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Methods For Solving Fuzzy Linear Programming Problem

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Committee Decision

Methods for Solving Fuzzy Linear Programming Problem

by

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Dedication

To my husband Banan Hajeh , my children Amro and Nazmi, my husband's father Nazmi Hajeh and husband's mother for their assistance, tolerance and endless support.

To my father Issam Thaher and my mother, there is no doubt in my mind that without there continued support I could not have completed my study.

To my brothers Saleh, Mohammed and Nasser, and my sisters. And to my friends Haneen, Hadeel and Lara.

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Abstract

Methods for Solving Fuzzy Linear Programming Problem

By

Anwaar Issam Thaher

In this thesis, some methods are used for solving fuzzy linear programming problem which are: Ranking function method, Alpha- cut method and Linear system method. Each method is introduced in details and some examples on them. These methods converting the fuzzy linear programming problem to crisp linear programming Problem which can be easily solved with many ways specially simplex method.

Contents

<i>Committee Decision</i>	ii
<i>Dedication</i>	ii
<i>Acknowledgements</i>	iv
Abstract	v
1. <i>Introduction</i>	1
2. <i>Fuzzy Sets And Fuzzy Numbers</i>	4
2.1 Fuzzy Set	4
2.2 Fuzzy Numbers	5
2.2.1 Triangular Fuzzy Number	6
2.2.2 Trapezoidal Fuzzy Numbers	8
2.2.3 Hexagonal Fuzzy Number	9
2.2.4 Octagonal Fuzzy Number	11
3. <i>Fully Fuzzy Linear Programming with Equality Constraints</i>	14
3.1 Generalized Inverse	14
3.2 Fuzzy Linear System For Solving Triangular Fully Fuzzy Linear Programming Problem	15
3.2.1 Applications	18

3.3	Fuzzy Linear Systems with Trapezoidal Fuzzy Numbers	20
3.3.1	Some Applications	23
3.4	Fuzzy Linear System For Solving Hexagonal Fully Fuzzy Linear Programming Problem	24
3.4.1	Some Examples	27
3.5	Solving Octagonal Fully Fuzzy Linear Programming Problem using Fully Fuzzy Linear System	29
4.	<i>Ranking Function Method For Solving Fully Fuzzy Linear Programming Problems</i>	36
4.1	Ranking Functions	36
4.2	Ranking Function Method	38
4.3	Applications on Ranking Function	39
4.3.1	Ranking Function Method For Solving Octagonal Fuzzy Linear Programming Problem	45
5.	<i>Alpha-Cut Method For Solving Fuzzy Linear Programming Problem</i>	49
5.1	Alpha-Cut	49
5.2	Interval Linear Programming Problem	50
5.2.1	Interval Arithmetics	50
5.2.2	Interval Linear Programming Problem	51
5.3	Applications on Alpha-cut Method	56
	<i>Conclusion</i>	63
	Abstract(Arabic)	68

Chapter 1

Introduction

The concepts of fuzzy sets and numbers was first introduced by Zadeh [30]. Linear programming plays an important role in operations research techniques and it has many real-word applications such as: agricultural economics, assignment problems, banking and finance, environment and management. For more applications we refer the reader to [11, 24].

The well-known crisp linear programming problem is defined by:

$$\begin{aligned} \text{Optimize } Z &= CX \\ \text{s.t} & \\ AX &\{ \leq, \geq, = \} b \\ X &\geq 0 \end{aligned} \tag{1.1}$$

where C and b are constant vectors of size $1 \times m$ and $n \times 1$ respectively, A is the coefficient matrix of size $n \times m$ and X is an $m \times 1$ solution of the problem.

Crisp linear programming has been investigated by many authors and they introduced many methods for solving these [8, 10, 16, 18, 27].

In this work, we will deal with fuzzy linear programming problem(FLPP) and fully

fuzzy linear programming (FFLPP), so some or all of the matrices and vectors A , c , b will be replaced by matrices whose entries are fuzzy numbers. Thus, the fuzzy linear programming problem can be formulated as follows:

$$\begin{aligned}
 \text{Optimize } Z &= \sum_{i=1}^m \hat{c}_i \otimes \hat{x}_i \\
 &s.t \\
 \sum_{j=1}^m \hat{a}_{ij} \otimes \hat{x}_j &\{\leq, \geq, =\} \hat{b}_i, \quad i = 1, \dots, n \\
 \hat{x}_j &\geq 0, \quad j = 1, \dots, m.
 \end{aligned} \tag{1.2}$$

where \hat{c}_i , \hat{b}_i , \hat{x}_j and \hat{a}_{ij} are fuzzy numbers.

Fuzzy linear programming problem is a problem whose at least one of its components (variables, coefficients of constraints, coefficients of the objective function and right hand sides) is a fuzzy number. This type of problem is proposed by Bellman and Zadeh [14].

Many methods are developed for solving fuzzy linear programming problem. In [7, 9], the solution of the fuzzy linear programming problem is introduced in which the technical coefficients and right hand sides are fuzzy numbers. But in [28], the solution was introduced when the coefficients of the constraints and the coefficients in the objective function are fuzzy numbers. The solution of linear programming problem in which decision variables, cost coefficients and right hand sides are fuzzy numbers in [29]. The Penalty Method [20] solves the fuzzy linear programming when the right hand sides are fuzzy numbers. In [17, 21], they used the simplex method for solving the fuzzy linear programming problem.

If all of the components of the linear programming problem are fuzzy numbers then it is called Fully Fuzzy Linear Programming Problem. A method called Bound and Decomposition is illustrated in [11] to solve the fully fuzzy linear programming problem and

it is also solved in [1] by another methods. In [3] the fully fuzzy linear programming problem has solved with LR fuzzy numbers. [25] has solved the triangular fully fuzzy linear programming problem. The trapezoidal fully fuzzy linear programming problem was solved in [12, 13, 31].

Besides the current chapter, this thesis consists of five chapters. First chapter is the introduction. In the second chapter, some basic notion in the theory of fuzzy sets and number are introduced. Further, the definitions of triangular, trapezoidal, hexagonal and octagonal fuzzy numbers are presented. In fact, we have applied the theory introduced and generalized in [23] in solving such problems. Further, we were able to extend this theory to octagonal fuzzy numbers. In chapter four, the ranking function method has been used to solve fuzzy linear programming problem with the fourth fuzzy numbers. And finally in chapter five, another method was introduced, which is an alpha-cut method for solving fuzzy linear programming problem.

Chapter 2

Fuzzy Sets And Fuzzy Numbers

In this chapter, some basic definitions about fuzzy numbers and fuzzy sets are introduced.

This chapter consists of two sections. The first section is talking about fuzzy set and its definition. And the second section talking about fuzzy numbers and it consists of five subsections, first subsection is about the definition of the fuzzy number and the rest subsections are about the definitions of each type and some operations on them.

2.1 Fuzzy Set

We start with some definitions on fuzzy set.

Definition 2.1.1 [15] *Membership function*

For a set A , we define a membership function by

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \quad (2.1)$$

Definition 2.1.2 *Fuzzy Set [15]*

A fuzzy set is a nonempty set which maps each element belongs to a nonempty set; X to $[0, 1]$ by membership function

$$\mu_A : X \longrightarrow [0, 1]$$

Definition 2.1.3 *Normal Set [15]* A fuzzy set A is called normal if there is at least one point x belongs to X such that $\mu_A(x) = 1$.

Definition 2.1.4 *Convex Set:* A fuzzy set A is convex if for any x_1, x_2 belong to X and $\lambda \in [0, 1]$ we have:

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min \mu_A(x_1), \mu_A(x_2)$$

Now ,we will use the above definitions to define the fuzzy number.

2.2 Fuzzy Numbers

Here, we will talk about the definition of fuzzy numbers and the definition of each type

.

Definition 2.2.1 *Fuzzy Number [15]* A fuzzy set A is a fuzzy number if it is satisfying the following conditions:

- (i) *Convex fuzzy set*
- (ii) *Normalized fuzzy set*
- (iii) *The membership function is piecewise continuous*
- (iv) *The membership function defined on the set of real numbers.*

Definition 2.2.2 A fuzzy number A is called positive (negative) denoted by $A > 0$ ($A < 0$) if it's membership function $\mu_A(X)$ satisfies $\mu_A(X) = 0, \forall x \leq 0$ ($\forall x \geq 0$).

Now, we will mention some fuzzy numbers about there definitions and some arithmetics on them.

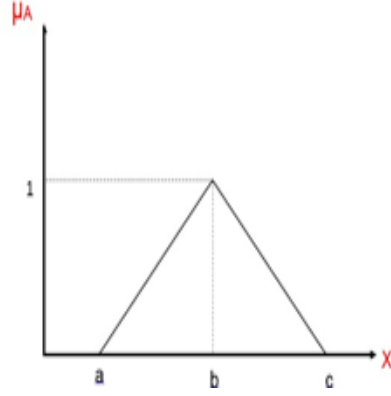


Fig. 2.1: Triangular Fuzzy Number

2.2.1 Triangular Fuzzy Number

Here in this subsection we will introduce the definition of triangular fuzzy number in two ways and some operators on them.

Definition 2.2.3 [22] *Triangular Fuzzy Number*

A fuzzy number $A = (a, b, c)$ is a triangular fuzzy number which interpreted as membership function and holds the following condition:

(i) a to b is increasing function

(ii) b to c is decreasing function

(iii) $a \leq b \leq c$ with

$$\mu_A(x) = \begin{cases} 0, & \text{otherwise} \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b} & b \leq x \leq c \end{cases} \quad (2.2)$$

Arithmetic operations:

Let $A = (a, b, c)$ and $B = (e, f, g)$ be two triangular fuzzy numbers. Then:

(i) $A + B = (a + e, b + f, c + g)$

$$(ii) \quad -A = (-c, -b, -a)$$

$$(iii) \quad A - B = (a - g, b - f, c - e)$$

$$(iv) \quad B \geq 0 \text{ if and only if } e \geq 0, f \geq 0, g \geq 0$$

$$(v) \quad \text{for any } B \geq 0,$$

$$A \times B = \begin{cases} (ae, bf, cg) & a \geq 0 \\ (ag, bf, cg) & a < 0, c \geq 0 \\ (eg, bf, ce) & c < 0 \end{cases} \quad (2.3)$$

But, if we use another definition of triangular fuzzy number using left and right shift, the triangular fuzzy number $A = (a, \alpha, \beta)$ is a triangular fuzzy number whose membership function is

$$\mu_A(x) = \begin{cases} 1 - \frac{a-x}{\alpha}, & a - \alpha \leq x \leq a \\ 1 - \frac{x-a}{\beta}, & a \leq x \leq a + \beta \\ 0, & \text{otherwise} \end{cases} \quad (2.4)$$

where a is the mean value of A , α and β are left and right spreads respectively.

Definition 2.2.4 [26] *Non-negative Triangular Fuzzy Number*

A fuzzy number is non-negative if and only if $\mu_A(x) = 0, \forall x < 0$. Then, a triangular fuzzy number $A = (a, \alpha, \beta)$ is non-negative if $a - \alpha \geq 0$.

Here some operations on triangular fuzzy numbers in this shape :

Let $A = (a, \alpha, \beta), B = (b, \gamma, \delta)$ be two triangular fuzzy numbers then:

$$(i) \quad A + B = (a + b, \alpha + \gamma, \beta + \delta)$$

$$(ii) \quad A - B = (a - b, \alpha + \delta, \beta + \gamma)$$

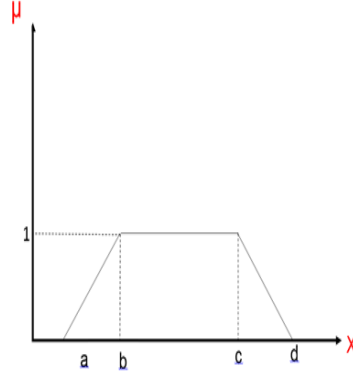


Fig. 2.2: Trapezoidal Fuzzy Number

(iii) If λ is scalar then,

$$\lambda A = \begin{cases} (\lambda a, \lambda \alpha, \lambda \beta), & \lambda \geq 0 \\ (\lambda a, \lambda \beta, \lambda \alpha), & \lambda \leq 0 \end{cases} \quad (2.5)$$

(iv) $A \cdot B = (ab, a\alpha + b\alpha, a\delta + b\beta)$

(v) A is symmetric if $\alpha = \beta$

2.2.2 Trapezoidal Fuzzy Numbers

This is the second subsection in this section which talking about trapezoidal fuzzy number.

Definition 2.2.5 [2] *Trapezoidal Fuzzy Number*

A trapezoidal fuzzy number $A = (a, b, c, d)$ is a fuzzy number which has a membership function as:

$$\mu_A(x) = \begin{cases} 0 & x \leq a, x \geq d \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & b \leq x \leq c \\ \frac{x-d}{c-d} & c < x < d \end{cases} \quad (2.6)$$

Definition 2.2.6 [17] *Trapezoidal Fuzzy Number With Left And Right Shift*

A trapezoidal fuzzy number $A = (m, n, \alpha, \beta)$ is a fuzzy number defined by it's corresponding membership function which is:

$$\mu_A(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{x-n}{\beta}, & n \leq x \leq n + \beta \\ 0 & otherwise \end{cases} \quad (2.7)$$

where α, β are left and right shift respectively

Some arithmetic operations on trapezoidal fuzzy numbers: Let $A = (m, n, \alpha_1, \beta_1)$ and $B = (p, q, \alpha_2, \beta_2)$ be two trapezoidal fuzzy numbers and $c \in R$. Then :

- (i) If $c \geq 0$, then $cA = (cm, cn, c\alpha_1, c\beta_1)$
- (ii) If $c < 0$, then $cA = (cn, cm, c\beta_1, c\alpha_1)$
- (iii) $A + B = (m + p, n + q, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$
- (iv) $A \cdot B = (mp, nq, m\alpha_2 + p\alpha_1, n\beta_2 + q\beta_1)$

Now, the definition of hexagonal fuzzy number will be introduced with it's usual definition and with left-right shift definition and mention some arithmetics on them.

2.2.3 Hexagonal Fuzzy Number

In this current subsection, we will talk about the definitions of hexagonal fuzzy number and some operations on them.

Definition 2.2.7 [6] *A fuzzy number A is a hexagonal fuzzy number denoted by $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ where it's components are real numbers and it's membership func-*

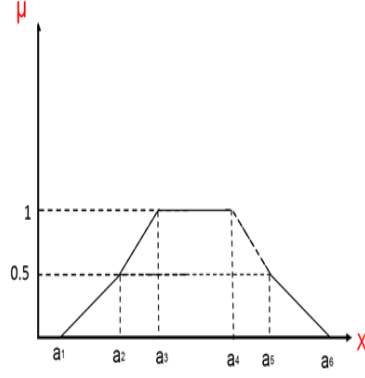


Fig. 2.3: Hexagonal Fuzzy Number

tion μ_A is given by:

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < a_1 \\ \frac{1}{2} \left(\frac{x-a_1}{a_2-a_1} \right) & a_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{1}{2} \left(\frac{x-a_2}{a_3-a_2} \right) & a_2 \leq x \leq a_3 \\ 1 & a_3 \leq x \leq a_4 \\ 1 - \frac{1}{2} \left(\frac{x-a_4}{a_5-a_4} \right) & a_4 \leq x \leq a_5 \\ \frac{1}{2} \left(\frac{a_6-x}{a_6-a_5} \right) & a_5 \leq x \leq a_6 \\ 0 & x > a_6 \end{cases} \quad (2.8)$$

Now, using α and β the definition become:

Definition 2.2.8 Let $m \leq n$, $\alpha_1, \beta_1, \alpha_2,$ and β_2 such that $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2$. A non zero hexagonal fuzzy number denoted by $A = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ is a fuzzy number

whose membership function is

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < m - \alpha_1 \\ \frac{1}{2} \left(\frac{x-m+\alpha_1}{\alpha_1-\alpha_2} \right) & m - \alpha_1 \leq x \leq m - \alpha_2 \\ 1 + \frac{1}{2} \left(\frac{x-m}{\alpha_2} \right) & m - \alpha_2 \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{1}{2} \left(\frac{x-n}{\beta_2} \right) & n \leq x \leq n + \beta_2 \\ -\frac{1}{2} \left(\frac{x-n-\beta_1}{\beta_1-\beta_2} \right) & n + \beta_2 \leq x \leq n + \beta_1 \\ 0 & x > n + \beta_1 \end{cases} \quad (2.9)$$

Let $B = (p, q, \alpha_{21}, \beta_{21}, \alpha_{22}, \beta_{22})$ be hexagonal fuzzy number, then $A \cdot B = (mp, nq, m\alpha_{21} + p\alpha_1, n\beta_{21} + q\beta_1, m\alpha_{22} + p\alpha_1, n\beta_{22} + q\beta_2)$

The last fuzzy number we will talk about is an Octagonal fuzzy number.

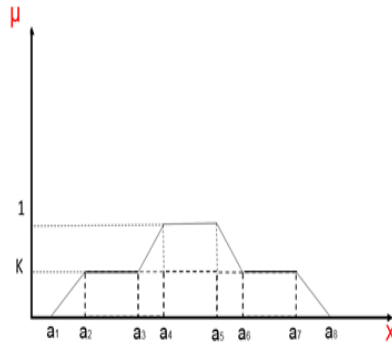
2.2.4 Octagonal Fuzzy Number

The octagonal fuzzy number can be defined as follows:

Definition 2.2.9 *Octagonal Fuzzy Number [5]* A fuzzy number is an octagonal fuzzy number denoted by (a, b, c, d, e, f, g, h) where a, b, c, d, e, f, g are real numbers and it's

membership function is given by:

$$\mu_A(x) = \begin{cases} 0, & x \leq a, x \geq h \\ k\left(\frac{x-a}{b-a}\right), & b \leq x \leq c \\ k & c \leq x \leq d \\ k + (1-k)\left(\frac{x-c}{d-c}\right) & d \leq x \leq e \\ 1 & e \leq x \leq f \\ k + (1-k)\left(\frac{f-x}{f-e}\right) & f \leq x \leq g \\ k & g \leq x \leq h \\ k\left(\frac{h-x}{f-e}\right) & \end{cases} \quad (2.10)$$



Now, when we do some calculations we can define the octagonal fuzzy number using left and right shift as follows:

Definition 2.2.10 *Octagonal Fuzzy Number*

A fuzzy number $A = (d, e, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$ is an octagonal fuzzy number if it's

membership function given by:

$$\mu_A(x) = \begin{cases} \frac{k}{\alpha_3 - \alpha_2}(x - d + \alpha_3) & d - \alpha_3 \leq x \leq d - \alpha_2 \\ k & d - \alpha_2 \leq x \leq d - \alpha_1 \\ 1 + \frac{x - d - k(d - x)}{\alpha_1} & d - \alpha_1 \leq x \leq d \\ 1 & d \leq x \leq e \\ 1 + \frac{(k-1)(x-e)}{\beta_1} & e \leq x \leq e + \beta_1 \\ k & e + \beta_1 \leq x \leq e + \beta_2 \\ -\frac{k}{\beta_3 - \beta_2}(x - e - \beta_3) & e + \beta_2 \leq x \leq e + \beta_3 \end{cases} \quad (2.11)$$

$$A_\alpha = \begin{cases} \frac{\alpha(\alpha_3 - \alpha_2)}{k} + d - \alpha_3, \frac{\alpha(\beta_3 - \beta_2)}{k} - e - \beta_3 & 0 < \alpha \leq k \\ \frac{(\alpha-1)\alpha_1}{k+1} + d, \frac{(\alpha-1)\beta_1}{k-1} + e & k \leq \alpha \leq 1 \end{cases} \quad (2.12)$$

In the following chapters, we introduce various methods to optimize the solution of problem (1.2) with triangular, trapezoidal, hexagonal and octagonal fuzzy numbers.

Chapter 3

Fully Fuzzy Linear Programming with Equality Constraints

In this chapter, we introduce some concepts that have been studied . These concepts will be used to optimize fully fuzzy linear programming with equality constraints. Since the coefficient matrix of the constraints need not be invertible, the method of generalized inverse is one of the useful technique to approximate the inverse of such matrices.

3.1 Generalized Inverse

Definition 3.1.1 *If A is an $m \times n$ matrix, and G is an $n \times m$ matrix then G is a generalized inverse of A if it satisfies the property $AGA = A$.*

When A is a square invertible matrix then $G = A^{-1}$.

The following algorithm describes the process of finding the Generalized inverse of a non-square matrix A .

1. Choose any non-singular sub-matrix H of size k .
2. Calculate $(H^{-1})^T$.
3. In A , replace the elements of sub-matrix H by the elements of $(H^{-1})^T$ and the rest entries by zeros to get new matrix \tilde{A} .

4. The generalized inverse $G = (\tilde{A})^T$.

From the above algorithm, the generalized inverse is not unique since it depends on the choice of the sub matrix H .

3.2 Fuzzy Linear System For Solving Triangular Fully Fuzzy Linear Programming Problem

As mentioned in chapter two, $A \otimes B = (mn, m\alpha_2 + n\alpha_1, m\beta_2 + n\beta_1)$ where, $A = (m, \alpha_1, \beta_1)$ and $B = (n, \alpha_2, \beta_2)$ are triangular fuzzy numbers.

Let

$$(a_{ij} \otimes x_j) + \dots + (a_{in} \otimes x_n) = b_i$$

be a linear system of triangular fuzzy numbers then it can be written as :

$$\Sigma((a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j)) = (b_i, g_i, h_i)$$

By using product of two triangular fuzzy numbers:

$$\Sigma(a_{ij}x_j, a_{ij}y_j + \alpha_{ij}x_j, a_{ij}z_j + \beta_{ij}x_j) = (b_i, g_i, h_i)$$

So

$$\begin{aligned}\Sigma a_{ij}x_j &= b_i \\ \Sigma(a_{ij}y_j + \alpha_{ij}x_j) &= g_i \\ \Sigma(a_{ij}z_j + \beta_{ij}x_j) &= h_i\end{aligned}$$

(3.1)

This can be written as a system of equations:

$$\begin{aligned}
AX &= B \\
AY + MX &= g \\
AZ + NX &= h
\end{aligned}$$

This can be written by a block matrix S as:

$$S = \begin{bmatrix} A & 0 & 0 \\ M & A & 0 \\ N & 0 & A \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \end{bmatrix}$$

Then the solution is:

$$\begin{aligned}
X &= A^{-1}b \\
Y &= A^{-1}(g - MA^{-1}b) \\
Z &= A^{-1}(h - NA^{-1}b)
\end{aligned}$$

where, $A = (a_{ij})$, $X = (x_j)$, $Y = (y_j)$, $M = (\alpha_{ij})$ and $N = (\beta_{ij})$

Definition 3.2.1 The linear system $SX = C$ where $S = \begin{bmatrix} A & 0 & 0 \\ M & A & 0 \\ N & 0 & 0 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

, $C = \begin{bmatrix} b \\ g \\ h \end{bmatrix}$ is called associated linear system of the fuzzy linear system $A \otimes X = B$. So,

$$\text{the solution is: } X = \begin{bmatrix} A^{-1}b \\ A^{-1}(g - MA^{-1}b) \\ A^{-1}(h - NA^{-1}b) \end{bmatrix}$$

where, $A = (a_{ij})$, $M = (\alpha_{ij})$, $N = (\beta_{ij})$, $x = (x_1, \dots, x_n)^T$, $y = (y_1, \dots, y_n)^T$, $z = (z_1, \dots, z_n)^T$, $b = (b_1, \dots, b_n)^T$, $g = (g_1, \dots, g_n)^T$ and $h = (h_1, \dots, h_n)^T$ Note: The last solution exists only if A is invertible.

For more, read in [19].

Let $OptZ = CX$

subject to

$$AX = b$$

$$X \geq 0$$

be fully fuzzy linear programming problem where, C is triangular fuzzy vector, X is fuzzy variable and A, b are fuzzy matrix and vector respectively.

We will solve the linear system $AX = b$ by the solution which we are mentioned and then substituting the solution (values of X) in the objective function to get the value of Z .

If A is singular, we can't use the previous method. We will approximate the solution of the linear system using the generalized inverse, so the solution will be:

$$X = \begin{bmatrix} Gb \\ G(g - MGb) \\ G(h - NGb) \end{bmatrix}$$

where G is the generalized inverse of A .

And then put the solution in the objective function to get the value of Z .

Now, we will take the case of non-square coefficient matrix.

Here, the matrix A will be non-square.

$$\begin{bmatrix} A & 0 & 0 \\ M & A & 0 \\ N & 0 & A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \end{bmatrix}$$

Definition 3.2.2 The vector x that minimizes $\|AX - b\|$ is the solution to the normal equation $A^TAX = A^Tb$. So, $X = (A^T A)^{-1}b$

The vector x is the least square solution to $AX = b$

If $(A^T A)^{-1}$ exist, then the solution is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (A^T A)^{-1}A^Tb \\ (A^T A)^{-1}A^T(g - M(A^T A)^{-1}A^Tb) \\ (A^T A)^{-1}A^T(h - N(A^T A)^{-1}A^Tb) \end{bmatrix}$$

But, if $A^T A$ is singular then:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} GA^Tb \\ GA^T(g - MGA^Tb) \\ GA^T(h - NGA^Tb) \end{bmatrix}$$

where G is the generalized inverse of $A^T A$.

3.2.1 Applications

Here in this section we will introduce some examples on using the previous method.

Example 3.2.1 $\max Z = (1, 4, 6)x_1 \oplus (3, 4, 5)x_2$

subject to

$$(3, 6, 2)x_1 \oplus (4, 6, 1)x_2 = (7, 8, 9)$$

$$(0, 1, 2)x_1 \oplus (5, 1, 6)x_2 = (5, 7, 1)$$

$$x_1, x_2 \geq 0$$

$$\text{Let } A = (a_{ij}) = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}, M = (\alpha_{ij}) = \begin{bmatrix} 6 & 6 \\ 1 & 1 \end{bmatrix},$$

$$N = (\beta_{ij}) = \begin{bmatrix} 2 & 1 \\ 2 & 6 \end{bmatrix}, b = (7, 5)^T, g = (8, 7)^T \text{ and } h = (9, 1)^T .$$

$|A| = 15 \neq 0$ so, A is invertible.

$$A^{-1} = \begin{bmatrix} 1/3 & -4/15 \\ 0 & 1/5 \end{bmatrix},$$

$$A^{-1}b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A^{-1}(g - MA^{-1}b) = \begin{bmatrix} -8/3 \\ 1 \end{bmatrix}, A^{-1}(h - NA^{-1}b) = \begin{bmatrix} 58/15 \\ -7/5 \end{bmatrix}$$

The solution is:

$$X = \begin{bmatrix} A^{-1}b \\ A^{-1}(g - MA^{-1}b) \\ A^{-1}(h - NA^{-1}b) \end{bmatrix} \text{ Therefore, } x_1 = (1, -8/3, 58/15), x_2 = (1, 1, -7/5)$$

and $Z = (4, 26/3, 160/15)$

Example 3.2.2

$$\max Z = (1, 2, 3)x_1 \oplus (0, 1, 2)x_2$$

s.t

$$(6, 8, 3)x_1 \oplus (7, 9, 2)x_2 = (5, 7, 1)$$

$$(5, 8, 1)x_1 \oplus (9, 11, 5)x_2 = (4, 5, 6)$$

$$(6, 9, 2)x_1 \oplus (5, 7, 1)x_2 = (1, 2, 3)$$

$$x_1, x_2 \geq 0$$

$$A = (a_{ij}) = \begin{bmatrix} 6 & 7 \\ 5 & 9 \\ 6 & 5 \end{bmatrix}, M = (\alpha_{ij}) = \begin{bmatrix} 8 & 9 \\ 8 & 11 \\ 9 & 11 \end{bmatrix}, N = (\beta_{ij}) = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 2 & 1 \end{bmatrix}, b = (b_i) = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix},$$

$$g = (g_i) = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix} \text{ and } h = (h_i) = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 97 & 117 \\ 117 & 155 \end{bmatrix}, |A^T A| = 1346 \neq 0$$

$$(A^T A)^{-1} = \begin{bmatrix} .12 & -.09 \\ -.09 & .07 \end{bmatrix}. \text{ So, } (A^T A)^{-1}b = \begin{bmatrix} -.12 \\ .28 \end{bmatrix}$$

$$(A^T A)^{-1}A^T(g - M(A^T A)^{-1}A^Tb) = \begin{bmatrix} .19 \\ .03 \end{bmatrix}, (A^T A)^{-1}A^T(h - N(A^T A)^{-1}A^Tb) = \begin{bmatrix} -.12 \\ .25 \end{bmatrix}$$

$$X = \begin{bmatrix} (A^T A)^{-1}A^Tb \\ (A^T A)^{-1}A^T(g - M(A^T A)^{-1}A^Tb) \\ (A^T A)^{-1}A^T(h - N(A^T A)^{-1}A^Tb) \end{bmatrix}$$

. Then, $x_1 = (-.12, .19, -.12)$, $x_2 = (.28, .03, .25)$ and

$$Z = (-.12, .33, .02)$$

3.3 Fuzzy Linear Systems with Trapezoidal Fuzzy Numbers

This section uses the product of two trapezoidal fuzzy numbers defined in chapter one to solve the fuzzy LPP.

This section consists of two parts :first part when the constraints and the right hand sides are square (size $n \times n$), and the second part when the size is $m \times n$.

As we defined before,

$$A \times B = (mp, nq, m\alpha_2 + p\alpha_1, n\beta_2 + q\beta_1) \text{ where } A = (m, n, \alpha_1, \beta_1) \text{ and } B = (p, q, \alpha_2, \beta_2)$$

$$\sum_{j=1}^n (a_{ij} \times x_j) = (b_i, g_i, h_i, k_i)$$

$$\sum_{j=1}^n (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}) \times (x_j, y_j, z_j, w_j) = (b_i, g_i, h_i, k_i)$$

$$\sum_{j=1}^n a_{ij}x_j = b_i$$

$$\sum_{j=1}^n b_{ij}y_j = g_i$$

$$\sum_{j=1}^n (a_{ij}z_j + \alpha_{ij}x_j) = h_i$$

$$\sum_{j=1}^n (b_{ij}w_j + \beta_{ij}y_j) = k_j.$$

This will lead to the following equations:

$$Ax = b$$

$$By = g$$

$$Az + Mx = h$$

$$Bw + Ny = k$$

where $A = (a_{ij})$, $B = (b_{ij})$, $M = (\alpha_{ij})$, $N = (\beta_{ij})$.

The block representation of the equation is:

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A & 0 \\ 0 & N & 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \\ k \end{bmatrix}$$

Theorem 3.3.1 *The block matrix $S =$*

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A & 0 \\ 0 & N & 0 & B \end{bmatrix}$$

is invertible if and only if A and B are invertible

If A and B are invertible matrices, then the unique solution of

$$Ax = b \text{ is } x = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ A^{-1}(h - MA^{-1}b) \\ B^{-1}(k - NB^{-1}g) \end{bmatrix}$$

Thus, for solving the FFLPP with trapezoidal fuzzy numbers, we will solve the constraints using the previous method and then substitute the values of x_i s in the objective function to get the value of z as we can see in the next subsection. Now we will take the case in which the block matrix is singular. So we will use the generalized inverse of A and B . The solution is:

$$x = G_1 b$$

$$y = G_2 g$$

$$z = G_1(h - MG_1 b)$$

$$w = G_2(k - NG_2 g)$$

where A, B, M, N are square matrices and G_1, G_2 are the generalized inverses of A and B respectively.

And to solve the FFLPP of trapezoidal fuzzy numbers of the form:

$$Opt Z = c_1 x_1 + \dots + c_n x_n$$

subject to

$$AX \{ \leq, \geq, = \} b$$

$$X \geq 0$$

where $A = (a_{ij})$ is $n \times n$ trapezoidal fuzzy matrix.

$b = (b_i)$ $n \times 1$ trapezoidal fuzzy vector.

$x = (x_j)$ trapezoidal fuzzy vector.

and c_1, \dots, c_n trapezoidal fuzzy numbers.

We use the values of x_j which we obtained by solving the FFLS and substitute them in the objective function. FFLPP which associated coefficients matrices A, B, M, N are

$m \times n$ matrices will be studying here.

If $(A^T A)^{-1}, (B^T B)^{-1}$ exist, then the solution is:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (A^T A)^{-1} A^T b \\ (B^T B)^{-1} B^T g \\ (A^T A)^{-1} A^T (h - M(A^T A)^{-1} A^T b) \\ (B^T B)^{-1} B^T (k - N(B^T B)^{-1} B^T g) \end{bmatrix}$$

Otherwise, if at least one of the matrices $A^T A, B^T B$ is singular then the solution is:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} G_1 A^T b \\ G_2 B^T g \\ G_1 A^T (h - M G_1 A^T b) \\ G_2 B^T (k - N G_2 B^T g) \end{bmatrix}$$

where G_1 the generalized inverse of $A^T A$ and G_2 the generalized inverse of $B^T B$.

3.3.1 Some Applications

Also this section consists about some examples on the previous method which are:

Example 3.3.1

$$\max z = (4, 6, 2, 3)x_1 + (5, 6, 3, 4)x_2$$

s.t

$$(2, 3, 1, 4)x_1 + (3, 4, 2, 6)x_2 = (10, 11, 3, 4)$$

$$(4, 6, 2, 9)x_1 + (6, 8, 5, 8)x_2 = (9, 12, 3, 5)$$

$$x_1, x_2 \geq 0$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}, M = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, N = \begin{bmatrix} 4 & 6 \\ 9 & 8 \end{bmatrix},$$

$$b = \begin{bmatrix} 10 \\ 9 \end{bmatrix}, g = \begin{bmatrix} 11 \\ 12 \end{bmatrix}, h = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \text{ and } k = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$|A| = 0, |B| = 0 \Rightarrow B \Rightarrow A$ and B are singular

$$G_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix} \text{ the generalized inverse of } A$$

$$G_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix} \text{ the generalized inverse of } B$$

$$G_1b = \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix},$$

$$G_2g = \begin{bmatrix} \frac{11}{3} \\ 0 \end{bmatrix}$$

$$G_1(h - MG_1b) = \begin{bmatrix} \frac{1}{8} \\ 0 \end{bmatrix}, G_2(k - NG_2g) = \begin{bmatrix} -32/9 \\ 0 \end{bmatrix}$$

Then, $x_1 = (5/2, 11/3, 1/8, -32/9)$, $x_2 = (0, 0, 0, 0)$ and $Z = (10, 22, 11/2, -31/3)$

3.4 Fuzzy Linear System For Solving Hexagonal Fully Fuzzy Linear Programming Problem

We studied before the definition of hexagonal fuzzy numbers with left and right shift and the product of two hexagonal fuzzy numbers.

Let $A = (m, n, \alpha_{11}, \beta_{11}, \alpha_{12}, \beta_{12})$, $B = (p, q, \alpha_{21}, \beta_{21}, \alpha_{22}, \beta_{22})$ be two hexagonal fuzzy numbers then:

1. $A \oplus B = (m + p, n + q, \alpha_{11} + \alpha_{21}, \beta_{11} + \beta_{21}, \alpha_{12} + \alpha_{22}, \beta_{12} + \beta_{22})$

2. $A \otimes B = (mp, nq, m\alpha_{21} + p\alpha_{11}, n\beta_{21} + q\beta_{11}, m\alpha_{22} + p\alpha_{12}, n\beta_{22} + q\beta_{12})$

Let $AX = B$ be FFLS such that $A = (a_{ij})$, $B = (b_j)$ are hexagonal fuzzy numbers where,

$$a_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij})$$

$$x_j = (x_j, y_j, z_j, w_j, u_j, v_j)$$

and $b_j = (b_j, g_j, h_j, k_j, l_j, t_j)$ then:

$$a_{ij}x_j = (a_{ij}x_j, b_{ij}y_j, a_{ij}z_j + \alpha_{ij}x_j, b_{ij}w_j + \beta_{ij}y_j, a_{ij}u_j + \gamma_{ij}x_j, b_{ij}v_j + \delta_{ij}y_j)$$

Let $AX = B$

$$(a_{ij}x_j) + \dots + (a_{in}x_n) = b_j$$

$$\Sigma a_{ij}x_j = b_j$$

$$\Sigma b_{ij}y_j = g_j$$

$$\Sigma(a_{ij}z_j + \alpha_{ij}x_j) = h_j$$

$$\Sigma(h_{ij}w_j + \beta_{ij}y_j) = k_j$$

$$\Sigma(a_{ij}u_j + \gamma_{ij}x_j) = l_j$$

$$\Sigma(b_{ij}v_j + \delta_{ij}y_j) = t_j$$

So, it become:

$$Ax = b$$

$$By = g$$

$$Az + Mx = h$$

$$Bw + Ny = k$$

$$Au + Tx = l$$

$$Bv + Uy = t$$

where $A = (a_{ij})$, $B = (b_{ij})$, $M = (\alpha_{ij})$, $N = (\beta_{ij})$, $T = (\gamma_{ij})$, $U = (\delta_{ij})$ and $t =$

(t_1, \dots, t_n)

$$S = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ M & 0 & A & 0 & 0 & 0 \\ 0 & N & 0 & B & 0 & 0 \\ T & 0 & 0 & 0 & A & 0 \\ 0 & U & 0 & 0 & 0 & B \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \\ w \\ u \\ v \end{bmatrix}, C = \begin{bmatrix} b \\ g \\ h \\ k \\ l \\ t \end{bmatrix}$$

If A and B are invertible, then the solution of $SX = B$ is:

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ A^{-1}(h - MA^{-1}b) \\ B^{-1}(k - NB^{-1}g) \\ A^{-1}(l - TA^{-1}b) \\ B^{-1}(t - UB^{-1}g) \end{bmatrix} \quad \text{If } A \text{ and } B \text{ are singular matrices, then:}$$

$$X = \begin{bmatrix} G_1b \\ G_2g \\ G_1(h - MG_1b) \\ G_2(k - NG_2g) \\ G_1(l - TG_1b) \\ G_2(t - UG_2g) \end{bmatrix}$$

where, G_1, G_2 are the generalized inverses of A and B respectively. And the solution of the hexagonal FFLPP is given by substituting the values of x which obtained from solving the FLS in the objective function to get the value of Z .

If the matrices A, B, M, N, V and T are non-square matrices, then we will look at the singularity of $A^T A$ and $B^T B$.

If $A^T A$ and $B^T B$ are non-singular then:

$$X = \begin{bmatrix} (A^T A)^{-1} A^T b \\ (B^T B)^{-1} B^T g \\ (A^T A)^{-1} A^T (h - M(A^T A)^{-1} A^T b) \\ (B^T B)^{-1} B^T (k - N(B^T B)^{-1} B^T g) \\ (A^T A)^{-1} A^T (l - T(A^T A)^{-1} A^T b) \\ (B^T B)^{-1} B^T (t - U(B^T B)^{-1} B^T g) \end{bmatrix}$$

is the solution of the FLS $AX = B$

And if $(A^T A)$ and $(B^T B)$ are singular, then the solution is:

$$X = \begin{bmatrix} G_1 A^T b \\ G_2 B^T g \\ G_1 A^T (h - M G_1 A^T b) \\ G_2 B^T (k - N G_2 B^T g) \\ G_1 A^T (l - T G_1 A^T b) \\ G_2 B^T (t - U G_2 B^T g) \end{bmatrix}$$

3.4.1 Some Examples

Simillerly, here are two examples about the method.

Example 3.4.1

$$\max Z = (1, 2, 3, 4, 5, 6)x_1 \oplus (7, 8, 0, 10, 11, 12)x_2$$

s.t

$$(8, 9, 6, 7, 5, 6)x_1 \oplus (4, 7, 3, 2, 2, 1)x_2 = (11, 12, 13, 14, 15, 16)$$

$$(5, 10, 4, 3, 3, 2)x_1 \oplus (3, 4, 2, 4, 1, 3)x_2 = (1, 2, 3, 5, 7, 8)$$

$$x_1, x_2 \geq 0$$

$$A = \begin{bmatrix} 8 & 4 \\ 5 & 3 \end{bmatrix}, B = \begin{bmatrix} 9 & 7 \\ 10 & 4 \end{bmatrix}, M = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}, N = \begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix}, T = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}, U = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$$

$$|A| = 4 \neq 0 \text{ and } |B| = -4 \neq 0$$

Since A and B are invertible, then the solution is:

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ A^{-1}(h - M A^{-1}b) \\ B^{-1}(k - N B^{-1}g) \\ A^{-1}(l - T A^{-1}b) \\ B^{-1}(t - U B^{-1}g) \end{bmatrix}$$

$$\text{where } b = \begin{bmatrix} 11 \\ 1 \end{bmatrix}, g = \begin{bmatrix} 12 \\ 2 \end{bmatrix}, h = \begin{bmatrix} 13 \\ 3 \end{bmatrix}, k = \begin{bmatrix} 14 \\ 5 \end{bmatrix}, l = \begin{bmatrix} 15 \\ 7 \end{bmatrix} \text{ and } t = \begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3/4 & -1 \\ -5/4 & 2 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} -1 & 7/4 \\ 5/2 & -9/4 \end{bmatrix}$$

$$A^{-1}b = \begin{bmatrix} 29/4 & -47/4 \end{bmatrix}, B^{-1}g = \begin{bmatrix} -17/2 \\ 51/2 \end{bmatrix}$$

$$A^{-1}(h - MA^{-1}b) = \begin{bmatrix} 87/2 \\ 175/16 \end{bmatrix}, B^{-1}(k - NB^{-1}g) = \begin{bmatrix} -976/2 \\ 1647/8 \end{bmatrix}$$

$$A^{-1}(l - TA^{-1}b) = \begin{bmatrix} 75/16 \\ -141/16 \end{bmatrix}, B^{-1}(t - UB^{-1}g) = \begin{bmatrix} -1053/8 \\ 1757/8 \end{bmatrix}$$

The solution is:

$$x_1 = (29/4, -17/2, 87/2, -976/2, 75/16, -1053/8),$$

$$x_2 = (-47/4, 51/2, 175/16, 1647/8, -141/16, 1757/8)$$

$$Z = (-75, 87, 36.06, 1019.5, -230, 1748.75)$$

Example 3.4.2

$$\max Z = (2, 5, 7, 9, 10, 11)x_1 \oplus (0, 1, 2, 3, 4, 5)x_2$$

s.t

$$(5, 7, 8, 9, 10, 11)x_1 \oplus (3, 4, 5, 6, 7, 8)x_2 = (0, 1, 2, 3, 4, 7)$$

$$(5, 7, 8, 11, 13, 14)x_1 \oplus (3, 4, 6, 8, 10, 12)x_2 = (3, 5, 8, 9, 10, 11)$$

$$x_1, x_2 \geq 0$$

$$A = \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 \\ 7 & 4 \end{bmatrix}, \quad M = \begin{bmatrix} 8 & 5 \\ 8 & 6 \end{bmatrix}, \quad N = \begin{bmatrix} 9 & 6 \\ 11 & 8 \end{bmatrix}$$

$$\begin{aligned}
T &= \begin{bmatrix} 10 & 7 \\ 13 & 10 \end{bmatrix}, & U &= \begin{bmatrix} 11 & 8 \\ 14 & 12 \end{bmatrix}, & b &= \begin{bmatrix} 0 \\ 3 \end{bmatrix}, & g &= \begin{bmatrix} 1 \\ 5 \end{bmatrix}, & h &= \begin{bmatrix} 2 \\ 8 \end{bmatrix} \\
k &= \begin{bmatrix} 3 \\ 9 \end{bmatrix}, & l &= \begin{bmatrix} 4 \\ 10 \end{bmatrix}, & t &= \begin{bmatrix} 7 \\ 11 \end{bmatrix} \\
|A| &= |B| = 0 \\
G_1 &= \begin{bmatrix} 0 & 1/3 \\ 0 & 0 \end{bmatrix}, & G_2 &= \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \end{bmatrix} \\
G_1 b &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, & G_2 g &= \begin{bmatrix} 5/4 \\ 0 \end{bmatrix} \\
G_1(h - MG_1 b) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, & G_2(k - NG_2 g) &= \begin{bmatrix} -9/16 \\ 0 \end{bmatrix} \\
G_1(l - TG_1 b) &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}, & G_2(t - UG_2 g) &= \begin{bmatrix} -3/18 \\ 0 \end{bmatrix} \\
x_1 &= (1, 5/4, 0, -9/16, -1, -13/8) & x_2 &= (0, 0, 0, 0, 0, 0) \\
Z &= (2, 25/4, 7, 135/16, 8, 35/8)
\end{aligned}$$

3.5 Solving Octagonal Fully Fuzzy Linear Programming Problem using Fully Fuzzy Linear System

In chapter two we talked about the definition of octagonal fuzzy number using left and right spread and the product of two octagonal fuzzy number which is:

$$A \otimes B = (am, bn, a\gamma_1 + m\alpha_1, b\delta_1 + n\beta_1, a\gamma_2 + m\alpha_2, b\delta_2 + n\beta_2, a\gamma_3 + m\alpha_3, b\delta_3 + n\beta_3)$$

where

$$A = (a, b, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$$

$$B = (m, n, \gamma_1, \delta_1, \gamma_2, \delta_2, \gamma_3, \delta_3)$$

consider the fuzzy linear system

$A \otimes X = B$ such that each entry of $A = (a_{ij})$ and $B = (b_1, \dots, b_n)^T$ is an octagonal fuzzy number and the unknown $X = (x_1, \dots, x_n)^T$

$$X_j = (x_j, y_j, z_j, w_j, u_j, v_j, k_j, l_j)$$

$$b_j = (b_i, g_i, h_i, t_i, m_i, n_i, o_i, p_i)$$

$$a_{ij} = (a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \zeta_{ij}, \lambda_{ij})$$

Then the equation $AX = B$ become:

$$(a_{ij} \otimes x_j) + \dots + (a_{in} \otimes x_n) = b_i$$

$$\sum_{j=1}^n (a_{ij} x_j) = b_i$$

$$\sum_{j=1}^n ((a_{ij}, b_{ij}, \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \zeta_{ij}, \lambda_{ij}) \otimes (x_j, y_j, z_j, w_j, u_j, v_j, k_j, l_j))$$

$$\sum_{j=1}^n (a_{ij} x_j, b_{ij} y_j, a_{ij} z_j + x_j \alpha_{ij}, b_{ij} w_j + \beta_{ij} y_j, a_{ij} u_j + x_j \gamma_{ij}, b_{ij} v_j + y_j \delta_{ij}, a_{ij} k_j + x_j \zeta_{ij}, b_{ij} l_j + y_j \lambda_{ij})$$

So, the system become:

$$\sum_{j=1}^n a_{ij} x_j = b_i$$

$$\sum_{j=1}^n b_{ij} y_j = g_i$$

$$\sum_{j=1}^n a_{ij} z_j + x_j \alpha_{ij} = h_i$$

$$\sum_{j=1}^n b_{ij} w_j + y_j \beta_{ij} = t_i$$

$$\sum_{j=1}^n a_{ij} u_j + x_j \gamma_{ij} = m_i$$

$$\sum_{j=1}^n b_{ij} v_j + y_j \delta_{ij} = n_i$$

$$\sum_{j=1}^n a_{ij} k_j + x_j \zeta_{ij} = o_i$$

$$\sum_{j=1}^n b_{ij} \zeta_j + y_j \lambda_{ij} = p_i$$

we get the following algebraic linear system:

$$AX = b$$

$$By = g$$

$$Az + Mx = h$$

$$Bw + Ny = t$$

$$Au + Tx = m$$

$$Bv + Uy = n$$

$$Ak + Vx = o$$

$$Bl + Qy = p$$

where $A = (a_{ij}), B = (b_{ij}), M = (\alpha_{ij}), N = (\beta_{ij}), T = (\gamma_{ij}), U = (\delta_{ij}), V = (\zeta_{ij}),$
 $Q = (\lambda_{ij}), b = (b_1, \dots, b_n)^T, g = (g_1, \dots, g_n)^T, h = (h_1, \dots, h_n)^T, t = (t_1, \dots, t_n)^T,$
 $m = (m_1, \dots, m_n)^T, n = (n_1, \dots, n_n)^T, o = (o_1, \dots, o_n)^T$ and $p = (p_1, \dots, p_n)^T$

And we can write it using matrices as:

$$S = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 0 & 0 \\ M & 0 & A & 0 & 0 & 0 & 0 & 0 \\ 0 & N & 0 & B & 0 & 0 & 0 & 0 \\ T & 0 & 0 & 0 & A & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 & B & 0 & 0 \\ V & 0 & 0 & 0 & 0 & 0 & A & 0 \\ 0 & Q & 0 & 0 & 0 & 0 & 0 & B \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \\ w \\ u \\ v \\ k \\ l \end{bmatrix}, C = \begin{bmatrix} b \\ g \\ h \\ t \\ m \\ n \\ o \\ p \end{bmatrix}$$

Solving FFLS with $n \times n$ matrix:

FLLS with invertible coefficient matrix:

The matrix S is invertible iff A and B are invertible

If S is invertible, then the unique solution of

$$Opt(z) = cx$$

subject to

$SX = B$ is given by:

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ A^{-1}(h - MA^{-1}b) \\ B^{-1}(t - NB^{-1}g) \\ A^{-1}(m - TA^{-1}b) \\ B^{-1}(n - UB^{-1}g) \\ A^{-1}(o - VA^{-1}b) \\ B^{-1}(p - QB^{-1}g) \end{bmatrix}$$

This solution of the FFLS we use it for solving the FFLPP by substituting the value of each x in the objective function to get the value of each x in the value of z

Now, if A and B are singular then:

$$X = \begin{bmatrix} G_1 b \\ G_2 g \\ G_1(h - MG_1 b) \\ G_2(t - NG_2 g) \\ G_1(m - TG_1 b) \\ G_2(n - UG_2 g) \\ G_1(o - VG_1 b) \\ G_2(p - QG_2 g) \end{bmatrix}$$

A, B, M, N, T, U, V and Q are non-square matrices:

In this case we will look at $A^T A$ and $B^T B$.

If $A^T A$ and $B^T B$ are non-singular then:

$$X = \begin{bmatrix} (A^T A)^{-1} A^T b \\ (B^T B)^{-1} B^T g \\ (A^T A)^{-1} A^T (h - N(A^T A)^{-1} A^T b) \\ (B^T B)^{-1} B^T (t - N(B^T B)^{-1} B^T g) \\ (A^T A)^{-1} A^T (m - T(A^T A)^{-1} A^T b) \\ (B^T B)^{-1} B^T (n - U(B^T B)^{-1} B^T g) \\ (A^T A)^{-1} A^T (o - V(A^T A)^{-1} A^T b) \\ (B^T B)^{-1} B^T (p - Q(B^T B)^{-1} B^T g) \end{bmatrix}$$

then we substitute the value of each x in the objective function to get the value of Z .

And if the matrices $(A^T A)$ and $(B^T B)$ are singular then:

$$X = \begin{bmatrix} G_1 A^T b \\ G_2 B^T g \\ G_1 A^T (h - M G_1 A^T b) \\ G_2 B^T (t - N G_2 B^T g) \\ G_1 A^T (m - T G_1 A^T b) \\ G_2 B^T (n - U G_2 B^T g) \\ G_1 A^T (o - V G_1 A^T b) \\ G_2 B^T (p - Q G_2 B^T g) \end{bmatrix}$$

Example 3.5.1 $\max Z = (1, 3, 4, 6, 8, 9, 10, 11)x_1 \oplus (0, 2, 4, 6, 8, 10, 12, 13)x_2$

subject to

$$(8, 9, 6, 7, 5, 6, 3, 4)x_1 \oplus (4, 7, 3, 2, 2, 1, 3, 5)x_2 = (11, 12, 13, 4, 5, 6, 7, 8)$$

$$(5, 10, 4, 3, 3, 2, 1, 5)x_2 \oplus (3, 4, 2, 4, 1, 3, 5, 6)x_2 = (4, 6, 8, 10, 12, 13, 14, 15)$$

$$x_1, x_2 \geq 0$$

$$\begin{aligned} A &= \begin{bmatrix} 8 & 4 \\ 5 & 3 \end{bmatrix}, & B &= \begin{bmatrix} 9 & 7 \\ 10 & 4 \end{bmatrix}, & M &= \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \\ N &= \begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix}, & T &= \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}, & U &= \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix} \\ V &= \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}, & Q &= \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix} \\ b &= \begin{bmatrix} 11 \\ 4 \end{bmatrix}, & g &= \begin{bmatrix} 12 \\ 6 \end{bmatrix}, & h &= \begin{bmatrix} 13 \\ 8 \end{bmatrix} \\ t &= \begin{bmatrix} 4 \\ 10 \end{bmatrix}, & m &= \begin{bmatrix} 5 \\ 12 \end{bmatrix}, & n &= \begin{bmatrix} 6 \\ 13 \end{bmatrix}, & o &= \begin{bmatrix} 7 \\ 14 \end{bmatrix} \text{ and } p = \begin{bmatrix} 8 \\ 15 \end{bmatrix} \\ |A| &= 4, & |B| &= -34 \end{aligned}$$

$$x_1 = (17/4, -3/17, 47/4, -1.459, -8.56, -.53, 6.875, 1.97)$$

$$x_2 = (-23/4, 87/17, -75/4, -4.101, 15.94, -.115, -10.88, -8.89)$$

$$Z = (17/4, 9.71, -28, -33.356, 59, 04, -5.92, -61.86, -93.44)$$

Chapter 4

Ranking Function Method For Solving Fully Fuzzy Linear Programming Problems

The aim of the current chapter is to solve fully fuzzy linear programming problems using the ranking function method. More precisely, the ranking function method is applied on triangular, trapezoidal, hexagonal and octagonal fully fuzzy linear programming problems.

4.1 Ranking Functions

Ranking fuzzy numbers has a great advantage in the decision problems that deal with fuzzy numbers, because ordering fuzzy numbers are completely different than the case of crisps.

The ranking function is a function maps the set of fuzzy numbers into the set of real number and denoted by $R(\hat{A})$. Many researchers have been introduce various types of ranking functions to solve linear programming problems with fuzzy parameters. in this chapter, we focus on the following definition of ranking function.

Definition 4.1.1 [3] *A ranking function is a function $R : F(R) \longrightarrow R$ where $F(R)$ is a set of fuzzy numbers defined on set of real numbers which maps each fuzzy number*

into the real line, where a natural order exists.

In order to use the ranking function we need the following properties about the using of ranking function:

1. $a > b$ if and only if $R(a) > R(b)$
2. $a < b$ if and only if $R(a) < R(b)$
3. $a = b$ if and only if $R(a) = R(b)$

Here are some shapes of ranking functions of the fuzzy numbers:

Definition 4.1.2 [22] *If $A = (a, b, c)$ a triangular fuzzy number , then the ranking function of A is $R(A) = \frac{a+2b+c}{4}$*

Definition 4.1.3 *Ranking Function Of Trapezoidal Fuzzy Number [17]*

If $A = (a, b, c, d)$ trapezoidal fuzzy number ,then the ranking function of A is: $R(A) = a + b + .5(d - c)$

Definition 4.1.4 [6] **Ranking Function of Hexagonal Fuzzy Number**

Let $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ be a hexagonal fuzzy number. The ranking functions of A are defined as follows:

1. $R(A) = \frac{a_1+2a_2+a_3+a_4+2a_5+a_6}{4}$
2. $R(A) = \frac{a_1+a_2+a_3+a_4+a_5+a_6}{4}$

$$3. R(A) = \frac{a_1+a_2+a_3+a_4+a_5+a_6}{6}$$

And the last definition in this section is about the ranking function of octagonal fuzzy number.

Definition 4.1.5 *Ranking Function OF Octagonal Fuzzy Number [5]*

If $A = (a, b, c, d, e, f, g, h)$ an octagonal fuzzy number ,then:

$$R(A) = \frac{2a+3b+4c+5d+5e+4f+3g+2h}{28} \left(\frac{7}{28}\right) \text{ is the ranking function of } A$$

4.2 Ranking Function Method

This section introducing the solution of fully fuzzy linear programming problems. Namely, the vectors $\hat{C} = [\hat{c}_j]_{1 \times n}$, $\hat{b} = [\hat{b}_i]_{m \times 1}$ are fuzzy numbers, $A = [\hat{a}_{ij}]_{m \times n}$ is a matrix whose entries are fuzzy numbers. And the solution vector $X = [\hat{x}_j]_{1 \times n}$ is of a fuzzy vector. Thus, problem(1.2) will have the following form:

$$\begin{aligned} \text{Optimize } Z &= \sum_{j=1}^n (p_j, q_j, s_j, \dots) \otimes (x_j, y_j, z_j, \dots) \\ \text{s.t} & \\ & \sum_{j=1}^n (a_{ij}, b_{ij}, c_{ij}, \dots) \otimes (x_j, y_j, z_j, \dots) \leq (b_i, g_i, h_i), \quad i = 1, \dots, m \\ & \hat{x}_j \geq 0, \quad j = 1, \dots, m. \end{aligned}$$

By definition(4.1.1), the objective function is

$$\text{Opt}Z = R\left(\sum_{j=1}^n (p_j, q_j, r_j, \dots) \otimes (x_j, y_j, z_j, \dots)\right)$$

On the other hand, the constraint can be simplified as follows:

$$\sum_{j=1}^n (a_{ij}, b_{ij}, c_{ij}, \dots) \otimes (x_j, y_j, z_j, \dots) \leq (b_i, g_i, h_i, \dots)$$

$$\left(\sum_{j=1}^n a_{ij}x_j, \sum_{j=1}^n b_{ij}y_j, \sum_{j=1}^n c_{ij}z_j, \dots \right) \leq (b_i, g_i, h_i, \dots).$$

The components comparison will give the following crisp linear programming problem:

$$\text{Optimize } Z = R\left(\sum_{j=1}^n (p_j, q_j, s_j, \dots) \otimes (x_j, y_j, z_j, \dots)\right)$$

s.t

$$\sum_{j=1}^n a_{ij}x_j \leq b_i$$

$$\sum_{j=1}^n b_{ij}y_j \leq g_i$$

$$\sum_{j=1}^n c_{ij}z_j \leq h_i$$

.

.

.

$$x_j \geq 0, y_j \geq 0, z_j \geq 0, \dots, j = 1, \dots, m.$$

And then we solve it using simplex method.

4.3 Applications on Ranking Function

In the previous section, the ranking function method was presented. And in this section, some examples will be taken using this method.

First example is about solving Triangular FFLPP.

Example 4.3.1 Let $\hat{c}_1 = (1, 2, 3)$, $\hat{c}_2 = (7, 8, 9)$, $\hat{b}_1 = (4, 5, 7)$, $\hat{b}_2 = (6, 9, 11)$

$$a_{\hat{1}1} = (1, 2, 5), a_{\hat{1}2} = (1, 3, 4), a_{\hat{2}1} = (2, 4, 6), a_{\hat{2}2} = (1, 2, 3).$$

Then problem has the following form: $\max Z = \frac{(x_1+7x_2+4y_1+16y_2+3z_1+9z_2)}{4}$ subject to

$$x_1 + x_2 \leq 4, \quad 2y_1 + 3y_2 \leq 5, \quad 5z_1 + 4z_2 \leq 7$$

$$2x_1 + x_2 \leq 8, \quad 4y_1 + 2y_2 \leq 9, \quad 6z_1 + 3z_2 \leq 11.$$

By simplex method, the solution of this problem $\hat{x}_1 = (0, 0, 0)$, $\hat{x}_2 = (4, 1.66, 1.7)$

and the optimum solution is $\hat{Z} = (1, 2, 3) \otimes (0, 0, 0) \oplus (7, 8, 9) \otimes (4, 1.66, 1.7) = (28, 13.28, 15.3)$

Now, we will move to the next example which using the ranking function of trapezoidal fuzzy numbers.

Example 4.3.2 *Solving Trapezoidal FFLPP Using Ranking Function Method*

To solve the trapezoidal FFLPP we will use the same steps as we used in the previous section but the ranking function will be changed as we can see in the following:

$$\max Z = (1, 2, 3, 4)x_1 \oplus (7, 8, 9, 10)x_2$$

s.t

$$(1, 2, 5, 7)x_1 \oplus (1, 3, 4, 5)x_2 \leq (4, 5, 7, 9)$$

$$(2, 4, 6, 8)x_1 \oplus (1, 2, 3, 4)x_2 \leq (6, 9, 11, 14)$$

$$x_1, x_2 \geq 0.$$

Let $x_1 = (x_1, y_1, z_1, w_1)$, $x_2 = (x_2, y_2, z_2, w_2)$

$$\max Z = (1, 2, 3, 4) \otimes (x_1, y_1, z_1, w_1) \oplus (7, 8, 9, 10) \otimes (x_2, y_2, z_2, w_2)$$

subjectto

$$(1, 2, 5, 7) \otimes (x_1, y_1, z_1, w_1) \oplus (1, 3, 4, 5) \otimes (x_2, y_2, z_2, w_2) \leq (4, 5, 7, 9)$$

$$(2, 4, 6, 8) \otimes (x_1, y_1, z_1, w_1) \oplus (1, 2, 3, 4) \otimes (x_2, y_2, z_2, w_2) \leq (6, 9, 11, 14)$$

$$(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) \geq 0.$$

By using addition of trapezoidal fuzzy numbers we get:

$$\max Z = (x_1, 2y_1, 3z_1, 4w_1) \oplus (7x_2, 8y_2, 9z_2, 10w_2)$$

subjectto

$$(x_1, 2y_1, 5z_1, 7w_1) \oplus (x_2, 3y_2, 4z_2, 5w_2) \leq (4, 5, 7, 9)$$

$$(2x_1, 4y_1, 5z_1, 8w_1) \oplus (x_2, 2y_2, 3z_2, 4w_2) \leq (6, 9, 11, 14)$$

$$(x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) \geq 0.$$

By the multiplication we get:

$$\max Z = (x_1 + 7x_2, 2y_1 + 8y_2, 3z_1 + 9z_2, 4w_1 + 10w_2)$$

s.t

$$(x_1 + x_2, 2y_1 + 3y_2, 5z_1 + 4z_2, 7w_1 + 5w_2) \leq (4, 5, 7, 9)$$

$$(2x_1 + x_2, 4y_1 + 2y_2, 5z_1 + 3z_2, 8w_1 + 4w_2) \leq (6, 9, 11, 14).$$

Using $R(A) = a + b + .5(d - c)$

$$\max Z = x_1 + 7x_2 + 2y_1 + 8y_2 + 2w_1 + 5w_2 - (3/2)z_1 - (9/2)z_2$$

s.t

$$x_1 + x_2 \leq 4$$

$$, \quad 2y_1 + 3y_2 \leq 5$$

$$, \quad 5z_1 + 4z_2 \leq 7$$

$$, \quad 7w_1 + 5w_2 \leq 9$$

$$2x_1 + x_2 \leq 6$$

$$, \quad 4y_1 + 2y_2 \leq 9$$

$$, \quad 5z_1 + 3z_2 \leq 11$$

$$, \quad 8w_1 + 4w_2 \leq 14.$$

Using simplex method to get:

$$x_1 = 0, x_2 = 4$$

$$y_1 = 0, y_2 = 1.66$$

$$z_1 = 0, z_2 = 0$$

$$w_1 = 0, w_2 = 1.8.$$

$$\begin{aligned}
\text{So, } x_1 &= (0, 0, 0, 0) \\
x_2 &= (4, 1.66, 0, 1.8) \\
Z &= (1, 2, 3, 4) \otimes (0, 0, 0, 0) \oplus (7, 8, 9, 10) \otimes (4, 1.66, 0, 1.8) \\
&= (28, 13.33, 0, 18).
\end{aligned}$$

Example 4.3.3 *Solving FFLPP with Hexagonal Fuzzy Numbers using Ranking Function Method*

$$\begin{aligned}
\max Z &= (1, 2, 3, 4, 5, 6)x_1 \oplus (7, 8, 9, 10, 11, 12)x_2 \\
\text{subject to.} \\
(1, 2, 5, 7, 9, 11)x_1 \oplus (1, 3, 4, 5, 6, 8)x_2 &\leq (4, 5, 7, 9, 10, 12) \\
(2, 4, 6, 8, 10, 12)x_1 \oplus (1, 2, 3, 4, 6, 8)x_2 &\leq (6, 9, 11, 14, 17, 20) \\
x_1, x_2 &\geq 0.
\end{aligned}$$

Convert to a standard form:

$$\begin{aligned}
\max Z &= (1, 2, 3, 4, 5, 6)x_1 \oplus (7, 8, 9, 10, 11, 12)x_2 \\
\text{subject to} \\
(1, 2, 5, 7, 9, 11)x_1 \oplus (1, 3, 4, 5, 6, 8)x_2 \oplus (1, 1, 1, 1, 1, 1)s_1 &\leq (4, 5, 7, 9, 10, 12) \\
(2, 4, 6, 8, 10, 12)x_1 \oplus (1, 2, 3, 4, 6, 8)x_2 \oplus (1, 1, 1, 1, 1, 1)s_2 &\leq (6, 9, 11, 14, 17, 20) \\
x_1, x_2, s_1, s_2 &\geq 0.
\end{aligned}$$

Using the ranking function(1): $R(A) = \frac{a_1+2a_2+a_3+a_4+2a_5+a_6}{4}$

$$\max Z = 7x_1 + 14x_2$$

subject to

$$9x_1 + 9x_2 + 2s_1 = (4, 5, 7, 9, 10, 12)$$

$$9x_1 + 8x_2 + 2s_2 = (6, 9, 11, 14, 17, 20)$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

By simplex method:

$$x_1 = (0, 0, 0, 0, 0, 0)$$

$$x_2 = (.44, .55, .77, 1, 1.1, 1.3)$$

$$Z = (6.2, 7.7, 10.8, 14, 15.5, 18.6).$$

Now, we will use the second ranking function : $R(A) = \frac{a_1+a_2+a_3+a_4+a_5+a_6}{4}$:

$$\max Z = (21/4)x_1 + (57/4)x_2$$

s.t

$$(35/4)x_1 + (27/4)x_2 + (3/2)s_1 = (4, 5, 7, 9, 10, 12)$$

$$(21/2)x_1 + 6x_2 + (3/2)s_2 = (6, 9, 11, 14, 17, 20)$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

then the solution become:

$$x_1 = (0, 0, 0, 0, 0, 0)$$

$$x_2 = (.5, .7, 1.03, 1.3, 1.4, 1.7)$$

$$\text{and } Z = (8.4, 10.5, 14.7, 19, 21.1, 25.3).$$

And the last ranking function will give the values of x_1, x_2, Z as:

$$x_1 = (0, 0, 0, 0, 0, 0)$$

$$x_2 = (8/9, 10/9, 14/9, 2, 20/9, 8/3)$$

$$\text{and } Z = (76/9, 95/9, 133/9, 19, 190/9, 76/3).$$

4.3.1 Ranking Function Method For Solving Octagonal Fuzzy Linear Programming Problem

In the current subsection, the solution of Octagonal FLPP will be introduced.

Let

$$\text{Optimize } Z = CX$$

s.t

$$AX \{ \leq \geq = \} b \tag{4.1}$$

$$X \geq 0$$

be a fuzzy linear programming problem where $C = [c_j]$ crisp row vector, $A = [a_{ij}]$ is

an $m \times n$ octagonal fuzzy matrix, $b = (b_1, \dots, b_m)$ is an m-dimensional octagonal fuzzy number and $X = (x_1, x_2, \dots)$ is an n-dimensional vector. Then, we can write the previous FLPP as:

$$OptZ = CX$$

s.t

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_n \leq b_i$$

$$x_1, \dots, x_n \geq 0$$

$$i = 1, \dots, m.$$

where: $a_{i_1} = (a_{i_{11}}, a_{i_{12}}, \dots, a_{i_{18}})$, \dots , $a_{i_n} = (a_{i_{n1}}, \dots, a_{i_{n8}})$ and $b_i = (b_{i_1}, \dots, b_{i_8})$.

By using the ranking function the problem become:

$$OptZ = c_1x_1 + \dots + c_nx_n$$

s.t

$$\begin{aligned} & 2(a_{i_{11}}x_n + \dots + a_{i_{n1}}x_1) + 3(a_{i_{12}}x_1 + \dots + a_{i_{n2}}x_n) + 4(a_{i_{13}}x_1 + \dots + a_{i_{n3}}x_n) \\ & + 5(a_{i_{14}}x_1 + \dots + a_{i_{n4}}x_n) + 5(a_{i_{15}}x_1 + \dots + a_{i_{n5}}x_n) + 4(a_{i_{16}}x_1 + \dots + a_{i_{n6}}x_n) \\ & + 3(a_{i_{17}}x_1 + \dots + a_{i_{n7}}x_n) + 2(a_{i_{18}}x_1 + \dots + a_{i_{n8}}x_n) \\ & \leq 2b_{i_1} + 3b_{i_2} + 4b_{i_3} + 5b_{i_4} + 5b_{i_5} + 4b_{i_6} + 3b_{i_7} + 2b_{i_8} \end{aligned}$$

$$x_1, \dots, x_n \geq 0$$

$$i = 1, \dots, m.$$

Now, it will be converted to a crisp linear programming problem which can be solved using simplex method.

Example 4.3.4

$$\max Z = 5x_1 + 7x_2$$

s.t

$$(1, 3, 4, 5, 7, 9, 10, 12)x_1 \oplus (1, 2, 3, 4, 5, 6, 7, 8)x_2 \leq (11, 12, 13, 14, 15, 16, 17, 18)$$

$$(2, 4, 6, 8, 10, 12, 14, 16)x_1 \oplus (0, 2, 5, 8, 11, 14, 17, 20)x_2 \leq (20, 22, 23, 26, 27, 28, 30, 32)$$

$$x_1, x_2 \geq 0$$

Using the ranking function to get the following crisp linear programming problem:

$$\max Z = 5x_1 + 7x_2$$

s.t

$$2(x_1 + x_2) + 3(3x_1 + 2x_2) + 4(4x_1 + 3x_2) + 5(5x_1 + 4x_2) + 5(7x_1 + 5x_2)$$

$$+ 4(9x_1 + 6x_2) + 3(10x_1 + 7x_2) + 2(12x_1 + 8x_2) \leq (22 + 36 + 52 + 70 + 75 + 84 + 91 + 30)$$

$$2(2x_1 + 0x_2) + 3(4x_1 + 2x_2) + 4(6x_1 + 5x_2) + 5(8x_1 + 8x_2) + 5(10x_1 + 11x_2) +$$

$$4(12x_1 + 14x_2) + 3(14x_1 + 17x_2) + 2(16x_1 + 20x_2)$$

$$\leq (40 + 66 + 92 + 130 + 135 + 112 + 90 + 64)$$

$$x_1, x_2 \geq 0.$$

$$\max Z = 5x_1 + 7x_2$$

s.t

$$177x_1 + 126x_2 \leq 466$$

$$252x_1 + 268x_2 \leq 729$$

$$x_1, x_2 \geq 0.$$

By simplex method:

$$x_1 = 0, x_2 = 2.7 \text{ and } Z = 18.9$$

Chapter 5

Alpha-Cut Method For Solving Fuzzy Linear Programming Problem

This chapter is devoted to solve FFLPP using α -cut method. The idea of the current method is to convert the given problem into a crisp by replacing the fuzzy numbers by their α -cuts and then the intervals inequality technique.

5.1 Alpha-Cut

In this section, the definition of alpha-cut denoted by α -cut is introduced and then we present the general formulas of α -cuts of the fuzzy numbers that mentioned in chapter two.

Definition 5.1.1 [15] *The alpha cut or alpha level set of a fuzzy set is a crisp set defined as follows: $A_\alpha = \{x : \forall x \in X, \mu_A(x) \geq \alpha\}$.*

The α -cut of the fuzzy numbers are:

1. If $A = (a, b, c)$ triangular fuzzy number, the α -cut of A is:

$$A_\alpha = [a + (b - a)\alpha, c - (c - b)\alpha]$$

2. Let $A = (a, b, c, d)$ trapezoidal fuzzy number, then $[A\alpha = a + \alpha(b - a), d - \alpha(d - c)]$ is the alpha-cut of A

3.

$$A_\alpha = \begin{cases} 2\alpha(\alpha_1 - \alpha_2) + m - \alpha_1, -2\alpha(\beta_1 - \beta_2) + n - \beta_1, & \alpha \in [0, 0.5] \\ 2\alpha_2(\alpha_1 - 1) + m, -2\beta_2(\alpha - 1) + n & \alpha \in [0.5, 1] \end{cases} \quad (5.1)$$

where $A = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ hexagonal fuzzy number.

4. The α -cut of an octagonal fuzzy number $= (d, e, \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3)$

$$A_\alpha = \begin{cases} \frac{\alpha(\alpha_3 - \alpha_2)}{k} + d - \alpha_3, \frac{\alpha(\beta_3 - \beta_2)}{k} - e - \beta_3, & 0 \leq \alpha \leq k \\ \frac{(\alpha - 1)\alpha_1}{k + 1} + d, \frac{(\alpha - 1)\beta_1}{k - 1} + e & k \leq \alpha \leq 1 \end{cases} \quad (5.2)$$

5.2 Interval Linear Programming Problem

In the current section, we introduce some arithmetic concepts related to the intervals and linear programming with interval coefficients.

5.2.1 Interval Arithmetics

The operations on intervals can be defined as follows:

Definition 5.2.1 [4] Let $A = [a, b]$, $B = [c, d]$ be two intervals, then:

1. $A + B = [a + c, b + d]$

$$2. A - B = [a - d, b - c]$$

$$3. A.B = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$4. \frac{A}{B} = [\min(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}), \max(\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d})]$$

$$5. kA = [ka, kb]$$

5.2.2 Interval Linear Programming Problem

This subsection will illustrate the interval method for solving the FLPP.

Solving Interval Linear Programming Problem(ILPP) with \geq constrains

Theorem 5.2.1 [2] *For the interval inequality*

$\sum_{j=1}^n [a_j, \bar{a}_j] \geq [b, \bar{b}]$ where $\forall_j, \sum_{j=1}^n a'_j x_j \geq \underline{b}, \sum_{j=1}^n a''_j x_j \geq \bar{b}$ are the largest and Smallest feasible regions respectively,

where

$$a'_j = \begin{cases} \bar{a}_j, & x_j \geq 0 \\ \underline{a}_j & x_j \leq 0 \end{cases} \quad (5.3)$$

$$a''_j = \begin{cases} \underline{a}_j, & x_j \geq 0 \\ \bar{a}_j & x_j \leq 0 \end{cases} \quad (5.4)$$

Theorem 5.2.2 [2] for ILPP:

$$\begin{aligned} \min z &= \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j \\ \text{s.t} & \\ & \sum_{j=1}^n [\underline{a}_j, \bar{a}_j] x_j \geq [\underline{b}, \bar{b}] \end{aligned}$$

the best and worst optimum obtain by solving the following problems respectively :

$$\begin{aligned} \min z &= \sum_{j=1}^n \acute{c}_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n \acute{a}_j x_j \geq \underline{b}_i, i = 1, 2, \dots, m \end{aligned}$$

$$\begin{aligned} \min z &= \sum_{j=1}^n \acute{c}_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n \acute{a}_{ij} x_j \geq \bar{b}_i, i = 1, 2, \dots, m \end{aligned}$$

where:

$$a'_{ij} = \begin{cases} \bar{a}_{ij}, & x_j \geq 0 \\ \underline{a}_{ij}, & x_j \leq 0 \end{cases} \quad (5.5)$$

$$a''_j = \begin{cases} \underline{a}_{ij}, & x_j \geq 0 \\ \bar{a}_{ij}, & x_j \leq 0 \end{cases} \quad (5.6)$$

$$c'_j = \begin{cases} \underline{c}_j, & x_j \geq 0 \\ \bar{c}_j, & x_j \leq 0 \end{cases} \quad (5.7)$$

$$c''_j = \begin{cases} \bar{c}_j, & x_j \geq 0 \\ \underline{c}_j, & x_j \leq 0 \end{cases} \quad (5.8)$$

Theorem 5.2.3 [2]

If the objective function is change to \max , then the best and worst optimum values obtain by the following:

$$\begin{aligned} \max z &= \sum_{j=1}^n c''_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n a'_{ij} x_j \geq \underline{b}_i, i = 1, 2, \dots, m \end{aligned}$$

$$\begin{aligned} \max \underline{z} &= \sum_{j=1}^n c'_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n a''_{ij} x_j \geq \bar{b}_i \end{aligned}$$

Solving ILPP with \leq constants

Theorem 5.2.4 [2]

For ILPP:

$$\begin{aligned} \min z &= \sum_{j=1}^n [\underline{c}_j, \bar{c}_j] x_j \\ \text{s.t} & \\ & \sum_{j=1}^n [\underline{a}_{ij}, \bar{a}_{ij}] x_j \leq [\underline{b}_i, \bar{b}_i] \end{aligned}$$

The best and worst optimum values obtains by solving the following:

$$\begin{aligned} \min \underline{z} &= \sum_{j=1}^n \underline{c}_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n \underline{a}_{ij} x_j \leq \underline{b}_i \end{aligned}$$

$$\begin{aligned} \min z &= \sum_{j=1}^n \underline{c}_j'' x_j \\ \text{s.t} & \\ & \sum_{j=1}^n \underline{a}_{ij} x_j \leq \underline{b}_i, i = 1, 2, \dots, m \end{aligned}$$

$\underline{c}_j, \underline{c}_j'', \underline{a}_{ij}$ and \underline{a}_{ij}'' are as before.

Theorem 5.2.5 [2]

If the objective function is changed to max.

Then the best and worst optimum values obtained by:

$$\begin{aligned} \max z &= \sum_{j=1}^n c_j'' x_j \\ \text{s.t} & \\ & \sum_{j=1}^n a_{ij}'' x_j \leq \bar{b}_i \end{aligned}$$

$$\begin{aligned} \max \underline{z} &= \sum_{j=1}^n \underline{c}_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n \underline{a}_{ij} x_j \leq \underline{b}_i \end{aligned}$$

Now, consider the following FLPP

$$\begin{aligned} \text{opt}(z) &= \sum_{j=1}^n c_j x_j \\ \text{s.t} & \\ & \sum_{j=1}^n a_{ij} x_j \leq b_i, \sum_{j=1}^n a_{ij} x_j \geq b_i, x_j \geq 0 \end{aligned}$$

where c_j, a_{ij}, b_i are fuzzy number numbers, then we can use the alpha-cut of each fuzzy number to convert the FLPP to an Interval Linear Programming Problem(ILPP) and using the previous theorems to convert it to crisp linear programming problem which can be solved by simplex method as we can see in the next section

5.3 Applications on Alpha-cut Method

This section consists of four examples about the method which are:

Example 5.3.1

$$\begin{aligned}maxz &= (1, 2, 3)x_1 + (7, 8, 9)x_2 \text{ s.t} \\ &(1, 2, 5)x_1 + (1, 3, 4)x_2 \leq (4, 5, 7) \\ &(2, 4, 6)x_1 + (1, 2, 3)x_2 \leq (6, 9, 11) \\ &x_1, x_2 \geq 0 \\ &\alpha = 0.5\end{aligned}$$

$$\alpha_{(1,2,3)} = [1.5, 2.5], \alpha_{(7,8,9)} = [7.5, 8.5], \alpha_{(1,2,5)} = [1.5, 3.5], \alpha_{(1,3,4)} = [2, 3.5], \alpha_{(4,5,7)} = [4.5, 6], \alpha_{(2,4,6)} = [3, 5], \alpha_{(1,2,3)} = [1.5, 2.5] \text{ and } \alpha_{(6,9,11)} = [7.5, 10]$$

The problem become:

$$\begin{aligned}maxz &= [1.5, 2.5]x_1 + [7.5, 8.5]x_2 \\ &\text{s.t} \\ &[1.5, 3.5]x_1 + [2, 3.5]x_2 \leq [4.5, 6] \\ &[3, 5]x_1 + [1.5, 2.5]x_2 \leq [7.5, 10] \\ &x_1, x_2 \geq 0\end{aligned}$$

using interval method we get:

The best optimum solution :

$$\begin{aligned} \max \bar{z} &= \sum_{j=1}^n c_j'' x_j \\ \text{s.t} & \\ & \sum_{j=1}^n a_{ij}'' x_j \leq \bar{b}_i \end{aligned}$$

where

a_j'' and c_j'' are as said before

$$\begin{aligned} \max \bar{z} &= 2.5x_1 + 8.5x_2 \\ \text{s.t} & \\ & 1.5x_1 + 2x_2 \leq 6 \\ & 3x_1 + 1.5x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

using simplex method we get:

$$x_1 = 0, x_2 = 3$$

$$z = (21, 24, 27)$$

And the worst optimum solution is:

$$\begin{aligned} \max z &= \sum_{j=1}^n c_j x_j \\ \text{s.t} \\ \sum_{j=1}^n a'_{ij} x_j &\geq b_i \end{aligned}$$

where a'_j and c'_j are as mentioned before.

$$\begin{aligned} \max z &= 1.5x_1 + 3.5x_2 \leq 4.5 \\ \text{s.t} \\ 3.5x_1 + 3.5x_2 &\leq 4.5 \\ 5x_1 + 2.5x_2 &\leq 7.5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Therefore, $x_1 = 0, x_2 = 1.28$ and $z = (8.96, 10.24, 11.52)$

The next example is about trapezoidal fuzzy number

Example 5.3.2

$$\begin{aligned} \max z &= (1, 2, 3, 4)x_1 + (7, 8, 9, 10)x_2 \\ \text{s.t} \\ (1, 2, 5, 7)x_1 + (1, 2, 3, 4)x_2 &\leq (4, 5, 7, 9) \\ (2, 4, 6, 8)x_1 + (1, 2, 3, 4)x_2 &\leq (6, 9, 11, 14) \\ x_1, x_2 &\geq 0 \end{aligned}$$

using $\alpha = 1$

since $\alpha = 1$:

$$\alpha_{(1,2,3,4)} = [1 + \alpha, 4 - \alpha] = [2, 3], \alpha_{(7,8,9,10)} = [7 + \alpha, 10 - \alpha] = [8, 9], \alpha_{(1,2,5,7)} = [2, 5], \alpha_{(1,3,4,5)} = [3, 4], \alpha_{(4,5,7,9)} = [5, 7], \alpha_{(2,4,6,8)} = [4, 6], \alpha_{(1,2,3,4)} = [2, 3] \text{ and } \alpha_{(6,9,11,14)} = [9, 11]$$

By substituting these intervals and using the previous theorems the best and worst optimum solutions are: The best optimum solution is:

$$\begin{aligned} \max \bar{z} &= 3x_1 + 9x_2 \\ \text{s.t} & \\ &2x_1 + 3x_2 \leq 7 \\ &4x_1 + 2x_2 \leq 11 \\ &x_1, x_2 \geq 0 \end{aligned}$$

Hence, $x_1 = 0, x_2 = 2.333$ and $z = (16.331, 18.664, 20.997, 23.33)$

And the worst optimum solution is:

$$\begin{aligned} \max \underline{z} &= 2x_1 + 8x_2 \\ \text{s.t} & \\ &5x_1 + 4x_2 \leq 5 \\ &6x_1 + 3x_2 \leq 9 \\ &x_1, x_2 \geq 0 \end{aligned}$$

And hence, $x_1 = 0, x_2 = 1.25$,

$z = (8.75, 10, 11.25, 12.5)$

By using $\alpha = 0.5$, the solution is:

The best optimum solution is, $x_1 = 0, x_2 = 4 \implies z = (28, 32, 36, 40)$

And the worst optimum is: $x_1 = 0, x_2 = 1.25 \implies z = (2.75, 10, 11.25, 12.5)$

Example 5.3.3

$$\max z = (1, 2, 3, 4, 5, 6)x_1 + (7, 8, 9, 10, 11, 12)x_2$$

s.t

$$(1, 2, 5, 7, 9, 11)x_1 + (1, 3, 4, 5, 6, 8)x_2 \leq (4, 5, 7, 9, 10, 12)$$

$$(2, 4, 6, 8, 10, 12)x_1 + (1, 2, 3, 4, 6, 8)x_2 \leq (6, 9, 11, 14, 17, 20)$$

$$x_1, x_2 \geq 0$$

using $\alpha = 1$

since $\alpha = 1$:

$$\alpha_{(1,2,3,4,5,6)} = [6(0) + 1, 2(6)(0) + 2] = [1, 2], \alpha_{(7,8,9,10,11,12)} = [7, 8], \alpha_{(1,2,5,7,9,11)} = [1, 2],$$

$$\alpha_{(1,3,4,5,6,8)} = [1, 3], \alpha_{(4,5,7,9,10,12)} = [4, 5], \alpha_{(2,4,6,8,10,12)} = [2, 4], \alpha_{(1,2,3,4,6,8)} = [1, 2] \text{ and}$$

$$\alpha_{(6,9,11,14,17,20)} = [6, 9].$$

since the problem is maximum and the constraints \leq then:

The best optimum solution:

$$\begin{aligned}
 \max z &= 2x_1 + 8x_2 \\
 \text{s.t} & \\
 &x_1 + x_2 \leq 5 \\
 &2x_1 + x_2 \leq 9 \\
 &7x_1, x_2 \geq 0
 \end{aligned}$$

$$x_1 = 0, x_2 = 5, z = (35, 40, 45, 50, 55, 60)$$

And similarly, the worst optimum solution is: $x_1 = 0, x_2 = \frac{4}{3}$

$$\underline{z} = (\frac{28}{3}, \frac{32}{3}, 12, \frac{40}{3}, \frac{44}{3}, 16)$$

And the last example about octagonal fuzzy number:

Example 5.3.4

$$\max z = (1, 2, 3, 4, 5, 6, 7, 8)x_1 + (9, 10, 11, 12, 13, 14, 15, 16)x_2$$

s.t

$$(2, 4, 6, 8, 10, 12, 14, 16)x_1 + (3, 6, 9, 12, 15, 18, 21, 24)x_2 \leq (4, 5, 7, 8, 10, 11, 12, 13)$$

$$7(0, 1, 3, 4, 5, 7, 8, 9)x_1 + (1, 2, 3, 7, 8, 9, 10, 12)x_2 \leq (4, 6, 7, 9, 12, 14, 16, 17)$$

$$7x_1, x_2 \geq 0$$

using $\alpha = 1, k = 0.6$

$$\begin{aligned}
 \alpha_{(1,2,3,4,5,6,7,8)} &= [1, 2], \alpha_{(9,10,11,12,13,14,15,16)} = [9, 10], \alpha_{(2,4,6,8,10,12,14,16)} = [2, 4], \alpha_{(3,6,9,12,15,18,21,24)} = \\
 &[3, 6], \alpha_{(4,5,7,8,10,11,12,13)} = [4, 5], \alpha_{(0,1,3,4,5,7,8,9)} = [0, 1] \alpha_{(1,2,3,7,8,9,10,12)} = [1, 2] \text{ and } \alpha_{(4,6,7,9,12,14,16,17)} = \\
 &[4, 6]
 \end{aligned}$$

The best optimum solution: $x_1 = 0, x_2 = \frac{5}{3}$ and $z = (15, 16.6, 18.3, 20, 21.6, 23.3, 25, 26.6)$

And the worst optimum solution is: $x_1 = 0, x_2 = \frac{2}{3}, z = (6, 6.6, 7.3, 8, 8.6, 9.3, 10, 10.5)$

Conclusion

In this thesis, we studied three methods for solving fuzzy linear programming problem. The first method is about solving fully fuzzy linear programming problem with equality in constraints which we are used the method explained in [23] for solving the constraints and then used the solution of constraints to give the value of Z . Second method is the ranking function method. In this method, the ranking function of each fuzzy number is deduced to convert each fuzzy equation to a crisp equation and then solving it by simplex method. The last method is "Alpha-Cut Method" which converting the fuzzy linear programming problem to an interval linear programming problem and then converting it to a crisp linear programming problem as in [2] which can be solved using simplex method.

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ملخص

طرق حل مسائل البرمجة الخطية الضبابية أنوار عصام ظاهر

في هذه الأطروحة، سنستخدم بعض الطرق لحل مسائل البرمجة الخطية الضبابية وفي كل فصل سيتم تناول كل طريقه بالتفصيل مع ذكر بعض الأمثلة عليها. وهذه الطرق تحول مسائل البرمجة الخطية الى مسائل برمجة عادية التي يمكن حلها بسهولة باستخدام عدة طرق.