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Compatible Factorization and Triad Design for Complete Graph of Some Orders

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This thesis was submitted in partial fulfillment of the requirements for the master's degree in Applied Mathematics

February/2022

Thesis Approval

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Declaration

My name is Mais Jaber, and I am a student of the AAUP university number 201720227. I confirm that I worked on this Master's thesis myself. And I declare that I have complied with all regulations, instructions, Arab American University standards of Academic codes of conduct. I also adhere to the Dean's Council regulations in withdrawing their confirmation of this degree in case of any violations.

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Dedication

All Thanks for who provided me with their support to achieve my work successfully. I dedicate my simple work for the dearest people to me: my beloved parents, my husband, my brothers, my parents-in-law, my brothers and sisters-in-law, my grandmother, and my grandfather, who will be remembered at every step.

Acknowledgements

I would like to express my special thanks to Dr. Abdelhalim Ziqan and Dr. Tareq Abu Saa for the continuous support accomplishing this thesis; for their patience, motivation, and immense knowledge. Their guidance, encouragement, and invaluable suggestions helped me in all the time of writing of this thesis. There are a number of people without whom this thesis might not have been written, and to whom I am greatly indebted; to my mother and father who have been a source of encouragement and inspiration to me throughout my life. I do appreciate my friends, and colleagues, who assisted, advised, and supported my research. Last but not the least; I would like to thank my family: my parents and my brothers and my husband for supporting me spiritually throughout writing this thesis.

Abstract

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In this work, we study Compatible Factorization $CF(v)$ and Triad design $TD(v)$ for some orders v, where $v = 6n + 5$. These are methods of arranging distinct triples (listing and counting triangles) on v objects with some properties. Previous studies on $TD(v)$ reported its existence when $v = 6n + 1$ and $v = 6n + 5$ and $TD(7)$ were developed by using a brute-force method. Furthermore, a starter of triad design, $STD(v) = SCF(v) \cup \overline{SCF(v)}$. Additionally, in this thesis, new technique for $STD(v)$ algorithms, known as the "Generalized Interval Method - GIM" constructed, by analyzing the pattern of the triples by illustrating the cases $v = 5$, 11 , 17, 23 and 29. At the end, this technique, lists the element of $TD(6n + 5)$ by repeated addition of 1 (mod v) from the $STD(v)$.

We focus on the construction of triad design for complete graph K_n. We construct a new method for developing a triad design on v objects $TD(v)$ that counts and list all triangles in K_n. This method depends on analyzing the triples to construct the starter. We illustrate the method by considering the cases $v =$

$11, 17, 23$ and 29. Then we conclude the general case of $v = 6n + 5$. We illustrate our results by building $TD(11)$, $TD(17)$ and $TD(23)$.

Contents

Introduction

A triangle in an undirected graph is a set of three vertices such that each possible edge between them is present in the graph. Combinatorial design is defined as a method of selecting subsets from a finite set in such a way that some limitations or conditions are satisfied [4], [3], [16]

In the past, the fundamental graph problem of triangle counting and listing has been studied intensively from a theoretical point of view [12]. Recently, triangle counting has also become a widely used tool in network analysis [13]. Due to the very large size of networks like the Internet, WWW or social networks, the efficiency of algorithms for triangle counting and listing is an important issue [15]. Counting and listing triangles in a graph has a practical importance in complex network analysis. For example, triangles are well-studied subgraphs [15], [11], [17]

Compatible factorization was covered through several article or researches, such as: [10], the authors exmined Three-fold Triple Systems. Compatible factorization was explained by W.D. Wallis in [15]. In [1], Triad design of orders 7, 13 and 19 has been considered to provide the design for the general case $v = 6n+1$, by analysing the triples to construct a starter for $TD(v)$, denoted by $STD(v)$, using interval techniques of the triples in the starter.

In this thesis, we build on previous studies and develope Triad Design for order $v = 6n + 5$, which exists if v = 1 or 5 (mod 6) [7], [14] . Our aim is to find the triad design of order $v = 6n+5$. Further, we generalize the method of finding the triad design for all $v = 6n + 5$. Firstly, we present the definition of Triad Design, Compatible Factorization, starter and their properties. After that, we construct the triad design of some orders. Also we generalize the method of finding the triad design for all $v = 6n + 5$.

In Chapter 1, we recall some of the definitions and proprieties for the Triad Design and Compatible Factorization. Chapter 2 will be dedicated for constructing the triad design of orders $11, 17$ and 23 using the method that mentioned in [1]. Afterwards, we present the Triad Design of orders 23 and 29 in Chapter 3. In chapter 4, we consider the general case of the construction of $STD(v)$, where $v = 6n + 5$. Finally, we conclude this thesis and suggest future research directions

Chapter One

Compatible Factorization and Triad Design

In this chapter, some definitions and notations which are needed throughout this thesis are introduced. We give properties of triples and compatible factorization of a complete undirected graph of order v.

1.1 Compatible Factorization

Definition 1.1.1 *[5] A graph G is a pair of sets. $G = (V, E)$, where V is a finite non-empty set of elements called vertices, and E is a set of unordered pair of distinct vertices called edges.*

In a graph G with the set of vertices $V = \{v_1, v_2, \ldots, v_n\}$, the pairs $\{v_i, v_j\} \in E$ if and only if there is a line in G which connect v_i and v_j. The edge $\{v_i, v_j\}$ is commonly written as $v_i v_j$. In this case v_i and v_j are adjacent or neighbours and the graph is called an undirected graph. A simple graph is an undirected graph without graph loops or multiple edges.

Definition 1.1.2 *[19] A complete graph is a simple, undirected graph in which each pair of vertices are connected. A complete graph with n vertices is denoted by K_n. A triangle is a complete graph K_3 as explained in Fig 1.1 .*

Definition 1.1.3 *A 1-factor of a graph is a set of edges in which every vertex of the graph appears exactly once. Further, a near 1-factor in K_{2n-1} is a set of $n-1$ edges which covers all but one vertex. [2]*

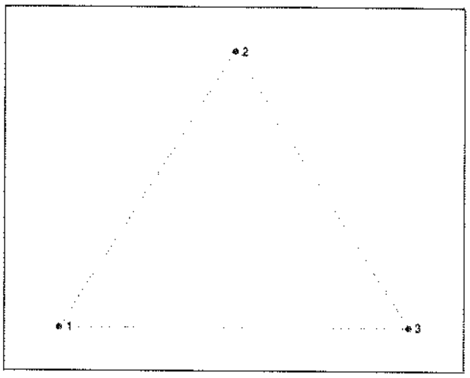

Fig. 1.1: Triangle K_3

Definition 1.1.4 *[18] A compatible factorization of a graph of order v, denoted by $CF(v)$ is a $v \times \frac{v-1}{2}$ array that satisfies the following conditions:*

1. *The entries in row m form a near-1 factor.*

2. *The triples associated with the rows contain no repetitions.*

Definition 1.1.5 *[1] The starter of compatible factorization denoted by $SCF(v)$ is the set of triples that generates all the triples in the design by repeated addition of 1 modulo v.*

Example 1.1.1 *If $v = 5$, that is a set of 5 points (vertices) labeled 1, 2, 3, 4, 5, then the compatable factorization $CF(5)$ is illustrated in Table 1.1. On appending C_1 with C_2, C_1 with C_3, we obtain 10 distinct triples in $CF(5)$ that represent all triangles in K_5 that shown in Fig 1.2*

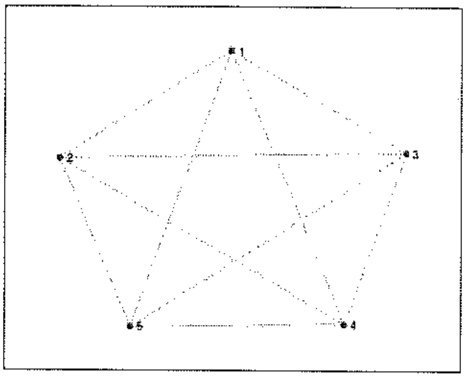

Fig. 1.2: K_5

Tab. 1.1: $CF(5)$

C_1	C_2	C_3
1	2 5	3 4
2	3 1	4 5
3	4 2	5 1
4	5 3	1 2
5	1 4	2 3

The shaded row is the starter of compatible factorization $SCF(5)$. *The rows below obtained by adding 1(mod5) to each number in the starter row.*

Example 1.1.2 *If* $v = 7$, *that is a set of 7 points labeled 1, 2, 3, 4, 5, 6, 7, then the compatable factorization* $CF(7)$ *is illustrated in Table 1.2 and the graph of* K_7 *is shown in Fig 1.3. On appending* C_1 *with* C_2, C_1 *with* C_3, C_1 *with* C_4, *we obtain 21 distinct triples in* $CF(7)$.

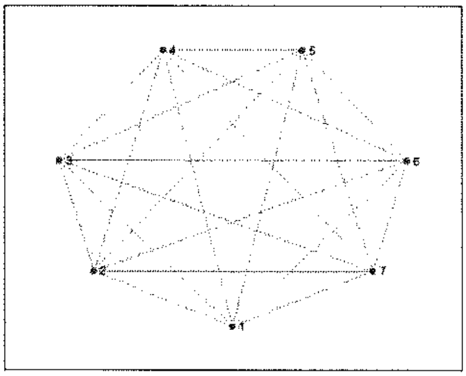

Fig. 1.3: K_7

Tab. 1.2: $CF(7)$

C_1	C_2	C_3	C_4
1	2 7	3 6	4 5
2	3 1	4 7	5 6
3	4 2	5 1	6 7
4	5 3	6 2	7 1
5	6 4	7 3	1 2
6	7 5	1 4	2 3
7	1 6	2 5	3 4

Tabel 1.1 gives $CF(7)$, which are 21 triangles. But in K_7, we have $\binom{7}{3} = 35$ triangles. To obtain all triangles in K_7 which are 35 triangles, we need to introduce the completion of compatible factorization $CF(v)$. So we introduce the

following section.

1.2 Triad Design

Definition 1.2.1 *[1] A triad design on v objects, TD(v) is a method for listing $\binom{v}{3}$ distinct triples . It is a triple system formed from compatible factorization CF(v) such that:*

1. *Row m contains $\frac{v-1}{2}$ triples, among which object m meets every other object precisely once, and contains also other distinct triples*
2. *Each triple occurs exactly once in the design*
3. *No two elements occur together in two or more triples in any row.*

Let us denote the completion of $CF(v)$ by $\overline{CF(v)}$ where $v = 6n+5$, then the triad design of graph of order v is $TD(v) = CF(v) \cup \overline{CF(v)}$. The number of triples in $TD(v)$ is $|TD(v)| = |CF(v)| + |\overline{CF(v)}|$.

Definition 1.2.2 *[1] [9] The starter of triad design denoted by $STD(v)$ is the set of triples that generates all the triples in the design by repeated addition of 1 modulo v., i.e.*

$$STD(v) = SCF(v) \cup \overline{SCF(v)}$$

Example 1.2.1 *For example 1.1.2 the completion can be organized as in table 1.3*

Tab. 1.3: $TD(7)$

C_1	C_2	C_3	C_4	$\overline{CF(v)}$	
1	2 7	3 6	4 5	7 6 4	5 3 2
2	3 1	4 7	5 6	1 7 5	6 4 3
3	4 2	5 1	6 7	2 1 6	7 5 4
4	5 3	6 2	7 1	3 2 7	1 6 5
5	6 4	7 3	1 2	4 3 1	2 7 6
6	7 5	1 4	2 3	5 4 2	3 1 7
7	1 6	2 5	3 4	6 5 3	4 2 1

Remark 1.2.1 *For the graph of order $v = 6n + 1$,*

1. *From definition1.1.4, the number of triples in the compatible factorization,*

$$|CF(v)| = v \times \frac{v-1}{2} = (6n+1)(\frac{6n}{2}) = 3 \times v \times (\frac{v-1}{6}) = \frac{v^2 - v}{2} \qquad (1.1)$$

2. *The number of triples is given by*

$$|TD(v)| = \binom{v}{3} = \frac{(6n+1)(6n)(6n-1)}{6} = (n)(6n+1)(6n-1) = (v+1)(v)(v-2) \quad (1.2)$$

3. *From equations 1.1 and 1.2, the number of triples of the compatible factorization is*

$$|\overline{CF(v)}| = 2n(6n+1)(3n-2) = \frac{v-1}{6}(v)(v-5) \qquad (1.3)$$

4. *The number of triangles in the starter of completion factorization [1], is*

$$|\overline{SCF(v)}| = 2n(3n-2) = \frac{v-1}{6} \times (v-5) \tag{1.4}$$

Remark 1.2.2 *We remark the following results for case $v = 6n+5$*

1. *From definition1.1.4, the number of triples in the compatible factorization,*

$$|CF(v)| = v \times \frac{v-1}{2} \tag{1.5}$$

2. *The number of triples is given by*

$$|TD(v)| = \binom{v}{3} = \frac{v(v-1)(v-2)}{6} \tag{1.6}$$

3. *From equations 1.5 and 1.6, the number of triples of the compatible factorization is*

$$|\overline{CF(v)}| = n(6n+5)(6n+4) = \frac{v-5}{6} \times v \times (v-1) \tag{1.7}$$

4. *The number of triangles in the starter of completion factorization [1], is*

$$|\overline{SCF(v)}| = 2n(3n+2) = \frac{v-5}{6} \times (v-1) \tag{1.8}$$

Definition 1.2.3 *[1] The $r-th$ element of $STD(v)$ are the $r-th$ numbers in each triple, denoted by $S_rTD(v)$, for $1 \leq r \leq 3$*

Example 1.2.2 *When $v = 7$, from table 1.3 $STD = \{1\,2\,7, 1\,3\,6, 1\,4\,5, 7\,6\,4, 5\,3\,2\}$. So* $S_1TD = 1, 1, 1, 7, 5$, $S_2TD = 2, 3, 4, 6, 3$, $S_3TD = 7, 6, 5, 4, 2$

Chapter Two

Triad Design of Orders 11 and 17

In this chapter, we introduce the triad design for 11 and 17, by constructing $STD(11)$ and $STD(17)$.

2.1 Triad Design of Order 11 $(6n+5, n=1)$

$$\begin{aligned} STD(11) &= SCF(11) \cup \overline{SCF(11)} \\ &= \{1\ 2\ 11,\ 1\ 3\ 10, 1\ 4\ 9, 1\ 5\ 8, 1\ 6\ 7\} \cup \{2\ 3\ 9,\ 11\ 10\ 4,\ 2\ 4\ 8,\ 11\ 9\ 5, \\ &\quad 2\ 5\ 7,\ 11\ 8\ 6,\ 3\ 4\ 7,\ 10\ 9\ 6,\ 4\ 5\ 7,\ 9\ 8\ 6\} \end{aligned}$$

Let T_k be the k-th triple in $STD(11)$, where $1 \leq k \leq 15$. So, $STD(11) = \{T_1, ..., T_{15}\}$. Let $[S_rTD(11)]_k$ be the k-th element in $S_rTD(11)$ for $1 \leq r \leq 3$. $STD(11)$ in terms of $S_rTD(11)$ as shown in Table 2.1.

Tab. 2.1: $S_rTD(11)$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$S_1TD(11)$	1	1	1	1	1	2	11	2	11	2	11	3	10	4	9
$S_2TD(11)$	2	3	4	5	6	3	10	4	9	5	8	4	9	5	8
$S_3TD(11)$	11	10	9	8	7	9	4	8	5	7	6	7	6	7	6

It can be observed from Table 2.1 that k, the number of tribles in $STD(11)$, can be devided into three intervals (periods). These intervals and corresponding elements in $S_1TD(11)$ are summarized in Table 2.2.

Tab. 2.2: Intervals of k and the corresponding elements in $S_1TD(11)$

No. of intervals	Intervals of k	corresponding elements in $S_1TD(11)$
1	$1 \leq k \leq 5$	1, 1, 1, 1
2	$6 \leq k \leq 11$	2, 11, 2, 11, 2, 11
3	$12 \leq k \leq 15$	3, 10, 4, 9

Let f denotes the first number of the last interval. The formula

$y = \frac{1}{2}[k-f+mod(\frac{k+1}{2})+1]$ can produce the corresponding elements in $S_1TD(11)$ in the last interval. For example, if $k = 13$, then $f = 12$ because 13 in the last interval $12 \leq k \leq 15$. Hence,

$$[y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1] = \frac{1}{2}[13 - 12 + mod(\frac{13+1}{2}) + 1] = 1.$$

Therefore, the corresponding element in $S_1TD(11)$ for $k = 13$ is $11 - y = 11 - 1 = 10$, which is the same number as shown in Table 2.2.

Also, if $k = 14$, then $f = 12$ because 14 in the last interval $12 \leq k \leq 15$. Hence,

$$[y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1] = \frac{1}{2}[14 - 12 + mod(\frac{14+1}{2}) + 1] = 2.$$

Therefore, the corresponding element in $S_1TD(11)$ for $k = 14$ is $2 + y = 2 + 2 = 4$, which is the same number as shown in Table 2.2. Thus, the corresponding elements of $S_1TD(11)$ is as follows.

$$[S_1TD(11)]_k = \begin{cases} 1 & 1 \leq k \leq 5 \\ 2 & 6 \leq k \leq 11, \ k \ is \ even \\ 11 & 6 \leq k \leq 11, \ k \ is \ odd \\ 2+y & 12 \leq k \leq 15, \ k \ is \ even \\ 11-y & 12 \leq k \leq 15, \ k \ is \ odd \end{cases}$$

In similar way, intervals of k and the corresponding elements in $S_2TD(11)$ are shown in Table 2.3.

Tab. 2.3: Intervals of k and the corresponding elements in $S_2TD(11)$

No. of intervals	Intervals of k	corresponding elements in $S_2TD(11)$
1	$1 \leq k \leq 5$	2, 3, 4, 5, 6
2	$6 \leq k \leq 11$	3, 10, 4, 9, 5, 8
3	$12 \leq k \leq 15$	4, 9, 5, 8

Use the same formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$ to produce the corresponding elements in $S_2TD(11)$ in all intervals of k except the first one. Hence, the construction of $S_2TD(11)$ is the following.

$$[S_2TD(11)]_k = \begin{cases} 1+k & 1 \leq k \leq 5 \\ 2+y & 6 \leq k \leq 11, \ k \ is \ even \\ 11-y & 6 \leq k \leq 11, \ k \ is \ odd \\ 3+y & 12 \leq k \leq 15, \ k \ is \ even \\ 10-y & 12 \leq k \leq 15, \ k \ is \ odd \end{cases}$$

Finally, using the above discussion, intervals of k and the corresponding elements in $S_3TD(11)$ are shown in Table 2.4

Tab. 2.4: Intervals of k and the corresponding elements in $S_3TD(11)$

No. of intervals	Intervals of k	corresponding elements in $S_3TD(11)$
1	$1 \le k \le 5$	11, 10, 9, 8, 7
2	$6 \le k \le 11$	9, 4, 8, 5, 7, 6
3	$12 \le k \le 15$	7, 6, 7, 6

Use the same formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$ to generate the corresponding elements in all intervals of k except the first one. Hence, the construction of $S_3TD(11)$ is the following.

$$[S_3TD(11)]_k = \begin{cases} 12 - k & 1 \le k \le 5 \\ 10 - y & 6 \le k \le 11, \ k \ is \ even \\ 3 + y & 6 \le k \le 11, \ k \ is \ odd \\ 7 & 12 \le k \le 15, \ k \ is \ even \\ 6 & 12 \le k \le 15, \ k \ is \ odd \end{cases}$$

So we have $TD(11)$ as in the following table :

Tab. 2.5: $TD(11)$

k	1			2			3			4			5		
	1	**2**	**11**	**1**	**3**	**10**	**1**	**4**	**9**	**1**	**5**	**8**	**1**	**6**	**7**
	2	3	1	2	4	11	2	5	10	2	6	9	2	7	8
	3	4	2	3	5	1	3	6	11	3	7	10	3	8	9
	4	5	3	4	6	2	4	7	1	4	8	11	4	9	10
	5	6	4	5	7	3	5	8	2	5	9	1	5	10	11
	6	7	5	6	8	4	6	9	3	6	10	2	6	11	1
	7	8	6	7	9	5	7	10	4	7	11	3	7	1	2
	8	9	7	8	10	6	8	11	5	8	1	4	8	2	3
	9	10	8	9	11	7	9	1	6	9	2	5	9	3	4
	10	11	9	10	1	8	10	2	7	10	3	6	10	4	5
	11	1	10	11	2	9	11	3	8	11	4	7	11	5	6

k	6			7			8			9			10		
	2	**3**	**9**	**11**	**10**	**4**	**2**	**4**	**8**	**11**	**9**	**5**	**2**	**5**	**7**
	3	4	10	1	11	5	3	5	9	1	10	6	3	6	8
	4	5	11	2	1	6	4	6	10	2	11	7	4	7	9
	5	6	1	3	2	7	5	7	11	3	1	8	5	8	10
	6	7	2	4	3	8	6	8	1	4	2	9	6	9	11
	7	8	3	5	4	9	7	9	2	5	3	10	7	10	1
	8	9	4	6	5	10	8	10	3	6	4	11	8	11	2
	9	10	5	7	6	11	9	11	4	7	5	1	9	1	3
	10	11	6	8	7	1	10	1	5	8	6	2	10	2	4
	11	1	7	9	8	2	11	2	6	9	7	3	11	3	5
	1	2	8	10	9	3	1	3	7	10	8	4	1	4	6

k	11			12			13			14			15		
	11	**8**	**6**	**3**	**4**	**7**	**10**	**9**	**6**	**4**	**5**	**7**	**9**	**8**	**6**
	1	9	7	4	5	8	11	10	7	5	6	8	10	9	7
	2	10	8	5	6	9	1	11	8	6	7	9	11	10	8
	3	11	9	6	7	10	2	1	9	7	8	10	1	11	9
	4	1	10	7	8	11	3	2	10	8	9	11	2	1	10
	5	2	11	8	9	1	4	3	11	9	10	1	3	2	11
	6	3	1	9	10	2	5	4	1	10	11	2	4	3	1
	7	4	2	10	11	3	6	5	2	11	1	3	5	4	11
	8	5	3	11	1	4	7	6	3	1	2	4	6	5	1
	9	6	4	1	2	5	8	7	4	2	3	5	7	6	2
	10	7	5	2	3	4	9	8	5	3	4	6	8	7	3

2.2 Triad Design of Order 17 $(6n+5, n=2)$

In this section, we devide $STD(17)$ into intervals to construct formulas for $S_rTD(17)$, where $1 \leq r \leq 3$.

Tab. 2.6: $S_rTD(17)$, where $1 \leq r \leq 3$.

k	**1**	**2**	...	**8**	**9**	**10**	...	**19**	**20**	**21**	**22**	...	**27**
$S_1TD(17)$	1	1	...	1	17	2	...	17	2	16	3	...	16
$S_2TD(17)$	2	3	...	9	16	3	...	11	8	15	4	...	12
$S_3TD(17)$	17	16	...	10	15	4	...	13	8	15	4	...	12
k	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
$S_1TD(17)$	3	15	4	15	4	14	5	14	5	6	13	7	12
$S_2TD(17)$	7	14	5	13	6	13	6	12	7	12	7	11	8
$S_3TD(17)$	7	8	11	9	10	8	11	9	10	9	10	9	10

From Table 2.6, k can be devided into 6 intervals. These intervals and the corresponding elements in $S_1TD(17)$ are illustrated as follows in Table 2.7.

Tab. 2.7: Intervals of k and the corresponding elements in $S_rTD(17)$

No. of intervals	Intervals of k	Corresponding elements in $S_1TD(17)$
1	$1 \leq k \leq 8$	1, 1, 1, 1, 1, 1, 1, 1
2	$9 \leq k \leq 20$	17, 2, 17, 2, 17, 2, 17, 2, 17, 2, 17, 2
3	$21 \leq k \leq 28$	16, 3, 16, 3, 16, 3, 16, 3
4	$29 \leq k \leq 32$	15, 4, 15, 4
5	$33 \leq k \leq 36$	14, 5, 14, 5
6	$37 \leq k \leq 40$	6, 13, 7, 12

If we let f denotes the first number of the interval. We can use the formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$ to determine the corresponding elements in $S_1TD(17)$. From Table 2.7, the construction of $S_1TD(17)$ for $1 \leq k \leq 28$ is as follows .

$$[S_1TD(17)]_k = \begin{cases} 1 & 1 \leq k \leq 8 \\ 2 & 9 \leq k \leq 20, k \ is \ even \\ 17 & 9 \leq k \leq 20, k \ is \ odd \\ 3 & 21 \leq k \leq 28, k \ is \ even \\ 16 & 21 \leq k \leq 28, k \ is \ odd \end{cases}$$

The construction of $S_1TD(17)$ for $29 \leq k \leq 40$ is as follows .

$$[S_1TD(17)]_k = \begin{cases} 4 & 29 \leq k \leq 32, k \text{ is even} \\ 15 & 29 \leq k \leq 32, k \text{ is odd} \\ 5 & 33 \leq k \leq 36, k \text{ is even} \\ 14 & 33 \leq k \leq 36, k \text{ is odd} \\ y+5 & 37 \leq k \leq 40, k \text{ is even} \\ 14-y & 37 \leq k \leq 40, k \text{ is odd} \end{cases}$$

Similarly, Table 2.8 shows intervals of k and the corresponding elements in $S_2TD(17)$.

Tab. 2.8: Intervals of k and the corresponding elements in $S_2TD(17)$

No. of intervals	Intervals of k	Corresponding elements in $S_2TD(17)$
1	$1 \leq k \leq 8$	2, 3, 4, 5, 6, 7, 8, 9
2	$9 \leq k \leq 20$	16, 3, 15, 4, 14, 5, 13, 6, 12, 7, 11, 8
3	$21 \leq k \leq 28$	15, 4, 14, 5, 13, 6, 12, 7
4	$29 \leq k \leq 32$	14, 5, 13, 6
5	$33 \leq k \leq 36$	13, 6, 12, 7
6	$37 \leq k \leq 40$	12, 7, 11, 8

Using the same formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$ to produce the corresponding elements in $S_2TD(17)$ in all intervals of k except the first interval. The construction of $S_2TD(17)$ for $1 \leq k \leq 15$ is as follows.

$$[S_2TD(17)]_k = \begin{cases} k+1 & 1 \leq k \leq 8 \\ y+2 & 9 \leq k \leq 20, k \textit{ is even} \\ 17-y & 9 \leq k \leq 20, k \textit{ is odd} \\ y+3 & 21 \leq k \leq 28, k \textit{ is even} \\ 16-y & 21 \leq k \leq 28, k \textit{ is odd} \\ y+4 & 29 \leq k \leq 32, k \textit{ is even} \\ 15-y & 29 \leq k \leq 32, k \textit{ is odd} \\ y+4 & 29 \leq k \leq 32, k \textit{ is even} \\ 15-y & 29 \leq k \leq 32, k \textit{ is odd} \\ y+5 & 33 \leq k \leq 36, k \textit{ is even} \\ 14-y & 33 \leq k \leq 36, k \textit{ is odd} \\ y+6 & 37 \leq k \leq 40, k \textit{ is even} \\ 13-y & 37 \leq k \leq 40, k \textit{ is odd} \end{cases}$$

Finally, in the same discussion, intervals of k and the corresponding elements in $S_3TD(17)$ are shown in Table 2.9.

Tab. 2.9: Intervals of k and the corresponding elements in $S_3TD(17)$

No. of intervals	Intervals of k	Corresponding elements in $S_3TD(17)$
1	$1 \leq k \leq 8$	17, 16, 15, 14, 13, 12, 11, 10
2	$9 \leq k \leq 20$	4, 15, 5, 14, 6, 13, 7, 12, 8, 11, 9, 10
3	$21 \leq k \leq 28$	6, 13, 7, 12, 8, 11, 9, 10
4	$29 \leq k \leq 32$	8, 11, 9, 10
5	$33 \leq k \leq 36$	8, 11, 9, 10
6	$37 \leq k \leq 40$	9, 10, 9, 10

Using the same formula, the construction of $S_3TD(17)$ is the following.

$$[S_3TD(17)]_k = \begin{cases} 18-k & 1 \leq k \leq 8 \\ 16-y & 9 \leq k \leq 20, k\ is\ even \\ y+3 & 9 \leq k \leq 20, k\ is\ odd \\ 14-y & 21 \leq k \leq 28, k\ is\ even \\ y+5 & 21 \leq k \leq 15, k\ is\ odd \\ 12-y & 29 \leq k \leq 32, k\ is\ even \\ y+7 & 29 \leq k \leq 32, k\ is\ odd \\ 12-y & 33 \leq k \leq 36, k\ is\ even \\ y+7 & 33 \leq k \leq 36, k\ is\ odd \\ 10 & 37 \leq k \leq 40, k\ is\ even \\ 9 & 37 \leq k \leq 40, k\ is\ odd \end{cases}$$

Chapter Three

Triad Design For 23 and 29

In this chapter, we introduce the triad design of order 23 and 29.

3.1 Triad design of order 23 ($6n + 5, n = 3$)

In this section , we will introduce the triad design of order 23 in the same method of $TD(17)$. The Table 3.1, summarize $STD(23)$ in terms of $S_rTD(23)$ and the triple number k.

Tab. 3.1: $S_rTD(23)$, where $1 \leq r \leq 3$.

k	1	...	11	12	13	...	28	29	30	31	...	42	43	44
$S_1TD(23)$	1	...	1	2	23	...	2	23	3	22	...	3	22	4
$S_2TD(23)$	2	...	12	3	22	...	11	14	4	21	...	10	15	5
$S_3TD(23)$	23	...	13	21	4	...	13	12	19	6	...	13	12	17
k	45	...	52	53	54	55	56	57	58	59	60	61	62	63
$S_1TD(23)$	21	...	4	21	5	20	5	20	5	20	6	19	6	19
$S_2TD(23)$	20	...	9	16	6	19	7	18	8	17	7	18	8	17
$S_3TD(23)$	8	...	13	12	15	10	14	11	13	12	15	10	14	11
k	64	65	66	67	68	69	70	71	72	73	74	75	76	77
$S_1TD(23)$	6	19	7	18	7	18	8	17	8	17	9	16	10	15
$S_2TD(23)$	9	16	8	17	9	16	9	16	10	15	10	15	11	14
$S_3TD(23)$	13	12	14	11	13	12	14	11	13	12	13	12	13	12

Now, From Tables 3.1 , k can be divided into 9 intervals. These intervals and the corresponding elements in $S_1TD(23)$ are shown in Table 3.2.

Tab. 3.2: Intervals of k and the corresponding elements in $S_1TD(23)$

No. of intervals	Intervals of k	Corresponding elements in $S_1TD(23)$
1	$1 \leq k \leq 11$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
2	$12 \leq k \leq 29$	2, 23, 2, 23, 2, 23, 2, 23, 2, 23, 2, 23, 2, 23, 2, 23, 2, 23
3	$30 \leq k \leq 43$	3, 22, 3, 22, 3, 22, 3, 22, 3, 22, 3, 22, 3, 22
4	$44 \leq k \leq 53$	4, 21, 4, 21, 4, 21, 4, 12, 4, 21
5	$54 \leq k \leq 59$	5, 20, 5, 20, 5, 20
6	$60 \leq k \leq 65$	6, 19, 6, 19, 6, 19
7	$66 \leq k \leq 69$	7, 18, 7, 18
8	$70 \leq k \leq 73$	8, 17, 8, 17
9	$74 \leq k \leq 77$	9, 16, 10, 15

Using the formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$ to produce the corresponding elements in $S_1TD(23)$. So, the construction of $S_1TD(23)$ for is the following.

$$[S_1TD(23)]_k = \begin{cases} 1 & 1 \leq k \leq 11 \\ 2 & 12 \leq k \leq 29,\ k\ is\ even \\ 23 & 12 \leq k \leq 29,\ k\ is\ odd \\ 3 & 30 \leq k \leq 43,\ k\ is\ even \\ 22 & 30 \leq k \leq 43,\ k\ is\ odd \\ 4 & 44 \leq k \leq 53,\ k\ is\ even \\ 21 & 44 \leq k \leq 53,\ k\ is\ odd \\ 5 & 54 \leq k \leq 59,\ k\ is\ even \\ 20 & 54 \leq k \leq 59,\ k\ is\ odd \\ 6 & 60 \leq k \leq 65,\ k\ is\ even \\ 19 & 60 \leq k \leq 65,\ k\ is\ odd \\ 7 & 66 \leq k \leq 69,\ k\ is\ even \\ 18 & 66 \leq k \leq 69,\ k\ is\ odd \\ 8 & 70 \leq k \leq 73,\ k\ is\ even \\ 17 & 70 \leq k \leq 73,\ k\ is\ odd \\ 8+y & 74 \leq k \leq 77,\ k\ is\ even \\ 17-y & 74 \leq k \leq 77,\ k\ is\ odd \end{cases}$$

Similarly, Table 3.3 shows intervals of k and the corresponding elements in $S_2TD(23)$.

Tab. 3.3: Intervals of k and the corresponding elements in $S_2TD(23)$

No. of intervals	Intervals of k	Corresponding elements in $S_2TD(23)$
1	$1 \leq k \leq 11$	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
2	$12 \leq k \leq 29$	3, 22, 4, 21, 5, 20, 6, 19, 7, 18, 8, 17, 9, 16, 10, 15, 11, 14
3	$30 \leq k \leq 43$	4, 21, 5, 20, 6, 19, 7, 18, 8, 17, 9, 16, 10, 15
4	$44 \leq k \leq 53$	5, 20, 6, 19, 7, 18, 8, 17, 9, 16
5	$54 \leq k \leq 59$	6, 19, 7, 18, 8, 17
6	$60 \leq k \leq 65$	7, 18, 8, 17, 9, 16
7	$66 \leq k \leq 69$	8, 17, 9, 16
8	$70 \leq k \leq 73$	9, 16, 10, 15
9	$74 \leq k \leq 77$	10, 15, 11, 14

Using the same formula above, the construction of $S_2TD(23)$ as follows.

$$[S_2TD(23)]_k = \begin{cases} k+1 & 1 \leq k \leq 11 \\ 2+y & 12 \leq k \leq 29, k \text{ } is \text{ } even \\ 23-y & 12 \leq k \leq 29, k \text{ } is \text{ } odd \\ 3+y & 30 \leq k \leq 43, k \text{ } is \text{ } even \\ 22-y & 30 \leq k \leq 43, k \text{ } is \text{ } odd \\ 4+y & 44 \leq k \leq 53, k \text{ } is \text{ } even \\ 21-y & 44 \leq k \leq 53, k \text{ } is \text{ } odd \\ 5+y & 54 \leq k \leq 59, k \text{ } is \text{ } even \\ 20-y & 54 \leq k \leq 59, k \text{ } is \text{ } odd \end{cases}$$

$$[S_2TD(23)]_k = \begin{cases} 6+y & 60 \leq k \leq 65, k \text{ is even} \\ 19-y & 60 \leq k \leq 65, k \text{ is odd} \\ 7+y & 66 \leq k \leq 69, k \text{ is even} \\ 18-y & 66 \leq k \leq 69, k \text{ is odd} \\ 8+y & 70 \leq k \leq 73, k \text{ is even} \\ 17-y & 70 \leq k \leq 73, k \text{ is odd} \\ 8+y & 74 \leq k \leq 77, k \text{ is even} \\ 17-y & 74 \leq k \leq 77, k \text{ is odd} \end{cases}$$

Finally, similar to the above work, intervals of k and the corresponding elements in $S_3TD(23)$ are shown in Table 3.4

Tab. 3.4: Intervals of k and the corresponding elements in $S_2TD(23)$

No. of intervals	Intervals of k	Corresponding elements in $S_3TD(23)$
1	$1 \leq k \leq 11$	23, 22, 21, 20, 19, 18, 17, 16, 15, 14, 13
2	$12 \leq k \leq 29$	21, 4, 20, 5, 19, 6, 18, 7, 17, 8, 16, 9, 15, 10, 14, 11, 13, 12
3	$30 \leq k \leq 43$	19, 6, 18, 7, 17, 8, 16, 9, 15, 10, 14, 11, 13, 12
4	$44 \leq k \leq 53$	17, 8, 16, 9, 15, 10, 14, 11, 13, 12
5	$54 \leq k \leq 59$	15, 10, 14, 11, 13, 12
6	$60 \leq k \leq 65$	15, 10, 14, 11, 13, 12
7	$66 \leq k \leq 69$	14, 11, 13, 12
8	$70 \leq k \leq 73$	14, 11, 13, 12
9	$74 \leq k \leq 77$	13, 12, 13, 12

In the same way, the construction of $S_3TD(23)$ is the following.

$$[S_3TD(23)]_k = \begin{cases} 24-k & 1 \le k \le 11 \\ 22-y & 12 \le k \le 29, k\ is\ even \\ 3+y & 12 \le k \le 29, k\ is\ odd \\ 20-y & 30 \le k \le 43, k\ is\ even \\ 5+y & 30 \le k \le 43, k\ is\ odd \\ 18-y & 44 \le k \le 53, k\ is\ even \\ 7+y & 44 \le k \le 53, k\ is\ odd \\ 16-y & 54 \le k \le 59, k\ is\ even \\ 9+y & 54 \le k \le 59, k\ is\ odd \\ 16-y & 60 \le k \le 65, k\ is\ even \\ 9+y & 60 \le k \le 65, k\ is\ odd \\ 15-y & 66 \le k \le 69, k\ is\ even \\ 10+y & 66 \le k \le 69, k\ is\ odd \\ 15-y & 70 \le k \le 73, k\ is\ even \\ 10+y & 70 \le k \le 73, k\ is\ odd \\ 13 & 74 \le k \le 77, k\ is\ even \\ 12 & 74 \le k \le 77, k\ is\ odd \end{cases}$$

3.2 Triad Design For 29 ($6n + 5, n = 4$)

In this section, we analyse and divide $STD(29)$ into 12 intervals to construct formulas for $S_rTD(29)$, where $1 \leq r \leq 3$. Let $[S_rTD(29)]_k$ be the k-th element in $S_rTD(29)$ for $1 \leq r \leq 3$. We can summarize $STD(29)$ in terms $S_rTD(29)$ as shown in Table 3.5 .

Tab. 3.5: $S_rTD(29)$, where $1 \leq r \leq 3$.

k	1	2	3	...	13	14	15	16	...	37	38	39	40	...	57
$S_1TD(29)$	1	1	1	...	1	1	29	2	...	29	2	28	3	...	28
$S_2TD(29)$	2	3	4	...	14	15	28	3	...	17	14	27	4	...	18
$S_3TD(29)$	29	28	27	...	17	16	4	27	...	15	16	6	25	...	15
k	58	59	60	...	73	74	75	76	...	85	86	87	88	...	93
$S_1TD(29)$	3	27	4	...	27	4	26	5	...	26	5	25	6	...	25
$S_2TD(29)$	13	26	5	...	19	12	25	6	...	20	11	24	7	...	21
$S_3TD(29)$	16	8	23	...	15	16	10	21	...	15	16	12	19	...	15
k	94	95	96	...	101	102	103	104	105	106	107	108	109	110	111
$S_1TD(29)$	6	24	7	...	24	7	23	8	23	8	23	8	22	9	22
$S_2TD(29)$	10	23	8	...	20	11	22	9	21	10	20	11	21	10	20
$S_3TD(29)$	16	12	19	...	15	16	13	18	14	17	15	16	13	18	14
k	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126
$S_1TD(29)$	9	22	9	21	10	21	10	20	11	20	11	20	12	19	13
$S_2TD(29)$	11	19	12	20	11	19	12	19	12	18	13	19	12	18	13
$S_3TD(29)$	17	15	16	14	17	15	16	14	17	15	16	16	15	16	15

Now, from Table 3.5, k can be divided into 12 intervals. These intervals and the corresponding elements in $S_1TD(29)$ are shown in Table 3.2.

Tab. 3.6: Intervals of k and the corresponding elements in $S_1TD(29)$

No. of intervals	Intervals of k	Corresponding elements in $S_1TD(29)$
1	$1 \leq k \leq 14$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
2	$15 \leq k \leq 38$	29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2, 29, 2
3	$39 \leq k \leq 58$	28, 3, 28, 3, 28, 3, 28, 3, 28, 3, 28, 3, 28, 3, 28, 3, 28, 3, 28, 3
4	$59 \leq k \leq 74$	27, 4, 27, 4, 27, 4, 27, 4, 27, 4, 27, 4, 27, 4, 27, 4
5	$75 \leq k \leq 86$	26, 5, 26, 5, 26, 5, 26, 5, 26, 5, 26, 5
6	$87 \leq k \leq 94$	25, 6, 25, 6, 25, 6, 25, 6
7	$95 \leq k \leq 102$	24, 7, 24, 7, 24, 7, 24, 7
8	$103 \leq k \leq 108$	23, 8, 23, 8, 23, 8
9	$109 \leq k \leq 114$	22, 9, 22, 9, 22, 9
10	$115 \leq k \leq 118$	21, 10, 21, 10
11	$119 \leq k \leq 122$	20, 11, 20, 11
12	$123 \leq k \leq 126$	20, 12, 19, 13

.

Using the formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$ to produce the corresponding elements in $S_1TD(29)$. So, the construction of $S_1TD(29)$ is the following.

$$[S_1TD(29)]_k = \begin{cases} 1 & 1 \le k \le 14 \\ 2 & 15 \le k \le 38, k\ is\ even \\ 29 & 15 \le k \le 38, k\ is\ odd \\ 3 & 39 \le k \le 58, k\ is\ even \\ 28 & 39 \le k \le 58, k\ is\ odd \\ 4 & 59 \le k \le 74, k\ is\ even \\ 27 & 59 \le k \le 74, k\ is\ odd \\ 5 & 75 \le k \le 86, k\ is\ even \\ 26 & 75 \le k \le 86, k\ is\ odd \\ 6 & 87 \le k \le 94, k\ is\ even \\ 25 & 87 \le k \le 94, k\ is\ odd \\ 7 & 95 \le k \le 102, k\ is\ even \\ 24 & 95 \le k \le 102, k\ is\ odd \\ 8 & 103 \le k \le 108, k\ is\ even \\ 23 & 103 \le k \le 108, k\ is\ odd \\ 9 & 109 \le k \le 114, k\ is\ even \\ 22 & 109 \le k \le 114, k\ is\ odd \\ 10 & 115 \le k \le 118, k\ is\ even \\ 21 & 115 \le k \le 118, k\ is\ odd \\ 11 & 119 \le k \le 122, k\ is\ even \\ 20 & 119 \le k \le 122, k\ is\ odd \\ 11+y & 123 \le k \le 126, k\ is\ even \\ 21-y & 123 \le k \le 126, k\ is\ odd \end{cases}$$

Similarly, Table 2.8 shows intervals of k and the corresponding elements in $S_2TD(17)$.

Tab. 3.7: Intervals of k and the corresponding elements in $S_2TD(29)$

No. of intervals	Intervals of k	Corresponding elements in $S_2TD(29)$
1	$1 \leq k \leq 14$	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
2	$15 \leq k \leq 38$	28, 3, 27, 4, 26, 5, 25, 6, 24, 7, 23, 8, 22, 9, 21, 10, 20, 11, 19, 12, 18, 13, 17, 14
3	$39 \leq k \leq 58$	27, 4, 26, 5, 25, 6, 24, 7, 23, 8, 22, 9, 21, 10, 20, 11, 19, 12, 18, 13
4	$59 \leq k \leq 74$	26, 5, 25, 6, 24, 7, 23, 8, 22, 9, 21, 10, 20, 11, 19, 12
5	$75 \leq k \leq 86$	25, 6, 24, 7, 23, 8, 22, 9, 21, 10, 20, 11
6	$87 \leq k \leq 94$	24, 7, 23, 8, 22, 9, 21, 10
7	$95 \leq k \leq 102$	23, 8, 22, 9, 21, 10, 20, 11
8	$103 \leq k \leq 108$	22, 9, 21, 10, 20, 11
9	$109 \leq k \leq 114$	21, 10, 20, 11, 19, 12
10	$115 \leq k \leq 118$	20, 11, 19, 12
11	$119 \leq k \leq 122$	19, 12, 18, 13
12	$123 \leq k \leq 126$	13, 20, 12, 19

Use the same formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{4}) + 1]$ to produce the corresponding elements in $S_2TD(29)$ in all intervals of k except the first one. Hence, the construction of $S_2TD(29)$ is the following.

$$[S_2TD(29)]_k = \begin{cases} k+1 & 1 \le k \le 14 \\ 2+y & 15 \le k \le 38,\ k \textit{ is even} \\ 29-y & 15 \le k \le 38,\ k \textit{ is odd} \\ 3+y & 39 \le k \le 58,\ k \textit{ is even} \\ 28-y & 39 \le k \le 58,\ k \textit{ is odd} \\ 4+y & 59 \le k \le 74,\ k \textit{ is even} \\ 27-y & 59 \le k \le 74,\ k \textit{ is odd} \\ 5+y & 75 \le k \le 86,\ k \textit{ is even} \\ 26-y & 75 \le k \le 86,\ k \textit{ is odd} \\ 6+y & 87 \le k \le 94,\ k \textit{ is even} \\ 25-y & 87 \le k \le 94,\ k \textit{ is odd} \\ 7+y & 95 \le k \le 102,\ k \textit{ is even} \\ 24-y & 95 \le k \le 102,\ k \textit{ is odd} \\ 8+y & 103 \le k \le 108,\ k \textit{ is even} \\ 23-y & 103 \le k \le 108,\ k \textit{ is odd} \\ 9+y & 109 \le k \le 114,\ k \textit{ is even} \\ 22-y & 109 \le k \le 114,\ k \textit{ is odd} \\ 10+y & 115 \le k \le 118,\ k \textit{ is even} \\ 21-y & 115 \le k \le 118,\ k \textit{ is odd} \\ 11+y & 119 \le k \le 122,\ k \textit{ is even} \\ 20-y & 119 \le k \le 122,\ k \textit{ is odd} \\ 11+y & 123 \le k \le 126,\ k \textit{ is even} \\ 20-y & 123 \le k \le 126,\ k \textit{ is odd} \end{cases}$$

Finally, using the above discussionm intervals of k and the corresponding elements in $S_3TD(29)$ are shown in Table 3.8

Tab. 3.8: Intervals of k and the corresponding elements in $S_3TD(29)$

No. of intervals	Intervals of k	Corresponding elements in $S_3TD(29)$
1	$1 \leq k \leq 14$	29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16
2	$15 \leq k \leq 38$	4, 27, 5, 26, 6, 25, 7, 24, 8, 23, 9, 22, 10, 21, 11, 20, 12, 19, 13, 18, 14, 17, 15, 16
3	$39 \leq k \leq 58$	6, 25, 7, 24, 8, 23, 9, 22, 10, 21, 11, 20, 12, 19, 13, 18, 14, 17, 15, 16
4	$59 \leq k \leq 74$	8, 23, 9, 22, 10, 21, 11, 20, 12, 19, 13, 18, 14, 17, 15, 16
5	$75 \leq k \leq 86$	10, 21, 11, 20, 12, 19, 13, 18, 14, 17, 15, 16
6	$87 \leq k \leq 94$	12, 19, 13, 18, 14, 17, 15, 16
7	$95 \leq k \leq 102$	12, 19, 13, 18, 14, 17, 15, 16
8	$103 \leq k \leq 108$	13, 18, 14, 17, 15, 16
9	$109 \leq k \leq 114$	13, 18, 14, 17, 15, 16
10	$115 \leq k \leq 118$	14, 17, 15, 16
11	$119 \leq k \leq 122$	14, 17, 15, 16
12	$123 \leq k \leq 126$	16, 15, 16, 15

In the same way, the construction of $S_3TD(29)$ is the following.

$$[S_3TD(29)]_k = \begin{cases} 30-k & 1 \le k \le 14 \\ 28-y & 15 \le k \le 38,\ k\ is\ even \\ 3+y & 15 \le k \le 38,\ k\ is\ odd \\ 26-y & 39 \le k \le 58,\ k\ is\ even \\ 5+y & 39 \le k \le 58,\ k\ is\ odd \\ 24-y & 59 \le k \le 74,\ k\ is\ even \\ 7+y & 59 \le k \le 74,\ k\ is\ odd \\ 22-y & 75 \le k \le 86,\ k\ is\ even \\ 9+y & 75 \le k \le 86,\ k\ is\ odd \\ 20-y & 87 \le k \le 94,\ k\ is\ even \\ 11+y & 87 \le k \le 94,\ k\ is\ odd \\ 20-y & 95 \le k \le 102,\ k\ is\ even \\ 11+y & 95 \le k \le 102,\ k\ is\ odd \\ 19-y & 103 \le k \le 108,\ k\ is\ even \\ 12+y & 103 \le k \le 108,\ k\ is\ odd \\ 19-y & 109 \le k \le 114,\ k\ is\ even \\ 12+y & 109 \le k \le 114,\ k\ is\ odd \\ 18-y & 115 \le k \le 118,\ k\ is\ even \\ 13+y & 115 \le k \le 118,\ k\ is\ odd \\ 18-y & 119 \le k \le 122,\ k\ is\ even \\ 13+y & 119 \le k \le 122,\ k\ is\ odd \\ 16 & 123 \le k \le 126,\ k\ is\ even \\ 15 & 123 \le k \le 126,\ k\ is\ odd \end{cases}$$

Chapter Four

Developing Interval Construction Method for $v = 6n + 5$

The existence of triad design, where $v = 6n + 5$, and other related results concerning this case have been presented in chapter one. A new method called the "Generalized Interval Method-GIM" for constructing $STD(v)$ is presented. We develop $TD(v)$ for the general case $v = 6n + 5$ by analyzing the patterns of the triples in $STD(v)$.

4.1 Developing Interval Construction for $STD(v)$, $v = 6n + 5$

Now we are ready to consider the general case of the construction of $STD(v)$, where $v = 6n + 5$. The process, as one can conclude from the previous discussions, depends on analyzing the triples in $STD(v)$ using interval techniques of the number of triples.

From the intervals construction for $STD(11)$ in Sections 1.2, and from $STD(17)$, The following table of intervals for cases $v = 11, 17, 23, ..., 6n+5$ is established. It is the result of analyzing the patterns of the triples in the designs of the previous cases. Some theorems and results are deduced from Table 4.1

Tab. 4.1: Intervals of $STD(6n+5)$

intervals of k	$v=11(n=1)$	$v=17(n=2)$	$v=23(n=3)$	...	$v=6n+5$
	$1\leq k\leq 5$	$1\leq k\leq 8$	$1\leq k\leq 11$	...	$1\leq k\leq 3n+2$
	$6\leq k\leq 11$	$9\leq k\leq 20$	$12\leq k\leq 29$	...	$3n+3\leq k\leq 9n+2$
	$12\leq k\leq 15$	$21\leq k\leq 28$	$30\leq k\leq 43$	...	$9n+3\leq k\leq 15n-2$
		$29\leq k\leq 32$	$44\leq k\leq 53$	...	$15n-1\leq k\leq 21n-10$
		$33\leq k\leq 36$	$54\leq k\leq 59$	...	$21n-9\leq k\leq 27n+2-24$
		$37\leq k\leq 40$	$60\leq k\leq 65$	...	...
			$66\leq k\leq 69$	...	...
			$70\leq k\leq 73$	...	...
			$74\leq k\leq 77$		...
					...
Total No. of intervals	3	6	9	...	$3n$

Theorem 4.1.1 *The number of intervals of $STD(6n+5)$ is equal to $3n$.*

Proof: It is obvious from Table 4.1 of intervals of $STD(6n+5)$ that pairs (n, number of intervals) are (1,3), (2, 6), (3, 12), ... which is a linear function of slope equal 3. Therefore, The number of intervals of $STD(6n+5)$ is equal to $3n$.
□

The general rules of the intervals for the general case $v=6n+5$ are:

1. The first interval is $1\leq k\leq 3n+2$

2. Other intervals except the last one are explaind as follows: Let i denote the interval number starting from second interval until the second to the

last one. It can be observed from Theorem 4.1.1, that the values of i are $1 \le i \le 3n-2$. That is, $i = 1$ is the second interval, $i = 2$ is the third interval, ..., $i = 3n - 2$ is the interval second to the last one. The last numbers of each interval, denoted by l, are constructed as shown in Table 4.2

Tab. 4.2: The rules of the last number of the intervals

interval	the last number of the interval
$3n+3 \le k \le 9n+2$	$l = 9n+2 = [6(1)+3]n+2-[2(1)^2-2(1)]$
$9n+3 \le k \le 15n-2$	$l = 15n-2 = [6(2)+3]n+2-[2(2)^2-2(2)]$
$15n-1 \le k \le 21n-10$	$l = 21n-10 = [6(3)+3]n+2-[2(3)^2-2(3)]$

Clearly, the general rule for the last number is $l = (6(i)+3)n+2-(2(i)^2-2(i))$, where $1 \le i \le n+1$.

Now, if $n+2 \le i \le 3n-2$, then from Table 4.1, the difference between the last numbers of two successive intervals is smaller than the case when $1 \le i \le n+1$. Consequently, a value denoted by a_t, defined below, must be added to obtain the desired last numbers of the intervals. Therefore, the last number of the general interval when $n+2 \le i \le 3n-2$ is equal to $l + a_t = (6i+3)n+2-(2i^2-2i)+a_t$. If $i = n+1+t$, then clearly $1 \le t \le 2n-3$.

$$\text{Define } a_t = \begin{cases} 0 & t = 0 \\ 4 & t = 1 \\ a_{t-2} + (6t-2) & t \ge 2 \end{cases}$$

Similarly, the first number of each interval, denoted by f, are constructed in Table 4.3

Tab. 4.3: The rules of the last number of the intervals

interval	the first number of the interval
$3n+3 \leq k \leq 9n+2$	$f = 3n+3 = [6(1-1)+3]n+2-[2(1-1)^2-2(1-1)]$
$9n+3 \leq k \leq 15n-2$	$f = 9n+3 = [6(2-1)+3]n+2-[2(2-1)^2-2(2-1)]+1$
$15n-1 \leq k \leq 21n-10$	$f = 15n-1 = [6(3-1)+3]n+2-[2(3-1)^2-2(3-1)]+1$

The general rule for the first number is $f = [6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1$

For the same reason above, if $n+2 \leq i \leq 3n-2$, then a value denoted by a_{t-1}, defined above, must be added to obtain the desired first numbers of intervals. Therefore, the first number of the general intervals when $n+2 \leq i \leq 3n-2$ is equal to

$$f + a_{t-1} = [6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1$$

, where $i = n+1+t$, $1 \leq t \leq 2n-3$.

3. The last interval is $l+1+a_t \leq k \leq (3n+2)(2n+1)$, where $t = 2n-3$.

4.2 Generalized Interval Method - GIM for $STD(6n + 5)$

Regarding the above discussion and notations, the patterns of the intervals for the general case $v = 6n + 5$ are:

1. The first interval is $1 \leq k \leq 3n + 2$.

2. Other intervals except the last one are explained as follows:

   - $f \leq k \leq l$ when $1 \leq i \leq n + 1$, and

   - $f + a_{t-1} \leq k \leq l + a_t$ when $n+2 \leq i \leq 3n-2$, $i = n+1+t$, $1 \leq t \leq 2n-3$. Also $a_0 = 0, a_1 = 4$, and if $t \geq 2$, then $a_t = a_{t-2} + (6t - 2)$.

3. The last interval is $l + 1 + a_t \leq k \leq (3n + 2)(2n + 1)$, where $t = 2n - 3$.

Example 4.2.1 *Let $n = 2$, that is $v = 17$. By Theorem 4.1.1, the number of intervals is equal to 6. Clearly, $1 \leq i \leq 4$, and $t = 1$. Applying the (GIM) to obtain the intervals for the case $v = 17$.*

1. *The first interval is $1 \leq k \leq 3n + 2$, that is $1 \leq k \leq 8$.*

2. *We use formula 2 part(1) of the (GIM) for $1 \leq i \leq 3$, because $n = 3$. If $i = 1$, then second interval is*

$$f \leq k \leq l$$

$$[6(i-1)+3]n + 2 - [2(i-1)^2 - 2(i-1)] + 1 \leq k \leq (6(i)+3)n + 2 - (2(i)^2 - 2(i))$$

$$[6(1-1)+3]2+2-[2(1-1)^2-2(1-1)]+1 \leq k \leq l = (6(1)+3)2+2-(2(1)^2-2(1))$$

$$9 \leq k \leq 20$$

If $i = 2$, then third interval is

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(2-1)+3]2+2-[2(2-1)^2-2(2-1)]+1 \leq k \leq l = (6(2)+3)2+2-(2(2)^2-2(2))$$

$$21 \leq k \leq 28$$

If $i = 3$, then fourth interval is

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(3-1)+3]2+2-[2(3-1)^2-2(3-1)]+1 \leq k \leq l = (6(3)+3)2+2-(2(3)^2-2(3))$$

$$29 \leq k \leq 32$$

Now: $i \geq n+1 = 3$, so formula 2 part 2 of the (GIM) will be used. If $i = 4$ then $t = 1$. Hence, the fifth interval is

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-2(2(i)^2-2(i))+a_t$$

$$[6(4-1)+3]2+2-[2(4-1)^2-2(4-1)]+a_{1-1}+1 \leq k \leq (6(4)+3)2+2-2(2(4)^2-2(4))+a_1]$$

$$33 \leq k \leq 36$$

3. *The last interval is $l+1+a_t \leq k \leq (3n+2)(2n+1)$, where $t = 1$, that is*

$$37 \leq k \leq 40$$

We are now in a position to give the constructions of $S_rTD(v)$, for $1 \leq r \leq 3$ and

$v = 6n + 5$. Let T_k be the $k-th$ triple in $STD(v)$, then $1 \leq k \leq (3n+2)(2n+1)$. Let $[S_rTD(v)]_k$ be the $k-th$ element in $S_rTD(v)$ for $1 \leq r \leq 3$. If i denotes the interval number starting from the second interval until the second to the last one, then by Theorem 4.1.1, $1 \leq i \leq 3n-2$. Let t be a variable appears when $i \geq n+2$, that is when $n+2 \leq i \leq 3n-2$. If $i = n+1+t$, then clearly $1 \leq t \leq 2n-3$.

Define $a_t = \begin{cases} 0 & t = 0 \\ 4 & t = 1 \\ a_{t-2} + (6t-2) & t \geq 2 \end{cases}$

We set in addition to the above notations, The formula $y = \frac{1}{2}[k - f + mod(\frac{k+1}{2}) + 1]$.

Theorem 4.2.1 *Following the above notations , the construction of $S_rTD(v)$ for $1 \leq r \leq 3$ are as follows:*

$$[S_1TD(v)]_k = \begin{cases} 1 & 1 \leq k \leq 3n+2 \\ i+1 & f \leq k \leq l, 1 \leq i \leq n+1, k \; is \; even \\ 6n+6-i & f \leq k \leq l, 1 \leq i \leq n+1, k \; is \; odd \\ i+1 & f+a_{t-1} \leq k \leq l+a_t, i=n+1+t, k \; is \; even \\ 6n+6-i & f+a_{t-1} \leq k \leq l+a_t, i=n+1+t, k \; is \; odd \\ & n+2 \leq i \leq 3n-2, 1 \leq t \leq 2n-3 \\ 3n-1+y & l+a_t+1 \leq k \leq (3n+2)(2n+1), k \; is \; even \\ 3n+8-y & l+a_t+1 \leq k \leq (3n+2)(2n+1), k \; is \; odd \end{cases}$$

$$[S_2TD(v)]_k = \begin{cases} k+1 & 1 \le k \le 3n+2 \\ i+1+y & f \le k \le l,\ 1 \le i \le n+1,\ k\ is\ even \\ 6n+6-i-y & f \le k \le l,\ 1 \le i \le n+1,\ k\ is\ odd \\ i+1+y & f+a_{t-1} \le k \le l+a_t,\ i=n+1+t,\ k\ is\ even \\ 6n+2-i-y & f+a_{t-1} \le k \le l+a_t,\ i=n+1+t,\ k\ is\ odd \\ & n+2 \le i \le 3n-2,\ 1 \le t \le 2n-3 \\ 3n-1+y & l+a_t+1 \le k \le (3n+2)(2n+1),\ k\ is\ even \\ 3n+8-y & l+a_t+1 \le k \le (3n+2)(2n+1),\ k\ is\ odd \end{cases}$$

$$[S_3TD(v)]_k = \begin{cases} 6n+6-k & 1 \le k \le 3n+2 \\ 6n+6-2i-y & f \le k \le l,\ 1 \le i \le n+1,\ k \textit{ is even} \\ 2i+1+y & f \le k \le l,\ 1 \le i \le n+1,\ k \textit{ is odd} \\ 6n+6-2i-y+C_j & f+a_{t-1} \le k \le l+a_t,\ i=n+1+t,\ k \textit{ is even} \\ 2i+1+y-C_j & f+a_{t-1} \le k \le l+a_t,\ i=n+1+t,\ k \textit{ is odd} \\ & n+2 \le i \le 3n-4,\ 1 \le t \le 2n-4 \\ 3n+4 & l+a_t+1 \le k \le (3n+2)(2n+1),\ k \textit{ is even} \\ 3n+3 & l+a_t+1 \le k \le (3n+2)(2n+1),\ k \textit{ is odd} \end{cases}$$

Where $C_j = C_{j-1} + 1 + mod(\frac{i+1}{2})$, $j = i - n$ *and* $C_j = 0$, *when* $j \le 1$

Proof: We prove the constructions related to $S_1TD(v)$. The proofs of the others two are similar.

(a) If $1 \le k \le 3n+2$, then $[S_1TD(v)]_k$ are the first elements of $CF(v)$ which are always equal to 1.

(b) From the construction of $[S_1TD(11)]_k$, we can see that if k is even, and $i = 1$

then the value of $[S_1TD(v)]_k$ is 2. Also the values of $[S_1TD(17)]_k$ are 2,3,4, when $1 \le i \le 3$. We conclude that the values of $[S_1TD(v)]_k$ are $i+1$, when $1 \le i \le n+1$

(c) Similarly, if k is odd, and $i = 1$, then the value of $[S_1TD(11)]_k$ is 11. Also the value of $[S_1TD(17)]_k$ are 17, 16, 15, when $1 \le i \le 3$. Hence the values of $[S_1TD(v)]_k$ are $v+1-i = 6n+6-i$, when $1 \le i \le n$.

(d), (e) Similar to (b) and (c), the values of $[S_1TD(v)]_k$ are the same when $n+2 \le i \le 3n-2$.

(f) Clearly, if $i = n+1+t$ and $n+2 \le i \le 3n-2$, then $n+2 \le n+1+t \le 3n-2$, that is $1 \le t \le 2n-3$.

(g) For the last interval $l + a_t + 1 \le k \le (3n+2)(2n+1)$, $t = 2n-3$ and k is even, we see that the value of $[S_1TD(11)]_k$ is equal to $2+y$. Also the value of $[S_1TD(17)]_k$ is equal to $5+y$. Because of (1,2) and (2,5), we can deduce easily that $[S_1TD(v)]_k = 3n-1+y$, where y is the rule defined above. □

Example 4.2.2 *Let $n = 5$ that is $v = 35$. By Theorem 4.1.1, the number of intervals is equal to 15. Clearly, ,$1 \le i \le 6$, and $1 \le t \le 7$. Applying the (GIM) to obtain the intervals for the case $v = 35$.*

1. *The first interval is $1 \le k \le 3n+2$, that is $1 \le k \le 17$.*

2. *We use formula 2 part(1) of the (GIM) for $1 \le i \le 13$, because $n = 5$. If $i = 1$, then second interval is*

$$f \le k \le l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \le k \le (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(1-1)+3]5+2-[2(1-1)^2-2(1-1)]+1 \le k \le l = (6(1)+3)5+2-(2(1)^2-2(1))$$

$$18 \le k \le 47$$

If $i = 2$, *then third interval is*

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(2-1)+3]5+2-[2(2-1)^2-2(2-1)]+1 \leq k \leq l = (6(2)+3)5+2-(2(2)^2-2(2))$$

$$48 \leq k \leq 73$$

If $i = 3$, *then fourth interval is*

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(3-1)+3]5+2-[2(3-1)^2-2(3-1)]+1 \leq k \leq l = (6(3)+3)5+2-(2(3)^2-2(3))$$

$$74 \leq k \leq 95$$

If $i = 4$, *then fifth interval is*

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(4-1)+3]5+2-[2(4-1)^2-2(4-1)]+1 \leq k \leq l = (6(4)+3)5+2-(2(4)^2-2(4))$$

$$96 \leq k \leq 113$$

If $i = 5$, *then sixth interval is*

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(5-1)+3]5+2-[2(5-1)^2-2(5-1)]+1 \leq k \leq l = (6(5)+3)5+2-(2(5)^2-2(5))$$

$$114 \leq k \leq 127$$

If $i = 6$, *then Seventh interval is*

$$f \leq k \leq l$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))$$

$$[6(6-1)+3]5+2-[2(6-1)^2-2(6-1)]+1 \leq k \leq l = (6(6)+3)5+2-(2(6)^2-2(6))$$

$$128 \leq k \leq 137$$

Now: $i \geq n+1 = 6$, *so formula* 2 *part* 2 *of the (GIM) will be used. If* $i = 7$ *then* $t = 1$. *Hence, the eighth interval is*

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(7-1)+3]5+2-[2(7-1)^2-2(7-1)]+a_{1-1}+1 \leq k \leq (6(7)+3)5+2-(2(7)^2-2(7))+a_1]$$

$$138 \leq k \leq 147$$

If $i = 8$ *then* $t = 2$. *Hence, the eighth interval is*

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(8-1)+3]5+2-[2(8-1)^2-2(8-1)]+a_{2-1}+1 \leq k \leq (6(8)+3)5+2-(2(8)^2-2(8))+a_2]$$

$$148 \leq k \leq 155$$

If $i = 9$ *then* $t = 3$. *Hence, the tenth interval is*

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(9-1)+3]5+2-[2(9-1)^2-2(9-1)]+a_{3-1}+1 \leq k \leq (6(9)+3)5+2-(2(9)^2-2(9))+a_3]$$

$$156 \leq k \leq 163$$

If $i = 10$ *then* $t = 4$. *Hence, the eleventh interval is*

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(10-1)+3]5+2-[2(10-1)^2-2(10-1)]+a_{4-1}+1 \leq k \leq (6(10)+3)5+2-(2(10)^2-2(10))+a_4]$$

$$164 \leq k \leq 169$$

If $i = 11$ *then* $t = 5$. *Hence, the twelfth interval is*

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(11-1)+3]5+2-[2(11-1)^2-2(11-1)]+a_{5-1}+1 \leq k \leq (6(11)+3)5+2-(2(11)^2-2(11))+a_5]$$

$$170 \leq k \leq 175$$

If $i = 12$ *then* $t = 6$. *Hence, the thirteen interval is*

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(12-1)+3]5+2-[2(12-1)^2-2(12-1)]+a_{6-1}+1 \leq k \leq (6(12)+3)5+2-(2(12)^2-2(12))+a_6]$$

$$176 \leq k \leq 179$$

If $i = 13$ then $t = 7$. Hence, the fourteen interval is

$$f + a_{t-1} \leq k \leq l + a_t$$

$$[6(i-1)+3]n+2-[2(i-1)^2-2(i-1)]+a_{t-1}+1 \leq k \leq (6(i)+3)n+2-(2(i)^2-2(i))+a_t$$

$$[6(13-1)+3]5+2-[2(13-1)^2-2(13-1)]+a_{7-1}+1 \leq k \leq (6(13)+3)5+2-(2(13)^2-2(13))+a_7]$$

$$180 \leq k \leq 183$$

3. *The last interval is $l + f + a_t \leq k \leq (3n+2)(2n+1)$, where $t = 2(5) - 3 = 7$, that is*

$$184 \leq k \leq 187$$

Example 4.2.3 *Let n=5, that is v=35. In example 4.2.2, we constructed the intervals of STD(35). In this example we want to build* $= S_\tau TD(35)$ *using Theorem 4.2.1. Thus we have* $S_1TD(35)$ *as follows*

$$[S_1TD(35)]_k = \begin{cases} 1 & 1 \le k \le 17 \\ 2 & 18 \le k \le 47,\ k\ is\ even \\ 35 & 18 \le k \le 47,\ k\ is\ odd \\ 3 & 48 \le k \le 73,\ k\ is\ even \\ 34 & 48 \le k \le 73,\ k\ is\ odd \\ 4 & 74 \le k \le 95,\ k\ is\ even \\ 33 & 74 \le k \le 95,\ k\ is\ odd \\ 5 & 96 \le k \le 113,\ k\ is\ even \\ 32 & 96 \le k \le 113,\ k\ is\ odd \\ 6 & 114 \le k \le 127,\ k\ is\ even \\ 31 & 114 \le k \le 127,\ k\ is\ odd \\ 7 & 128 \le k \le 137,\ k\ is\ even \\ 30 & 128 \le k \le 137,\ k\ is\ odd \\ 8 & 138 \le k \le 147,\ k\ is\ even \\ 29 & 138 \le k \le 147,\ k\ is\ odd \end{cases} \quad [S_1TD(35)]_k = \begin{cases} 9 & 148 \le k \le 155,\ k\ is\ even \\ 28 & 148 \le k \le 155,\ k\ is\ odd \\ 10 & 156 \le k \le 163,\ k\ is\ even \\ 27 & 156 \le k \le 163,\ k\ is\ odd \\ 11 & 164 \le k \le 169,\ k\ is\ even \\ 26 & 164 \le k \le 169,\ k\ is\ odd \\ 12 & 170 \le k \le 175,\ k\ is\ even \\ 25 & 170 \le k \le 175,\ k\ is\ odd \\ 13 & 176 \le k \le 179,\ k\ is\ even \\ 24 & 176 \le k \le 179,\ k\ is\ odd \\ 14 & 180 \le k \le 183,\ k\ is\ even \\ 23 & 180 \le k \le 183,\ k\ is\ odd \\ 23-y & 184 \le k \le 187,\ k\ is\ even \\ 14+y & 184 \le k \le 187,\ k\ is\ odd \end{cases}$$

In the same way, we have $[S_2TD(35)]_k$ *as follows*

$$[S_2TD(35)]_k = \begin{cases} k+1 & 1 \leq k \leq 17 \\ 2+y & 18 \leq k \leq 47, k \; is \; even \\ 35-y & 18 \leq k \leq 47, k \; is \; odd \\ 3+y & 48 \leq k \leq 73, k \; is \; even \\ 34-y & 48 \leq k \leq 73, k \; is \; odd \\ 4+y & 74 \leq k \leq 95, k \; is \; even \\ 33-y & 74 \leq k \leq 95, k \; is \; odd \\ 5+y & 96 \leq k \leq 113, k \; is \; even \\ 32-y & 96 \leq k \leq 113, k \; is \; odd \\ 6+y & 114 \leq k \leq 127, k \; is \; even \\ 31-y & 114 \leq k \leq 127, k \; is \; odd \\ 7+y & 128 \leq k \leq 137, k \; is \; even \\ 30-y & 128 \leq k \leq 137, k \; is \; odd \\ 8+y & 138 \leq k \leq 147, k \; is \; even \\ 29-y & 138 \leq k \leq 147, k \; is \; odd \end{cases}$$

$$[S_2TD(35)]_k = \begin{cases} 9+y & 148 \leq k \leq 155, k\ is\ even \\ 28-y & 148 \leq k \leq 155, k\ is\ odd \\ 10+y & 156 \leq k \leq 163, k\ is\ even \\ 27-y & 156 \leq k \leq 163, k\ is\ odd \\ 11+y & 164 \leq k \leq 169, k\ is\ even \\ 26-y & 164 \leq k \leq 169, k\ is\ odd \\ 12+y & 170 \leq k \leq 175, k\ is\ even \\ 25-y & 170 \leq k \leq 175, k\ is\ odd \\ 13+y & 176 \leq k \leq 179, k\ is\ even \\ 24-y & 176 \leq k \leq 179, k\ is\ odd \\ 14+y & 180 \leq k \leq 183, k\ is\ even \\ 23-y & 180 \leq k \leq 183, k\ is\ odd \\ 14+y & 184 \leq k \leq 187, k\ is\ even \\ 23-y & 184 \leq k \leq 187, k\ is\ odd \end{cases}$$

In the same way, we have $[S_2TD(35)]_k$ *as follows*

$$[S_3TD(35)]_k = \begin{cases} 36-k & 1 \le k \le 17 \\ 34-y & 18 \le k \le 47, k \text{ is even} \\ 3+y & 18 \le k \le 47, k \text{ is odd} \\ 32-y & 48 \le k \le 73, k \text{ is even} \\ 5+y & 48 \le k \le 73, k \text{ is odd} \\ 30-y & 74 \le k \le 95, k \text{ is even} \\ 7+y & 74 \le k \le 95, k \text{ is odd} \\ 28-y & 96 \le k \le 113, k \text{ is even} \\ 9+y & 96 \le k \le 113, k \text{ is odd} \\ 26-y & 114 \le k \le 127, k \text{ is even} \\ 11+y & 114 \le k \le 127, k \text{ is odd} \\ 24-y & 128 \le k \le 137, k \text{ is even} \\ 13+y & 128 \le k \le 137, k \text{ is odd} \\ 24-y & 138 \le k \le 147, k \text{ is even} \\ 13+y & 138 \le k \le 147, k \text{ is odd} \end{cases} \quad [S_3TD(35)]_k = \begin{cases} 23-y & 148 \le k \le 155, k \text{ is even} \\ 14+y & 148 \le k \le 155, k \text{ is odd} \\ 23-y & 156 \le k \le 163, k \text{ is even} \\ 14+y & 156 \le k \le 163, k \text{ is odd} \\ 22-y & 164 \le k \le 169, k \text{ is even} \\ 15+y & 164 \le k \le 169, k \text{ is odd} \\ 22-y & 170 \le k \le 175, k \text{ is even} \\ 15+y & 170 \le k \le 175, k \text{ is odd} \\ 21-y & 176 \le k \le 179, k \text{ is even} \\ 16+y & 176 \le k \le 179, k \text{ is odd} \\ 21-y & 180 \le k \le 183, k \text{ is even} \\ 16+y & 180 \le k \le 183, k \text{ is odd} \\ 19 & 184 \le k \le 187, k \text{ is even} \\ 18 & 184 \le k \le 187, k \text{ is odd} \end{cases}$$

Conclusions and Future work

We have constructed a new method for developing the triad design of orders 17, 23, 29 and 35. by analyzing the triples to construct the starter for $TD(v)$. For each case, first we divide the triples into number of intervals. Then for each interval we construct $S_1TD(v)$, $S_2TD(v)$ and $S_3TD(v)$. Furthermore, we use the formula $y = \frac{1}{2}[k - f + mod(\frac{k+n}{2}) + 1]$ to produce the corresponding elements in $S_1TD(v)$, $S_2TD(v)$ and $S_3TD(v)$ respectively. After that, derived the general rules of intervals of the general case $v = 6n+5$. From that, we construct $S_rTD(v)$ for $1 \leq r \leq 3$ that mentioned in Theorem 4.2.1. Also we applied the proposed method using numerical examples for the case $v = 35$ in order to test the results. Many topics can follow the work presented in this thesis. Here, we mention some of the topics that can be considered:

1. Construct the triad design of other orders.
2. Constructing the triad design for the cases $v = 6n$ or $v = 6n + 2$
3. Apply the mathematical results presented in this thesis for real life problems in different fields (e.g. economics and physics).

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ملخص

عامل متوافق وتصميم ثلاثي لرسوم كاملة ذات رتب معينة

ميس جابر

في هذا العمل ، قمنا بدراسة العوامل المتوافقة $CF(v)$ و التصميم الثلاثي $TD(v)$ لبعض العناصر v ، حيث $v = 6n+5$. هذه طرق لترتيب الثلاثيات المختلفة (سرد المثلثات وعدّها) على العناصر v مع بعض الخصائص. الدراسات السابقة على $TD(v)$ ذكرت وجودها عندما تم تطوير $v = 6n+1$ و $v = 6n+5$ باستخدام طريقة القوة الغاشمة. علاوة على ذلك ، بداية التصميم الثلاثي $STD(v) = SCF(v) \cup \overline{SCF(v)}$. بالإضافة إلى ذلك ، في هذه الأطروحة، تم إنشاء تقنية جديدة لخوارزميات $STD(v)$. والمعروفة باسم (طريقة الفاصل الزمني المعمم - GIM)، من خلال تحليل نمط الثلاثيات ومن خلال توضيح الحالات $v = 5, 11, 17, 23, 29$. في النهاية ، تسرد هذه التقنية عناصر $TD(6n+5)$ عن طريق الإضافة المتكررة لـ ١ (مد v) من $STD(v)$. نحن نركز على بناء تصميم ثلاثي للرسم البياني الكامل K_n . نقوم ببناء طريقة جديدة لتطوير تصميم ثلاثي على كائنات $TD(v)$ ، والتي تحسب وتسرد جميع المثلثات في K_n . تعتمد هذه الطريقة على تحليل الثلاثيات لبناء المبدئ. نوضح الطريقة من خلال النظر في الحالات $v = 11, 17, 23, 29$. ثم نختتم ل $v = 6n+5$ ،نوضح نتائجنا من خلال بناء $TD(11)$ و $TD(17)$ و $TD(23)$.