition 1.1.9, we recall that:

$$
\begin{aligned}
({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) &= \frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)} \sum_{k=(a-h)/h+1}^{t/h} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h \\
&= \frac{\nabla_h^{\rho}}{\rho^{1-\alpha}\Gamma(1-\alpha)} \sum_{k=a/h}^{t/h} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h.
\end{aligned}
$$

Now let

$$
S(t) = \sum_{k=a/h}^{t/h} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h.
$$

Using the assumption, we have $\nabla_h^{\rho} S(t) \geq 0$, that is

$$
\nabla_h^{\rho} S(t) = (1-\rho)S(t) + \rho\nabla_h S(t) \geq 0. \tag{3.1}
$$

For $\nabla_h S(t)$, we have

$$
\nabla_h S(t) = \frac{S(t)-S(t-h)}{h}
$$

By substituting $S(t)$and using basic definitions presented in chapter one, we can do the following:

$$
\begin{aligned}
\nabla_h S(t) = &\left[\sum_{k=a/h}^{t/h} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h\right.\\
&\left.-\sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h\right]\frac{1}{h}\\
&=\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t)\\
&+\left[\sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h\right.\\
&\left.-\sum_{k=a/h}^{t/h-1} \widehat{e}_p(t-h-h(k-1+\alpha),0)(t-h-\rho_h(kh))_h^{\overline{-\alpha}} f(kh)h\right]\frac{1}{h}\\
&=\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t)\\
&+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\widehat{e}_p(t-h(k-1+\alpha),0)(t-kh+h)_h^{\overline{-\alpha}}\right.\\
&\left.-\widehat{e}_p(t-h-h(k-1+\alpha),0)(t-kh)_h^{\overline{-\alpha}}\right]\frac{1}{h}\\
&=\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t)\\
&+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k+1-\alpha}\frac{\Gamma(\frac{t-kh+h}{h}-\alpha)}{\Gamma(\frac{t-kh+h}{h})}h^{-\alpha}\right.\\
&\left.-\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\frac{\Gamma(\frac{t-kh}{h}-\alpha)}{\Gamma(\frac{t-kh}{h})}h^{-\alpha}\right]\frac{1}{h}\\
&=\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(t)\\
&+\sum_{k=a/h}^{t/h-1} f(kh)h\left[\left(\frac{\rho}{\rho-(\rho-1)h}\right)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\frac{\Gamma(t/h-k+1-\alpha)}{\Gamma(t/h-k+1)}h^{-\alpha}\right.\\
&\left.-\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{t/h-k-\alpha}\frac{\Gamma(t/h-k-\alpha)}{\Gamma(t/h-k)}h^{-\alpha}\right]\frac{1}{h}
\end{aligned}
$$

$$
\begin{aligned}
&= \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) + \sum_{k=a/h}^{t/h-1} f(kh)h \left[\left(\frac{\rho}{\rho - (\rho - 1)h} \right) \left(\frac{t/h - k - \alpha}{t/h - k} \right) - 1 \right] \\
&\left(\frac{\rho}{\rho - (\rho - 1)h} \right)^{t/h-k-\alpha} \left(\frac{\Gamma(t/h - k - \alpha)}{\Gamma(t/h - k)} \right) \frac{h^{-\alpha}}{h} \\
&= \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) + \sum_{k=a/h}^{t/h-1} f(kh)h \left[\frac{\rho(t - hk - \alpha) - t + hk}{\rho - (\rho - 1)h} \right] \\
&\left(\frac{\rho}{\rho - (\rho - 1)h} \right)^{t/h-k-\alpha} \left(\frac{\Gamma(t/h - k + 1 + (-\alpha - 1))}{\Gamma(t/h - k + 1)} \right) h^{-\alpha-1} \\
&= \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) + \sum_{k=a/h}^{t/h-1} f(kh)h \left[t - hk - \alpha - \frac{t - hk}{\rho} \right] \\
&\widehat{e}_p(t - h(k - 1 + \alpha), 0)(t - \rho_h(kh))_h^{\overline{-\alpha-1}} \\
&= \widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) - \frac{1}{\rho} \sum_{k=a/h}^{t/h-1} [t - hk - \rho(t - hk - \alpha)] \\
&\widehat{e}_p(t - h(k - 1 + \alpha), 0)(t - \rho_h(kh))_h^{\overline{-\alpha-1}} f(kh)h. \qquad (3.2)
\end{aligned}
$$

Using similar steps on $(1 - \rho)S(t)$ in equation (3.1), we get

$$
\begin{aligned}
(1 - \rho)S(t) &= (1 - \rho) \sum_{k=a/h}^{t/h} \widehat{e}_p(t - h(k - 1 + \alpha), 0)(t - \rho_h(kh))_h^{\overline{-\alpha}} f(kh)h \\
&= (1 - \rho)h\widehat{e}_p(h - h\alpha, 0)(h)_h^{\overline{-\alpha}} f(t) + \sum_{k=a/h}^{t/h-1} [\rho(hk + h\alpha - t) + t - hk - h\alpha] \\
&\quad \widehat{e}_p(t - h(k - 1 + \alpha), 0)(t - \rho_h(kh))_h^{\overline{-\alpha-1}} f(kh)h. \qquad (3.3)
\end{aligned}
$$

Now substitute equations (3.2) and (3.3) in equation (3.1), we have

$$\begin{aligned}\nabla_h^\rho S(t) &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \Bigg[\sum_{k=a/h}^{t/h-1}[t-hk \\ &\quad -\rho(t-hk-\alpha)]\,\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \\ &\quad -\sum_{k=a/h}^{t/h-1}[\rho(hk+h\alpha-t)+t-hk-h\alpha]\,\widehat{e}_p(t-h(k-1+\alpha),0) \\ &\quad (t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h\Bigg] \qquad (3.4) \\ &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \sum_{k=a/h}^{t/h-1}\Bigg[[t-hk \\ &\quad -\rho(t-hk-\alpha)] - [\rho(hk+h\alpha-t)+t-hk-h\alpha]\Bigg] \\ &\quad \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h\end{aligned}$$

$$\begin{aligned}&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \sum_{k=a/h}^{t/h-1}[\rho\alpha-\rho h\alpha \\ &\quad +h\alpha]\,\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \\ &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\,((1-\rho)h+\rho)\sum_{k=a/h}^{t/h-1} \\ &\quad \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \geq 0 \qquad (3.5)\end{aligned}$$

When $t=a$, we get

$$\nabla_h^\rho S(a) \;=\; ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(a) \geq 0, \qquad (3.6)$$

Hence $f(a) \geq 0$.

When $t = a + h$, we have

$$\begin{aligned}
\nabla_h^\rho S(a+h) &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(a+h) - \alpha((1-\rho)h+\rho) \\
&\quad \widehat{e}_p(2h-h\alpha,0)(a+h-\rho_h(a))_h^{\overline{-\alpha-1}} f(a)h \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} f(a+h) - \alpha((1-\rho)h+\rho) \\
&\quad \widehat{e}_p(2h-h\alpha,0)(2h)_h^{\overline{-\alpha-1}} f(a)h \\
&= ((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\frac{h-h\alpha}{h}} \left(\frac{\Gamma(h/h-\alpha)}{\Gamma(h/h)}\right) h^{-\alpha} f(a+h) \\
&\quad - \alpha((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{\frac{2h-h\alpha}{h}} \left(\frac{\Gamma(2h/h-\alpha-1)}{\Gamma(2h/h)}\right) h^{-\alpha-1} f(a)h \\
&= ((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{1-\alpha} \left(\frac{\Gamma(1-\alpha)}{\Gamma(1)}\right) h^{-\alpha} f(a+h) \\
&\quad - \alpha((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{2-\alpha} \left(\frac{\Gamma(1-\alpha)}{\Gamma(2)}\right) h^{-\alpha} f(a) \\
&= ((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{1-\alpha} \Gamma(1-\alpha) h^{-\alpha} f(a+h) \\
&\quad - \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)((1-\rho)h+\rho)\left(\frac{\rho}{\rho-(\rho-1)h}\right)^{1-\alpha} \Gamma(1-\alpha) h^{-\alpha} f(a) \\
&\geq 0
\end{aligned}$$

Hence that $f(a+h) \geq \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) f(a)$

It is clear that by continuing inductively in the same way, we get that in general

$$f(t+h) \geq \left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) f(t)$$

. ☐

The monotonicity factor as a function of ρ is presented in Figure 3.1 for different values of h and different values of α.

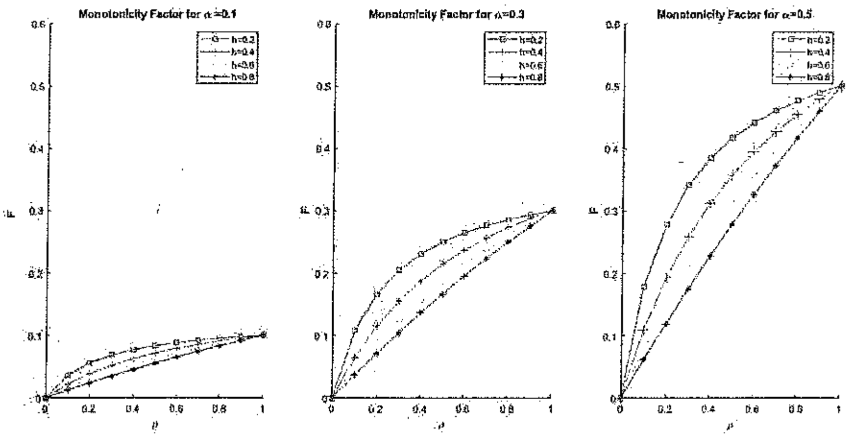

Fig. 3.1: The Monotonicity factor.

Theorem 3.1.2 *Assume that the function* $f : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *satisfies* $f(a) \geq 0$ *for* $0 < \alpha \leq 1$, $0 < h \leq 1$ *and* $0 < \rho \leq 1$. *If* f *is increasing on* $\mathbb{N}_{a,h}$ *then we have*

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \geq 0, \ \forall t \in \mathbb{N}_{a-h,h}$$

Proof: As shown in the proof of Theorem (3.1.1), we have

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) = \frac{1}{\rho^{1-\alpha}\Gamma(1-\alpha)} \nabla_h^{\rho} S(t), \quad t \in \mathbb{N}_{a-h,h}.$$

It is enough to show that $S(t)$ is increasing on $\mathbb{N}_{a,h}$ to reach our goal.
Now when $t = a$, we get

$$\nabla_h^{\rho} S(a) \;=\; ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha, 0)(h)_h^{\overline{-\alpha}} f(a).$$

By the assumptions $0 < h \leq 1$, $0 < \rho \leq 1$, $0 < \alpha \leq 1$ and $f(a) \geq 0$, then $\nabla_h^{\rho} S(a) \geq 0$
Now assume that $\nabla_h^{\rho} S(i) \geq 0$, $\forall i < t$, so we need to show that also $\nabla_h^{\rho} S(t) \geq 0$.

From the assumption that $f(t)$ is increasing, then

$$f(t) \geq f(t-h) \geq f(a) \geq 0, \forall t \in \mathbb{N}_{a,h}$$

Recalling equation (3.4),

$$\begin{aligned} \nabla_h^\rho S(t) &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\left((1-\rho)h+\rho\right)\sum_{k=a/h}^{t/h-1} \\ &\quad \widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \geq 0, \end{aligned}$$

we have

$$\begin{aligned} \nabla_h^\rho S(t) &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\left((1-\rho)h+\rho\right)\Big[\widehat{e}_p(t-h(t/h-1 \\ &\quad -1+\alpha),0)(t-\rho_h(t-h))_h^{\overline{-\alpha-1}}f(t-h)h + \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0) \\ &\quad (t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h\Big] \\ &= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\left((1-\rho)h+\rho\right)\Big[\widehat{e}_p(2h-h\alpha,0) \\ &\quad (2h)_h^{\overline{-\alpha-1}}f(t-h)h + \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(kh)h \\ &\quad - \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(t-h)h \\ &\quad + \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(t-h)h\Big] \end{aligned}$$

$$
\begin{aligned}
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\,((1-\rho)h+\rho)\Big[\widehat{e}_p(2h-h\alpha,0) \\
&(2h)_h^{\overline{-\alpha-1}}f(t-h)h + \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}} \\
&(f(t-h)-f(kh))h + \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(t-h)h\Big].
\end{aligned}
$$

Given that $f(t)$ is increasing, then $f(t-h)-f(kh)\geq 0, \forall k$, where

$$
k=\frac{a}{h},\frac{a}{h}+1,\ldots,\frac{t}{h}-2.
$$

hence, we can do the following

$$
\begin{aligned}
\nabla_h^\rho S(t) &\geq ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\,((1-\rho)h+\rho)\Big[\widehat{e}_p(2h-h\alpha,0) \\
&(2h)_h^{\overline{-\alpha-1}}f(t-h)h + \sum_{k=a/h}^{t/h-2}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(t-h)h\Big] \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - \alpha\,((1-\rho)h+\rho)\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h \\
&(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(t-h)h \\
&= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t) - ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} \\
&f(t-h) + ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t-h) \\
&-\alpha\,((1-\rho)h+\rho)\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}f(t-h)h
\end{aligned}
$$

$$= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}(f(t)-f(t-h))$$

$$+ ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}f(t-h)$$

$$-\alpha\,((1-\rho)h+\rho)\,f(t-h)h\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}$$

$$= ((1-\rho)h+\rho)\,\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}}(f(t)-f(t-h)) + ((1-\rho)h+\rho)\,f(t-h)$$

$$\Big[\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} - \alpha h\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h(k-1+\alpha),0)(t-\rho_h(kh))_h^{\overline{-\alpha-1}}\Big]$$

$$\geq ((1-\rho)h+\rho)\,f(t-h)\Big[\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} - \alpha h\sum_{k=a/h}^{t/h-1}\widehat{e}_p(t-h(k-1+\alpha),0)$$

$$(t-\rho_h(kh))_h^{\overline{-\alpha-1}}\Big]$$

$$= ((1-\rho)h+\rho)\,f(t-h)\Big[\widehat{e}_p(h-h\alpha,0)(h)_h^{\overline{-\alpha}} - \alpha h\Big(\widehat{e}_p(t-a+h-h\alpha),0)$$

$$(t-\rho_h(a))_h^{\overline{-\alpha-1}} + \widehat{e}_p(t-a-h\alpha),0)(t-\rho_h(a+h))_h^{\overline{-\alpha-1}} + \cdots + \widehat{e}_p(2h-h\alpha),0)$$

$$(t-\rho_h(t-h))_h^{\overline{-\alpha-1}}\Big)\Big]$$

$$= ((1-\rho)h+\rho)\,f(t-h)h^{-\alpha}\Big[\widehat{e}_p(h-h\alpha,0)\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \widehat{e}_p(t-a+h-h\alpha,0)$$

$$\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1)} - \widehat{e}_p(t-a-h\alpha,0)\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha-1)}{\Gamma(\frac{t-a}{h})} - \cdots - \widehat{e}_p(2h-h\alpha,0)$$

$$\alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)}\Big)\Big]$$

$$= ((1-\rho)h+\rho)\,f(t-h)h^{-\alpha}\Big[\widehat{e}_p(h-h\alpha,0)\frac{\Gamma(1-\alpha)}{\Gamma(1)} - \widehat{e}_p(2h-h\alpha,0)$$

$$\alpha\frac{\Gamma(1-\alpha)}{\Gamma(2)} - \widehat{e}_p(3h-h\alpha,0)\alpha\frac{\Gamma(2-\alpha)}{\Gamma(3)} - \cdots - \widehat{e}_p(t-a+h-h\alpha,0)$$

$$\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1))}\Big)\Big].$$

By adding the terms and continuing in the same manner, we get

$$\nabla_h^\rho S(t) \geq ((1-\rho)h+\rho)\, f(t-h)h^{-\alpha}\Big(\widehat{e}_p(t-a-h\alpha,0)\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}))} \\ -\widehat{e}_p(t-a+h-h\alpha,0)\alpha\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}+1))}\Big)$$

$$= ((1-\rho)h+\rho)\, f(t-h)h^{-\alpha}\widehat{e}_p(t-a-h\alpha,0)\frac{\Gamma(\frac{t-a}{h}-\alpha)}{\Gamma(\frac{t-a}{h}))}\Big(1- \\ -\left(\frac{\rho}{\rho-(\rho-1)h}\right)\alpha\frac{1}{\frac{t-a}{h}}\Big) \\ = (\rho+\frac{h(\rho-1)(t-a)}{a+h\alpha-t})f(t-h)\widehat{e}_p(t-a-h\alpha,0)h^{-\alpha}\frac{\Gamma(\frac{t-a+h}{h}-\alpha)}{\Gamma(\frac{t-a+h}{h}))} \\ = (\rho+\frac{h(\rho-1)(t-a)}{a+h\alpha-t})f(t-h)\widehat{e}_p(t-a-h\alpha,0)(t-a+h)^{\overline{-\alpha}} \geq 0.$$

Hence, the proof is completed.

□

Theorem 3.1.3 *Let a function* $f : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *satisfies* $f(a) > 0$ *and be strictly increasing on* $\mathbb{N}_{a,h}$, *where* $0 < \alpha \leq 1$, $0 < h \leq 1$ *and* $0 < \rho \leq 1$. *Then*

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) > 0, \forall t \in \mathbb{N}_{a-h,h}$$

Proof: The proof is in the same manner as the proof of 3.1.2 with paying attention to that the function f is strictly increasing. □

Theorem 3.1.4 *Let* $f : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *and suppose that* $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \leq 0$ *,for* $0 < \alpha \leq 1$, $0 < h \leq 1$ *and* $0 < \rho \leq 1 t \in \mathbb{N}_{a-h,h}$, . *Then* $f(t)$*is* $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) -$ *decreasing*

Proof: Let $g : \mathbb{N}_{a-h,h} \to \mathbb{R}$ be a function such that

$$g(t) = -f(t)$$

, then

$$({}_{a-h}\nabla_h^{\alpha,\rho} g)(t) = ({}_{a-h}\nabla_h^{\alpha,\rho}(-f))(t) = -({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \geq 0$$

By applying Theorem (3.1.1) to $g(t)$, g is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) - increasing$.
Hence, f is$\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) - decreasing$ □

Theorem 3.1.5 *let a function* $f : \mathbb{N}_{a-h,h} \to \mathbb{R}$ *satisfies* $f(a) \leq 0$ *for* $0 < \alpha \leq 1$, $0 < h \leq 1$ *and* $0 < \rho \leq 1$. *If f is decreasing on* $\mathbb{N}_{a,h}$ *then we have*

$$({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \leq 0, \ \forall t \in \mathbb{N}_{a-h,h}$$

Proof: The proof follows by applying Theorem (3.1.2) to $g(t) = -f(t)$
if $f(a) \leq 0$ then $g(a) \geq 0$
$f(t)$is decreasing , then $g(t)$ is increasing
Then, by Theorem (3.1.2) we have $({}_{a-h}\nabla_h^{\alpha,\rho} g)(t) \geq 0, \ \forall t \in \mathbb{N}_{a-h,h}$
Hence , $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \leq 0, \ \forall t \in \mathbb{N}_{a-h,h}$ □

3.2 Conclusions and Future Work

First, the definition of nabla fractional proportional h-sums and Reiman-Liouville (RL) fractional proportional h-differences on the time scale $h\mathbb{Z}$ have been presented. Some applications on old monotonicity analysis of fractional proportional differences and fractional h-differences and an application on the h-fractional difference version of the mean value theorem have been detected. Finally, the monotonicity analysis of fractional proportional h-differences have been proved if $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \geq 0$, then $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) - increasing$,$\forall t \in \mathbb{N}_{a-h,h}$, also if $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \leq 0$, then $f(t)$ is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right) - decreasing$,$\forall t \in \mathbb{N}_{a-h,h}$. Notice that the monotonicity factor which is $\left(\frac{\alpha\rho}{\rho-(\rho-1)h}\right)$, affected by the discretization step h and ρ. Moreover, if f is increasing satisfying $f(a) \geq 0$, then we have $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \geq 0, \ \forall t \in \mathbb{N}_{a-h,h}$, If f is strictly increasing on $\mathbb{N}_{a,h}$ satisfying that $f(a) > 0$ then we have $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) > 0, \ \forall t \in \mathbb{N}_{a-h,h}$, also if f is decreasing on $\mathbb{N}_{a,h}$ satisfying that $f(a) \leq 0$ then we have $({}_{a-h}\nabla_h^{\alpha,\rho} f)(t) \leq 0, \ \forall t \in \mathbb{N}_{a-h,h}$.

As future work, we will extend our work for $\alpha > 1$ and $\rho > 1$.

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ملخص

تحليل التزايد والتناقص للفروق النسبية المتناسبة h

إيمان سعدي

العديد من المسائل العلمية والمسائل في علم الهندسة بالامكان حلها باستخدام حساب التفاضل والتكامل الكسري المنفصل والمتصل . مؤخراً ، فإن الأبحاث تتركز حول تحليل التزايد والتناقص للفروق الكسرية.

في هذا العمل ، نقدم تعريف مجموع نابلا المتناسب الكسري h وتعريف فرق ريمان - ليوفيل المتناسب الكسري h ، حيث $0 < \alpha \leq 1$ و $0 < \rho \leq 1$ ، بعض التطبيقات على الدراسات السابقة في موضوع تحليل التزايد والتناقص للفروق الكسرية المتناسبة والفروق الكسرية ال h جزئية . بالإضافة لذلك ، نقدم نظريات مثبتة في موضوع تحليل التزايد والتناقص للفروق النسبية المتناسبة h حيث $0 < \alpha \leq 1$ و $0 < \rho \leq 1$.