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Fully Fuzzy Linear Systems Via Alpha-cuts

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Thesis Approval

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Declaration

The work in this thesis,unless otherwise referenced,is the researcher's own work and has not been submitted elsewhere for any other degree or qualification

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Date: 16/11/2021

Dedication

To my parents for their loving and supporting.

To my dear husband for his unlimited patience and continuous support.

To my kind family, Iyad, Haneen, Zead, Saja and Sujood.

To my sweety son Zain-Aldeen .

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Abstract

Fully Fuzzy Linear Systems Via Alpha-cuts

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In this work, we studied fully fuzzy linear systems with triangular, trapezoidal and hexagonal fuzzy numbers. In literature, the product of two fuzzy numbers using $\alpha - cuts$ has an unpreferable amount, so many authors went to the approximated multiplication formula. Following approximated multiplication, the multiplication of two positive fuzzy numbers need not be positive and in other times leads to a fuzzy number that is not of the same type. Our concern in this work was to use the exact value of the multiplication of two fuzzy numbers and insert suitable condition(s) to guarantee the uniqueness of the solution of a fully fuzzy system with triangular, trapezoidal and hexagonal fuzzy numbers. We proposed a structure for the solution of the fully fuzzy linear system in each case. We then illustrated our results using a number of numerical examples.

Contents

Chapter 1

Introduction

Linear algebraic systems play an important role in many branches of science, statistics and engineering [31]. In applications which sometimes deal with inaccurate parameters, it is necessary to find a system that can describe uncertain data, to replace the crisp system. Therefore, many authors have investigated the solution of such type of systems deeply to develop mathematical tools and numerical proceduresosed solutions and prop. Zadeh was the first who introduced the definition of fuzzy set theory [37,38]. Since then, a big amount of research work have been dedicated to build up the basics of fuzzy set theory [10]. Fuzzy numbers are fuzzy subsets of the real numbers. In this sense, fuzzy numbers simulates the physical phenomena in a more realistic way. Many applications have made use of the fuzzy concept including computer programming, decision making and communications [3, 40].

The fuzzy arithmetic operations are essential to the development of fuzzy mathematics. Different approaches have been adopted in the literature to define fuzzy basic operations such as addition, subtraction and multiplication [8, 9]. While adding and subtracting two fuzzy numbers are fairly simple, multiplication still have some difficulties. Most of the researchers use approximation to get more straightforward answer by enforcing some conditions to the $\alpha - cut$ multiplication definition [8]. However, approximating the results of multiplication will logically lead to more fuzziness and higher levels of uncertainties in the resulting fuzzy number [34]. A number of research papers discussed extensively this problem and proposed several computational procedures [14, 15, 35].

Solving fuzzy linear systems of equations (FLS) is one field of applied mathematics that shows up in various scientific applications especially when the coefficients of such systems are imprecise and can only be given by fuzzy numbers. The solution of FLS was first introduced by Friedman et al. [12] by proposing a general method for solving an $n \times n$ system which includes a crisp coefficients matrix with an arbitrary fuzzy number vector as right-hand side. Since then, many methods were proposed to solve FLS computationally (the reader is advised to turn to [2,5] for a review of some methods). A fuzzy linear system where all the coefficients of the system are fuzzy numbers is called a fully fuzzy linear system (FFLS). Dehghan and Hashemi in [7] extended the Adomian decomposition method to solve FFLS. In [27], FFLS with triangular fuzzy numbers is discussed and a method is proposed using Gauss-Jordan Elimination. Malkawi et al. [26] proposed a modified associated linear system for solving FFLS where the fuzzy numbers are hexagonal and positive. In [17], a new algorithm is proposed to solve general FFLS where there is no restrictions on the sign of the parameters nor the variables in the linear system. Kumar et al. [21] proposed a simplified computational method to solve the FFLS that overcomes the non-negativity condition in two ways, one is by eliminating the non-negativity constraint from the coefficient matrix while the other eliminates the non-negativity constraint on the solution vector. Another approach that proposes solving trapezoidal FFLS using partitioning is presented in [16]. In [23], a computational method for solving

FFLS using row reduced echelon form was applied.

The authors in [30] presented a numerical algorithm for solving fuzzy systems of linear equations based on homotopy perturbation method. Chandrasekaran [6] solved fuzzy linear system by singular value decomposition method. The authors in [36] introduced ST decomposition procedure to solve fully fuzzy linear systems. In [29] the authors presented a new representation of interval arithmetic, they used it to develop algorithms to solve fuzzy linear system and fuzzy linear system with both triangular and trapezoidal type of fully fuzzy numbers. In [31] the authors solved the fully fuzzy linear system consisting of positive fuzzy numbers using QR decomposition method. In [1] the authors proposed a semi-iterative method to find solution of the fully fuzzy linear systems. The authors in [11] used the arithmetic operations on fuzzy numbers that introduced by Kaffman and found a positive fuzzy solution for the fully fuzzy linear system of equations. Nasseri and Zehmakkesh [28] proposed Huang method for computing a nonnegative solution of the fully fuzzy linear system of equations. Kumar and others [19] gave a new approach for solving fully fuzzy linear systems. In the same time solution of fully fuzzy linear system with arbitrary coefficient was introduced by them [20] . Several methods were used for solving fully fuzzy linear systems with trianglar fuzzy numbers have been introduced by many authors [42] [22] [4] [25].

In this work, we use the alpha-cuts to define the product two fuzzy numbers without dropping any terms. Also, we find a suitable condition that guarantees not only the existence of the solution, but also a positive one. In this work, we discuss fully fuzzy linear system with triangular, trapezoidal and hexagonal fuzzy numbers, respectively. Firstly, we will present the definition of each fuzzy number, the addition and the multiplication using the alpha cut. After that, we solve the fully fuzzy linear systems with the fuzzy numbers when the coefficients matrix is square invertible and singular. Moreover, we give the suitable conditions that guarantee not only the existence of the solution, but also unique one. We deal in the last of each chapter with the multiplication of two fuzzy numbers by applying the norm theorem and writing the new certain conditions that obtain existence positive solution. In addition, we present some examples for the fully fuzzy linear systems.

The reminder of this thesis is organized as follows: In Chapter 1, we recall some of the definitions and properties for the fuzzy numbers. We follow that by presenting the main operations on fuzzy numbers. Chapter 2 is dedicated for solving fully fuzzy linear systems using triangular fuzzy numbers. Afterwords, we present the proposed solution for fully fuzzy linear systems with trapezoidal fuzzy numbers in Chapter 3. Chapter 4 explains the solution of fully fuzzy linear systems for hexagonal fuzzy numbers. Finally, we conclude this thesis and suggest future research directions.

Chapter 2

Fuzzy Set and Fuzzy Number

In this chapter, we start by introducing some important basic definitions on fuzzy sets. We then move to present operations of fuzzy sets and fuzzy relations and alpha cut approach.

2.1 Fuzzy Set

Fuzzy set theory is a branch that deals precisely with imprecision and ambiguity. In 1965, Zadeh introduced the concept of fuzzy sets [38, 39] and discussed the main operations related to it. In this section, we define fuzzy sets and membership functions with their related operations.

2.1.1 Definitions and Notations

Definition 2.1.1 *[24](Membership function): Let X be a universal set and A subset of X, the membership function is $\mu_A : X \to \{0, 1\}$ such that*

$$X_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \tag{2.1}$$

Definition 2.1.2 *Let X be a fuzzy set, the membership function of X is μ_A :*

$X \to [0,1]$, *where the value of* $\mu_A(x)$ *at x shows the grade of membership of x in A.*

If X is discrete then the membership function can be written $A = \{(x, \mu_A(x)) : x \in X\}$ or $A = \sum\limits_i \frac{\mu_A(x_i)}{x_i}$. For example, $A = \{(1, 0.3), (2, 0.5)\}$. The symbol '$\sum$' means not addition but it mean the usual union between sets. On the other hand, if X ic continuous, then the set A can be written $A = \int \frac{\mu_A(x)}{x}$

2.1.2 Operation of Fuzzy Set

Lets start by defining some of the properties related to fuzzy sets.

Definition 2.1.3 *A fuzzy set A is empty if and only if* $\mu_A(x) = 0, \forall x \in X$.

Definition 2.1.4 *[24] Two fuzzy sets A and B are equivalent,denoted by* $A = B$ *if and only if* $\mu_A(x) = \mu_B(x), \forall x \in X$. *And if* $\mu_A(x) \neq \mu_B(x), for some x \in X$ *then* $A \neq B$.

Definition 2.1.5 *[24]* $A \subseteq B$ *if and only if* $\mu_A(x) \leq \mu_B(x), \forall x \in X$.

Definition 2.1.6 *[24] The complement set of A is denoted by* $\bar{A}$ *has a membership function* $\mu_{\bar{A}}(x) = 1 - \mu_A(x), \forall x \in X$.

Definition 2.1.7 *[24] The union of two fuzzy sets A and B is* $A \cup B$ *which is a fuzzy set whose membership function is defined by* $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X$.

Definition 2.1.8 *[24] The intersection of two fuzzy sets A and B is* $A \cap B$ *which is a fuzzy set whose membership function is defined by* $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$.

Clearly, we can extend many rules that holds for crisp sets to fuzzy by using operations of union, complement and intersection. Here we provide some rules:

(1) Involution: The complement set of $\bar{A} = A$.

(2) Commutative rule: $A \cup B = B \cup A$.

$$A \cap B = B \cap A$$

(3) Associative rule: $(A \cup B) \cup C = A \cup (B \cup C)$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

(4) Distributive rule: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(5) De Morgan's law: the complement set $A \cup B$ is $\bar{A} \cap \bar{B}$ and complement set for $A \cap B$ is $\bar{A} \cup \bar{B}$.

2.1.3 Alpha Cuts

Now, we move to introduce the $\alpha-$ cut definition and relation to fuzzy sets.

Definition 2.1.9 *[41] The set $A_\alpha = \{x \in X : \mu_A(x) \geq \alpha\}$is called the $\alpha-$ cut which is a crisp set.*

Example 2.1.1 *Let* $A = \{(2, 0.5), (3, 1), (5, 0.7), (7, 0.4), (9, 0.2)\}$ *then:* $A_0 = \{2, 3, 5, 7, 9\}$. $A_{0.2} = \{2, 3, 5, 7, 9\}$, $A_{0.4} = \{2, 3, 5, 7\}$, $A_{0.5} = \{2, 3, 5\}$, $A_{0.7} = \{3, 5\}$, $A_1 = \{3\}$.

Definition 2.1.10 *[41] A fuzzy set A is convex if $x_1, x_2 \in X$, and $\lambda \in [0, 1]$ then*

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\},.$$

Alternatively, a fuzzy set is convex if and only if each α-cut is convex.

2.2 Fuzzy Number

In this section, the definition and basic operationS on fuzzy numbers are presented. Also, the definitions of ianula, trapezoidal and hexagonal fuzzy numbers are stated.

2.2.1 DefinitionS and Notations

Definition 2.2.1 *[41] A fuzzy number A is a fuzzy set satisfies the following conditions:*

1. Its membership function is defined on the real number.

2. Its membership function is piece-wise continuous.

3. Convex fuzzy set.

4. Normalized fuzzy set (i.e $\exists x \in \mathbf{R}, \mu_A(x) = 1$)

Definition 2.2.2 *[24] :A fuzzy number A is called positive (negative) denoted by $A > 0 (A < 0)$ if its membership function $\mu_A(x)$ satisfies $\mu_A(x) = 0, \forall x \leq 0 (\forall x \geq 0)$.*

2.2.2 Operations on Fuzzy Numbers

Definition 2.2.3 *[41] The maximum of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A \vee B}(z) = \bigvee_{z=x \vee y} (\mu_A(x) \wedge \mu_B(y)), \forall z \in X$.*

Definition 2.2.4 *[41] The minimum of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A \vee B}(z) = \bigvee_{z=x \wedge y} (\mu_A(x) \wedge \mu_B(y)), \forall z \in X$.*

Definition 2.2.5 *[41] The Addition of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A+B}(z) = \bigvee_{z=x+y} (\mu_A(x) \wedge \mu_B(y)), \forall z \in X$.*

Definition 2.2.6 *[41] The subtraction of two fuzzy numbers* A *and* B *is a fuzzy set and the membership function is* $\mu_{A-B}(z) = \bigvee_{z=x-y} (\mu_A(x) \wedge \mu_B(y)), \forall z \in X.$

Definition 2.2.7 *[41] The multiplication of two fuzzy numbers* A *and* B *is a fuzzy set and the membership function is* $\mu_{A*B}(z) = \bigvee_{z=x*y} (\mu_A(x) \wedge \mu_B(y)), \forall z \in X.$

Definition 2.2.8 *[41] The division of two fuzzy numbers* A *and* B *is a fuzzy set and the membership function is* $\mu_{A/B}(z) = \bigvee_{z=x/y} (\mu_A(x) \wedge \mu_B(y)), \forall z \in X.$

2.2.3 Triangular Fuzzy Numbers

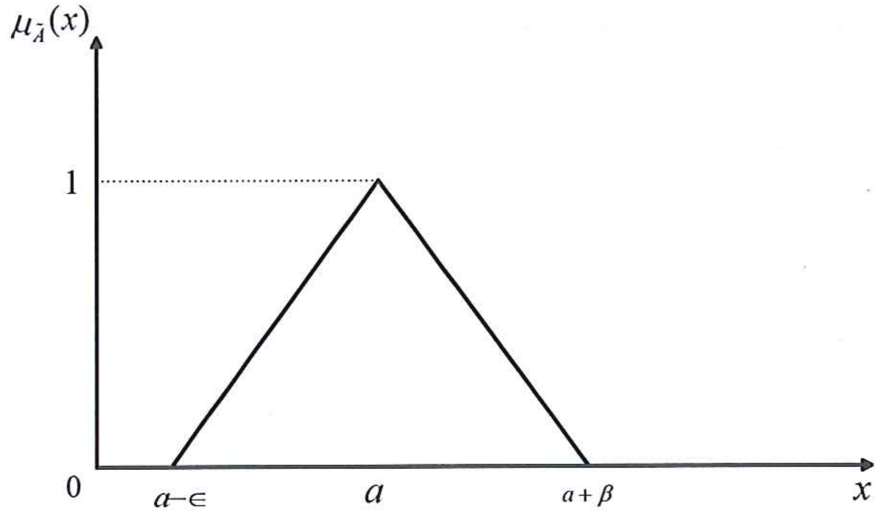

Fig. 2.1: Triangular fuzzy number $\tilde{A} = (a, \epsilon, \beta)$.

Definition 2.2.9 *[33] A triangular fuzzy number* $\widehat{A}$ *is a fuzzy number defined by three real numbers* a_1, a_2 *and* a_3 *with* $a_1 < a_2 < a_3$, $\widehat{A} = (a_1, a_2, a_3)$ *, whose membership function is given by:*

$$\widehat{A}(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3. \end{cases}$$

Figure2.1 shows the triangular number with uncertainty interval.

Definition 2.2.10 *[13] If $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ are triangular fuzzy numbers then:*

1) Addition: $A \oplus B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

2) Symmetric image: $-A = (-a_3, -a_2, -a_1)$

3) multiplication: $\widehat{A} \otimes \widehat{B} = (a_1b_1, a_2b_2, a_3b_3)$

The set $A_\alpha = \{x \in [a_1, a_3], \mu_{\widehat{A}}(x) \geq \alpha\}$ is the $\alpha - cut$ of the fuzzy number which is a crisp set. If $A_\alpha = [a_{1\alpha}, a_{3\alpha}]$, $a_{1\alpha}$ and $a_{3\alpha}$ can be obtained by solving $\frac{a_{1\alpha}-a_1}{a_2-a_1} = \alpha$ and $\frac{a_3-a_{3\alpha}}{a_3-a_2} = \alpha$. Therefore, the $\alpha - cut$ of a triangular fuzzy number $\widehat{A}$ is

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]. \quad (2.2)$$

So its core and support are a_2 and $A_0 = [a_1, a_3]$ respectively.

Definition 2.2.11 *$\widehat{A} = (a_1, a_2, a_3)$ and $\widehat{B} = (b_1, b_2, b_3)$ are both triangular fuzzy numbers then, from (2.2) we have*

$$\begin{aligned} A_\alpha + B_\alpha &= [(a_2 - a_1 + b_2 - b_1)\alpha + a_1 + b_1, -(a_3 - a_2 - b_3 - b_2) + a_3 + b_3] \\ A_\alpha - B_\alpha &= [(a_2 - a_1 + b_3 - b_2)\alpha + a_1 - b_3, -(a_3 - a_2 - b_2 - b_1)\alpha + a_3 - b_1] \end{aligned}$$

2.2.4 Trapeziodal Fuzzy Number

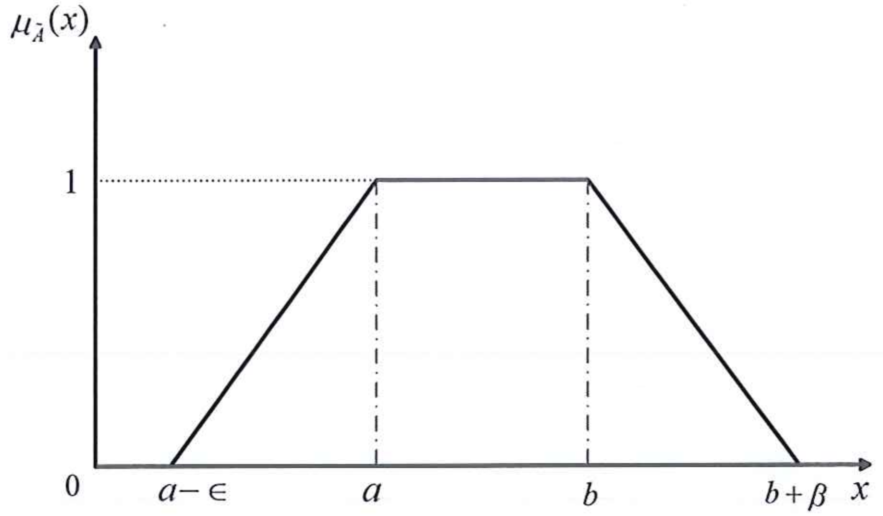

Fig. 2.2: Trapezoidal fuzzy number $\tilde{A} = (a, b, \epsilon, \beta)$.

Definition 2.2.12 *[24] Trapezoidal fuzzy number is a fuzzy number represented by four real numbers* $a_1 \leq a_2 \leq a_3 \leq a_4$ *denoted by* $\tilde{A} = (a_1, a_2, a_3, a_4)$ *and whose membership function is*

$$\mu_A(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases} \tag{2.3}$$

I Figure 2.2, an example for a trapezoidal number is given.

Definition 2.2.13 *[24] If* $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$ *are trapezoidal fuzzy numbers then:*

1) Addition: $A \oplus B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$

2) Symmetric image: $-A = (-a_4, -a_3, -a_2, -a_1)$

3) multiplication: $A \otimes B = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4)$

Definition 2.2.14 *[24] A trapezoidal fuzzy number* $A = (a_1, a_2, a_3, a_4)$ *is said to be zero trapezoidal fuzzy number if and only if* $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$.

Definition 2.2.15 *[24] Two fuzzy numbers* $A = (a_1, a_2, a_3, a_4)$ *and* $B = (b_1, b_2, b_3, b_4)$ *are equal if and only if* $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4$.

2.2.5 Hexagonal Fuzzy Numbers

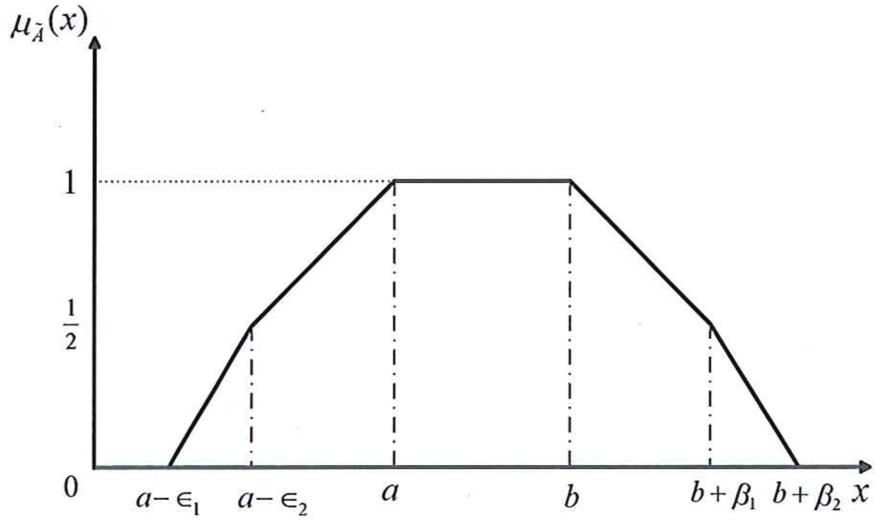

Fig. 2.3: Hexagonal fuzzy number $\tilde{A} = (a, b, \epsilon_1, \beta_1, \epsilon_2, \beta_2)$.

Definition 2.2.16 *[32] A hexagonal fuzzy number* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$*,where*

$a_1, a_2, a_3, a_4, a_5, a_6$ *are real numbers whose membership function* $\mu_{\hat{A}}(x)$ *is*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < a_1 \\ \frac{1}{2}(\frac{x-a_1}{a_2-a_1}) & a_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{1}{2}(\frac{x-a_2}{a_3-a_2}) & a_2 \leq x \leq a_3 \\ 1 & a_3 \leq x \leq a_4 \\ 1 - \frac{1}{2}(\frac{x-a_4}{a_5-a_4}) & a_4 \leq x \leq a_5 \\ \frac{1}{2}(\frac{a_6-x}{a_6-a_5}) & a_5 \leq x \leq a_6 \\ 0 & x > a_6 \end{cases} \tag{2.4}$$

Figure 2.3 depicts a hexagonal fuzzy number.

Definition 2.2.17 *[32] A hexagonal fuzzy number* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ *is positive (negative) if* $a_i > 0$ $(a_i < 0)$ *for* $i = 1, 2, 3, 4, 5, 6$.

Example 2.2.1 $\hat{A} = (1, 2, 5, 7, 3, 4)$ *is positive.*

$\hat{A} = (-10, -8, -5, -4, -2, -1)$ *is negative.*

Definition 2.2.18 *[32] Let* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ *,*$\hat{B} = (b_1, b_2, b_3, b_4, b_5, b_6)$ *be two hexagonal fuzzy number,* $\hat{A}$= $\hat{B}$ *if only if* $a_i = b_i$ *for* $i = 1, 2, 3, 4, 5, 6$.

Definition 2.2.19 *[32] :If* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ *is hexagonal fuzzy number then* $-\hat{A} = (-a_6, -a_5, -a_4, -a_3, -a_2, -a_1)$ *which is the symmetric image of* $\hat{A}$ *is also a hexagonal fuzzy number.*

Definition 2.2.20 *[32](Operations of Hexagonal Fuzzy numbers) Let* $\hat{A}$ *,*$\hat{B}$ *be two hexagonal fuzzy numbers then :*

Addition: $\hat{A} \oplus \hat{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6)$

Symmetric image: -$\hat{A} = (-a_6, -a_5, -a_4, -a_3, -a_2, -a_1)$

Multiplication: $\hat{A} \otimes \hat{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6)$

Example 2.2.2 *Let* $\hat{A} = (1,3,4,7,8,10)$, $\hat{B} = (2,5,7,11,13,17)$ *be two fuzzy numbers then :*

$\hat{A} \oplus \hat{B} = (3,8,11,18,21,27)$

$\hat{A} \otimes \hat{B} = (2,15,28,77,104,170)$

$-\hat{A} = (-10,-8,-7,-4,-3,-1)$.

Using α- cut arithmetic and the left-right spreads, we give a new definition to hexagonal fuzzy numbers as follows.

Definition 2.2.21 *Let* $m \leq n$, $\alpha_1, \beta_1, \alpha_2$, *and* β_2 *such that* $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2$. *A non-zero hexagonal fuzzy number denoted by* $\hat{A} = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *is a fuzzy number whose membership function is given by*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < m - \alpha_1 \\ \frac{1}{2}(\frac{x-m+\alpha_1}{\alpha_1-\alpha_2}) & m - \alpha_1 \leq x \leq m - \alpha_2 \\ 1 + \frac{1}{2}(\frac{x-m}{\alpha_2}) & m - \alpha_2 \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{1}{2}(\frac{x-n}{\beta_2}) & n \leq x \leq n + \beta_2 \\ -\frac{1}{2}(\frac{x-n-\beta_1}{\beta_1-\beta_2}) & n + \beta_2 \leq x \leq n + \beta_1 \\ 0 & x > n + \beta_1 \end{cases} \quad (2.5)$$

Definition 2.2.22 *A hexagonal fuzzy number* $\hat{A} = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *is positive if* $m - \alpha_1 \geq 0$.

Definition 2.2.23 *The hexagonal fuzzy numbers* $\hat{A} = (m, n, \alpha_{11}, \beta_{11}, \alpha_{12}, \beta_{12})$, *and* $\hat{B} = (p, q, \alpha_{21}, \beta_{21}, \alpha_{22}, \beta_{22})$ *are equal if* $m = p$, $n = q$, $\alpha_{11} = \alpha_{21}, \beta_{11} = \beta_{21}, \alpha_{12} = \alpha_{22}, \beta_{12} = \beta_{22}$.

Remark 2.2.1 *Hexagonal fuzzy number* $\hat{A}_H$ *is the order quadruple* $P_1(u)$, $Q_1(v)$, $Q_2(v)$, $P_2(u)$ *for* $u \in [0, 0.5]$, $v \in [0.5, 1]$ *such that* $P_1(u) = \frac{1}{2}\frac{u-m+\alpha_1}{\alpha_1-\alpha_2}$, $P_2(u) =$

$-\frac{1}{2}\frac{u-n-\beta_1}{\beta_1-\beta_2}$, $Q_1(v) = 1 + \frac{1}{2}\frac{v-m}{\alpha_2}$ *and* $Q_2(v) = 1 - \frac{1}{2}(\frac{v-n}{\beta_2})$. *In fact, if*

$P_1(x) = \frac{1}{2}\frac{x-m+\alpha_1}{\alpha_1-\alpha_2} = \alpha$ *then* $x = 2\alpha(\alpha_1 - \alpha_2) + m - \alpha_1$,

$P_2(x) = -\frac{1}{2}\frac{x-n-\beta_1}{\beta_1-\beta_2} = \alpha$ *then* $x = -2\alpha(\beta_1 - \beta_2) + n + \beta_1$,

$Q_1(x) = 1 + \frac{1}{2}\frac{x-m}{\alpha_2} = \alpha$ *then* $x = 2\alpha_2(\alpha - 1) + m$

$Q_2(x) = 1 - \frac{1}{2}(\frac{x-n}{\beta_2}) = \alpha$ *then* $x = 2\beta_2(1 - \alpha) + n$.

Using α-cut arithmetics and the above remark, we conclude that the α-cut of hexagonal fuzzy numbers $\hat{A}_H$ and $\hat{B}_H$ are

$$A_\alpha = \begin{cases} [2\alpha(\alpha_{11} - \alpha_{12}) + m - \alpha_{11}, -2\alpha(\beta_{11} - \beta_{12}) + n + \beta_{11}] & \alpha \in [0, 0.5) \\ [2\alpha_{12}(\alpha - 1) + m, 2\beta_{12}(1 - \alpha) + n] & \alpha \in [0.5, 1] \end{cases}$$

$$B_\alpha = \begin{cases} [2\alpha(\alpha_{21} - \alpha_{22}) + p - \alpha_{21}, -2\alpha(\beta_{21} - \beta_{22}) + q + \beta_{21}] & \alpha \in [0, 0.5) \\ [2\alpha_{22}(\alpha - 1) + p, 2\beta_{22}(1 - \alpha) + q] & \alpha \in [0.5, 1] \end{cases}$$

So, for $\alpha = 0$,

$$\begin{aligned} A_0 + B_0 &= [(m - \alpha_{11}) + (p - \alpha_{21}), (n + \beta_{11}) + (q + \beta_{21})] \\ &= [(m + p - (\alpha_{11} + \alpha_{21})), (n + q + (\beta_{11} + \beta_{21}))], \end{aligned}$$

for $\alpha = 0.5$,

$$\begin{aligned} A_{0.5} + B_{0.5} &= [(m - \alpha_{12}) + (p - \alpha_{22}), (n + \beta_{12}) + (q + \beta_{22})] \\ &= [(m + p - (\alpha_{12} + \alpha_{22})), (n + q + (\beta_{12} + \beta_{22}))], \end{aligned}$$

and for $\alpha = 1$,$A_1 + B_1 = [m + p, n + q]$. So, we get the sum of two hexagonal fuzzy numbers

$$\hat{A} \oplus \hat{B} = [m + p, n + q, \alpha_{11} + \alpha_{21}, \beta_{11} + \beta_{21}, \alpha_{12} + \alpha_{22}, \beta_{12} + \beta_{22}]. \tag{2.6}$$

Example 2.2.3 *let* $\hat{A} = (4,6,2,3,1,2)$,$\hat{B} = (8,10,5,7,3,6)$. *Then*

$$A_\alpha = \begin{cases} [2\alpha+2, -2\alpha+9] & \alpha \in [0,0.5) \\ [2\alpha+2, 10-4\alpha] & \alpha \in [0.5,1] \end{cases}$$

$$B_\alpha = \begin{cases} [4\alpha+3, -2\alpha+17] & \alpha \in [0,0.5) \quad \alpha \in [0,0.5) \\ [6\alpha+2, 22-12\alpha] & \alpha \in [0.5,1] \quad \alpha \in [0.5,1] \end{cases}$$

$$A_\alpha + B_\alpha = \begin{cases} [6\alpha+5, -4\alpha+26] & \alpha \in [0,0.5) \\ [8\alpha+4, 32-16\alpha] & \alpha \in [0.5,1]. \end{cases}$$

Then, $\hat{A} \oplus \hat{B} = (12,16,7,10,4,8)$

Chapter 3

Triangular Fully Fuzzy Linear systems

3.1 Triangular Fuzzy Numbers and $\alpha - cuts$

Definition 3.1.1 *[33] Let n be a real number, α_1 and β_1 are positive numbers. A non-zero fuzzy number is a triangular fuzzy number denoted by $\hat{A} = (n, \alpha_1, \beta_1)$ is a fuzzy number whose membership is given by*

$$\mu_{\widehat{A}}(x) = \begin{cases} 0, & x < n - \alpha_1 \\ \frac{x-n}{\alpha_1} + 1, & n - \alpha_1 \leq x \leq n \\ \frac{n-x}{\beta_1} + 1, & n \leq x \leq n + \beta_1 \\ 0, & x > n + \beta_1 \end{cases}$$

.

The α-cut of a triangular fuzzy number is the crisp set $A_\alpha = \{x | x \in X, \mu_{\widehat{A}}(x) \geq \alpha\}$. If $A_\alpha = [a_{1\alpha}, a_{3\alpha}]$ then, $a_{1\alpha}$ and $a_{3\alpha}$ can be obtained by solving $\frac{a_{1\alpha}-a_1}{a_2-a_1} = \alpha$ and $\frac{a_3-a_{3\alpha}}{a_3-a_2} = \alpha$. Therefore,

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \tag{3.1}$$

Now let $\widehat{A} = (a_1, a_2, a_3) = (n, \alpha_1, \beta_1)$, then from (3.1) we have

$$
\begin{aligned}
A_\alpha &= [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \\
&= [(n - n + \alpha_1)\alpha + n - \alpha_1, -(n + \beta_1 - n)\alpha + n + \beta_1] \\
&= [\alpha\alpha_1 + n - \alpha_1, -\beta_1\alpha + n + \beta_1] \\
&= [(\alpha - 1)\alpha_1 + n, (1 - \alpha)\beta_1 + n]
\end{aligned}
$$

Addition and Multiplication of Two Triangular Fuzzy Numbers:

Let $\widehat{A} = (n, \alpha_1, \beta_1)$ and $\widehat{B} = (m, \alpha_2, \beta_2)$ be two triangular fuzzy numbers, then $A_\alpha = [(\alpha - 1)\alpha_1 + n, (1 - \alpha)\beta_1 + n]$ and $B_\alpha = [(\alpha - 1)\alpha_2 + m, (1 - \alpha)\beta_2 + m]$. From the definition of intervals addition we have, for $\alpha = 0$

$$
\begin{aligned}
A_0 + B_0 &= [(m - \alpha_2) + (n - \alpha_1), (n + \beta_1) + (m + \beta_2)] \\
&= [m + n - (\alpha_1 + \alpha_2), m + n + (\beta_1 + \beta_2)]
\end{aligned}
$$

Similarly, calculating the sum of $\alpha - cuts$ of the two numbers at any α will give the $\alpha - cut$ of the following triangular fuzzy number

$$
\widehat{A} \oplus \widehat{B} = (m + n, \alpha_1 + \alpha_2, \beta_1 + \beta_2) \tag{3.2}
$$

Also, we can define the product of two triangular fuzzy number using the itervals multiplication and the $\alpha - cuts$ as follows: for $\alpha = 0$

$$
\begin{aligned}
A_0 \times B_0 &= [(n - \alpha_1)(m - \alpha_2), (n + \beta_1)(m + \beta_2)] \\
&= [(nm - m\alpha_1 - n\alpha_2 + \alpha_1\alpha_2), (nm + n\beta_2 + m\beta_1 + \beta_1\beta_2)] \\
&= [nm - (m\alpha_1 + n\alpha_2 - \alpha_1\alpha_2), nm + (n\beta_2 + m\beta_1 + \beta_1\beta_2)]
\end{aligned}
$$

Therefore the product of $\widehat{A} = (n, \alpha_1, \beta_1)$ and $\widehat{B} = (m, \alpha_2, \beta_2)$ which both are positive is given by the formula

$$\widehat{A} \otimes \widehat{B} = (nm, m\alpha_1 + n\alpha_2 - \alpha_1\alpha_2, n\beta_2 + m\beta_1 + \beta_1\beta_2) \tag{3.3}$$

3.2 Fully Fuzzy Linear System with Triangular Fuzzy Numbers

In this section, we deal with fully fuzzy linear systems where the associated linear systems have a square invertible matrix.

Consider the fuzzy linear system

$$\mathbb{A} \otimes \mathbb{X} = \mathbb{B}$$

such that each entry of $\mathbb{A} = (\widehat{a}_{ij})_{2n\times 2n}$ and $\mathbb{B} = (\widehat{b}_1, ..., \widehat{b}_n)^T$ are triangular fuzzy numbers, and $\mathbb{X} = (\widehat{x}_1, ..., \widehat{x}_n)^T$ is unknown .If $\widehat{a}_{ij} = (a_{ij}, \alpha_{ij}, \beta_{ij})^T$ and $\widehat{x}_j = (x_j, y_j, z_j)^T$ then, for each i

$$(\widehat{a}_{ij} \otimes \widehat{x}_j) \oplus ... \oplus (\widehat{a}_{in} \otimes \widehat{x}_n) = \widehat{b}_i$$

Where

$$\widehat{b}_i = (b_i, h_i, k_i)$$

From equations (3.2) and (3.3) we have

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}x_j &= b_i \\ \sum_{j=1}^{n}(\alpha_{ij}x_j + a_{ij}y_j - \alpha_{ij}y_j) &= h_i \\ \sum_{j=1}^{n}(\beta_{ij}x_j + a_{ij}z_j + \beta_{ij}z_j) &= k_i\end{aligned} \tag{3.4}$$

Let $A = [a_{ij}]$, $M = [\alpha_{ij}]$, $N = [\beta_{ij}]$, $x = (x_1, ..., x_n)^T$, $y = (y_1, ..., y_n)^T$ and $z = (z_1, ..., z_n)^T$ then (3.4) gives the following algebraic systems

$$\begin{gathered}Ax = b \\ Ay + Mx - My = h \\ Az + Nx + Nz = k\end{gathered} \tag{3.5}$$

The block representation of (3.5) is

$$\begin{bmatrix} A & 0 & 0 \\ M & A-M & 0 \\ N & 0 & A+N \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ h \\ k \end{bmatrix} \tag{3.6}$$

Definition 3.2.1 *The linear system* $SX = C$ *where* $X = (x, y, z)^T$, $C = (b, h, k)^T$ *and* $S = \begin{bmatrix} A & 0 & 0 \\ M & A-M & 0 \\ N & 0 & A+N \end{bmatrix}$ *is called the associated linear system of the fully fuzzy system* $\mathbb{A} \otimes \mathbb{X} = \mathbb{B}$

To solve the fuzzy linear system $\mathbb{A} \otimes \mathbb{X} = \mathbb{B}$, it is enough to solve the associated linear system $SX = C$.

Since the solution of the fully fuzzy system is totally dependent on the associated matrix S, this work focuses on two cases regarding the associated matrix S and these cases are:

- **The matrices $A, A - M$ and $A + N$ are invertible, and**
- **The more general case where we make sure matrix A is invertible such that $\|A^{-1}M\|_\infty$ and $\|B^{-1}N\|_\infty$ are less than 1.**

Remark 3.2.1 *The block matrix S is invertible if and only if the matrices A, $A - M$ and $A + N$ are invertible.*

Lemma 3.2.1 *If S is invertible then the unique solution of the system $SX = C$ is given by*

$$X = \begin{bmatrix} A^{-1}b \\ (A-M)^{-1}(h - MA^{-1}b) \\ (A+N)^{-1}(k - NA^{-1}b) \end{bmatrix} \tag{3.7}$$

Proof: Apply elementary row operations on the associated matrix S, to get

$$S^{-1} = \begin{bmatrix} A^{-1} & 0 & 0 \\ -MA^{-1}(A-M)^{-1} & (A-M)^{-1} & 0 \\ -NA^{-1}(A+N)^{-1} & 0 & (A+N)^{-1} \end{bmatrix}$$

Therefore,

$$
\begin{aligned}
x &= A^{-1}b \\
y &= -MA^{-1}(A-M)^{-1}b+(A-M)^{-1}h \\
&= (A-M)^{-1}(h-MA^{-1}b) \\
z &= -NA^{-1}(A+N)^{-1}b+(A+N)^{-1}k \\
&= (A+N)^{-1}(k-NA^{-1}b)
\end{aligned}
$$

□

Example 3.2.1 *Consider the following fuzzy linear system:*

$$
\begin{aligned}
(59,48,39)\otimes(x_1,y_1,z_1)\oplus(33,28,29)\otimes(x_2,y_2,z_2) &= (49,44,49) \\
(44,28,48)\otimes(x_1,y_1,z_1)\oplus(72,40,70)\otimes(x_2,y_2,z_2) &= (68,47,95).
\end{aligned}
$$

Then,

$A=\begin{bmatrix} 59 & 33 \\ 44 & 72 \end{bmatrix}$, $M=\begin{bmatrix} 48 & 28 \\ 28 & 40 \end{bmatrix}$, $N=\begin{bmatrix} 39 & 29 \\ 48 & 70 \end{bmatrix}$, $b=\begin{bmatrix} 49 \\ 68 \end{bmatrix}$, $h=\begin{bmatrix} 44 \\ 47 \end{bmatrix}$ and $k=\begin{bmatrix} 49 \\ 95 \end{bmatrix}$.

From equation(3.7) we have

$$x = A^{-1}b = \begin{bmatrix} 59 & 33 \\ 44 & 72 \end{bmatrix}^{-1} \begin{bmatrix} 49 \\ 68 \end{bmatrix} = \begin{bmatrix} 0.4592 \\ 0.6638 \end{bmatrix}$$

$$y = \begin{bmatrix} 11 & 5 \\ 16 & 32 \end{bmatrix}^{-1} \left(\begin{bmatrix} 44 \\ 47 \end{bmatrix} - \begin{bmatrix} 48 & 28 \\ 28 & 40 \end{bmatrix} \begin{bmatrix} 59 & 33 \\ 44 & 72 \end{bmatrix}^{-1} \begin{bmatrix} 49 \\ 68 \end{bmatrix} \right) = \begin{bmatrix} 0.2570 \\ 0.1087 \end{bmatrix}$$

$$z = \begin{bmatrix} 98 & 62 \\ 92 & 142 \end{bmatrix}^{-1} \left(\begin{bmatrix} 49 \\ 95 \end{bmatrix} - \begin{bmatrix} 39 & 29 \\ 48 & 70 \end{bmatrix} \begin{bmatrix} 59 & 33 \\ 44 & 72 \end{bmatrix}^{-1} \begin{bmatrix} 49 \\ 68 \end{bmatrix} \right) = \begin{bmatrix} 0.0047 \\ 0.1835 \end{bmatrix}$$

Hence $\widehat{x}_1 = (0.4592, 0.2570, 0.0047)$, $\widehat{x}_2 = (0.6638, 0.1087, 0.1835)$. For the other case that mentioned above, the following theorem is very needed, and will be used frequently in the coming chapters.

Theorem 3.2.2 *[18] If T is a bounded linear operator on a Banach space X such that $\|T\| < 1$, then $(I - T)^{-1}$ exists on X.*

So, to guarantee the uniqueness of the solution of fuzzy system, we stated the following corollary.

Corollary 3.2.1 *If the matrix A is invertible such that $\|A^{-1}M\|_\infty$ and $\|A^{-1}N\|_\infty$ are less than 1, then the associated system $SX = C$ has a unique solution.*

Proof: From theorem(3.6), we have $x = A^{-1}b$. Also, $(A - M)y = h - Mx$, multiply by A^{-1} to get $(I - A^{-1}M)y = A^{-1}(h - Mx)$. From (3.2.2), $y = (I - A^{-1}M)^{-1}A^{-1}(h - Mx)$. Similarly, $z = (I + A^{-1}N)^{-1}A^{-1}(k - Nx)$. □

Example 3.2.2 *Consider the following fuzzy linear system:*

$$(0.40, 0.005, 0.009) \otimes (x_1, y_1, z_1) \oplus (0.44, 0.001, 0.002) \otimes (x_2, y_2, z_2) = (0.68, 0.05, 0.39)$$

$$(0.29, 0.005, 0.008) \otimes (x_1, y_1, z_1) \oplus (0.22, 0.012, 0.013) \otimes (x_2, y_2, z_2) = (0.43, 0.037, 0.25)$$

$$A = \begin{bmatrix} 0.4 & 0.44 \\ 0.29 & 0.22 \end{bmatrix} \quad M = \begin{bmatrix} 0.005 & 0.001 \\ 0.005 & 0.012 \end{bmatrix} \quad N = \begin{bmatrix} 0.009 & 0.002 \\ 0.008 & 0.013 \end{bmatrix}$$

$$b = \begin{bmatrix} 0.68 \\ 0.43 \end{bmatrix} \quad h = \begin{bmatrix} 0.05 \\ 0.037 \end{bmatrix} \quad k = \begin{bmatrix} 0.39 \\ 0.25 \end{bmatrix}$$

we get :

$\widehat{x}_1 = (1, 0.0342, 0.4072), \widehat{x}_2 = (0.6364, 0.0703, 0.4823)$

where

$$\|A^{-1}M\|_\infty = 0.1556, \|A^{-1}N\|_\infty = 0.1722$$

Example 3.2.3 *Consider the following fuzzy linear system:*

$$(0.34, 0.015, 0.019) \otimes (x_1, y_1, z_1) \oplus (0.33, 0.011, 0.002) \otimes (x_2, y_2, z_2) = (0.98, 0.15, 0.29)$$

$$(0.29, 0.005, 0.015) \otimes (x_1, y_1, z_1) \oplus (0.22, 0.012, 0.013) \otimes (x_2, y_2, z_2) = (0.75, 0.11, 0.25)$$

$$\text{A=} \begin{bmatrix} 0.34 & 0.33 \\ 0.29 & 0.22 \end{bmatrix} \text{M=} \begin{bmatrix} 0.015 & 0.011 \\ 0.005 & 0.012 \end{bmatrix} \text{N=} \begin{bmatrix} 0.019 & 0.002 \\ 0.015 & 0.013 \end{bmatrix}$$

$$\text{b=} \begin{bmatrix} 0.98 \\ 0.75 \end{bmatrix} \text{h=} \begin{bmatrix} 0.15 \\ 0.11 \end{bmatrix} \text{k=} \begin{bmatrix} 0.29 \\ 0.25 \end{bmatrix}$$

we get :

$$\widehat{x}_1 = (1.55263, 0.1744, 0.5227), \widehat{x}_2 = (1.3971, 0.1726, 0.2125)$$

where

$$\|A^{-1}M\|_\infty = 0.1694, \|A^{-1}N\|_\infty = 0.2211$$

Example 3.2.4 *Consider the following fuzzy linear system:*

$$(0.64, .015, 0.029) \otimes (x_1, y_1, z_1) \oplus (0.13, 0.021, 0.012) \otimes (x_2, y_2, z_2) = (0.58, 0.25, 0.39)$$

$$(0.59, 0.025, 0.015) \otimes (x_1, y_1, z_1) \oplus (0.22, 0.012, 0.013) \otimes (x_2, y_2, z_2) = (0.75, 0.31, 0.45)$$

A= $\begin{bmatrix} 0.64 & 0.13 \\ 0.59 & 0.22 \end{bmatrix}$ M= $\begin{bmatrix} 0.015 & 0.021 \\ 0.025 & 0.012 \end{bmatrix}$ N= $\begin{bmatrix} 0.029 & 0.012 \\ 0.015 & 0.013 \end{bmatrix}$

b= $\begin{bmatrix} 0.58 \\ 0.75 \end{bmatrix}$ h= $\begin{bmatrix} 0.25 \\ 0.31 \end{bmatrix}$ k= $\begin{bmatrix} 0.39 \\ 0.45 \end{bmatrix}$

we get :

$$\widehat{x}_1 = (0.4696, 0.1673, 0.3252), \widehat{x}_2 = (2.1498, 0.8555, 0.9367)$$

where

$$\|A^{-1}M\|_\infty = 0.1850, \|A^{-1}N\|_\infty = 0.1365$$

Example 3.2.5 *Consider the following fuzzy linear system:*

$$(0.84, .0025, 0.039) \otimes (x_1, y_1, z_1) \oplus (0.13, 0.021, 0.012) \otimes (x_2, y_2, z_2) = (0.88, 0.25, 0.49)$$

$$(0.29, 0.015, 0.005) \otimes (x_1, y_1, z_1) \oplus (0.22, 0.012, 0.013) \otimes (x_2, y_2, z_2) = (0.75, 0.21, 0.45)$$

A= $\begin{bmatrix} 0.84 & 0.13 \\ 0.29 & 0.22 \end{bmatrix}$ M= $\begin{bmatrix} 0.025 & 0.021 \\ 0.015 & 0.012 \end{bmatrix}$ N= $\begin{bmatrix} 0.039 & 0.012 \\ 0.005 & 0.013 \end{bmatrix}$

b= $\begin{bmatrix} 0.88 \\ 0.75 \end{bmatrix}$ h= $\begin{bmatrix} 0.25 \\ 0.21 \end{bmatrix}$ k= $\begin{bmatrix} 0.49 \\ 0.45 \end{bmatrix}$

we get :

$$\widehat{x}_1 = (0.6533, 0.1360, 0.2601), \widehat{x}_2 = (2.5479, 0.6356, 1.4458)$$

where

$$\|A^{-1}M\|_\infty = 0.0635, \|A^{-1}N\|_\infty = 0.0989$$

Chapter 4

Trapezoidal Fully Fuzzy Linear systems

In this chapter, fully fuzzy linear systems with trapezoidal fuzzy numbers are considered.

4.1 Arithmetic Operations on Trapezoidal Fuzzy Numbers

Definition 4.1.1 *Let $m \leq n$ be real numbers and γ, β are positive numbers. A non-zero fuzzy number is a trapezoidal fuzzy number denoted by $\hat{A} = (m, n, \gamma, \beta)$ is a fuzzy number whose membership function is given by*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x \leq m - \gamma \\ 1 - \frac{m-x}{\gamma} & m - \gamma \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{x-n}{\beta} & n \leq x \leq n + \beta \\ 0 & x \geq n + \beta \end{cases} \tag{4.1}$$

Definition 4.1.2 *$\hat{A} = (m, n, \gamma, \beta)$ is positive if and only if $m - \gamma > 0$.*

Definition 4.1.3 *Let $\widehat{A} = (m, n, \alpha_1, \beta_1)$, $\widehat{B} = (p, q, \alpha_2, \beta_2)$ be two trapezoidal fuzzy numbers if $\hat{A}$ is identically equal to B only if $m = p$ $n = q, \alpha_1 = \alpha_2, \beta_1 = \beta_2$.*

Using definition (4.1.1) and α-cuts we can define the addition and multiplication of two trapezoidal fuzzy numbers as follows: Now, let $\widehat{A} = (m, n, \alpha_1, \beta_1)$, $\widehat{B} = (p, q, \alpha_2, \beta_2)$ then $A_\alpha = [\alpha_1(\alpha - 1) + m, \beta_1(1 - \alpha) + n]$, and $B_\alpha = [\alpha_2(\alpha - 1) + p, \beta_2(1 - \alpha) + q]$.

When $\alpha = 0, A_0 = [m - \alpha_1, n + \beta_1]$ and $B_0 = [p - \alpha_2, q + \beta_2]$. So
$A_0 + B_0 = [(m - \alpha_1) + (p - \alpha_2), (n + \beta_1) + (q + \beta_2)] = [m + p - (\alpha_1 + \alpha_2), n + q + (\beta_1 + \beta_2)]$
When $\alpha = 1$, $A_1 = [m, n], B_1 = [p, q]$ and $A_1 + B_1 = [m + p, n + q]$. Therefore,

$$\hat{A} \oplus \hat{B} = (m + p, n + q, \alpha_1 + \alpha_2, \beta_1 + \beta_2) \tag{4.2}$$

Similarly, for $\alpha = 0$,

$$\begin{aligned} A_0 \times B_0 &= [(m - \alpha_1)(p - \alpha_2), (n + \beta_1)(q + \beta_2)] \\ &= [(mp - m\alpha_2 - \alpha_1 p + \alpha_1\alpha_2), (nq + n\beta_2 + \beta_1 q + \beta_1\beta_2)] \\ &= [mp - (m\alpha_2 + \alpha_1 p - \alpha_1\alpha_2), nq + (n\beta_2 + q\beta_1 + \beta_1\beta_2)]. \end{aligned}$$

Hence, the multiplication of two positive fuzzy numbers can be

$$\hat{A} \otimes \hat{B} = (mp, nq, m\alpha_2 + \alpha_1 p - \alpha_1\alpha_2, n\beta_2 + \beta_1 q + \beta_1\beta_2) \tag{4.3}$$

Definition 4.1.4 *Let $\hat{A} = (m, n, \alpha_1, \beta_1)$ be a trapezoidal fuzzy number, the scalar multiplication is defined as follows*

$$c \times \hat{A} = \begin{cases} (cm, cn, c\alpha, c\beta), & c \geq 0 \\ (cn, cm, -c\beta, -c\alpha), & c < 0 \end{cases} \tag{4.4}$$

Definition 4.1.5 *A trapezoidal fuzzy number $\hat{A} = (m, n, \alpha_1, \beta_1)$ is identically crisp zero number if $m = n = \alpha_1 = \beta_1 = 0$*

4.2 Linear Systems with Trapezoidal Fuzzy Numbers

In this section, we deal with fully fuzzy linear system where the associated linear system have a square invertible matrix. Consider the fuzzy linear system $\mathbb{A}\otimes\mathbb{X} = \mathbb{B}$ such that $\mathbb{A} = (\widehat{a_{ij}})_{4n\times 4n}$ and $\mathbb{B} = (\widehat{b}_1, ..., \widehat{b}_n)^T$ is a trapezoidal fuzzy number, and $\mathbb{X} = (\widehat{x}_1, ..., \widehat{x}_n)^T$ is unknown variable. Let $\widehat{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij})^T$ and $\widehat{x}_j = (x_j, y_j, z_j, w_j)^T$. Then for each i, we have

$$(\widehat{a}_{ij} \otimes \widehat{x}_j) \oplus ... \oplus (\widehat{a}_{in} \otimes \widehat{x}_n) = \widehat{b}_i, \tag{4.5}$$

where

$$\widehat{b}_i = (b_i, g_i, h_i, k_i)^T$$

where

$$\widehat{a}_{ij} \otimes \widehat{x}_j = (a_{ij}x_j, b_{ij}y_j, a_{ij}z_j + \alpha_{ij}x_j - \alpha_{ij}z_j, b_{ij}w_j + \beta_{ij}y_j + \beta_{ij}w_j).$$

From equation (4.5) we have the following crisp linear equations for each i

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij}x_j &= b_i \\
\sum_{j=1}^{n} b_{ij}y_j &= g_i \\
\sum_{j=1}^{n} (a_{ij}z_j + \alpha_{ij}x_j - \alpha_{ij}z_j) &= h_i \\
\sum_{j=1}^{n} (b_{ij}w_j + \beta_{ij}y_j + \beta_{ij}w_j) &= k_i
\end{aligned} \tag{4.6}$$

. This will lead to the following algebraic systems of equations

$$\begin{aligned} Ax &= b \\ By &= g \\ Az + Mx - Mz &= h \\ Bw + Ny + Nw &= k \end{aligned} \tag{4.7}$$

where $A = (a_{ij}), B = (b_{ij}), M = (\alpha_{ij}), N = (\beta_{ij})$ are square matrices of size n. Further, $x = (x_1, ..., x_n)^T, y = (y_1, ..., y_n)^T, z = (z_1, ..., z_n)^T, w = (w_1, ...w_n)^T, b = (b_1, ..., b_n)^T, g = (g_1, ..., g_n)^T, h = (h_1, ..., h_n)^T$and $k = (k_1, ..., k_n)^T$ are column vectors. The associated block representation system (3.7) is

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A-M & 0 \\ 0 & N & 0 & B+N \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \\ k \end{bmatrix}$$

Definition 4.2.1 *The linear system* $SX = C$ *where* $X = (x, y, z, w)^T$, $C = (b, h, g, k)^T$ *and*

$$S = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A-M & 0 \\ 0 & N & 0 & B+N \end{bmatrix}$$

The associated linear system $SX = C$ and the fuzzy system $\mathbb{A} \otimes \mathbb{X} = \mathbb{B}$ are equivalent. In other words, to solve the fuzzy linear system we need to solve the associated linear system.

In this chapter as we have seen in the previous one, the main purpose is to deal with we two cases concerning the associated matrix S which are:

- **The matrices $A, B, A-M$ and $B+N$ are invertible,**
- **The matrices A and B are invertible such that $\|A^{-1}M\|_\infty$ and $\|B^{-1}N\|_\infty$ are less than 1.**

For the first case, we assume that the associated matrix has a non-singular matrices on the main diagonal.

Remark 4.2.1 *The block matrix S in is invertible if and only if the matrix A, B, $A-M$ and $B+N$ are invertible.*

Proof: Let $S = \begin{bmatrix} H & 0 \\ L & F \end{bmatrix}$, where $H = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ and $F = \begin{bmatrix} A-M & 0 \\ 0 & B+N \end{bmatrix}$
then

$$\begin{aligned} |S| &= |H|\,|F| \\ &= |A|\,|B|\,|A-M|\,|B+N| \end{aligned}$$

□

Theorem 4.2.1 *If A, B, $A-M$ and $B+N$ are invertible matrices, then the unique solution of the associated system is given by:*

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ (A-M)^{-1}(h - MA^{-1}b) \\ (B+N)^{-1}(k - NB^{-1}g) \end{bmatrix} \tag{4.8}$$

Proof: Since A, B, $A-M$ and $B+N$ are invertible matrices, the inverse of the matrix S can be obtained by elementary row operations as follows:

$$S^{-1} = \begin{bmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 \\ -A^{-1}(A-M)^{-1}M & 0 & (A-M)^{-1} & 0 \\ 0 & -B^{-1}(B+N)^{-1}N & 0 & (B+N)^{-1} \end{bmatrix}$$

□

Example 4.2.1 *Consider the following system*

$$(6.2, 7.0, 0.24, 0.81) \otimes (x_1, y_1, z_1, w_1) \oplus (2.5, 2.6, 0.88, 1.21) \otimes (x_2, y_2, z_2, w_2) = (4.2, 7.2, 1.8, 5.64)$$
$$(2.5, 2.7, 0.18, 0.55) \otimes (x_1, y_1, z_1, w_1) \oplus (3.1, 3.3, 1.21, 0.19) \otimes (x_2, y_2, z_2, w_2) = (3.3, 6.1, 1.51, 2.5).$$

Then $A = \begin{bmatrix} 6.2 & 2.5 \\ 2.5 & 3.1 \end{bmatrix}$ $B = \begin{bmatrix} 7.0 & 2.6 \\ 2.7 & 3.3 \end{bmatrix}$ $M = \begin{bmatrix} 0.24 & 0.88 \\ 0.18 & 1.21 \end{bmatrix}$ $N = \begin{bmatrix} 0.81 & 1.21 \\ 0.55 & 0.19 \end{bmatrix}$
$b = \begin{bmatrix} 4.2 \\ 3.3 \end{bmatrix}$ $g = \begin{bmatrix} 7.2 \\ 6.1 \end{bmatrix}$ $h = \begin{bmatrix} 1.8 \\ 1.51 \end{bmatrix}$ $k = \begin{bmatrix} 5.64 \\ 2.5 \end{bmatrix}$. From (4.8), we get
$\widehat{x}_1 = (0.3678, 0.4913, 0.1498, 0.3185)$ and $\widehat{x}_2 = (0.7679, 1.4465, 0.0884, 0.2635)$

Corollary 4.2.1 *Let A and B be two invertible matrices such that $\|A^{-1}M\|_\infty$ and $\|B^{-1}N\|_\infty$ are less than 1. Then the associated system $SX = C$ has a unique solution.*

Proof: Since T and B are invertible, the associated system can be reduced as

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ T^{-1}M & 0 & I - T^{-1}M & 0 \\ 0 & B^{-1}N & 0 & I + B^{-1}N \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} T^{-1}b \\ B^{-1}g \\ T^{-1}h \\ B^{-1}k \end{bmatrix}.$$

Therefore, $x = T^{-1}b$, $y = B^{-1}g$ and since $\|A^{-1}M\|_\infty$ and $\|B^{-1}N\|_\infty$ are less than

1, we have

$$z = \left(I - T^{-1}M\right)^{-1}\left(T^{-1}h - T^{-1}MT^{-1}b\right) \quad (4.9)$$

$$w = \left(I + B^{-1}N\right)^{-1}\left(B^{-1}k - B^{-1}NB^{-1}g\right) \quad (4.10)$$

□

Example 4.2.2

$$(0.52, 0.42, 0.051, 0.21) \otimes \widehat{x}_1 \oplus (0.25, 0.25, 0.18, 0.19) \otimes \widehat{x}_2 = (0.52, 0.82, 0.32, 0.564)$$
$$(0.15, 0.26, 0.018, 0.22) \otimes \widehat{x}_1 \oplus (0.29, 0.33, 0.021, 0.19) \otimes \widehat{x}_2 = (0.33, 0.81, 0.48, 0.555)$$

$$A = \begin{bmatrix} 0.52 & 0.25 \\ 0.15 & 0.29 \end{bmatrix} \quad B = \begin{bmatrix} 0.42 & 0.25 \\ 0.26 & 0.33 \end{bmatrix} \quad M = \begin{bmatrix} 0.051 & 0.18 \\ 0.018 & 0.021 \end{bmatrix} \quad N = \begin{bmatrix} 0.21 & 0.19 \\ 0.22 & 0.19 \end{bmatrix}$$
$$b = \begin{bmatrix} 0.52 \\ 0.33 \end{bmatrix} \quad g = \begin{bmatrix} 0.82 \\ 0.81 \end{bmatrix} \quad h = \begin{bmatrix} 0.32 \\ 0.48 \end{bmatrix} \quad k = \begin{bmatrix} 0.564 \\ 0.555 \end{bmatrix}$$

we get $\widehat{x}_1 = (0.6028, 0.9253, 0.0529, 0.0978), \widehat{x}_2 = (0.8261, 1.7255, 1.6536, -0.0449)$, where$\|A^{-1}M\|_\infty = 0.5052$ and $\|B^{-1}N\|_\infty = 0.9266$.

Example 4.2.3

$$(0.52, 0.42, 0.024, 0.081) \otimes \widehat{x}_1 \oplus (0.25, 0.35, 0.088, 0.121) \otimes \widehat{x}_2 = ((0.42, 0.72, 0.18, 0.45)$$
$$(0.25, 0.27, 0.018, 0.055) \otimes \widehat{x}_1 \oplus (0.29, 0.33, 0.121, 0.019) \otimes \widehat{x}_2 = (0.33, 0.61, 0.151, 0.32)$$

$$A = \begin{bmatrix} 0.52 & 0.25 \\ 0.25 & 0.29 \end{bmatrix} \quad B = \begin{bmatrix} 0.42 & 0.35 \\ 0.27 & 0.33 \end{bmatrix} \quad M = \begin{bmatrix} 0.024 & 0.088 \\ 0.018 & 0.121 \end{bmatrix} \quad N = \begin{bmatrix} 0.081 & 0.121 \\ 0.055 & 0.019 \end{bmatrix}$$
$$b = \begin{bmatrix} 0.42 \\ 0.33 \end{bmatrix} \quad g = \begin{bmatrix} 0.72 \\ 0.61 \end{bmatrix} \quad h = \begin{bmatrix} 0.18 \\ 0.151 \end{bmatrix} \quad k = \begin{bmatrix} 0.45 \\ 0.32 \end{bmatrix}$$

we get: $\widehat{x}_1 = (0.4451, 0.5465, 0.1950, -1.9105), \widehat{x}_2 = (0.7542, 1.4014, 0.0383, 2.5336)$

where $\|A^{-1}M\|_{\infty}= 0.5015$ and $\|B^{-1}N\|_{\infty}= 0.9243$.

Chapter 5

Hexagonal Fully Fuzzy Linear Systems

In this chapter, fully fuzzy linear systems with hexagonal fuzzy numbers are considered.

5.1 Arithmetic Operations on Hexagonal Fuzzy Numbers

Definition 5.1.1 *Let $m \leq n$ be real numbers and $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2$ be positive numbers. A non-zero fuzzy number is a hexagonal fuzzy number denoted by $\widehat{A} = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ whose membership function is given by:*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < m - \alpha_1 \\ \frac{x-m+\alpha_1}{2(\alpha_1-\alpha_2)}, & m - \alpha_1 \leq x \leq m - \alpha_2 \\ 1 + \frac{x-m}{2(\alpha_2)}, & m - a_2 \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{x-n}{2(\beta_2)}, & n \leq x \leq n + \beta_2 \\ -\frac{x-n-\beta_1}{2(\beta_1-\beta_2)}, & n + \beta_2 \leq x \leq n + \beta_1 \\ 0 & x \geq n + \beta_1 \end{cases} \tag{5.1}$$

Definition 5.1.2 *A positive hexagonal fuzzy number denoted as $\widehat{A} = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ where $m \leq n, \alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2$ and $m - \alpha_1 \geq 0$.*

Definition 5.1.3 *let* $\widehat{A} = (m, n, \alpha_{11}, \beta_{11}, \alpha_{12}, \beta_{12}), \widehat{B} = (p, q, \alpha_{21}, \beta_{21}, \alpha_{22}, \beta_{22})$ *be two hexagonal fuzzy number if* $\hat{A}_H$ *is identically equal to* B_H *only if* $m = p$ $n = q, \alpha_{11} = \alpha_{21}, \beta_{11} = \beta_{21}, \alpha_{12} = \alpha_{22}, \beta_{12} = \beta_{22}$.

Using $\alpha - cut$ arithmetic and the definition of the hexagonal fuzzy number as shown, the $\alpha - cuts$ of the fuzzy numbers $\widehat{A} = (m, n, \alpha_{11}, \beta_{11}, \alpha_{12}, \beta_{12})$ and $\widehat{B} = (p, q, \alpha_{21}, \beta_{21}, \alpha_{22}, \beta_{22})$ can be defined as

$$A_\alpha = \begin{cases} [2\alpha(\alpha_{11} - \alpha_{12}) + m - \alpha_{11}, -2\alpha(\beta_{11} - \beta_{12}) + n + \beta_{11}], & 0 \le \alpha \le 0.5 \\ [2\alpha_{12}(\alpha - 1) + m, 2\beta_{12}(1 - \alpha) + n], & 0.5 \le \alpha \le 1 \end{cases}$$

$$B_\alpha = \begin{cases} [2\alpha(\alpha_{21} - \alpha_{22}) + p - \alpha_{21}, -2\alpha(\beta_{21} - \beta_{22}) + q + \beta_{21}], & 0 \le \alpha \le 0.5 \\ [2\alpha_{22}(\alpha - 1) + p, 2\beta_{22}(1 - \alpha) + q], & 0.5 \le \alpha \le 1 \end{cases}$$

For $\alpha = 0$, $A_0 = [m - \alpha_{11}, n + \beta_{11}]$, $B_0 = [p - \alpha_{21}, q + \beta_{21}]$,

$$\begin{aligned} A_0 + B_0 &= [(m - \alpha_{11}) + (p - \alpha_{21}), (n + \beta_{11}) + (q + \beta_{21})] \\ &= [m + p - (\alpha_{11} + \alpha_{21}), n + q + (\beta_{11} + \beta_{21})]. \end{aligned}$$

For $\alpha = 0.5$, $A_{0.5} = [m - \alpha_{12}, n + \beta_{12}]$, $B_{0.5} = [p - \alpha_{22}, q + \beta_{22}]$ then

$$\begin{aligned} A_{0.5} + B_{0.5} &= [(m - \alpha_{12}) + (p - \alpha_{22}), (n + \beta_{12}) + (q + \beta_{22})] \\ &= [(m + p - (\alpha_{12} + \alpha_{22})), (n + q + (\beta_{12} + \beta_{22}))]. \end{aligned}$$

For $\alpha = 1$, $A_1 = [m, n]$ $B_1 = [p, q]$, then $A_1 + B_1 = [m + p, n + q]$. Therefore,

$$\widehat{A} \oplus \widehat{B} = (m + p, n + q, \alpha_{11} + \alpha_{21}, \beta_{11} + \beta_{21}, \alpha_{12} + \alpha_{22}, \beta_{12} + \beta_{22}). \tag{5.2}$$

Similarly, we can deduce the product of two hexagonal numbers as follows:

for $\alpha = 0$,

$$
\begin{aligned}
A_0 \times B_0 &= [(m-\alpha_{11}) \times (p-\alpha_{21}), (n+\beta_{11}) \times (q+\beta_{21})] \\
&= [mp - m\alpha_{21} - \alpha_{11}p + \alpha_{11}\alpha_{21}, nq + n\beta_{21} + \beta_{11}q + \beta_{11}\beta_{21}].
\end{aligned}
$$

For $\alpha = 0.5$,

$$
\begin{aligned}
A_{0.5} \times B_{0.5} &= [(m-\alpha_{12}) \times (p-\alpha_{22}), (n+\beta_{12}) \times (q+\beta_{22})] \\
&= [mp - m\alpha_{22} - \alpha_{12}p + \alpha_{12}\alpha_{22}, nq + n\beta_{22} + \beta_{12}q + \beta_{12}\beta_{22}].
\end{aligned}
$$

For $\alpha = 1$,$A_1 \times B_1 = [mp, nq]$. Therefore,the multiplication of two positive fuzzy numbers is given by:

$$
\begin{aligned}
\widehat{A} \otimes \widehat{B} = (mp, nq, m\alpha_{21} + p\alpha_{11} - \alpha_{11}\alpha_{21}, n\beta_{21} + \beta_{11}q + \beta_{11}\beta_{21}, \\
m\alpha_{22} + \alpha_{12}p - \alpha_{12}\alpha_{22}, n\beta_{22} + \beta_{12}q + \beta_{12}\beta_{22})
\end{aligned}
\tag{5.3}
$$

5.2 Linear Systems with Hexagonal Fuzzy Numbers

In this section, we consider fully fuzzy linear systems of hexagonal type $\mathbb{A} \otimes \mathbb{X} = \mathbb{B}$ where the associated linear system have a square matrix, such that $\widehat{A} = (\widehat{a_{ij}})_{6n\times 6n}$ and $\widehat{B} = (\widehat{b}_1, ..., \widehat{b}_n)^T$ is a hexagonal fuzzy number, and $\widehat{X} = (\widehat{x}_1, ..., \widehat{x}_n)^T$ is unknown variable. Let $\widehat{a}_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, \delta_{ij}, \zeta_{ij})^T$,$\widehat{b}_j = (b_j, g_j, h_j, k_j, l_j, m_j)^T$ and $\widehat{x}_j = (x_j, y_j, z_j, w_j, o_j, p_j)^T$. Then for each i, we have

$$
(\widehat{a}_{ij} \otimes \widehat{x}_j) \oplus ... \oplus (\widehat{a}_{in} \otimes \widehat{x}_n) = \widehat{b}_i, \tag{5.4}
$$

where

$$\widehat{a}_{ij} \otimes \widehat{x}_j = (a_{ij}x_j, b_{ij}y_j, a_{ij}z_j + \alpha_{ij}x_j - \alpha_{ij}z_j, b_{ij}w_j + \beta_{ij}y_j + \beta_{ij}w_j,$$
$$a_{ij}o_j - \delta_{ij}o_j + \delta_{ij}x_j, b_{ij}p_j + \zeta_{ij}y_j + \zeta_{ij}p_j).$$

From equation (5.4), we have the following crisp linear equations for each i

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij}x_j &= b_i \\
\sum_{j=1}^{n} b_{ij}y_j &= g_i \\
\sum_{j=1}^{n} a_{ij}z_j + \alpha_{ij}x_j - \alpha_{ij}z_j &= h_i \\
\sum_{j=1}^{n} b_{ij}w_j + \beta_{ij}y_j + \beta_{ij}w_j &= k_i \\
\sum_{j=1}^{n} a_{ij}o_j - \delta_{ij}o_j + \delta_{ij}x_j &= l_i \\
\sum_{j=1}^{n} b_{ij}p_j + \zeta_{ij}y_j + \zeta_{ij}p_j &= m_i. \qquad (5.5)
\end{aligned}$$

This will lead to the following algebraic systems of equations

$$\begin{aligned}
Ax &= b \\
By &= g \\
Az + Mx - Mz &= h \\
Bw + Ny + Nw &= k \\
Ao + Cx - Co &= l \\
Bp + Dy + Dp &= m \qquad (5.6)
\end{aligned}$$

where $A = (a_{ij})$, $B = (b_{ij})$, $M = (\alpha_{ij})$, $N = (\beta_{ij})$, $C = (\delta_{ij})$ and $D = (\zeta_{ij})$ are square matrices of size n. Further, $x = (x_1, ..., x_n)^T, y = (y_1, ..., y_n)^T, z =$

$(z_1, ..., z_n)^T, w = (w_1, ...w_n)^T, b = (b_1, ..., b_n)^T, g = (g_1, ..., g_n)^T, h = (h_1, ..., h_n)^T$, $k = (k_1, ..., k_n)^T$, $l = (l_1, ..., l_n)^T$ and $m = (m_1, ..., m_n)^T$ are column vectors. The associated block representation system is

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ M & 0 & A-M & 0 & 0 & 0 \\ 0 & N & 0 & B+N & 0 & 0 \\ C & 0 & 0 & 0 & A-C & 0 \\ 0 & D & 0 & 0 & 0 & B+D \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ o \\ p \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \\ k \\ l \\ m \end{bmatrix}$$

The associated matrix can be written in block form as $S = \begin{bmatrix} S_{11} & 0 & 0 \\ S_{21} & S_{22} & 0 \\ S_{31} & 0 & S_{33} \end{bmatrix}$,

where $S_{11} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $S_{21} = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$, $S_{22} = \begin{bmatrix} A-M & 0 \\ 0 & B+N \end{bmatrix}$, $S_{31} = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ and $S_{33} = \begin{bmatrix} A-C & 0 \\ 0 & B+D \end{bmatrix}$.

Remark 5.2.1 *The matrix S is invertible if and only if S_{11}, S_{22} and S_{33} are invertible matrices. Further, the inverse of S is*

$$\begin{bmatrix} A^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 & 0 & 0 \\ -MA^{-1}(A-M)^{-1} & 0 & (A-M)^{-1} & 0 & 0 & 0 \\ 0 & -NB^{-1}(B+N)^{-1} & 0 & (B+N)^{-1} & 0 & 0 \\ -CA^{-1}(A-C)^{-1} & 0 & 0 & 0 & (A-C)^{-1} & 0 \\ 0 & -DB^{-1}(B+D)^{-1} & 0 & 0 & 0 & (B+D)^{-1} \end{bmatrix}$$

Corollary 5.2.1 *If the matrix S is invertible, the unique solution of the associ-*

ated system $SX = C$ is given by:

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ (A-M)^{-1}(h - MA^{-1}b) \\ (B+N)^{-1}(k - NB^{-1}g) \\ (A-C)^{-1}(l - CA^{-1}b) \\ (B+D)^{-1}(m - DB^{-1}g) \end{bmatrix}$$

Example 5.2.1 *Consider the system $\mathbb{A} \otimes \mathbb{X} = \mathbb{B}$ where*

$$\mathbb{A} = \begin{bmatrix} (0.1, 0.12, 0.05, 0.04, 0.04, 0.03) & (0.07, 0.08, 0.04, 0.05, 0.02, 0.01) \\ (0.06, 0.09, 0.03, 0.03, 0.01, 0.02) & (0.09, 0.11, 0.03, 0.03, 0.03, 0.02) \end{bmatrix},$$

$$\mathbb{B} = \begin{bmatrix} (1.22, 1.8, 1.12, 1.89, 0.76, 1.16) \\ (1.02, 1.07, 0.9, 1.59, 0.53, 1.22) \end{bmatrix}.$$

Then, $A = \begin{bmatrix} 0.1 & 0.07 \\ 0.06 & 0.09 \end{bmatrix}$ $B = \begin{bmatrix} 0.12 & 0.08 \\ 0.09 & 0.11 \end{bmatrix}$ $M = \begin{bmatrix} 0.05 & 0.04 \\ 0.03 & 0.03 \end{bmatrix}$ $N = \begin{bmatrix} 0.04 & 0.05 \\ 0.03 & 0.03 \end{bmatrix}$ $C = \begin{bmatrix} 0.04 & 0.02 \\ 0.01 & 0.03 \end{bmatrix}$ $D = \begin{bmatrix} 0.03 & 0.01 \\ 0.02 & 0.02 \end{bmatrix}$ $b = \begin{bmatrix} 1.22 \\ 1.02 \end{bmatrix}$ $g = \begin{bmatrix} 1.8 \\ 1.07 \end{bmatrix}$ $h = \begin{bmatrix} 1.12 \\ 0.9 \end{bmatrix}$ $k = \begin{bmatrix} 1.89 \\ 1.59 \end{bmatrix}$ $l = \begin{bmatrix} 0.76 \\ 0.53 \end{bmatrix}$ $m = \begin{bmatrix} 1.16 \\ 1.22 \end{bmatrix}$ and $|A| = 0.0048,\ |B| = 0.006,\ |A - M| = 0.0021,\ |A - C| = 0.0011,\ |B + N| = 0.0068$ and $|B + D| = 0.0096$. From (5.2.1) we get: $\widehat{x}_1 = (8, 10.2, 6.8571, 2.6824, 5.1818, 2.4146)$ and $\widehat{x}_2 = (6, 7.2, 4.5714, 5.3294, 0.1818, 4.6646)$

Example 5.2.2 *Let* $A = \begin{bmatrix} 0.3 & 0.05 \\ 0.06 & 0.09 \end{bmatrix}$, $B = \begin{bmatrix} 0.4 & 0.08 \\ 0.09 & 0.10 \end{bmatrix}$, $M = \begin{bmatrix} 0.05 & 0.04 \\ 0.03 & 0.03 \end{bmatrix}$, $N = \begin{bmatrix} 0.04 & 0.05 \\ 0.03 & 0.03 \end{bmatrix}$, $C = \begin{bmatrix} 0.04 & 0.02 \\ 0.01 & 0.02 \end{bmatrix}$, $D = \begin{bmatrix} 0.03 & 0.01 \\ 0.02 & 0.02 \end{bmatrix}$, $b = \begin{bmatrix} 2.22 \\ 1 \end{bmatrix}$, $g =$

$$\begin{bmatrix} 3.5 \\ 1.77 \end{bmatrix}, h = \begin{bmatrix} 1.12 \\ 0.7 \end{bmatrix} k = \begin{bmatrix} 1.99 \\ 1.59 \end{bmatrix}, l = \begin{bmatrix} 0.8 \\ 0.33 \end{bmatrix} \text{ and } m = \begin{bmatrix} 1 \\ 1.22 \end{bmatrix}.$$

Then $|A| = 0.024$, $|B| = 0.0328$, $|A - M| = 0.0147$, $|A - C| = 0.0167$, $|B + N| = 0.0416$ and $|B + D| = 0.0417$. From (5.2.1), we get $\widehat{x}_1 = (6.2417, 6.3537, 1.9560, 0.3026, 1.4932, 0.1427)$ $\widehat{x}_2 = (6.9500, 11.9817, 4.0929, 7.7202, 0.7704, 6.9799)$

Corollary 5.2.2 *Let A and B be two invertible matrices such that $\|A^{-1}M\|_\infty$, $\|A^{-1}C\|_\infty$ $\|B^{-1}N\|_\infty$ and $\|B^{-1}D\|_\infty$ are less than 1. Then the associated system $SX = C$ has a unique solution.*

Example 5.2.3 *Consider the system:*

$(0.25, 0.4, 0.15, 0.04, 0.04, 0.01) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.15, 0.20, 0.09, 0.05, 0.08, 0.002) \otimes (x_2, y_2, z_2, w_2, o_2, p_2) = (2, 4.5, 1.6, 0.99, 0.9, 0.077)$

$(0.06, 0.09, 0.03, 0.03, 0.001, 0.029) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.09, 0.10, 0.03, 0.03, 0.02, 0.018) \otimes (x_2, y_2, z_2, w_2, o_2 p_2) = (1, 1.8, 0.7, 0.6, 0.23, 0.38)$

$$A = \begin{bmatrix} 0.25 & 0.15 \\ 0.06 & 0.09 \end{bmatrix}, B = \begin{bmatrix} 0.4 & 0.20 \\ 0.09 & 0.10 \end{bmatrix}, M = \begin{bmatrix} 0.15 & 0.09 \\ 0.03 & 0.03 \end{bmatrix}, N = \begin{bmatrix} 0.04 & 0.05 \\ 0.03 & 0.03 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.04 & 0.08 \\ 0.001 & 0.02 \end{bmatrix}, D = \begin{bmatrix} 0.01 & 0.002 \\ 0.029 & 0.018 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, g = \begin{bmatrix} 4.5 \\ 1.8 \end{bmatrix}, h = \begin{bmatrix} 1.6 \\ 0.7 \end{bmatrix},$$

$$k = \begin{bmatrix} 0.99 \\ 0.6 \end{bmatrix}, l = \begin{bmatrix} 0.9 \\ 0.23 \end{bmatrix} \text{ and } m = \begin{bmatrix} 0.077 \\ 0.38 \end{bmatrix}$$

Then $\|A^{-1}M\|_\infty = 0.9333$, $\|B^{-1}N\|_\infty = 0.7227$, $\|A^{-1}C\|_\infty = 0.5667$, and $\|B^{-1}D\|_\infty = 0.8055$.

From (5.2.2), we get : $\widehat{x}_1 = (2.2222, 4.0909, 0.7937, 0.0892, 0.0368, 0.0060)$.

$\widehat{x}_2 = (9.6296, 14.3182, 5.3439, 0.2848, 0.4716, 0.0248)$

Example 5.2.4 *Consider the system:*

$(0.05, 0.08, 0.03, 0.03, 0.01, 0.005) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.02, 0.04, 0.01, 0.01, 0.01, 0.001) \otimes$

$(x_2, y_2, z_2, w_2, o_2, p_2) = (0.335, 0.6, 0.22, 0.49, 0.1, 0.07)$

$(0.02, 0.04, 0.01, 0.02, 0.01, 0.05) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.7, 0.8, 0.2, 0.2, 0.1, 0.002) \otimes$

$(x_2, y_2, z_2, w_2, o_2, p_2) = (0.77, 1.3, 0.29, 0.65, 0.166, 0.44)$

$$A = \begin{bmatrix} 0.05 & 0.02 \\ 0.02 & 0.7 \end{bmatrix}, B = \begin{bmatrix} 0.08 & 0.04 \\ 0.04 & 0.8 \end{bmatrix}, M = \begin{bmatrix} 0.03 & 0.01 \\ 0.01 & 0.2 \end{bmatrix}, N = \begin{bmatrix} 0.03 & 0.01 \\ 0.02 & 0.2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.1 \end{bmatrix}, D = \begin{bmatrix} 0.005 & 0.001 \\ 0.05 & 0.002 \end{bmatrix}, b = \begin{bmatrix} 0.335 \\ 0.77 \end{bmatrix}, g = \begin{bmatrix} 0.6 \\ 1.3 \end{bmatrix},$$

$$h = \begin{bmatrix} 0.22 \\ 0.29 \end{bmatrix}, k = \begin{bmatrix} 0.49 \\ 0.65 \end{bmatrix}, l = \begin{bmatrix} 0.1 \\ 0.166 \end{bmatrix} \text{ and } m = \begin{bmatrix} 0.07 \\ 0.44 \end{bmatrix}$$

we get: $\widehat{x}_1 = (6.3324, 6.8590, 1.0091, 2.4167, 0.6855, 0.3681)$,

$\widehat{x}_2 = (0.9191, 1.2821, 0.0655, 0.1114, 0.0065, 0.0765)$ Where $\|A^{-1}M\|_\infty = 0.6879$, $\|B^{-1}N\|_\infty =$ 0.3718, $\|A^{-1}C\|_\infty = 0.3410$, and $\|B^{-1}D\|_\infty = 0.0628$

Example 5.2.5 *Consider the system:*

$(0.25, 0.4, 0.15, 0.04, 0.04, 0.01) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.15, 0.20, 0.09, 0.05, 0.08, 0.002) \otimes$

$(x_2, y_2, z_2, w_2, o_2, p_2) = (2, 4.5, 1.6, 0.99, 0.9, 0.77)$

$(0.06, 0.09, 0.03, 0.03, 0.001, 0.029) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.09, 0.10, 0.03, 0.03, 0.02, 0.018) \otimes$

$(x_2, y_2, z_2, w_2, o_2 p_2) = (1, 1.8, 0.7, 0.6, 0.23, 0.38)$

$$A = \begin{bmatrix} 0.25 & 0.15 \\ 0.06 & 0.09 \end{bmatrix}, B = \begin{bmatrix} 0.4 & 0.20 \\ 0.09 & 0.10 \end{bmatrix}, M = \begin{bmatrix} 0.15 & 0.09 \\ 0.03 & 0.03 \end{bmatrix}, N = \begin{bmatrix} 0.04 & 0.05 \\ 0.03 & 0.03 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.04 & 0.08 \\ 0.001 & 0.02 \end{bmatrix}, D = \begin{bmatrix} 0.01 & 0.002 \\ 0.029 & 0.018 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} g = \begin{bmatrix} 4.5 \\ 1.8 \end{bmatrix}, h = \begin{bmatrix} 1.6 \\ 0.7 \end{bmatrix},$$

$$k = \begin{bmatrix} 0.99 \\ 0.6 \end{bmatrix}, l = \begin{bmatrix} 0.9 \\ 0.23 \end{bmatrix}, m = \begin{bmatrix} 0.077 \\ 0.38 \end{bmatrix}$$ $\|A^{-1}M\|_\infty = 0.9333$, $\|B^{-1}N\|_\infty = 0.7227$,

$\|A^{-1}C\|_\infty = 0.5667$, and $\|B^{-1}D\|_\infty = 0.8055$. From (5.2.2), we get : $\widehat{x}_1 = (2.2222, 4.0909, 0.7937, 0.089$

$\widehat{x}_2 = (9.6296, 14.3182, 5.3439, 0.2848, 0.4716, 0.0248)$

Example 5.2.6 *Consider the system:*

$(0.3, 0.4, 0.05, 0.04, 0.04, 0.03) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.05, 0.08, 0.04, 0.05, 0.02, 0.01) \otimes (x_2, y_2, z_2, w_2, o_2, p_2) = (2.22, 3.5, 1.12, 1.99, 0.8, 1)$

$(0.06, 0.09, 0.03, 0.03, 0.02, 0.02) \otimes (x_1, y_1, z_1, w_1, o_1, p_1) \oplus (0.9, 0.10, 0.03, 0.03, 0.02, 0.02) \otimes (x_2, y_2, z_2, w_2, o_2, p_2) = (1, 1.77, 0.7, 1.59, 0.33, 1.22)$

$$A = \begin{bmatrix} 0.3 & 0.05 \\ 0.06 & 0.9 \end{bmatrix}, B = \begin{bmatrix} 0.4 & 0.08 \\ 0.09 & 0.10 \end{bmatrix}, M = \begin{bmatrix} 0.05 & 0.04 \\ 0.03 & 0.03 \end{bmatrix}, N = \begin{bmatrix} 0.04 & 0.05 \\ 0.03 & 0.03 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.04 & 0.02 \\ 0.02 & 0.02 \end{bmatrix}, D = \begin{bmatrix} 0.01 & 0.01 \\ 0.02 & 0.02 \end{bmatrix}, b = \begin{bmatrix} 2.22 \\ 1 \end{bmatrix}, g = \begin{bmatrix} 3.5 \\ 1.77 \end{bmatrix}, h = \begin{bmatrix} 1.12 \\ 0.7 \end{bmatrix},$$

$$k = \begin{bmatrix} 1.99 \\ 1.59 \end{bmatrix}, l = \begin{bmatrix} 0.8 \\ 0.33 \end{bmatrix} \text{ and } m = \begin{bmatrix} 1.0 \\ 1.22 \end{bmatrix}$$

Then$\|A^{-1}M\|_\infty = 0.2921$, $\|B^{-1}N\|_\infty = 0.4848$, $\|A^{-1}C\|_\infty = 0.1948$, and $\|B^{-1}D\|_\infty = 0.3780$

From (5.2.2), we get: $\widehat{x}_1 = (7.2959, 6.3537, 2.9036, 0.3026, 1.8939, 0.1427)$,

$\widehat{x}_2 = (0.6247, 11.9817, 0.4313, 7.7202, 0.1089, 6.9799)$

Conclusions and Future work

Many methods to solve fully fuzzy linear systems use an approximated multiplication operation for two fuzzy numbers when using either the $\alpha - cut$ or extension principle approach. However, the approximated multiplication of two positive trapezoidal fuzzy numbers need not be positive. Moreover, one of the main practical usage of fuzzy numbers is to model inaccurate data. If we have data and its uncertainties best fit with trapezoidal fuzzy numbers, then we would like to keep this (trapezoidal) modeling throughout all computations while fixing/processing data. It does not make sense that we start modeling our data by trapezoidal fuzzy numbers and then at some point we get triangular fuzzy numbers, otherwise we start modeling by triangular fuzzy numbers and save calculations. This deficiency is mainly due to the approximated multiplication. In this thesis, We consider solving fully fuzzy linear systems of equations for triangular, trapezoidal and the hexagonal fuzzy numbers. We have used the multiplication operation for fuzzy numbers without any approximation or any added special conditions to eliminate some terms in the multiplication operation itself. We have proposed a solution mechanism and organized the steps as an algorithm. We have applied the proposed method using several numerical examples in order to test the results.

Many topics can follow the work presented in this thesis. Here, we mention some of the topics that can be considered:

1. Finding a more general approach to address fully fuzzy linear systems where the main idea can be to reduce the amount of uncertainty related to fuzzy numbers.

2. Solving fully fuzzy linear systems with octagonal fuzzy numbers since an octagonal fuzzy number is highly sophisticated and is getting high attention in research these days.

3. Apply the mathematical results presented in this thesis for real life problems in different fields (e.g. economics and physics).

Bibliography

[1] E. Abdolmaleki and S. Edalatpanah, *Chebyshev semi-iterative method to solve fully fuzzy linear systems*, Journal of Information and Computing Science, 9 (2014), pp. 067–074.

[2] T. Allahviranloo, *Numerical methods for fuzzy system of linear equations*, Applied mathematics and computation, 155 (2004), pp. 493–502.

[3] M. Anoop, K. B. Rao, and S. Gopalakrishnan, *Conversion of probabilistic information into fuzzy sets for engineering decision analysis*, Computers & structures, 84 (2006), pp. 141–155.

[4] N. Babbar, A. Kumar, and A. Bansal, *Solving fully fuzzy linear system with arbitrary triangular fuzzy numbers ({m,\ alpha,\ beta})*, Soft Computing, 17 (2013), pp. 691–702.

[5] J. Buckley and Y. Qu, *Solving systems of linear fuzzy equations*, Fuzzy sets and systems, 43 (1991), pp. 33–43.

[6] N. J. K. E. Chandrasekaran, *Solving fully fuzzy linear systems with trapezoidal fuzzy number matrices by singular value decomposition*, Intern. J. Fuzzy Mathematical Archive, 3 (2013), pp. 16–22.

[7] M. Dehghan and B. Hashemi, *Solution of the fully fuzzy linear systems using the decomposition procedure*, Applied Mathematics and Computation, 182 (2006), pp. 1568–1580.

[8] D. Dubois and H. Prade, *Operations on fuzzy numbers*, International Journal of systems science, 9 (1978), pp. 613–626.

[9] ——, *Fuzzy numbers: an overview*, in Readings in Fuzzy Sets for Intelligent Systems, Elsevier, 1993, pp. 112–148.

[10] D. J. Dubois, *Fuzzy sets and systems: theory and applications*, vol. 144, Academic press, 1980.

[11] R. Ezzati and A. Yousefzadeh, *Positive solution of non-square fully fuzzy linear system of equation in general form using least square method*, Journal of Linear and Topological Algebra, 3 (2014), pp. 23–33.

[12] M. Friedman, M. Ming, and A. Kandel, *Fuzzy linear systems*, Fuzzy sets and systems, 96 (1998), pp. 201–209.

[13] A. N. Gani and S. M. Assarudeen, *A new operation on triangular fuzzy number for solving fuzzy linear programming problem*, Applied Mathematical Sciences, 6 (2012), pp. 525–532.

[14] S. Gao, Z. Zhang, and C. Cao, *Multiplication operation on fuzzy numbers.*, JSW, 4 (2009), pp. 331–338.

[15] R. E. Giachetti and R. E. Young, *Analysis of the error in the standard approximation used for multiplication of triangular and trapezoidal fuzzy numbers and*

the development of a new approximation, Fuzzy Sets and Systems, 91 (1997), pp. 1–13.

[16] N. J. Karthik and E. Chandrasekaran, *Solving fully fuzzy linear systems with trapezoidal fuzzy number matrices by partitioning the block matrices*, Annals of Pure and Applied Mathematics, 8 (2014), pp. 261–267.

[17] H. G. Kocken, M. Ahlatcioglu, and I. Albayrak, *Finding the fuzzy solutions of a general fully fuzzy linear equation system*, Journal of Intelligent & Fuzzy Systems, 30 (2016), pp. 921–933.

[18] E. Kreyszig, *Introductory functional analysis with applications*, vol. 1, wiley New York, 1978.

[19] A. Kumar, N. Babbar, and A. Bansal, *A new approach for solving fully fuzzy linear systems*, Advances in Fuzzy Systems, 2011 (2011), p. 5.

[20] A. Kumar, A. Bansal, and N. Babbar, *Solution of fully fuzzy linear system with arbitrary coefficients*, International Journal of Applied Mathematics and Computation, 3 (2011), pp. 232–237.

[21] ——, *Fully fuzzy linear systems of triangular fuzzy numbers (a, b, c)*, International Journal of Intelligent Computing and Cybernetics, (2013).

[22] A. Kumar, A. Bansal, et al., *A new computational method for solving fully fuzzy linear systems of triangular fuzzy numbers*, Fuzzy Information and Engineering, 4 (2012), pp. 63–73.

[23] A. Kumar and A. B. Neetu, *A new method to solve fully fuzzy linear system*

with trapezoidal fuzzy numbers, Canadian Journal on Science and Engineering Mathematics, 1 (2010), pp. 45–56.

[24] K. H. Lee, *First course on fuzzy theory and applications*, vol. 27, Springer Science & Business Media, 2006.

[25] G. Malkawi, N. Ahmad, H. Ibrahim, and B. Alshmari, *Row reduced echelon form for solving fully fuzzy system with unknown coefficients*, Journal of Fuzzy Set Valued Analysis, 2014 (2014), pp. 1–18.

[26] G. Malkawi, I. Rida, and N. Ahmad, *An associated linear system approach for solving fully fuzzy linear system with hexagonal fuzzy number*, in 2018 Advances in Science and Engineering Technology International Conferences (ASET), IEEE, pp. 1–7.

[27] S. Muruganandam, K. A. Razak, and K. Rajakumar, *Solving fully fuzzy linear systems by gauss jordan elimination method*, in Journal of Physics: Conference Series, vol. 1362, IOP Publishing, 2019, p. 012087.

[28] S. Nasseri and F. Zahmatkesh, *Huang method for solving fully fuzzy linear system of equations*, The Journal of Mathematics and Computer Science, 1 (2010), pp. 1–5.

[29] S. Nayak and S. Chakraverty, *A new approach to solve fuzzy system of linear equations*, Journal of Mathematics and Computer Science, 7 (2013), pp. 205–212.

[30] M. Paripour, J. Saeidian, and A. Sadeghi, *A new approach to solve fuzzy system*

of linear equations by homotopy perturbation method, Journal of Linear and Topological Algebra (JLTA), 2 (2013), pp. 105–115.

[31] S. Radhakrishnan and P. Gajivaradhan, *A new approach to solve fully fuzzy linear system*, International Journal of Mathematical Archive (IJMA) ISSN 2229-5046, 5 (2014).

[32] P. Rajarajeshwari, A. S. Sudha, and R. Karthika, *A new operation on hexagonal fuzzy number*, International Journal of Fuzzy Logic Systems, 3 (2013), pp. 15–26.

[33] M. R. Seikh, P. K. Nayak, and M. Pal, *Generalized triangular fuzzy numbers in intuitionistic fuzzy environment*, International Journal of Engineering Research and Development, 5 (2012), pp. 08–13.

[34] N. G. Seresht and A. R. Fayek, *Computational method for fuzzy arithmetic operations on triangular fuzzy numbers by extension principle*, International Journal of Approximate Reasoning, 106 (2019), pp. 172–193.

[35] J. Vahidi and S. Rezvani, *Arithmetic operations on trapezoidal fuzzy numbers*, Journal Nonlinear Analysis and Application, 2013 (2013), pp. 1–8.

[36] V. Vijayalakshmi and R. Sattanathan, *St decomposition method for solving fully fuzzy linear systems using gauss jordan for trapezoidal fuzzy matrices*, in International Mathematical Forum, vol. 6, 2011, pp. 2245–2254.

[37] L. A. Zadeh, *Fuzzy sets*, Information and control, 8 (1965), pp. 338–353.

[38] ——, *The concept of a linguistic variable and its application to approximate reasoning—i*, Information sciences, 8 (1975), pp. 199–249.

[39] ——, *Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic*, Fuzzy sets and systems, 90 (1997), pp. 111–127.

[40] D. Zhang, Q.-L. Han, and X. Jia, *Network-based output tracking control for t–s fuzzy systems using an event-triggered communication scheme*, Fuzzy sets and systems, 273 (2015), pp. 26–48.

[41] H.-J. Zimmermann, *Fuzzy set theory—and its applications*, Springer Science & Business Media, 2011.

[42] ——, *Fuzzy sets, decision making, and expert systems*, vol. 10, Springer Science & Business Media, 2012.

ملخص

أنظمة خطية ضبابية بالكامل عبر α -cuts

صبرين أمين

في هذا العمل ، درسنا أنظمة خطية ضبابية بالكامل بأرقام ضبابية مثلثة وشبه منحرف وسداسية. في الأدب ، ناتج رقمين غامضين باستخدام α -cuts ل يحتوي على قيم غير مفضلة ، لذلك ذهب العديد من المؤلفين إلى صيغة الضرب التقريبية. بعد الضرب التقريبي ، لا يلزم أن يكون ضرب رقمين موجبين غامضين موجبًا وفي أوقات أخرى يؤدي إلى رقم غامض ليس من نفس النوع. ينصب اهتمامنا في هذا العمل على استخدام القيمة الدقيقة لضرب رقمين غامضين وإدخال شروط مناسبة لضمان الحصول على حل وحيد لنظام ضبابي بالكامل بأرقام ضبابية مثلثة وشبه منحرف وسداسية. نقترح هيكلًا لحل النظام الخطي الغامض بالكامل في كل حالة. ثم نوضح نتائجنا باستخدام عدد من الأمثلة العددية.