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Fast and Accurate Methods for Calculating Riemann Zeta Function

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Committee Decision

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”An Najah National University”

DEDICATION

To my parents,

To my life partner (Mohammad), and,

To everyone encouraged me to continue this work.

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ABSTRACT

The Riemann Zeta function $\zeta(s)$ occurs as an important tool in many fields of mathematics and physics. Because of the lack of the closed formulas for calculating this function, many fast and accurate approximation methods are developed and numerically verified.

Direct summation is used to evaluate $\zeta(s)$. Results show that decreasing s increases the error for fixed number of terms. To avoid the obstacle of slowing down of convergence in the case of small values of s, some fast algorithms are used and compared to direct calculations.

Due to its high efficiency, Borwein algorithm is deeply investigated and widely used in this thesis. Various classes of polynomials are incorporated in the algorithm. Results are compared. Functions of the form $p(x) = x^n(1-x)^n$ and the Gegenbauer functions give best approximations.

Contents

INTRODUCTION

The Riemann Zeta function $\zeta(s)$ is defined by the convergent series [89]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}, \ \operatorname{Re}(s) > 1 \tag{1}$$

Due to its valuable applications in many fields of mathematics and physics, Riemann Zeta function is encountered as the most important special function in mathematics [54]. It occurs, for example, in number theory with some other functions, like; Euler function $\phi(n)$ which counts the number of natural numbers less than n and relatively prime to it. The two functions are connected by the relation [54]

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$$

Riemann Zeta function arises also in the related formulas of Möbius function $\mu(n)$ [54]:

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

More substantially, the distribution of prime numbers is related in some way to the one of Riemann Zeta function's zeros [89]. In fact, $\zeta(s)$ has two types of zeros; trivial zeros that occur at negative even integers with no significance, and nontrivial zeros with real part assumed to equal $\frac{1}{2}$ [54]. Actually this assumption about the position of the nontrivial zeros is known as Riemann Hypothesis (RH) and it is considered as the greatest unsolved problem in mathematics. Hence, proving RH implies tremendous applications for the Riemann Zeta function, since knowing the distribution of $\zeta(s)$'s zeros helps in understanding the properties of the prime numbers, discovering their distribution, and therefore leads to better encryption systems which rely basically on primes [79]. Furthermore, the zeros of $\zeta(s)$ are connected to the eigenvalues of some random matrices, and thus $\zeta(s)$ is useful in modeling the energy of some heavy nuclei which are distributed like $\zeta(s)$'s zeros [18]. Thus, the Riemann Zeta function connects many different fields of mathematics and physics with each other.

It is used also in the context of the expectation of the relative frequency of an individual obeying Zipf's law with known frequency rank [48, 70]. Thus, if k is the rank variable, then the underlying discrete Zipfian or Zeta distribution has the probability mass function [70]

$$f_s(k) = \frac{k^{-s}}{\zeta(s)}, \ s > 1 \tag{2}$$

Zipf's law is commonly used to find out the probability of the appearance of some word in arbitrary text according to its rank of frequency in the tested corpus [7, 85, 93].

In quantum physics, Riemann Zeta function occurs in the derivation of the total Casimir energy for the electromagnetic waves between two parallel conducting plates in the vacuum [13]. Also, it contributes in enhancing path integral of the harmonic oscillator's spectrum by calculating the corresponding partition function [84]. It is also included in deriving Stefan Boltzmann Law [20]. Not far away, $\zeta(s)$ is appearing in identifying the distributions of bosonic and fermionic particles [22]. Riemann Zeta function is also useful in modeling neural networks to recognize handwritten texts, and for processing images [86].

It worths mentioning that, in general, the exact values of the Riemann Zeta function still unknown till this day. Thus, the mathematical world tends to find approximations for these values due to their real importance and applications. In order to obtain accurate approximations for $\zeta(s)$, many algorithms had been proposed in the past and in the present centuries [14, 17, 74, 76, 77].

In this thesis, many fast and accurate approximation methods will be presented and some numerical verifications are done in the case of some well known methods. These methods include basically; direct approach, Euler Maclaurin summation, and Borwein algorithms. Besides, many rapid convergent alternative series will be given in the case of some values of s. Manipulations of the cutoff parameters in the case of some methods are done, the behavior of these parameters is studied, and then some comparisons, plots, and preferences are made. In fact, we are concern more about Borwein algorithms [14], which works by using the coefficients of any polynomial that does not vanish at -1 in such formulas to obtain efficient approximations for $\zeta(s)$. Refinements for Borwein algorithms are fulfilled by inserting some polynomials into the algorithm to obtain more accurate approximations for $\zeta(s)$. Many classes of polynomials are used, they were organized into orthogonal, Horadam, and traditional polynomials. Comparisons between the studied polynomials, preferences between them are going to be shown, and also the suitable degree of some polynomials to a given accuracy is determined. The efficiency of Borwein algorithms is checked out in the case of each set of the presented polynomials by calculating the corresponding relative error and observing its behavior as the polynomial degree n increases. For the families of polynomials with free parameter (in addition to n), sometimes a bound of the additional parameter is determined, by which its family has neither desirable approximations nor preferable little errors as the ones obtained by the polynomials corresponding to this bound parameter.

We will restrict our research in this thesis to the case when $\mathrm{Im}(s) = 0$. The structure of the thesis is organized into four chapters; Chapter 1 gives a solid historical background of the Riemann Zeta function. Chapter 2 offers the necessary preliminaries needed for the subsequence chapters, which are basically about the polynomials used with Borwein algorithms for approximating Riemann Zeta function. In Chapter 3, many fast and accurate methods will be presented and numerically verified. In Chapter 4, the polynomials presented in Chapter 2 are included in Borwein algorithm and the resulted accuracy is observed.

Chapter 1

RIEMANN ZETA FUNCTION

In this chapter, we present the Riemann Zeta function (1) in depth by the means of the following sections.

1.1 From Dirichlet Series to Riemann Zeta function

As a powerful tool, the Dirichlet series arise frequently in the area of interest due to their central contribution in the analytic number theory. These series which were introduced early in the nineteenth century by the German mathematician Johan Dirichlet have the form [10]

$$D(s,a_n) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{1.1}$$

where $\{a_n\}$ is a sequence of complex numbers (its terms are often called coefficients), and s is a complex number.

An important feature of Dirichlet series that every series has a region $\{\mathrm{Re}(s) > constant1\}$ where it conditionally converges and another one $\{\mathrm{Re}(s) > constant2\}$ where the series is absolutely convergent. The convergent series play a pivotal role since the functions defined by the means of their sum are analytic [38].

Historically, Dirichlet series were used to prove the arithmetic progression of primes, the theorem that emphasizes the existence of infinitely many primes among the progression $a, a+k, a+2k, a+3k, \ldots$ where a and k are co-prime [29]. In fact, this theorem generalizes Euclid conjecture, the conjecture that was turned into theorem by Leonhard Euler [10].

As a particular instance, Riemann Zeta function arises from the most renowned and significant Dirichlet series in the analysis, which can be attained by taking the sequence $a_n = 1$ in (1.1) [38]. Moreover, if $\mathrm{Im}(s) = 0$, this gives the traditional Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \mathrm{Im}(s) = 0 \tag{1.2}$$

Series (1.2) was seriously considered by the Swiss mathematician Leonhard Euler, who devoted his time to evaluate its special case (when $\mathrm{Re}(s) = 2$), the so- called Basel Problem. In fact, the later problem had been raised to the mathematical community by Mengoli in 1644 when he failed to find the exact sum of the reciprocals of the squared natural numbers

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

Many mathematicians in the area took the challenge and tried to solve it, like Leibniz, Bernoulli and Basel, to name few. Unfortunately, they all failed and the problem did not meet its match until 1737. By when Euler announced his success in finding the outlook answer which is $\frac{\pi^2}{6}$ [32].

Since Basel problem is a particular case of the evaluation of Euler series (1.2), then the fulfilled success attained by Euler did not stop here, but rather this motivated him to find a general formula for the evaluation of (1.2) in case $\mathrm{Re}(s)$ is an even natural number. Moreover, he made good approximations (may reach six correct digits) in the case of odd natural numbers.

Later on, the Russian mathematician Pafnuty Chebychev extended the validity of series (1.2) to any complex number s with $\mathrm{Re}(s) > 1$ [89].

More substantially, in 1859, the German mathematician Bernhard Riemann introduced the Riemann Zeta function as the one that analytically extends the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad , \mathrm{Re}(s) > 1 \tag{1.3}$$

to the whole complex plane except at its unique simple pole $(s = 1)$ [12].

Although the Riemann Zeta function was inspired from Euler calculations, but the function is commonly known to Riemann. In fact, this makes sense, since the importance of the Riemann Zeta function and the consequence applications of it all come from the analytic continuation established by Riemann.

1.2 Features of the Riemann Zeta Function

The Riemann Zeta function $\zeta(s)$ is considered by many scholars to be the most significant special function in analysis and at the same time it is the function of mystery. This is because of many unknown and unverified facts concerning it, like the exact values of the Riemann Zeta function at odd integers and the Riemann Hypothesis as will be presented latterly in this thesis. This regular function can be defined on the region $\{s \in \mathbb{C}, \operatorname{Re}(s) > 1\}$ by the identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1.4}$$

Or by Euler Product formula [2]

$$\zeta(s) = \prod_{p:prime} \frac{1}{1-p^{-s}} \tag{1.5}$$

Or even by [2]

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{n=0}^{\infty} \frac{x^{s-1}}{e^x+1} \quad ,\Gamma \text{ is the Gamma function} \tag{1.6}$$

Beyond this region, many formulas will be developed for this aim in section (1.4).

It can be seen that (1.4) is absolutely convergent [59]. Further, like other Dirichlet series, (1.4) is uniformly convergent in any finite region within the region of definition. So, (1.2) is uniformly convergent for $\operatorname{Re} s \geqslant 1+\delta,\ \delta > 0$. This directly follows from Weierstrass criterion [90].

1.3 The Bridge to Analytic Number Theory

The Riemann Zeta function, which is the bedrock of the contemporary analytic number theory, is deeply related to primes' distribution, from where it occupies the importance it has [37]. In fact, the beginning of this connection refers to 1749 when Euler was trying to prove the infinity of the infinite sum of reciprocals of prime numbers

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$$

In his way to do that, he developed the so-called Euler product formula as a tool to reach his goal, he used [32]

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \frac{2^s 3^s 5^s 7^s \ldots}{(2^s-1)(3^s-1)(5^s-1)(7^s-1)\ldots} \tag{1.7}$$

Which is the same as formula (1.5). This formula was proved rigorously in 1876 by Leopold

Kronecker for $\text{Re}(s) > 1$ [89]. As an immediate corollary, there are infinite number of primes, the fact that already known and proved by Euclid. Besides, the primes are distributed in a fair way among the whole numbers. And the Riemann Zeta function does not have any zeros in the region $\{s, \text{ Re}(s) > 1\}$ [89].
The significance of the Riemann Zeta function is actually come from its participation in proving the Prime Number theorem (PNT). The conjecture that was formulated (in its modern form) by Gauss and Legendre and took about one century before coming theorem [51]. This theorem made a revolution in the analytic number theory according to the suggested pattern of primes' distribution that it assumes. This asymptotic distribution expressed by

$$\pi(x) \sim \frac{x}{log(x)} \tag{1.8}$$

where $\pi(x)$ is a function that counts the number of primes less than or equal to x.
The role of the Riemann Zeta function in proving PNT is referred to the paper published by Riemann in 1859, in which he discussed the number of primes less than a given value [54]. Although he did not prove PNT, but he connected for the first time the complex analysis to number theory and developed many ideas and tools used later to prove PNT.
It was Jacques Hadamard and Charles Vallée-Poussin who proved the PNT at the same time and separately in 1896. It worths mentioning that PNT was proved latterly using elementary methods rather than complex analysis or the theory of Riemann Zeta function by Selbreg and Erdős [89].

1.4 Analytic Continuation and Functional Equations

In general, the analytic continuation is enlarging the domain of an analytic function by which it remains analytic. The uniqueness of this continuation is guaranteed if the new analytic function (extension) is defined on a nonempty open connected set (domain) [3]. So, if the analytic function f is defined on the domain U, and the analytic function F is defined on the domain V such that $U \subseteq V \subseteq \mathbb{C}$ and $F(w) = f(w), \forall w \in U$. Then, we say that F is the analytic continuation of f to V.
In our case, $\zeta(s)$ is an analytic function which is defined by (1.2) on the open connected and of course nonempty set$\{s \in \mathbb{C}, \text{Re}(s) > 1\}$. Fortunately, it can be defined beyond this domain by several techniques using the idea of meromorphic (A function is called meromorphic if it is analytic everywhere except at a countable number of points) continuation provided by

Riemann to the whole complex plane. We will present that over two steps:

Step 1: Analytic continuation to the half plane$\{s, \mathrm{Re}(s) > 0,\ s \neq 1\}$

This can be done by using one of the following two techniques:

1. Using Dirichlet eta Function

The Dirichlet eta function or Zeta alternating series is an analytic function defined on the half plane $\{s, \mathrm{Re}(s) > 0, s \neq 1\}$ by [89]

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \tag{1.9}$$

Tricky, write the above series as

$$\eta(s) = (1 - 2^{1-s}) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

So,

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})} \eta(s)$$

which is an analytic continuation of the Riemann Zeta function to a larger region $\{s, \mathrm{Re}(s) > 0,\ s \neq 1\}$.

2. Using the formula [81]

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{t\}}{t^{s+1}} dt \tag{1.10}$$

where $\{t\}$ symbolizes the fractional part of t.

Since the above formula is valid in the half plane $\{s, \mathrm{Re}(s) > 0, s \neq 1\}$, so it provides an analytic continuation for $\zeta(s)$ to the desired half plane.

Step 2: Analytic continuation to $\mathbb{C}$ except at $(s = 1)$

Similarly, we give here two techniques to construct the analytic continuation to the whole complex plane.

1. Using Euler Maclaurin Summation formula (EMS)

The Riemann Zeta function can be approximated by EMS by [89]

$$\zeta(s)=\frac{1}{s-1}+\frac{1}{2}+\sum_{i=2}^{q}\frac{B_i}{i!}s(s+1)...(s+i-2)-\frac{1}{q!}s(s+1)...(s+q-1)\int_1^{+\infty}B_q(\{x\})x^{-s-q}dx \tag{1.11}$$

which is valid for $\mathrm{Re}(s)>-q$, where q is any natural number and B_j is the j^{th} Bernoulli number.

By choosing q large enough, so (1.11) establishes the analytic continuation of $\zeta(s)$ to the whole complex plane except at $(s=1)$ which is obvious from the first term in (1.11) [81].

2. Using the famous functional equation [54]

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-(\frac{1-s}{2})}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{1.12}$$

As given from step 1, $\zeta(s)$ can be extended to the region $\{s,\mathrm{Re}\,s>0,\, s\neq 1\}$, then by (1.12), $\zeta(s)$ can be defined on $\{s,\,\mathrm{Re}(s)<0, s\neq 1\}$.

The above functional equation is considered as a fundamental one, since many properties of $\zeta(s)$ arise from it. Formula (1.12) was proved by Riemann by more than one way, and today there are several many other proofs, Titchmarsh gave seven of them in his book [89]. It is remarkable to observe the symmetry of (1.12) respect to the line $\mathrm{Re}(s)=\frac{1}{2}$. However, the functional equation of $\zeta(s)$ has variant versions, it may be written as

$$\zeta(s)=2^s\pi^{(s-1)}sin(\frac{s\pi}{2})\Gamma(1-s)\zeta(1-s) \tag{1.13}$$

By replacing s by $(1-s)$, this yields

$$\zeta(1-s)=2^{(1-s)}\pi^{-s}cos(\frac{s\pi}{2})\Gamma(s)\zeta(s) \tag{1.14}$$

1.5 Values of the Riemann Zeta Function at Integers

As mentioned previously, the Riemann Zeta function $\zeta(s)$ was firstly evaluated at the positive integer $(\mathrm{Re}\,s=2)$ as a special case (Basel Problem) by Euler. Also, the later evaluated $\zeta(s)$ at every positive even integer s. But he could not make the same for odd integers, but rather, he gave approximations to $\zeta(s)$ for $\mathrm{Re}(s)=3,5,7,9$. Actually, the evaluation at these arguments is still an open problem till this day.

Euler assumed that

$$\zeta(s)=N.\pi^s$$

where N is interpreted as

$$N := \begin{cases} \text{rational number}, & s \text{ even} \\ \text{function of } \log 2, & s \text{ odd} \end{cases}$$

But, he never discovered the nature of N when the argument s is odd [12].
Values of $\zeta(s)$, for $s \in \mathbb{Z}$ will be given in details by the following subsections, while the methods for evaluating $\zeta(s)$ when $s \in \mathbb{R}$ are given in Chapter 3 and Chapter 4.

1.5.1 Values of $\zeta(s)$ at positive even integers

The value of the Riemann Zeta function at even natural numbers is given by [33]

$$\zeta(2k) = \frac{(-1)^{k-1} 4^k \pi^{2k} B_{2k}}{2(2k)!} \tag{1.15}$$

where k is positive integer and B_{2k} is the $2k^{th}$ Bernoulli number. Table 1.1 shows the value of $\zeta(2k)$ for $k = 1, 2, 3, 4, 5$.

k	$\zeta(2k)$
1	$\frac{\pi^2}{6}$
2	$\frac{\pi^4}{90}$
3	$\frac{\pi^6}{945}$
4	$\frac{\pi^8}{9450}$
5	$\frac{\pi^{10}}{93555}$

Table 1.1: Values of $\zeta(2k)$ for $k = 1, 2, 3, 4, 5$.

1.5.2 Values of $\zeta(s)$ at negative even integers

From the second version of $\zeta(s)$'s functional equation (1.13), it is easy to notice that the negative even integers are direct zeros of $\sin(\frac{s\pi}{2})$ and hence they are also zeros of $\zeta(s)$. So,

$$\zeta(-2k) = 0 \tag{1.16}$$

for every positive integer k.
Note that the previous result can be obtained also from the first version of the functional

equation (1.12), since

$$\Gamma(-\frac{2k}{2}) = 0$$

This is because Euler - Gamma function Γ has simple zeros at negative integers [81]. In fact, these direct zeros are called "trivial" or "simple" zeros of $\zeta(s)$.

1.5.3 Values of $\zeta(s)$ at positive odd integers

When s is positive odd integer, the exact values of $\zeta(s)$ are unknown till this day, and then there is no closed formula for calculating $\zeta(s)$ as in the case of even integers.
At odd integers, the Riemann Zeta function can be represented by the integral

$$\zeta(2k+1) = \frac{(-1)^{k+1}(2\pi)^{2k+1}}{2(2k+1)!}\int_0^1 B_{2k+1}(x)cot(\pi x)dx \tag{1.17}$$

Because of the importance of these values, many numerical methods were developed to approximate them. These methods compete in enhancing $\zeta(s)$ approximations and to which correct digit they are offered with. Some of these methods will be presented in Chapter 3.
Moreover, $\zeta(s)$ values are not verified whether they are rationals or irrational when s is positive odd integer, except $\zeta(3)$ which is proved by Apery (1978) to be an irrational number and it is some times known as Apery constant [9].
The same for other positive odd integers is not guaranteed. But, many theorems appear in the area about their irrationality, one of them refers to Rivoal (2000) which insures the existence of infinite number of irrational numbers among the sequence $\zeta(3), \zeta(5), \zeta(7), ...$ [33].
Rivoal gave another theorem in (2002) which guarantees the irrationality of one of the values $\zeta(5), \zeta(7), ..., \zeta(21)$ [33].
Another more precise theorem is established by V. Zudilin (2001), it says that one of the four values $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational [94].

1.5.4 Values of $\zeta(s)$ at negative odd integers

In contrast to the positive odd values, Riemann Zeta function values at negative odd integers has an explicit formula for evaluation, which is given by

$$\zeta(-2k+1) = \frac{-B_{2k}}{2k} \tag{1.18}$$

where k is a positive integer.
This can be attained if $2k$ is substituted instead of s in the functional equation (1.12) and using the fact [2]

$$\Gamma(\frac{1}{2}-k)=\frac{(-4)^{k}(k)!\sqrt{\pi}}{(2k)!} \tag{1.19}$$

1.6 Zeros of the Riemann Zeta function

Like any function, the zeros or the roots of Riemann Zeta function are the numbers at which the value of the function equals zero. Some of Riemann Zeta function's zeros can be determined easily from the functional equation as pointed out in (1.5.2), these zeros are called trivial zeros and they occur at negative even integers.
In fact, there are other zeros that are nontrivial, whose position is conjectured to be at the critical line $\text{Re}(s)=\frac{1}{2}$, but this has never been proved. This conjecture or more commonly hypothesis is known as Riemann Hypothesis (RH). It lies at the top of the Clay institute list as one of the greatest seven unsolved problems in mathematics [27].
As Euler Product formula (1.5) implies, the region $\{s \in \mathbb{C}, \text{Re}(s)>1\}$ has no zeros within it and by the symmetry of (1.12), it can be deduced that $\{s \in \mathbb{C}, \text{Re}(s)<0\}$ is also a free region of the nontrivial zeros of $\zeta(s)$.
Thus, the nontrivial zeros lie in the region $\{s \in \mathbb{C}, 0 \leq \text{Re}(s) \leq 1\}$, which is called the critical strip [89]. In fact, we are sure that there are an infinite number of those zeros in this strip by the conjecture established by Riemann and proved by Hadamard [67]. Another more specialize theorem was given by Hardy, who proved the existence of infinitely many s such that $\zeta(\frac{1}{2}+\text{Im}(s))=0$. In other words, he proved that there is infinite number of complex zeros in the critical line of $\zeta(s)$.
These nontrivial zeros are distributed in a symmetric way according to the critical line of $\zeta(s)$, which means that if ρ is a zero of $\zeta(s)$, i.e $\zeta(\rho)=0$, then $1-\rho$ is also a zero of $\zeta(s)$. In fact, $\bar{\rho}$ and $1-\bar{\rho}$ are also zeros of $\zeta(s)$ since $\zeta(\bar{s})=\overline{\zeta(s)}$, where $\overline{\rho}$ is the complex conjugate of ρ [81].
Numerically, the Riemann hypothesis was investigated by many scholars to be true. In particular, Riemann was the first one who calculated the first few of the nontrivial zeros of $\zeta(s)$ by hand and then established the Riemann Hypothesis [27]. Later, Gram used EMS formula

to carry out the calculations of those nontrivial zeros [67]. Today digital and super computers take the place and are used to calculate high number of zeros which may reach 10^{22} zeros [73]. All of the calculated zeros were found to lie on the critical line of $\zeta(s)$, but this does not solve the Riemann Hypothesis, but this may help in understanding it. So, the RH has not been proved or disproved until these days.
An illuminating result concerns the nontrivial zeros of the Riemann Zeta function was constructed by Riemann and is given by [81]

$$s(1-s)\Gamma(\frac{s}{2})\zeta(s) = e^{(-bs)}\prod(1-\frac{s}{\rho})e^{\frac{s}{\rho}} \tag{1.20}$$

where ρ runs over all nontrivial zeros in the critical strip, while the constant b is defined as $b = \log 2\pi - 1 - \frac{\gamma}{2}$, and γ is Euler's constant.

1.7 Generalized Riemann Zeta functions

There exist many generalizations of the Riemann Zeta function, that were obtained by different ways from (1.2). We will give some of these generalizations in details in the following subsections.

1.7.1 Hurwitz Zeta function

Hurwitz Zeta function is defined analogously as the Riemann Zeta function by including a real parameter α to the later [89]

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \tag{1.21}$$

where $0 < \alpha \leq 1$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$. It is remarkable to observe that $\zeta(s,1) = \zeta(s)$, which shows that the Riemann Zeta function is a special case of the Hurwitz Zeta function.
Actually, Hurwitz Zeta function does not share the most properties that the Riemann Zeta function has, but obeys Euler Product formula at particular cases only when ($\alpha = 1$ or $\alpha = \frac{1}{2}$).
Fortunately, it satisfies a certain functional equation [54]

$$\zeta(s,\alpha)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}(\sin\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\cos 2m\pi\alpha}{m^{1-s}}+\cos\frac{\pi s}{2}\sum_{m=1}^{\infty}\frac{\sin 2m\pi\alpha}{m^{1-s}}) \tag{1.22}$$

Equation (1.22) reduces to the functional equation of the Riemann Zeta function (1.12) when $\alpha = 1$. Further, the Hurwitz Zeta function can be generalized by [54]

$$L(s,\alpha,\lambda)=\sum_{n=0}^{\infty}\frac{e^{2\pi i\lambda n}}{(n+\alpha)^s} \tag{1.23}$$

which is called Lerch Zeta function.

1.7.2 Dirichlet L- functions

Series (1.2) is an example of series with the form [54]

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\ ,s\in\mathbb{C},\ \mathrm{Re}(s)>1 \tag{1.24}$$

where $\chi(n)$ is the Dirichlet character modulo m, $m \in \mathbb{N}$ [10]. $L(s,\chi)$ are called Dirichlet-L functions and are clearly homomorphic for $\mathrm{Re}(s) > 1$.

Obviously, by taking $m = 1$, and so $\chi(n) = 1, \forall n \in \mathbb{N}$, then equation (1.24) is reduced to (1.2), which is the series representation of the Riemann Zeta function. So,

$$\zeta(s)=L(s,1)$$

Dirichlet L-functions satisfy an analogous formula to Euler Product (1.5), given by [54]

$$\frac{1}{L(s,\chi)}=\prod_{p:prime}(1-\frac{\chi(p)}{p^s}),\ \mathrm{Re}(s)>1 \tag{1.25}$$

Besides, Dirichlet L-functions can be written in terms of Hurwitz Zeta function, this is shown by [54]

$$L(s,\chi)=\frac{1}{m^s}\sum_{l=1}^{m}\chi(l)\zeta(s,\frac{1}{m}) \tag{1.26}$$

By taking $m = 1$, this is another way to clarify the relation between Riemann Zeta function with Hurwitz Zeta function. This is clearly seen

$$L(s,1)=\zeta(s)=\zeta(s,1)$$

Chapter 2

ORTHOGONAL AND NON-ORTHOGONAL SETS OF POLYNOMIALS

Due to the importance of polynomials in all branches of science, including; mathematical physics, approximation theory, and quantum mechanics [34], many classes are presented in this chapter as major tools for the numerical work that will be handled Later. These classes include classical and discrete orthogonal polynomials. As classical orthogonal, we present Jacobi, Laguerre, and Bessel polynomials in details. For the discrete case, we present Charlier polynomials. Then, we offer a generalized case of the orthogonal polynomials, namely; Humbert polynomials and their generalizations as multiple d-orthogonal polynomials. Also, recently recognized polynomials, named Horadam polynomials, are exhibited with their main kinds. As the last presented group of polynomials, we give some traditional ones, like Bernoulli, Newton, and Bernstein Basis polynomials with their main formulas and properties. All polynomials presented in this chapter are incorporated in Borwein Algorithm.

2.1 Orthogonal Polynomials

A set of polynomials are considered to be orthogonal if and only if the inner product for pairwise distinct polynomials is zero with respect to a well defined positive conditioned function known as weight function [31, 63, 88].
In this section, we present classical and discrete orthogonal subclasses with their main remarkable polynomials.

2.1.1 Classical Orthogonal Polynomials

The set of the three well known orthogonal polynomials; Jacobi, Laguerre and Hermite, are used to be known as classical orthogonal class [63]. In 1949, Bessel polynomials were investigated to be a member of this class [62]. Thus, the new classification includes Jacobi, Laguerre, Bessel and Hermite polynomials.

The orthogonality of these polynomials $\{p_n(x)\}_{n=0}^{\infty}$ on the interval (a,b) is handled with respect to a positive continuous weight function $w(x)$ that satisfy [34]

$$\int_a^b w(x)p_n(x)p_m(x)dx = c\delta_{nm} \quad ,n,m = 0,1,2,... \tag{2.1}$$

where δ_{nm} is the Kronecker function and c is a constant. Such that

$$\int_a^b w(x)x^n dx < \infty$$

In general, the weight functions for the classical orthogonal polynomials occur as functions of continuous probability distributions. In particular, Beta, Gamma and the Normal probability density functions are weight functions for Jacobi, Laguerre and Hermite polynomials respectively [88]. For Bessel polynomials, in [62], their weight functions are derived as $w(x) = e^{-\frac{2}{x}}$.

A characterization of the classical orthogonal polynomials that they are defined via hyper geometric series and so obey second order differential equations of the form

$$A(x)y'' + B(x)y' + \lambda_n y = 0 \tag{2.2}$$

where the constants $A(x)$ and $B(x)$ depend only on x and λ_n depends only on n.

Another characterization insures that derivatives of the classical orthogonal polynomials are orthogonal [34]. Besides, all of these polynomials verify Rodrigues' formula.

In the following discussion, polynomials of Jacobi, Laguerre, and Bessel are manifested with their main features and equations.

2.1.1.1 Jacobi Polynomials

Jacobi polynomials $P_n^{\alpha,\beta}(x)$, which were introduced by Carl Jacobi, form one of the most important classes in mathematics due to their useful applications in different disciplines. They possessing many properties include [88]:

1. $P_n^{\alpha,\beta}(x)$ are orthogonal with respect to the weight function

$$B(\alpha,\beta) = (1-x)^{\alpha}(1+x)^{\beta}$$

Thus,

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{nm},\ n,m=0,1,2,... \tag{2.3}$$

where $\alpha, \beta > -1$ and $x \in [-1,1]$.

2. They are generated from the relation

$$\sum_{n=0}^{\infty} P_n^{\alpha,\beta}(x)t^n = 2^{\alpha+\beta}T^{-1}(1-t+T)^{-\alpha}(1+t+T)^{-\beta} \tag{2.4}$$

where

$$T = (1-2xt+t^2)^{-\frac{1}{2}} \tag{2.5}$$

3. They satisfy a second order differential equation of the form:

$$(1-x^2)y'' + (\beta - \alpha - (\alpha+\beta+2)x)y' + n(\alpha+\beta+n+1)y = 0 \tag{2.6}$$

4. They also have a Rodrigues formula, which can be considered as an alternative way to define Jacobi polynomials. This formula takes the form

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\right\} \tag{2.7}$$

5. Remarkably, they can be expressed by the explicit formula

$$P_n^{\alpha,\beta}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{k=0}^{n}\binom{n}{k}\frac{\Gamma(\alpha+\beta+n+k+1)}{2^k\Gamma(\alpha+k+1)}(x-1)^k \tag{2.8}$$

6. Recurrently, Jacobi polynomials are given by the three terms relation

$$\begin{aligned}2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{\alpha,\beta}(x) &= [(2n+\alpha+\beta+1)(\alpha^2-\beta^2)+(2n+\alpha+\beta)_3 x]P_n^{\alpha,\beta}(x)\\ &\quad -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{\alpha,\beta}(x)\end{aligned} \tag{2.9}$$

where $(k)_n$ is Pochhammer's symbol such that;$(k)_0 = 1$, $(k)_n = k(k+1)....(k+n-1)$ for $n \geq 1$.

Depending on the values of α and β, many types of polynomials are generated and considered as special cases of Jacobi polynomials, some of them are given in the following discussion.

1. Legendre Polynomials

When the free parameters in (2.8) satisfy $\alpha = \beta = 0$, Jacobi polynomials turn into the famous Legendre polynomials $\lambda_n(x)$, which can be expressed simply by [88]

$$\lambda_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} x^{(n-2k)} \tag{2.10}$$

They have the recurrence relation

$$4n(n+1)^2 \lambda_{n+1}(x) = (2n+1)[4n(n+1)x+1]\lambda_n(x) - 4n^2(n+1)\lambda_{n-1}(x) \tag{2.11}$$

2. T-Chebyshev polynomials

Chebyshev polynomials of first kind or more commonly T-Chebyshev polynomials $T_n(x)$ are obtained from Jacobi polynomials as a particular case by taking $\alpha = \beta = -\frac{1}{2}$ in (2.8). $T_n(x)$ are obtained explicitly by the formula [2, 88]

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{n-k} \binom{n-k}{k} (2x)^{n-2k} \tag{2.12}$$

They are generated also from the simple recurrence relation

$$2n(n+1)(2n-1)T_{n+1}(x) = 4n(n+1)(2n-1)xT_n(x) - 2(2n+1)(n-\frac{1}{2})^2 T_{n-1}(x) \tag{2.13}$$

where $T_0(x) = 1$, and $T_1(x) = x$.

3. U-Chebyshev Polynomials

Chebyshev polynomials of second kind or U-Chebychev polynomials are obtained by the substitution [2, 88]

$$U_n(x) = P_n^{\frac{1}{2},\frac{1}{2}}(x) \tag{2.14}$$

Thus they are given explicitly by

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k} \tag{2.15}$$

Recurrently, they are given by the relation

$$2(n+1)(n+2)(2n+1)U_{n+1}(x) = (2n+1)(2n+2)(2n+2)(2n+3)U_n(x) - 2(2n+3)(n+\frac{1}{2})^2 U_{n-1}(x) \tag{2.16}$$

where $U_0(x) = 1$, and $U_1(x) = 2x$.

4. V-Chebyshev Polynomials

Chebyshev polynomials of third kind or V-Chebyshev polynomials $V_n(x)$ refer to the special case of Jacobi polynomials with the parameters $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$ in (2.8), such that

$$V_n(x) = P_n^{-\frac{1}{2},\frac{1}{2}}(x) \tag{2.17}$$

They are given explicitly by [66]

$$V_n(x) = \frac{(2n)!}{(n!)^2 4^n} \sum_{k=0}^{n} \frac{2^k (n+k)!}{(n-k)!(2k)!} (x-1)^k \tag{2.18}$$

They are given by the recurrence relation

$$4n(n+1)^2 V_{n+1}(x) = 2n(2n+1)(2n+2)xV_n(x) - 4(n+1)(n^2 - \frac{1}{4})V_{n-1}(x) \tag{2.19}$$

where $V_0(x) = 1$, and $V_1(x) = x - \frac{1}{2}$.

5. W-Chebyshev Polynomials

This type of Chebyshev polynomials is symbolized by $W_n(x)$ which is considered as the fourth type of Chebyshev polynomials and commonly known as W-Chebyshev polynomials, they arise from the relation

$$W_n(x) = P_n^{\frac{1}{2},-\frac{1}{2}}(x) \tag{2.20}$$

Explicitly, $W_n(x)$ are given by the formula [66]

$$W_n(x) = \frac{(n+0.5)(2n)!}{(n!)^2 4^n} \sum_{k=0}^{n} \frac{(n+k)! 2^k}{(k+0.5)(n-k)!(2k)!} (x-1)^k \tag{2.21}$$

They are given simply by the three terms recurrence relation

$$4n(n+1)^2 W_{n+1}(x) = 2n(2n+1)(2n+2)xW_n(x) - 4(n+1)(n^2 - \frac{1}{4})W_{n-1}(x) \tag{2.22}$$

where $W_0(x) = 1$, and $W_1(x) = x + \frac{1}{2}$.

6. Gegenbauer Polynomials

Gegenbauer class of polynomials or the Ultraspherical polynomials $C_n^{\gamma}(x)$ are constructed by the replacement $\alpha = \beta = \gamma - \frac{1}{2}$ in (2.8).

This gives polynomials of the form [2]

$$C_n^{\gamma}(x) = P_n^{\gamma-\frac{1}{2},\gamma-\frac{1}{2}}(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \Gamma(n-k+\gamma)}{\Gamma(\gamma) k!(n-k)!} (2x)^{(n-2k)} \tag{2.23}$$

where $\gamma > -\frac{1}{2}$.

It is remarkable to observe that Legendre, T-Chebyshev, and U-Chebyshev polynomials are again special cases of Gegenbauer polynomials when $\gamma = \frac{1}{2}$, 0, and 1 respectively. Gegenbauer polynomials have the recurrence relation

$$(n+1)C_{n+1}^{\gamma}(x) = 2(n+\gamma)C_n^{\gamma}(x) - (n+2\gamma-1)C_{n-1}^{\gamma}(x), \;\; n = 1, 2, ... \tag{2.24}$$

where $C_0^{\gamma}(x) = 1$, and $C_1^{\gamma}(x) = 2\gamma x$.

2.1.1.2 Laguerre Polynomials

Laguerre polynomials $L_n(x)$, which are orthogonal on $[0, \infty)$ with respect to gamma distribution function e^{-x}, are particular instance of the so-called Generalized or Associated Laguerre polynomials $L_n^{\alpha}(x)$. Those functions are solutions of a second order differential equation of the form [2]

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \tag{2.25}$$

They are generated from the relation [2]

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) \frac{t^n}{n!} = \frac{1}{(1-t)^{\alpha+1}} e^{-\frac{xt}{1-t}} \tag{2.26}$$

Associated Laguerre polynomials are explicitly formulated by [2]

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+\alpha}{k+\alpha} x^k \quad , \alpha > -1 \tag{2.27}$$

where $\binom{n}{k} = \frac{\Gamma(n)}{\Gamma(k)\Gamma(n-k)}$.

The orthogonality of this set of polynomials can be fulfilled by taking $x^{\alpha} e^{-x}$ as a weight function [88]. Hence, Laguerre polynomials come a cross when the free parameter $\alpha = 0$, and then we can write

$$L_n(x) = L_n^0(x)$$

Recurrence relations of the Associated Laguerre polynomials include

$$L_n^{\alpha}(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x) \tag{2.28}$$

2.1.1.3 Bessel Polynomials

The orthogonality of Bessel polynomials $\beta_n(x)$ are noted firstly in the work of H.L.Krall [61] and verified then by Krall and Frink [6, 62].
Orthogonal Bessel polynomials are solutions of the second order differential equation [62]

$$x^2y'' + (2x+2)y' - n(n+1)y = 0 \tag{2.29}$$

Recurrently, they are obtained by [62]

$$\beta_{n+1}(x) = (2n+1)x\beta_n(x) + \beta_{n-1}(x) \tag{2.30}$$

Also, they are given explicitly by [62]

$$\beta_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}(\frac{x}{2})^k \tag{2.31}$$

2.1.2 Discrete Orthogonal Polynomials

Discrete orthogonal polynomials $\{p_n(x)\}$ are polynomials with discrete variables, that take their values at specific points and not on intervals [34]. The orthogonality of these polynomials is established by taking the weight function $w(x)$ as a jump function $j(x)$, whose values jump from one position to another at certain points $\{x_k\}$ [88]. This means that $\{x_k\}$ are the points of discontinuity of $j(x)$, which are indeed the possible values of the variable x for a polynomial $p_n(x)$. Thus if $p_n(x)$ and $p_m(x)$ are two orthogonal discrete polynomials within the same set, then [88]

$$\sum_k p_n(x_k)p_m(x_k)j(x_k) = c_n\delta_{mn} \tag{2.32}$$

where c_n is a constant depends on n and δ_{mn} is the Kronecker function. Where the jump function $j(x)$ is restricted to be positive, such that

$$\sum_k j(x_k) < \infty \tag{2.33}$$

A common property, or actually characterization of these polynomials emphasizes that all of them have analogue of Rodrigues's formula of the form [88]

$$p_n(x) = \frac{\Delta^n[j(x-n)\chi(x)\chi(x-1)...\chi(x-n+1)]}{\lambda_n j(x)} \tag{2.34}$$

where $\Delta f(x) = f(x+1) - f(x)$.

Examples on this type of polynomials include Charlier, Meixner, Krawtchouk, and Hahn polynomials. More properties and details about orthogonal discrete polynomials can be found in [2, 34, 49, 88]. For now, we shall present Charlier Polynomials in details (using the reference [34]).

Charlier, or Poisson Charlier $Ch_n^\alpha(x)$ are polynomials with one discrete variable, they were introduced by Charlier and they are orthogonal with respect to the discrete Poisson probability distribution function with mean α. Thus, Charlier's weight function is given by

$$j(x) = \frac{e^{-\alpha}\alpha^x}{x!} \tag{2.35}$$

and

$$\sum_{x=0}^{\infty} Ch_n^\alpha(x) Ch_m^\alpha(x) \frac{e^{-\alpha}\alpha^x}{x!} = \alpha^{-n} n! \delta_{nm} \quad , \alpha > 0 \tag{2.36}$$

where δ_{nm} is the Kronecker function.

Originally, these polynomials are generated by the relation

$$\sum_{n=0}^{\infty} Ch_n^\alpha(x) \frac{t^n}{n!} = e^{-t}(1 - \frac{t}{\alpha})^x \quad , |t| < \alpha \tag{2.37}$$

An interesting relation of Charlier polynomials is that

$$Ch_n^\alpha(x) = Ch_x^\alpha(n) \ , x \in \mathbb{N} \tag{2.38}$$

The Rodrigues's formula of Charlier polynomials takes the form

$$Ch_n^\alpha(x) = \frac{x!}{\alpha^x} \Delta^n \left[\frac{\alpha^{x-n}}{(x-n)!} \right] \tag{2.39}$$

Also, they obey the recurrence relations

$$\alpha Ch_{n+1}^\alpha(x) + (x - n - \alpha) Ch_n^\alpha(x) + n Ch_{n-1}^\alpha(x) = 0 \tag{2.40}$$

and

$$\alpha Ch_n^\alpha(x+1) + (n - x - \alpha) Ch_n^\alpha(x) + x Ch_{n-1}^\alpha(x-1) = 0 \tag{2.41}$$

Furthermore, they are connected to Laguerre polynomials by the relation

$$Ch_n^\alpha(x) = (-\alpha)^{-n} n! L_n^{x-n}(\alpha) \tag{2.42}$$

Explicitly, they are given by

$$Ch_n^{\alpha}(x) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \binom{x}{k} \frac{k}{\alpha^k} \quad ,x = 0,1,2,... \tag{2.43}$$

Or by

$$Ch_n^{\alpha}(x) = \sum_{k=0}^{\infty} (-1)^{n-k} \binom{n}{k} \frac{x^k}{\alpha^k} \quad ,x = 0,1,2,... \tag{2.44}$$

Full extent of these polynomials can be found in [2, 34, 56, 75, 78, 87, 88].

2.2 d-Orthogonal Polynomials

The concept of orthogonality that we deal with can be generalized in many ways. The multiple orthogonality takes apart as a famous generalization that raised lately in the eighteenth of the previous century [23, 24, 64, 91]. A special subclass of the multiple orthogonal polynomials is the set of d-orthogonal polynomials.

Definition 2.2.1. Polynomials of a set $\{p_n(x)\}_{n=0}^{\infty}$ are called d-orthogonal polynomials with respect to a vector function Ω with d elements ($d \in \mathbb{Z}^+$)

$$\varOmega = (\Omega_0, \Omega_1, ..., \Omega_{d-1})^T$$

if they satisfy

$$\begin{cases} \langle \Omega_i, p_n p_m \rangle = 0, & m > dn + i,\, n \in \mathbb{N} \\ \langle \Omega_i, p_n p_{dn+i} \rangle \neq 0, & n \in \mathbb{N} \end{cases}$$

where $i \in \{0, 1, ..., d-1\}$ and $< p_n(x), p_m(x) >$ denotes the inner product of the two functions $p_n(x)$ and $p_m(x)$ [23].

Note that, when $d = 1$, the above property is reduced to the well known condition of orthogonality (2.1) [23]. A famous example on the d-orthogonal polynomials is Humbert family and its generalizations given in the following discussion.

2.2.1 Humbert Polynomials

The set of the polynomials $\{h_n^{r,\gamma}(x)\}_{n=0}^{\infty}$ that are generated by [43, 64]

$$(1 - rxt + t^r)^{-\gamma} = \sum_{n=0}^{\infty} h_n^{r,\gamma}(x) t^n \tag{2.45}$$

are called Humbert polynomials, where r is a positive integer and $\gamma > -\frac{1}{2}$.

In the literature, variable r in (2.45) is replaced sometimes by $d+1$, where d refers to the factor of multiple orthogonality. Thus, $r = 2$ indicates to polynomials with standard orthogonality, which are indeed the Gegenbauer polynomials.

The explicit formula of these polynomials is given by

$$h_n^{r,\gamma}(x) = \sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \frac{(-1)^k (\gamma)_{n+(1-r)k} (rx)^{n-rk}}{k!(n-rk)!} \tag{2.46}$$

where $(\gamma)_k$ is Pochhammer's symbol and it is treated here as the rising factorial function. It is defined by $(\gamma)_0 = 1$, and $(\gamma)_k = (\gamma)(\gamma+1)(\gamma+2)...(\gamma+k-1)$ [36, 68].

As special cases of Humbert polynomials, we present the following polynomials.

1. Gegenbauer Polynomials

When $r = 2$ in (2.46), Humbert polynomials are reduced to Gegenbauer polynomials that were manifested in section (2.1.1.1), i.e

$$C_n^{\gamma}(x) = h_n^{2,\gamma}(x)$$

The explicit formula of Gegenbauer polynomials is given by equation (2.23).

2. Pincherle Polynomials

Pincherle polynomials $\rho_n(x)$ occur with the combination of parameters $\{r = 3, \gamma = -0.5\}$ in (2.46) [68]. Hence,

$$\rho_n(x) = h_n^{3,-0.5}(x)$$

Then, we have

$$\rho_n(x) = \sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \frac{(-1)^k (-0.5)_{n-2k} (3x)^{n-3k}}{k!(n-3k)!} \tag{2.47}$$

3. Horadam-Pethe Polynomials

Horadam Pethe polynomials $R_n^{\lambda}(x)$ are obtained by replacing x in (2.46) by $\frac{2x}{3}$ [46], that is

$$R_{n+1}^{\lambda}(x) = h_n^{3,\lambda}(\frac{2x}{3}) \tag{2.48}$$

These polynomials are given explicitly by

$$R_n^{\lambda}(x) = \sum_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} \frac{(-1)^k (\lambda)_{n-2k-1} (2x)^{n-3k-1}}{k!(n-3k-1)!} \tag{2.49}$$

where $\lambda > -\frac{1}{2}$.

4. Milovanović Dordevic polynomials

These polynomials $R_{n,r}^{\lambda}(x)$ extend the definition of Horadam-Pethe and many other polynomials, which are obtained by taking

$$R_{n,r}^{\lambda}(x) = h_{n,r}^{\lambda}(\frac{2x}{3}) \tag{2.50}$$

Thus,

$$R_{n+1}^{\lambda}(x) = R_{n,3}^{\lambda}(x) \tag{2.51}$$

which are Horadam-Pethe polynomials.
In fact, these polynomials originally are obtained from Gegenbauer polynomials, for more details, see [43].
The generating function of $R_{n,r}^{\lambda}(x)$, takes the form [69]

$$(1-2xt+t^r)^{-\lambda} = \sum_{n=0}^{\infty} R_{n,r}^{\lambda}(x) t^n \tag{2.52}$$

where $n \in \mathbb{N}$ and $\lambda > -\frac{1}{2}$.
Milovanović Dordevic polynomials can be expressed by [69]

$$R_{n,r}^{\lambda}(x) = \sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} (-1)^k \frac{(\lambda)_{n-(r-1)k}}{k!(n-rk)!} (2x)^{n-rk} \tag{2.53}$$

where $(\lambda)_n$ is Pochhammer's symbol.

2.2.2 Generalized Humbert Polynomials

The generating function

$$(C-rxt+yt^r)^p = \sum_{n=0}^{\infty} H_n(r,x,y,p,C) t^n \tag{2.54}$$

where r is a positive integer, defines more generalized set of polynomials called Generalized Humbert polynomials, which were studied for the first time by Gould [36].

In explicit way, they can be expressed by

$$H_n(r,x,y,p,C) = \sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \binom{p}{k} \binom{p-k}{n-r} C^{p-n+(r-1)k} y^k (-rx)^{n-rk} \tag{2.55}$$

Also, they are defined recurrently by the three-terms formula

$$CnH_n(x) - r(n-1-p)xH_{n-1}(x) + (n-r-rp)yH_{n-r}(x) = 0, n \geq m \geq 1. \tag{2.56}$$

In the following discussion, we present special cases of the Generalized Humbert polynomials.

1. Humbert polynomials

Humbert polynomials $h_n^{r,\gamma}(x)$ and consequently all their special cases are natural special cases of the Generalized Humbert polynomials and so can be generated from (2.55). For instance, Humbert polynomials are obtained by taking $y = C = 1$ and $p = -\gamma$ [31].

Therefore,

$$h_n^{r,\gamma}(x) = H_n(r,x,1,-\gamma,1) \tag{2.57}$$

2. Kinney Polynomials

Kinney polynomials $K_n^r(x)$ arise directly from (2.55), by the equality [31]

$$K_n^r(x) = H_n(r,x,1,-\frac{1}{r},1) \tag{2.58}$$

Hence,

$$K_n^r(x) = \sum_{k=0}^{\lfloor \frac{n}{r} \rfloor} \frac{\Gamma(1-\frac{1}{r})(-rx)^{n-rk}}{k!(n-m-k)!\Gamma(r-\frac{1}{r}-n+1)} \tag{2.59}$$

3. Byrd Polynomials

The set of Byrd polynomials $\Upsilon_n(x)$ are obtained by [4]

$$\Upsilon_{n+1}(x) = H_n(2,x,-1,-1,1) \tag{2.60}$$

as a special case of (2.55).

4. Liouville Polynomials

Liouville polynomials come from the equality [4]

$$l_n^{p,q}(x) = H_n(2,q,-1,-\frac{1}{2},p^2) \tag{2.61}$$

as a special case of (2.55).

2.3 Horadam Polynomials

The Fibonacci, Lucas, Pell, and many other sequences of numbers, can be generalized to the one satisfy the following Linear recurrence relation [39]

$$d_{n+2} = pd_{n+1} + qd_n, n \geqslant 0 \tag{2.62}$$

with the two initial numbers

$$d_0 = a, d_1 = b$$

where a, b, p, and q are integers. This sequence is known as Horadam sequence of numbers $d_n(a,b,p,q)$ and introduced by A. F. Horadam in 1965 [41, 47]. An interesting generalization of these numbers is Horadam polynomials.

In similar way, Horadam polynomials $d_n(x,a,b,p,q)$ can be obtained from the three-terms recurrence relation [31]

$$d_n(x,a,b,p,q) = pxd_{n-1}(x,a,b,p,q) + qd_{n-2}(x,a,b,p,q), n \geqslant 3 \tag{2.63}$$

with

$$d_1(x,a,b,p,q) = a, d_2(x,a,b,p,q) = bx \tag{2.64}$$

Horadam polynomials are generated by [47]

$$g(x,t) = \frac{a + xt(b - ap)}{1 - pxt - qt^2} = \sum_{n=0}^{\infty} d_n(x,a,b,p,q)t^n \tag{2.65}$$

Also, they are expressed explicitly by [47]

$$d_{n+1}(x) = a \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} (px)^{n-2k} q^k + (\frac{b}{p} - a) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (px)^{n-2k} q^k \tag{2.66}$$

Horadam sequence of numbers is an obvious particular case of Horadam polynomials and can be attained by substituting x=1 in (2.66). Furthermore, the famous particular cases of Horadam polynomials [47] are demonstrated in Table 2.1.

Polynomials	symbol	parameters' detection in (2.66)
Fermat of the first kind	$\phi_n(x)$	$d_n(x,1,1,1,-2)$
Fibonacci	$F_n(x)$	$d_n(x,1,1,1,1)$
Lucas	$\Psi_n(x)$	$d_n(x,2,1,1,1)$
Pell	$P_n(x)$	$d_n(x,1,2,2,1)$
Pell-Lucas	$Q_n(x)$	$d_n(x,2,2,2,1)$
Jacobsthal	$J_n(x)$	$d_n(1,1,1,1,2x)$
Jacobsthal-Lucas	$j_n(x)$	$d_n(1,2,1,1,2x)$
T-Chebychev	$T_n(x)$	$d_n(x,1,2,2,-1)$
U-Chebychev	$U_n(x)$	$d_n(x,1,1,2,-1)$

Table 2.1: Special Cases of Horadam Polynomials

2.3.1 Fermat Polynomials

There are two types of Fermat polynomials: Fermat polynomials of the first and of the second kinds. They are denoted historically by $\phi_n(x)$ and $\theta_n(x)$ as they appeared for the first time in A. F. Horadam papers [42]. In fact, the two kinds $\phi_n(x)$ and $\theta_n(x)$ generalize specific sequences of numbers that can be obtained respectively from the recurrence relations

$$\phi_n = 3\phi_{n-1} - 2\phi_{n-2}, \quad \phi_0 = 0,\ \phi_1 = 1,\ n \geqslant 2. \tag{2.67}$$

and

$$\theta_n = 3\theta_{n-1} - 2\theta_{n-2}, \quad \theta_0 = 2,\ \theta_1 = 3,\ n \geqslant 2. \tag{2.68}$$

where ϕ_n and θ_n are Fermat numbers of the first and of the second kinds respectively.
More generally, the corresponding polynomials are obtained recurrently by

$$\phi_n(x) = x\phi_{n-1}(x) - 2\phi_{n-2}(x), \quad \phi_0(x) = 0,\ \phi_1(x) = 1,\ n \geqslant 2. \tag{2.69}$$

and

$$\theta_n(x) = x\theta_{n-1}(x) - 2\theta_{n-2}(x), \quad \theta_0(x) = 2,\ \theta_1(x) = 3,\ n \geqslant 2. \tag{2.70}$$

Their generating functions are respectively

$$\frac{1}{1-xt+2t^2} = \sum_{n=0}^{\infty} \phi_n(x)t^n \tag{2.71}$$

$$\frac{1-2t^2}{1-xt+2t^2}=\sum_{n=0}^{\infty}\theta_n(x)t^n \tag{2.72}$$

Note that,

$$\sum_{n=0}^{\infty}\theta_n(x)t^n=\sum_n\phi_n(x)t^n-2t^2\sum_n\phi_n(x)t^{n-2},\quad \phi_0(x)=0,\ \phi_1(x)=1,\ n\geqslant 2$$

This can be interpreted as the following interesting relation between the second and the first Fermat polynomials kinds:

$$\theta_n(x)=\phi_n(x)-2\phi_{n-2}(x),\ n\geqslant 2 \tag{2.73}$$

The explicit formula of Fermat polynomials $\phi_n(x)$ is given by

$$\phi_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-2)^k(n-k)!x^{n-2k}}{k!(n-2k)!} \tag{2.74}$$

The first few of Fermat polynomials of the two kinds are given in Table 2.2:

n	$\phi_n(x)$	$\theta_n(x)$
0	1	2
1	x	x
2	x^2-2	x^2-4
3	x^3-4x	x^3-6x
4	x^4-6x^2+4	x^4-8x^2+8
5	x^5-8x^3+12x	x^5-10x^3+20x

Table 2.2: Fermat polynomials

As clarified, $\phi_n(x)$ and $\theta_n(x)$ are defined originally by A. F. Horadam, but the common Fermat polynomials nowadays are referred to those that generalize the famous Fermat numbers in number theory [1]

$$3,5,257,65537,...$$

which take the form

$$2^{2^n}+1\ \ ,n\in\mathbb{N}$$

The corresponding Fermat Polynomials are given by

$$\Im_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-2)^k(n-k)!(3x)^{n-2k}}{k!(n-2k)!} \tag{2.75}$$

2.3.2 Fibonacci Polynomials

The famous Fibonacci polynomials $F_n(x)$ were introduced by Catalan in 1883 [71], thus they are called sometimes Catalan's Fibonacci polynomials. They can be obtained by the formula

$$F_n(x) = d_n(x,1,1,1,1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1}, n \neq 0 \tag{2.76}$$

Alternatively, they can be expressed by the recurrence formula

$$F_n(x) = xF_{n-1} + F_{n-2}(x); \quad F_1(x) = 1,\ F_2(x) = x \tag{2.77}$$

$F_n(x)$ have the generating function $f(x,t)$:

$$f(x,t) = \frac{1}{1 - xt - t^2} = \sum_{n=1}^{\infty} F_n(x) t^n \tag{2.78}$$

Note that, $F_1(1) = 1, F_2(1) = 1, F_3(1) = 2, F_4(1) = 3, F_5(1) = 5, ...$ which are the known Fibonacci numbers.

2.3.3 Lucas Polynomials

The explicit formula of Lucas polynomials $\Psi_n(x)$ is given by

$$\Psi_n(x) = d_n(x,2,1,1,1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} \tag{2.79}$$

They have the recurrence relation [34]

$$\Psi_n(x) = x\Psi_{n-1} + \Psi_{n-2}, \quad \Psi_0(x) = 2, \Psi_1(x) = x. \tag{2.80}$$

Also, they are generated by [34]

$$\frac{2 - xt}{1 - xt - t^2} = \sum_{n=0}^{\infty} \Psi_n(x) t^n \tag{2.81}$$

It is easy to investigate that $\Psi_n(1)$ is the n^{th} Lucas number [71]. The sequence of Lucas numbers $\{2, 1, 3, 4, 7, 11, ...\}$ is like Fibonacci sequence in the sense of obtaining a term by the sum of the two immediate previous numbers. But, the difference here is that Lucas sequence starts with a term of value 2 rather 1.

2.3.4 Pell Polynomials

Pell polynomials $P_n(x)$ are obtained by

$$P_n(x) = d_n(x,1,2,2,1) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} (2x)^{n-2k-1}, n \neq 0 \tag{2.82}$$

They satisfy the three-term recurrence relation [53]

$$P_n(x) = 2xP_{n-1} + P_{n-2}, \tag{2.83}$$

where $P_1(x) = 1$, and $P_2(x) = 2x$, and they are generated by [53]

$$\frac{1}{1-2xt-t^2} = \sum_{n=1}^{\infty} P_n(x)t^n \tag{2.84}$$

The sequence of Pell numbers is $\{0,1,2,5,12,29,...\}$.

2.3.5 Pell-Lucas Polynomials

The Pell-Lucas polynomials $Q_n(x)$ are obtained by

$$Q_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}, n \neq 0 \tag{2.85}$$

Pell-Lucas polynomials possess the generating function [44]

$$\frac{2-2xt}{1-2xt-t^2} = \sum_{n=1}^{\infty} P_n(x)t^n \tag{2.86}$$

Moreover, they obey the recurrence relation [31]

$$Q_n(x) = 2xQ_{n-1} + Q_{n-2}, n \geqslant 2 \tag{2.87}$$

where $Q_0(x) = 2$, and $Q_1(x) = 2x$.

2.4 Traditional Polynomials

In this section, we give different polynomials with no main properties in common. In particular, we shall present Bernoulli, Newton, and Bernstein basis polynomials in details by revealing their main properties and formulas.

2.4.1 Bernoulli Polynomials

Bernoulli polynomials $B_n(x)$ occur as a generalization of Bernoulli numbers B_n, which are generated by the formula [31]

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \tag{2.88}$$

Bernoulli numbers were introduced firstly by Jacob Bernoulli and they appeared in the context of finite sums of powers' generalization

$$\sum_{k=0}^{n-1} k^m = \frac{1}{(m+1)} \left[\sum_{j=0}^{m} \binom{m+1}{j} n^{m+1-j} B_j \right] \tag{2.89}$$

The first few of Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$. However, $B_{2k+1} = 0, \forall k \geq 1$.

Bernoulli polynomials can be defined by the generating function [31]

$$f(x,t) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)t^n}{(n)!} \tag{2.90}$$

Remarkably, they share the properties [34]

1. $B_0(x) = 1$
2. $B_k{}'(x) = kB_{k-1}(x)$
3. $\int_0^1 B_k(x)dx = 0, \ \forall k \ \geq 1$

Explicitly, Bernoulli polynomials can be expressed by [31]

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \tag{2.91}$$

More properties of Bernoulli polynomials can be found in [2, 21, 30, 31, 34, 83, 88].

The first few of Bernoulli polynomials are

$B_0(x) = 1$

$B_1(x) = x - \frac{1}{2}$

$B_2(x) = x^2 - x + \frac{1}{6}$

$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$

$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}$

$B_5(x) = x^5 - \frac{5}{2}x^4 - \frac{5}{3}x^3 - \frac{1}{6}x.$

2.4.2 Newton Polynomials

Newton polynomials $N_n(x)$ are defined via a set of numbers called Stirling numbers of second kind [2], $S(n,k)$ symbolize these numbers which are given by the formula [80]

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \qquad ,S(n,1) = S(n,n) = 1 \tag{2.92}$$

Stirling numbers occur in the study of partitions and they describe the number of way by which we can partition a set of n items into k nonempty sets. We give the first few of $S(n,k)$ in Table 2.3.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
n=0	1	0	0	0	0	0
n=1	0	1	0	0	0	0
n=2	0	1	1	0	0	0
n=3	0	1	3	1	0	0
n=4	0	1	7	6	1	0
n=5	0	1	15	25	10	1

Table 2.3: $S(n,k)$ numbers.

Newton polynomials are defined explicitly by the simple formula [2]

$$N_n(x) = \sum_{k=0}^{n}\frac{(-1)^k S(n,n-k)}{n!}x^{n-k} \tag{2.93}$$

The first few of Newton polynomials are

$N_0(x) = 1$

$N_1(x) = x$

$N_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x$

$N_3(x) = \frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x$

$N_4(x) = \frac{1}{24}x^4 - \frac{1}{4}x^3 + \frac{7}{24}x^2 - \frac{1}{24}x$

$N_5(x) = \frac{1}{120}x^5 - \frac{1}{12}x^4 + \frac{5}{24}x^3 - \frac{1}{8}x^2 + \frac{1}{120}x$

2.4.3 Bernstein Basis Polynomials

Bernstein Basis polynomials $\Omega_n^\alpha(x)$ are heavily used in approximation theory, and they are the introductory to define Bernstein polynomials [88].

Bernstein Basis polynomials are defined explicitly by the formula [31]

$$\Omega_n^\alpha(x) = \binom{n}{\alpha} x^\alpha (1-x)^{n-\alpha} \qquad \alpha = 0, 1, 2, ..., n \tag{2.94}$$

The first few of Bernstein Basis polynomials are given in Table 2.4

	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$
n=0	1	0	0	0	0
n=1	$1-x$	x	0	0	0
n=2	$1-2x+x^2$	$2x-2x^2$	x^2	0	0
n=3	$1-3x+3x^2-x^3$	$3x-6x^2+3x^3$	$3x^2-3x^3$	x^3	0
n=4	$1-4x+6x^2-4x^3+x^4$	$4x-12x^2+12x^3-4x^4$	$6x^2-12x^3+6x^4$	$4x^3-4x^4$	x^4

Table 2.4: $\Omega_n^\alpha(x)$ polynomials.

Note that the sum of each row

$$\sum_{\alpha=0}^{n} \binom{n}{\alpha} x^\alpha (1-x)^{n-\alpha} \tag{2.95}$$

equals 1. This actually comes from Binomial theorem [10], that is

$$\sum_{\alpha=0}^{n} \binom{n}{\alpha} x^\alpha (1-x)^{n-\alpha} = (x+1-x)^n = 1 \tag{2.96}$$

In Chapter 3, fast and accurate methods for calculating $\zeta(s)$ will be presented. In Chapter 4, we will use the presented classes of polynomials in some formulas in order to get satisfying approximations for $\zeta(s)$.

Chapter 3

METHODS FOR EVALUATING RIEMANN ZETA FUNCTION

Unlike many other infinite series, Euler series (1.2) has no general explicit formula for evaluating $\zeta(s)$. Many methods were mentioned in section (1.5) for calculating $\zeta(s)$ values, when s is even integer (equation (1.15) and equation (1.16)), and when s is negative odd integer (equation (1.18)). For a general argument s, no similar formulas were found, but instead, many tremendous methods were established to approximate $\zeta(s)$ with a reasonable a mount of accuracy.

In this chapter, we shall be concern about the basic methods that are used to approximate $\zeta(s)$ for small values of s oftentimes. Intuitively, we firstly exhibit the direct approach for calculating $\zeta(s)$ by introducing a cutoff parameter for Euler series (1.2). Also, some comparisons are made between $\zeta(s)$ approximations obtained for small and reasonably large values of s by direct approach. Then, many fast series for $\zeta(2)$, $\zeta(3)$, $\zeta(4)$, $\zeta(5)$, and $\zeta(7)$ are given and some of them are used to enhance the approximations obtained by direct approach. Thus, it makes sense to compare the results obtained by these fast series with the ones derived directly at some s.

Also in this chapter, Euler Maclaurin summation method is presented with disclosing its historical importance and the main advantages/disadvantages associated with implementing it. Furthermore, some numerical verifications are carried out by controlling the induced cutoff parameters as well as some comparisons with earlier methods.

Finally and more substantially, we offer a set of three efficient algorithms introduced by P. Borwein which rely on accelerating eta series $\eta(s)$ (1.9) that is deeply and directly connected to $\zeta(s)$ series. Preliminary results are obtained by using only two of these algorithms and by making some comparisons to demonstrate their efficiency. We end this chapter with a preface for the next chapter by manifesting the work that will be carried out for developing Borwein algorithms.

It should be clarified that the exact values of $\zeta(s)$ are taken with 50 digits of accuracy and they were calculated originally by direct methods for very long time. These exact values are implemented in many softwares, like; Mathematica, Matlab, and Maple.

3.1 Direct Approach

The basic definition of $\zeta(s)$ given by formula (1.2), is the simplest approach to take for approximating $\zeta(s)$ with cutting the series off by finite number of terms, but it is not the best; specially for small values of s.

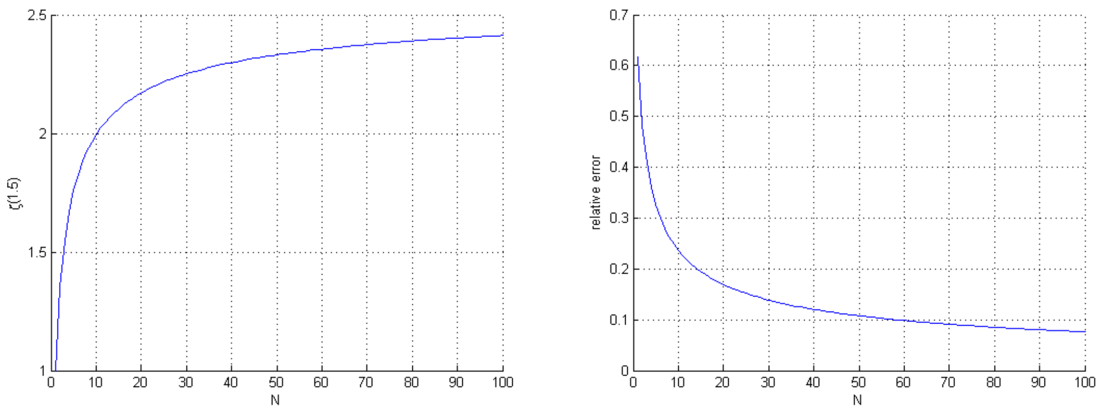
In order to calculate $\zeta(s)$ directly, we introduce a cutoff parameter N for Euler series, so that

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} \tag{3.1}$$

where N is large enough.

Note that equation (3.1) gives approximate values for $\zeta(s)$ depending on N and not exact values. For $s = 1.5, 2, 2.5, 3$, approximate values of $\zeta(s)$ are calculated using equation (3.1) with different values of N as well as the resulting relative error, this is exhibited by Fig. 3.1, Fig. 3.2, Fig. 3.3, and Fig. 3.4 respectively.

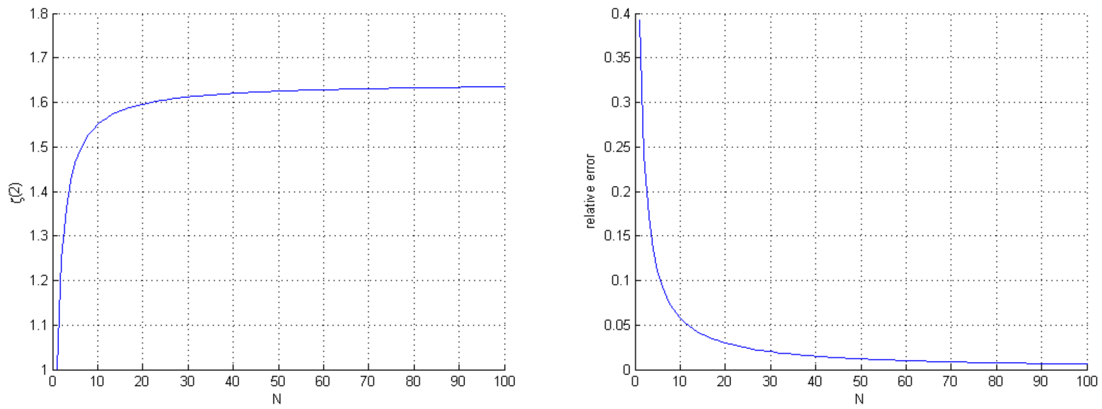
It is obvious that these approximations are not so good, specially those of $\zeta(1.5)$, since it requires 100 terms to obtain an approximation with a relative error around $\varepsilon = 0.1$.



(a) $\zeta(1.5)$ approximations versus N.

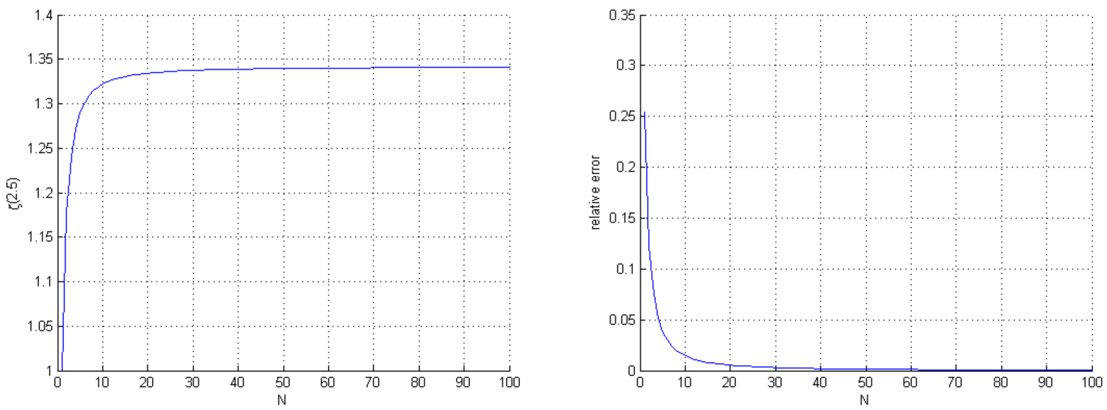
(b) Relative error versus N.

Figure 3.1: Approximating $\zeta(1.5)$ by direct approach using (3.1).



(a) $\zeta(2)$ approximations versus N. (b) Relative error versus N.

Figure 3.2: Approximating $\zeta(2)$ by direct approach using (3.1).



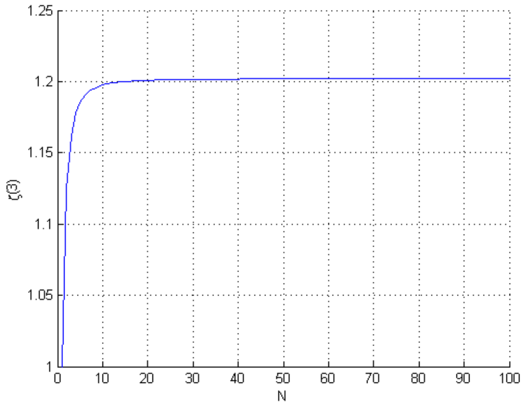
(a) $\zeta(2.5)$ approximations versus N. (b) Relative error versus N.

Figure 3.3: Approximating $\zeta(2.5)$ by direct approach using (3.1).

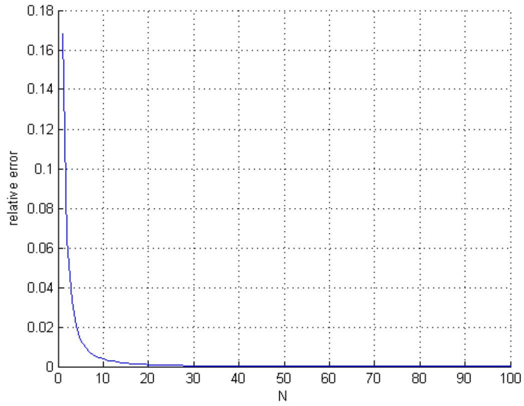
In fact, $\zeta(s)$ approximations obtained by direct approach become better when s gets larger. Fig. 3.5 depicts this behavior by introducing the relative error corresponding to different values of s and N.

For $N = 100$, Table 3.1 shows the values of relative error for approximating $\zeta(s)$ for different values of s.

Now, to see how the direct approach is acceptable for a sufficient large argument s, we offer $\zeta(s)$ approximations and the corresponding relative error as N increases in the case of $s = 5$ and $s = 7$ by Fig. 3.6 and Fig. 3.7 respectively.



(a) $\zeta(3)$ approximations versus N.



(b) Relative error versus N.

Figure 3.4: Approximating $\zeta(3)$ by direct approach using (3.1).

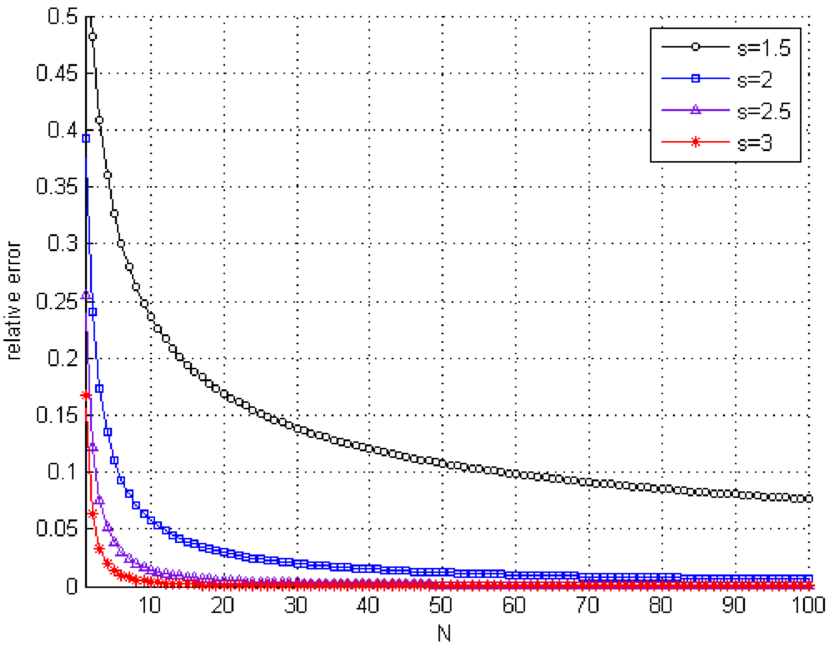
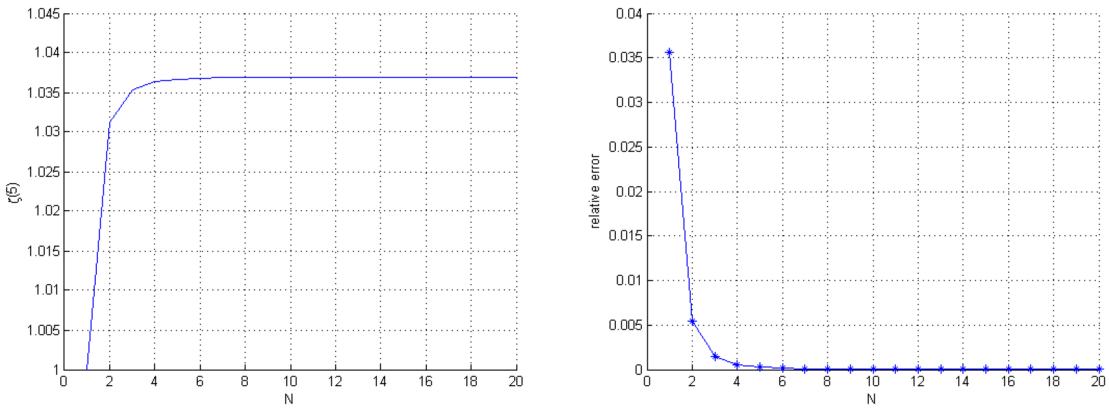


Figure 3.5: Comparison between $\zeta(s)$ approximations obtained by direct approach for different values of s.

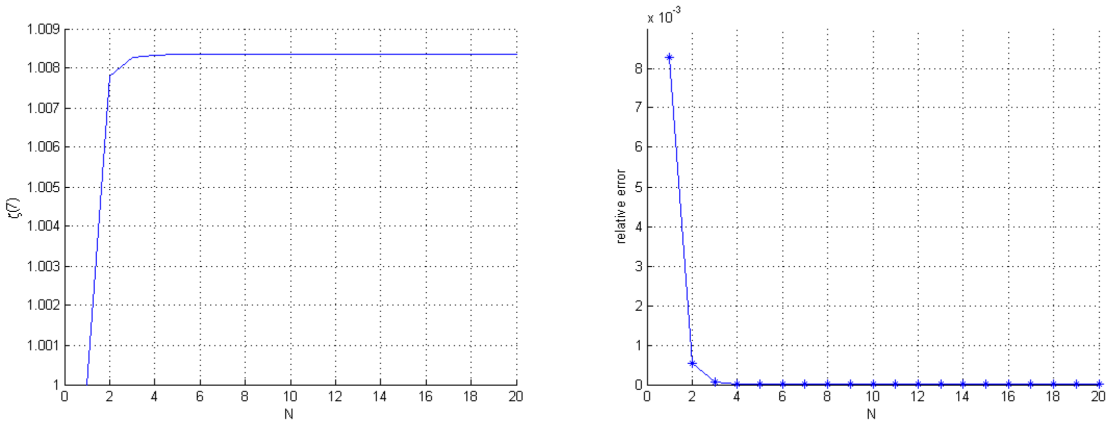
s	1.5	2	2.5	3
ε	7.64×10^{-2}	6.05×10^{-3}	4.93×10^{-4}	4.12×10^{-5}

Table 3.1: The values of relative error in approximating $\zeta(s)$ with direct approach using $N = 100$.



(a) $\zeta(5)$ approximations versus N. (b) Relative error versus N.

Figure 3.6: Approximating $\zeta(5)$ by direct approach using (3.1).



(a) $\zeta(7)$ approximations versus N. (b) Relative error versus N.

Figure 3.7: Approximating $\zeta(7)$ by direct approach using (3.1).

For instance, to approximate $\zeta(s)$ with relative error of $\varepsilon = 1.5 \times 10^{-3}$, Table 3.2 detects the required values of N for different s.

	$s = 1.5$	$s = 2$	$s = 2.5$	$s = 3$	$s = 5$	$s = 7$
N	257060	403	48	17	3	2

Table 3.2: Required N for approximating $\zeta(s)$ with $\varepsilon = 1.50 \times 10^{-3}$.

By plotting the values in Table 3.2, we obtain Fig. 3.8, which discloses obviously the inverse relationship between the values of N and s.

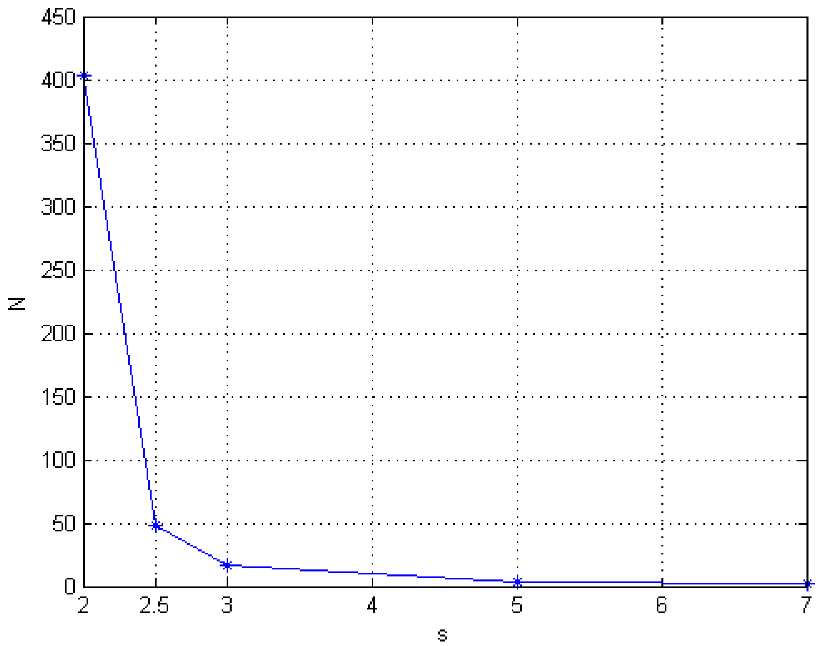


Figure 3.8: The inverse relationship between s and N in approximating $\zeta(s)$ using (3.1) with $\varepsilon = 1.5 \times 10^{-3}$.

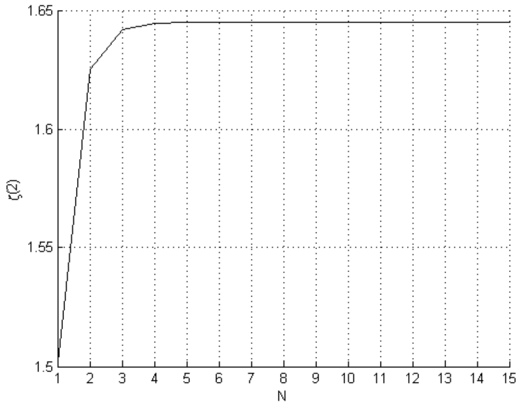
3.2 Fast Series for calculating $\zeta(s)$ at some integers

The need for big values of N to achieve some accuracy with approximating $\zeta(s)$ by direct approach, specially for small s, requires too much time and memory for computers to handle. Thus, some scholars tried to accelerate some specific particular cases of Euler series by finding equivalent infinite series that rapidly converge, so then smaller values of N are required for the new series.

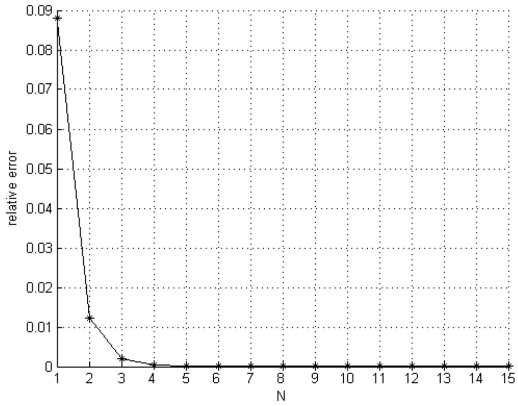
An example on rapid convergent series is the classical one devoted for accelerating $\zeta(2)$ given by [17, 82]

$$\zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \tag{3.2}$$

Using (3.2) with a cutoff parameter N, instead of (3.1) with $s = 2$, then $\zeta(2)$ approximations with the same value of N will be better. Fig. 3.9 presents $\zeta(2)$ approximations obtained with different values of N with the corresponding relative error using (3.2).



(a) $\zeta(2)$ approximations versus N.



(b) Relative error versus N.

Figure 3.9: Approximating $\zeta(2)$ by the accelerating series (3.2).

Table 3.3 gives the required values of N to approximate $\zeta(2)$ by both direct series (3.1) and the accelerating series (3.2).

ε	Euler Series	Accelerating Series
1×10^{-3}	553	4
1×10^{-5}	55266	7

Table 3.3: Required values of N for approximating $\zeta(2)$.

Fig. 3.10 shows how the relative error corresponding to $\zeta(2)$ approximations attained by the accelerating series (3.2) is obviously less than the one attained by direct series (3.1) for the same value of N.
Similar results were obtained for $\zeta(4)$, which has an analog to the accelerating series (3.2), which is given in [26] by the equation

$$\zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}} \tag{3.3}$$

Using series (3.3) with cutoff parameter N to approximate $\zeta(4)$, this gives the approximations demonstrated in Fig. 3.11 as well as the corresponding relative error for different values of N.

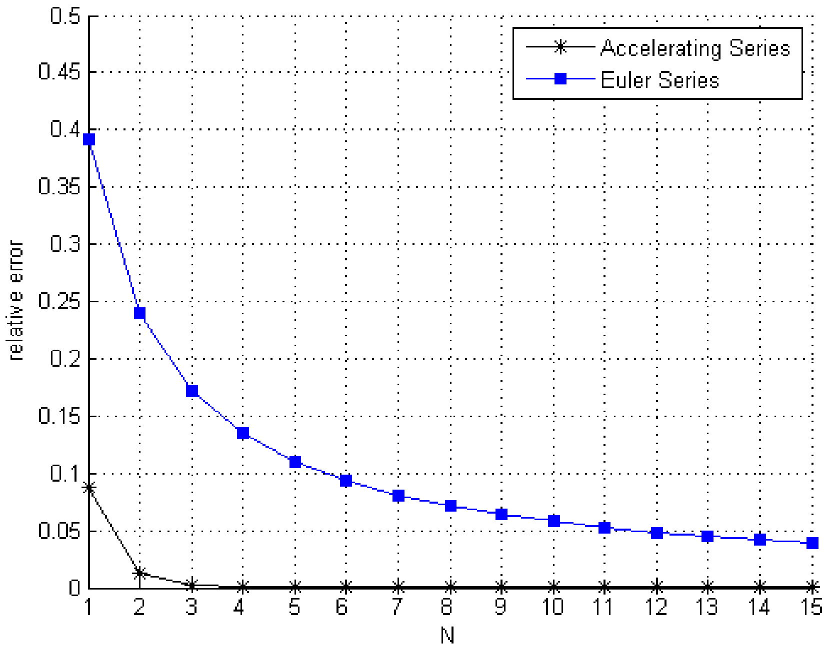
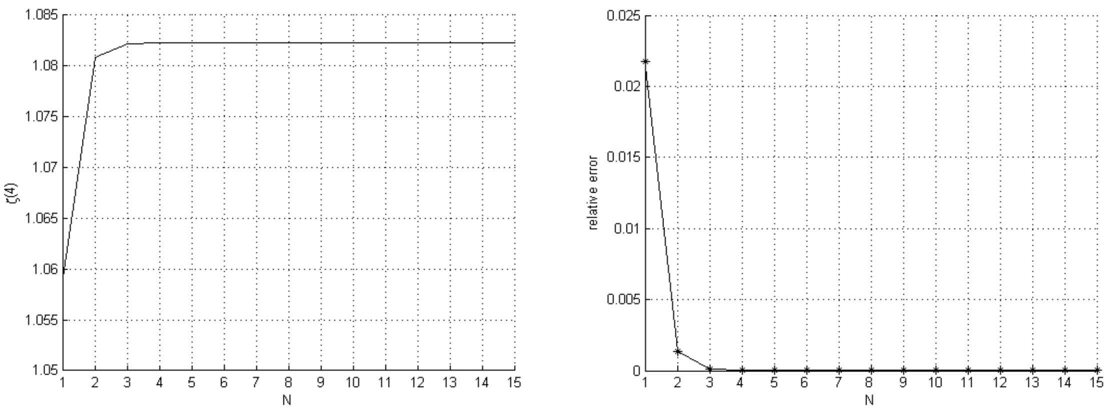


Figure 3.10: Comparison between $\zeta(2)$ approximations obtained by direct and accelerating series.



(a) $\zeta(4)$ approximations versus N.

(b) Relative error versus N.

Figure 3.11: Approximating $\zeta(4)$ by the accelerating series (3.3).

Table 3.4 gives the required values of N to approximate $\zeta(4)$ by both direct series (3.1) and the accelerating series (3.3).

ε	Euler Series	Accelerating Series
1×10^{-3}	7	3
1×10^{-5}	31	5

Table 3.4: Required values of N for approximating $\zeta(4)$.

Fig. 3.12 describes differences between $\zeta(4)$ approximations obtained by direct series (3.1) and the accelerating series (3.3).

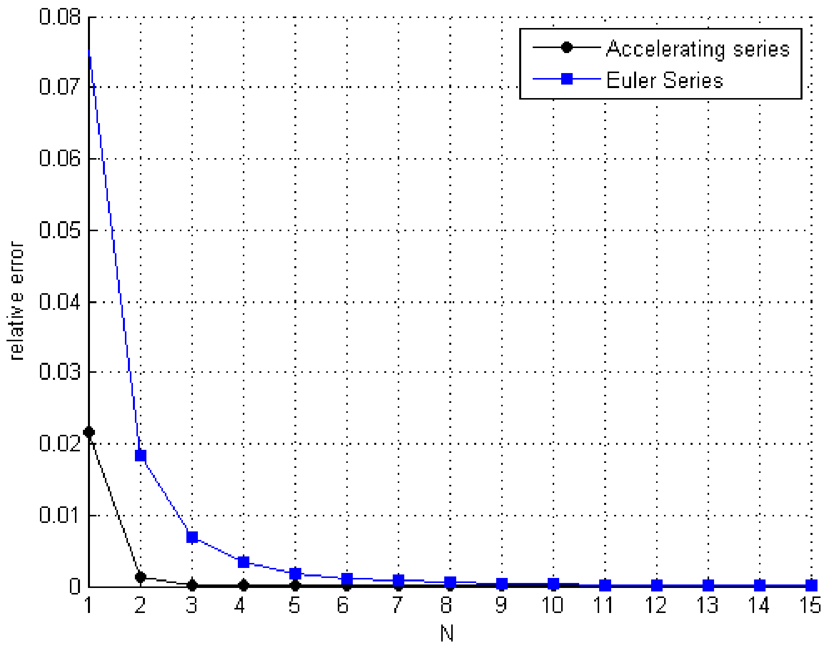


Figure 3.12: Comparison between $\zeta(4)$ approximations by direct series (3.1) and accelerating series (3.3).

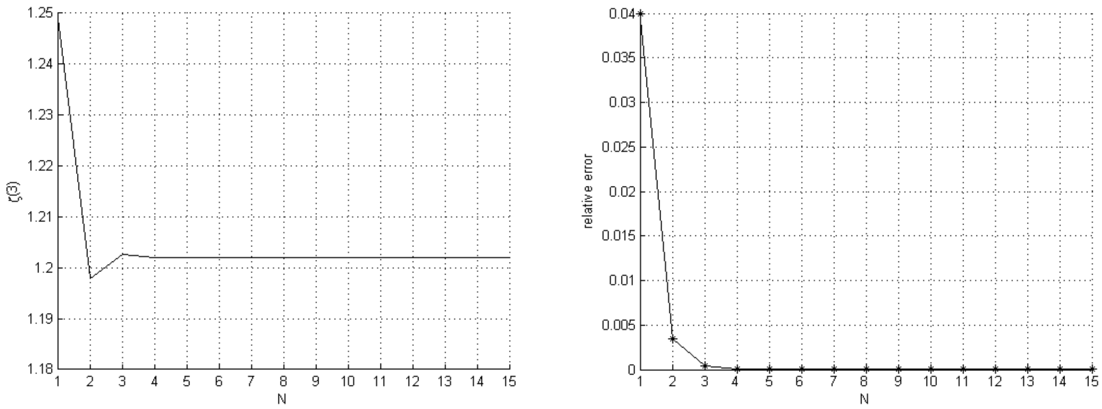
An analogue of the previous results for $\zeta(3)$ was established by Hjortnaes [17] by discovering the rapid series

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \tag{3.4}$$

The results obtained using Hjortnaes accelerating series (3.4) for approximating $\zeta(3)$ are exhibited in Fig. 3.13 for different values of N. A much more rapid series for approximating $\zeta(3)$ was given by Amdeberhan and Zeilberger [8] by

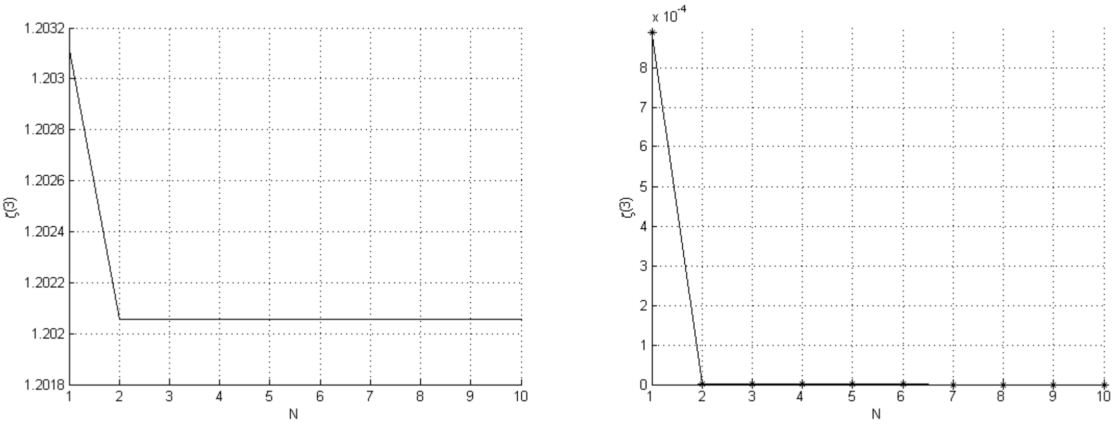
$$\zeta(3) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(205k^2 - 160k + 32)}{k^5\binom{2k}{k}^5} \tag{3.5}$$

which is associated with very high precision approximations for $\zeta(3)$. This is demonstrated in Fig. 3.14 by showing $\zeta(3)$ approximations and the corresponding relative error with different values of N.



(a) $\zeta(3)$ approximations versus N. (b) relative error versus N.

Figure 3.13: Approximating $\zeta(3)$ by Hjortnaes accelerating series (3.4).



(a) $\zeta(3)$ approximations versus N. (b) Relative error versus N.

Figure 3.14: Approximating $\zeta(3)$ by Amdeberhan and Zeilberger accelerating series (3.5).

In order to make a comparison between $\zeta(3)$ approximations obtained by the three presented series, we offer the required values of N for the desired accuracy ε by Table 3.5.

ε	Euler Series	Hjortnaes Series	Amdeberhan and Zeilberger Series
1×10^{-3}	20	3	1
1×10^{-5}	203	5	2

Table 3.5: Required values of N for approximating $\zeta(3)$.

Results in Table 3.5 can be presented in general for different values of N and ε. This is accomplished by Fig. 3.15.

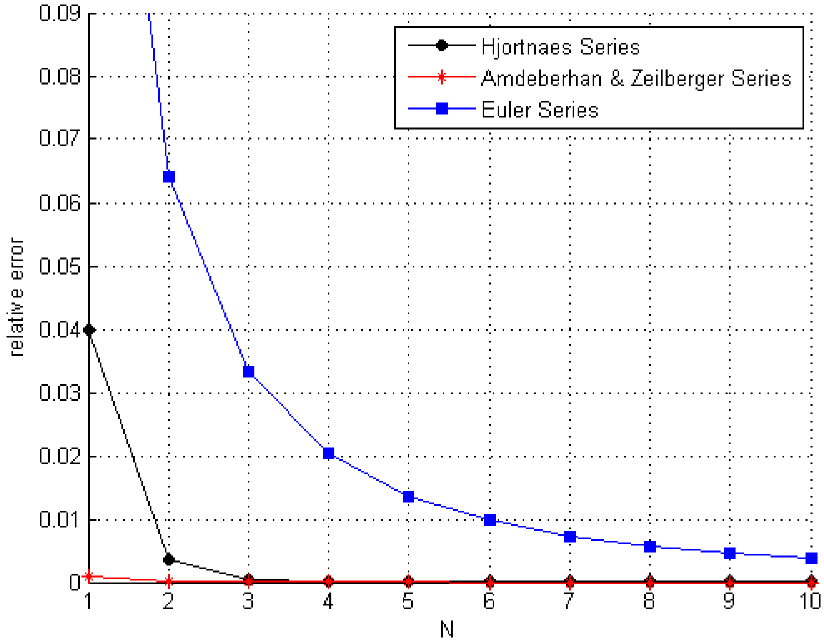
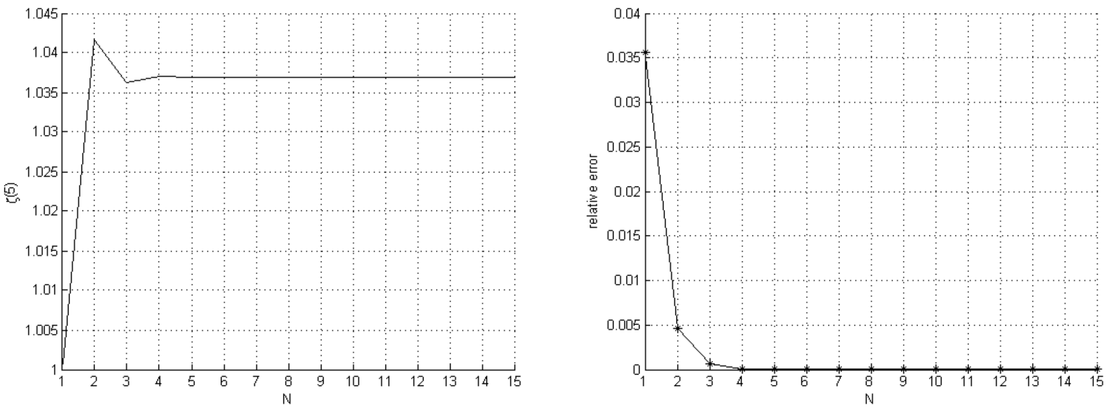


Figure 3.15: Comparison between $\zeta(3)$ approximations obtained by Euler, Hjortnaes, and Amdeberhan and Zeilberger Series

For calculating $\zeta(5)$, Koecher established the formula [57, 58]

$$\zeta(5) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} - \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2} \tag{3.6}$$

$\zeta(5)$ approximations and the corresponding relative error obtained by Koecher series (3.6) with different values of the cutoff parameter N are presented in Fig. 3.16.



(a) $\zeta(5)$ approximations versus N.

(b) Relative error versus N.

Figure 3.16: Approximating $\zeta(5)$ with Koecher Series (3.6).

A comparison between $\zeta(5)$ approximations obtained by Euler and Koecher series is given in Fig. 3.17 for different values of N. From the figure, it's clear that differences between values of relative error associated with Euler and Koecher series for the same N are little or at least not as in the case of using equations (3.2), (3.4), and (3.3) and as demonstrated in Fig. 3.2, Fig. 3.13, and Fig. 3.3. The reason is that, by increasing s, $\zeta(s)$ function becomes more and more short range one. This means that for large values of s, the need for using new methods rather than the direct one becomes less important, the same result for $s = 7$ is obtained by the following discussion.
J. Borwein and D. Bradley had established a similar series of (3.6) for $\zeta(7)$ [5, 15]

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^7\binom{2k}{k}} + \frac{25}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^3\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4} \tag{3.7}$$

Using Borwein-Bradley series (3.7) with different values of the cutoff parameter N, many approximations of $\zeta(7)$ are obtained and manifested in Fig. 3.18 as well as the corresponding relative error.

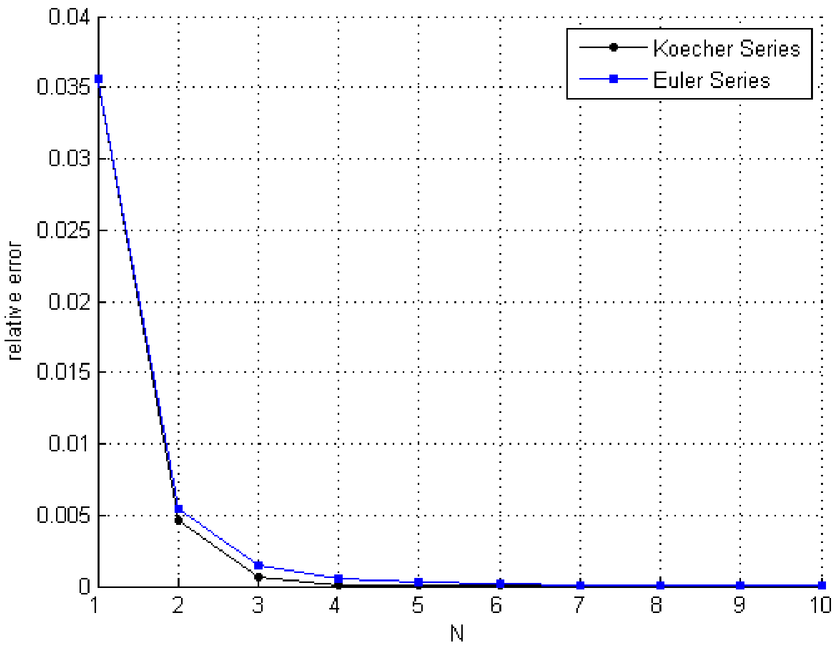
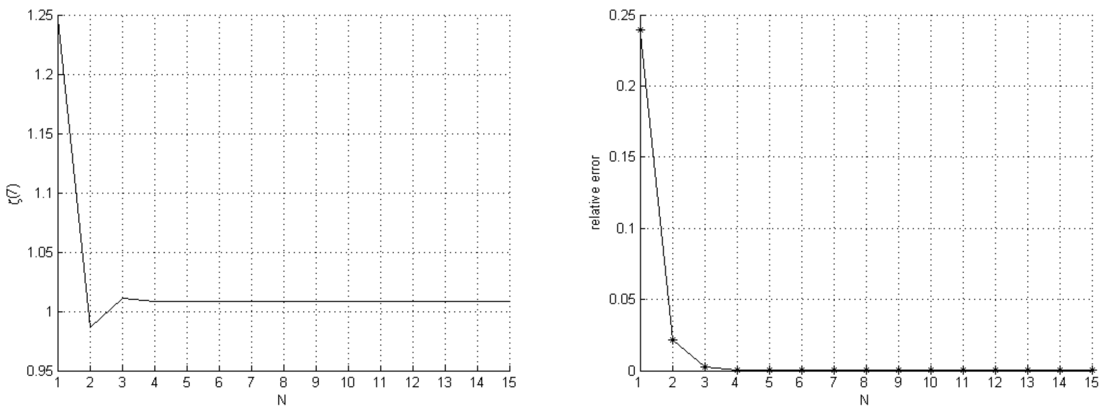


Figure 3.17: Comparison between $\zeta(5)$ approximations obtained by Euler and Koecher series.



(a) $\zeta(7)$ approximations versus N.

(b) Relative error versus N.

Figure 3.18: Approximating $\zeta(7)$ by Borwein-Bradley series (3.7).

Unlike all the previous established series (3.2), (3.3), (3.4), (3.5) and (3.6), the approximations obtained using Borwein-Bradley are worse than the ones taken by Euler series. This make sense, because the direct approach is sufficient for reasonable large arguments. Fig. 3.19 describes the differences between $\zeta(7)$ approximations resulted from using Borwein-Bradley and Euler series for different values of N. More rapid series can be found in [28, 35, 52, 55, 76].

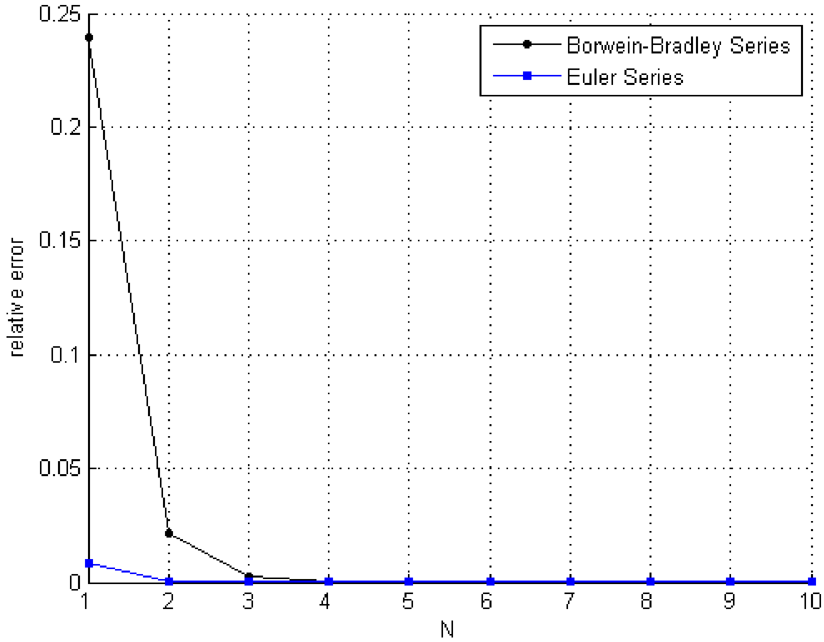


Figure 3.19: Comparison between $\zeta(7)$ approximations obtained by Euler and Borwein-Bradley series.

3.3 Euler Maclaurin Summation Method

Historically, Euler Maclaurin summation formula (EMS) was the first tool used by Euler to evaluate Riemann Zeta function and particularly to numerically solve Basel problem before solving it analytically [32].

Originally, this formula is used to evaluate infinite series by integrals and vise versa. This is only applicable when the derivative of the summand or the integrand exists. It was discovered separately by Leonhard Euler and Colin Maclaurin in the thirteenth of the eighteenth century. Actually, the former used it to evaluate infinite sums, where the later used it to calculate integrals.

To evaluate sums by integrals we have the formula [19, 74]:

$$\sum_{n=M}^{N} f(n) = \int_{M}^{N} f(x)dx - B_1(f(N) + f(M) + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(N) - f^{(2k-1)}(M)) \quad (3.8)$$

where B_k is the k^{th} Bernoulli number.

Neither Euler, nor Maclaurin gave an estimate for the error term. In fact, it was given later by S. D. Poisson by the formula

$$E = -\int_M^N \frac{B_k}{k!}\{1-t\} f^{(k)}(t)dt \tag{3.9}$$

where $\{t\}$is the fractional part of t [2].

Now, to calculate sums with number of terms $N \to \infty$, asymptotic expansions are obtained rather than explicit ones and EMS is applied on the reminder of these sums after calculating a certain number of terms directly.

In order to approximate $\zeta(s)$ by EMS method, we apply EMS formula to Riemann Zeta function remainder

$$\sum_{n>N} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1$$

where N is a large enough cutoff parameter. This gives

$$\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{-s}}{2} + \frac{N^{1-s}}{s-1} + \sum_{k=1}^{M} T_{k,N}(s) + E_{M,N}(s) \tag{3.10}$$

where

$$T_{k,N}(s) = \frac{B_{2k}}{(2k)!} N^{1-s-2k} \prod_{i=0}^{2k-2} (s+i) \tag{3.11}$$

and

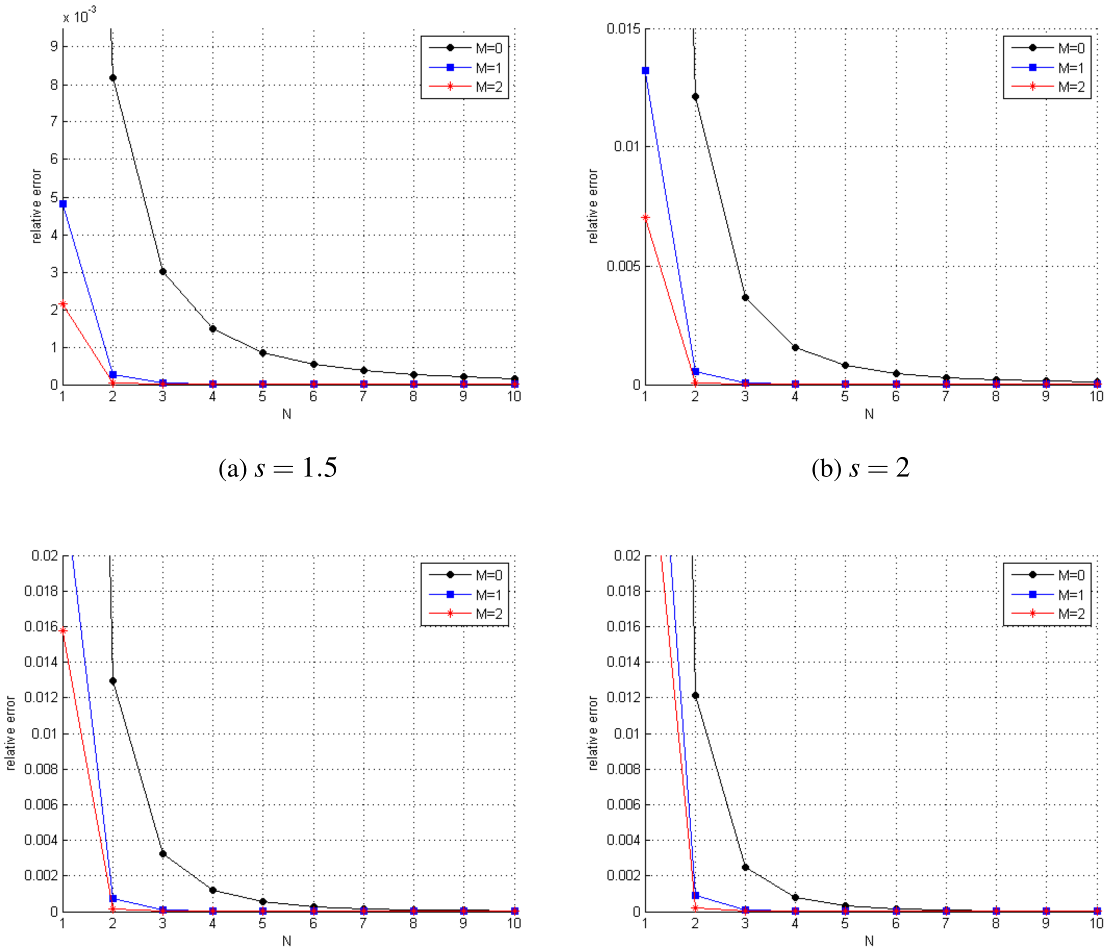
$$|E_{M,N(s)}| < |\frac{T_{M+1,N}(s)(s+2M+1)}{\mathrm{Re}(s)+2M+1}| \tag{3.12}$$

such that, M is another cutoff parameter; $M \geq 0$ and $N \geq 1$. Formula (3.10) is valid for any $s; \mathrm{Re}(s) > -2M-1$ [2].

By rescaling the cutoff parameters N and M, one can evaluate Riemann Zeta function for any value and to any needed precision [19].

Actually, EMS method was the workhorse for numerical evaluation of Riemann Zeta function until 1930. In fact, all calculations concerning zeros of Riemann Zeta function before the later date are carried out by using EMS formula, and hence EMS method was an important tool for numerical verification of Riemann Hypothesis [72].

Though many methods are developed later for calculating $\zeta(s)$, but EMS method is still used in very accurate calculations and it is implemented in many softwares, like Maple, Matlab, and Mathematica. This is because the error term of (3.10) is easily estimated. On the other hand, EMS method in not preferable in the context of efficiency, since the method requires doing $\mathrm{Im}(s)$ operations when calculating a single value of the form $\zeta(\frac{1}{2} + i\,\mathrm{Im}(s))$. Also, to obtain a new precision, one has to rescale the cutoff parameters of (3.10) every time [17]. To approximate $\zeta(s)$ with EMS method using equations (3.10) and (3.11) with error bound given by (3.12), we have to detect N and M values. In order to do that as best as possible, the behavior of the relative error when changing the values of N and M should be studied. Fig. 3.20 shows that for different values of s, the relative error associated with approximating $\zeta(s)$ using (3.10) and (3.11) decays as N increases at a fixed value of M. Also, from the same figures, it is obvious that $\zeta(s)$ approximations carried out with $M = 2$ are better than those obtained by $M = 1$, which in turn are better than the ones attained with $M = 0$.



(a) $s = 1.5$ (b) $s = 2$

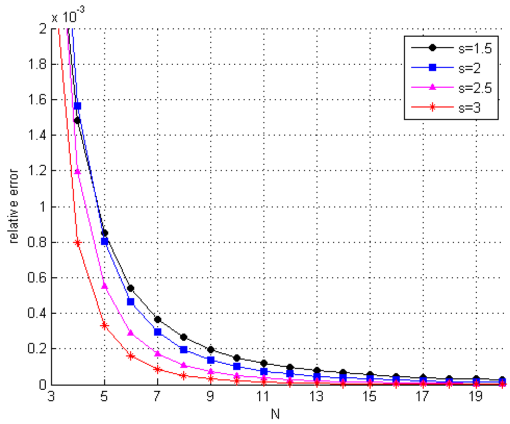
(c) $s = 2.5$ (d) $s = 3$

Figure 3.20: Comparisons between $\zeta(s)$ approximations obtained by EMS method at different values of M for $s = 1.5, 2, 2.5, 3$.

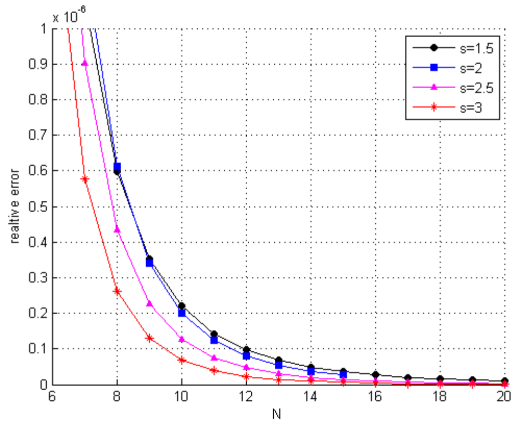
For M values, $\zeta(s)$ approximations obtained by large M values are preferable, but on the other hand N must be large enough. That is, the number of terms calculated directly in (3.10) should be reasonably large.

For example, $N = 1$, $M = 3$ are the best choices to approximate $\zeta(1.5)$ with relative error $\varepsilon = 2.14 \times 10^{-3}$. At $N = 4$, then $M = 12$ is chosen to obtain an approximation of $\zeta(1.5)$ with $\varepsilon = 6.45 \times 10^{-12}$.

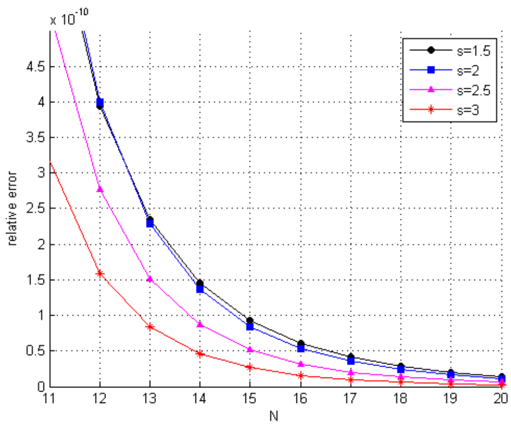
When using a fixed value of M, then $\zeta(s)$ approximations obtained by EMS method get better as argument s gets larger for sufficient values of N. Fig. 3.21 shows comparisons between the relative error associated with $\zeta(s)$ approximations for $s = 1.5, 2, 2.5, 3$ using EMS method with $M = 0, 1, 2$.



(a) $M = 0$



(b) $M = 1$



(c) $M = 2$

Figure 3.21: Comparison between $\zeta(s)$ approximation obtained by EMS method for $s = 1.5, 2, 2.5, 3$.

Fig. 3.22 exhibits the relative error associated with detecting a couple of M and N values in (3.10) and (3.11) for approximating $\zeta(1.5)$.

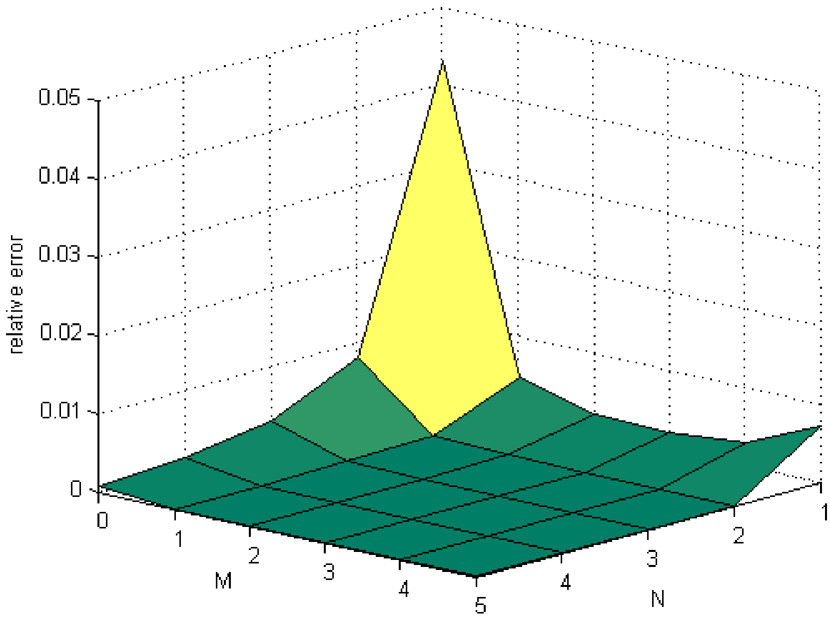


Figure 3.22: The relative error associated with approximating $\zeta(1.5)$ by EMS method versus different values of N and M.

Now, we are going to give some approximations and the corresponding relative error for $\zeta(1.5)$ and $\zeta(2)$ associated with some values of M and N by Table 3.6 and Table 3.7 respectively.

M	N	Approximate Value	ε
0	4	2.6085034803231490	1.40×10^{-3}
0	12	2.6121250153776868	9.6×10^{-5}
1	2	2.6130873445818115	2.70×10^{-4}
1	4	2.6124097303231490	1.30×10^{-5}
2	1	2.6067708333333333	2.14×10^{-3}
2	2	2.6122817216214753	3.6×10^{-5}
3	1	2.6175130208333333	1.96×10^{-3}
3	2	2.6124004071468820	9.60×10^{-6}

Table 3.6: Approximate values of $\zeta(1.5)$ and the corresponding relative error when using EMS method at different values of M and N.

M	N	Approximate Value	ε
0	4	1.6423611111111111	1.50×10^{-3}
0	11	1.6448090534805903	7.6×10^{-5}
1	2	1.6458333333333333	5.50×10^{-4}
1	3	1.6450617283950617	7.70×10^{-5}
2	1	1.6333333333333333	7.1×10^{-3}
2	2	1.6447916666666667	8.7×10^{-5}
3	1	1.6571428571428571	7.4×10^{-3}
3	2	1.6449776785714286	2.70×10^{-5}

Table 3.7: Approximate values of $\zeta(2)$ and the corresponding relative error when using EMS method at different values of M and N.

From Table 3.2, to obtain an accuracy of $\varepsilon = 1.5 \times 10^{-3}$ for approximating $\zeta(1.5)$ by Euler series, 257060 terms are needed. At the same time, EMS requires only three terms for series (3.10) to obtain a further better approximation ($\varepsilon = 1.4 \times 10^{-3}$). In the case of $\zeta(2)$, 403 terms are needed by direct approach with $\varepsilon = 1.5 \times 10^{-3}$, whereas only three terms are needed by EMS to obtain the same accuracy. This shows how Euler Maclaurin Summation method is efficient, which is obviously better than the direct approach.

3.4 Borwein Algorithms

In this section we present a set of efficient algorithms aim to approximate $\zeta(s)$ with a very high precision for arbitrary $s > 0$. These indeed are introduced by Peter Borwein [14]. The idea behind them is based on the connection between $\zeta(s)$ and eta Dirichlet series

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{3.13}$$

which is given by

$$\zeta(s) = \frac{1}{(1-2^{1-s})}\eta(s) \tag{3.14}$$

With respect to equation (3.13), and by making use of the methods constructed for approximating the alternating series of $\eta(s)$ which are known sometimes as Zeta alternating series, then $\zeta(s)$ can be approximated in straightforward way.

However, a set of efficient algorithms were developed by H. Cohen, F. Villegas, and D. Zagier [25] for approximating $\eta(s)$ with very high precision. The first algorithm that was presented, depends on using T-Chebyshev polynomials $T_n(2x-1)$ in such way that $\eta(s)$ approximations are associated with a relative accuracy of 5.828^{-n}. The second algorithm can be considered as a generalization of the first one, since it uses any polynomial that does vanish at -1.

Later on, P. Borwein had used these algorithms with equation (3.13) in order to make high precision approximations of $\zeta(s)$ [14]. The basic algorithm of Borwein makes use of the coefficients of a general polynomial $p_n(x)$ exactly like the second algorithm mentioned recently for approximating $\eta(s)$, except the division by $(1-2^{1-s})$. We now give Borwein algorithm [14] and then two special cases of the algorithm are discussed below.

Algorithm 3.4.1. *"Borwein Algorithm"*
let $p_n(x) = \sum_{k=0}^{n} a_k x^k$ *be a polynomial of degree n, such that;* $p_n(-1) \neq 1$. *By calculating*

$$c_j = (-1)^j \left(\sum_{k=0}^{j} (-1)^k a_k - p_n(-1) \right) \tag{3.15}$$

then

$$\zeta(s) = \frac{-1}{(1-2^{1-s})p_n(-1)} \sum_{j=0}^{n-1} \frac{c_j}{(1+j)^s} + \xi_n(s) \tag{3.16}$$

where

$$\xi_n(s) = \frac{1}{p_n(-1)(1-2^{1-s})\Gamma(s)} \int_0^1 \frac{p_n(x)|\log x|^{s-1}}{1+x} dx \tag{3.17}$$

where $\Gamma(s)$ *is gamma function and* $s \in \mathbb{R}; s > 0$.

Now, by taking $p_n(x) = T_n(2x-1)$, where $T_n(x)$ are the ordinary T-Chebychev polynomials given by equation (2.12). This implies the following special algorithm, which can be found also in [14]

Borwein Algorithm Case 3.4.1. *Calculate*

$$d_k = n \sum_{i=0}^{k} \frac{(n+i-1)!4^i}{(n-i)!(2i)!} \tag{3.18}$$

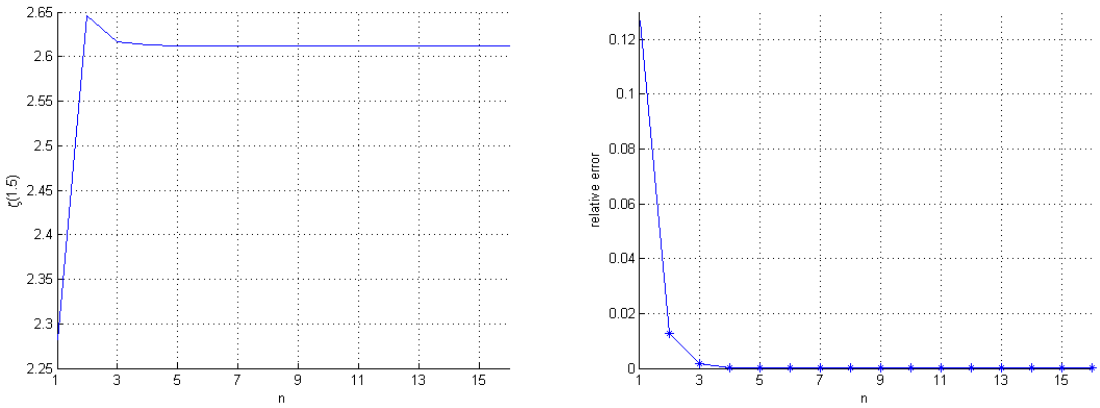
Then,

$$\zeta(s) = \frac{-1}{d_n(1-2^{1-s})} \sum_{k=0}^{n-1} \frac{(-1)^k (d_k - d_n)}{(k+1)^s} + \gamma_n(s) \quad , s \geq \frac{1}{2} \tag{3.19}$$

where

$$\gamma_n(s) \leq \frac{3}{(3+\sqrt{8})^n} \frac{1}{|1-2^{1-s}|} \tag{3.20}$$

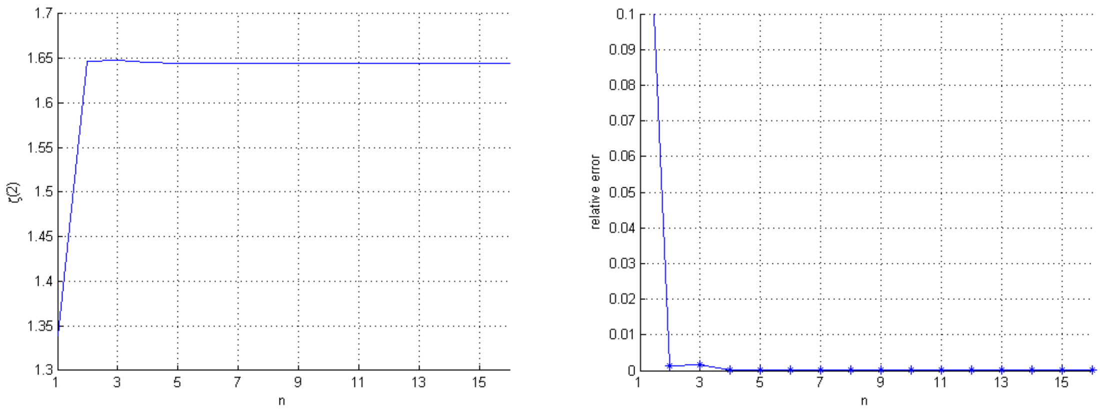
Using equation (3.18) and equation (3.19) to approximate $\zeta(s)$ for $s = 1.5, 2, 2.5, 3$, we obtain the approximations and the corresponding relative error presented in Fig. 3.23, Fig. 3.24, Fig. 3.25, and Fig. 3.26 respectively.



(a) $\zeta(1.5)$ approximations vesus n

(b) Relative error versus n.

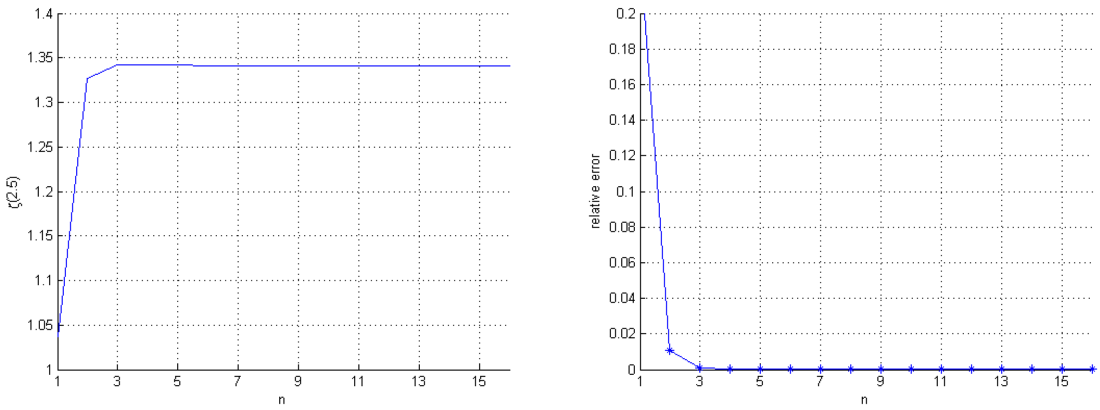
Figure 3.23: Approximating $\zeta(1.5)$ by Borwein Algorithm case 3.4.1.



(a) $\zeta(2)$ approximations vesus n

(b) Relative error versus n.

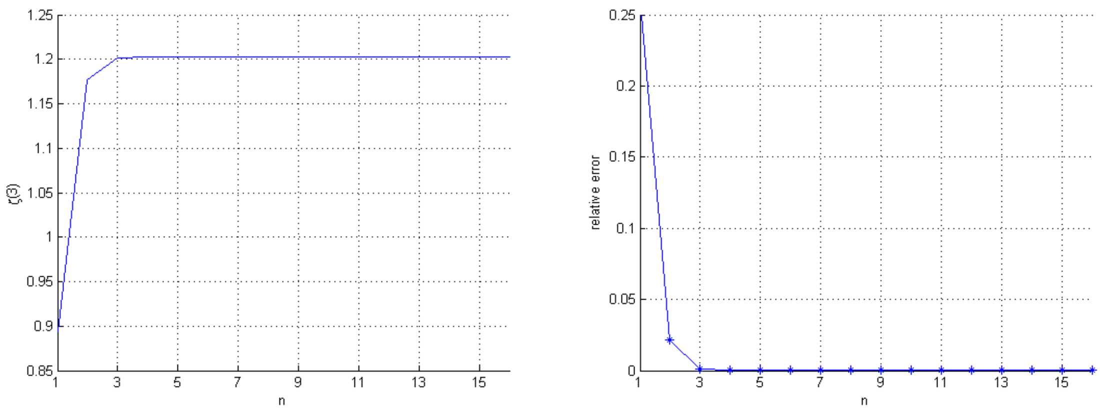
Figure 3.24: Approximating $\zeta(2)$ by Borwein Algorithm case 3.4.1.



(a) $\zeta(2.5)$ approximations vesus n

(b) Relative error versus n.

Figure 3.25: Approximating $\zeta(2.5)$ by Borwein Algorithm case 3.4.1.



(a) $\zeta(3)$ approximations vesus n

(b) Relative error versus n.

Figure 3.26: Approximating $\zeta(3)$ by Borwein Algorithm case 3.4.1.

Unlike earlier methods that were presented in the previous sections to approximate $\zeta(s)$, this algorithm is better for smaller values of s when using sufficient value of n. Fig. 3.27 describes this by plotting the relative error values associated with $\zeta(s)$ approximations for $s = 1.5, 2, 2.5, 3$ and by using degree $n = 10, 11, ..., 20$.
Now, if we take $p_{2n}(x) = x^n(1-x)^n$, the we have the special algorithm

Borwein Algorithm Case 3.4.2. *Calculate*

$$e_j = (-1)^j \left[\sum_{k=0}^{j-n} \frac{n!}{k!(n-k)!} - 2^n \right] \tag{3.21}$$

Then,

$$\zeta(s) = \frac{-1}{2^n(1-2^{1-s})} \sum_{j=0}^{2n-1} \frac{e_j}{(j+1)^s} + \gamma_n(s) \quad , s > 0 \tag{3.22}$$

where

$$|\gamma_n(s)| \leq \frac{1}{8^n} \frac{1}{|1-2^{1-s}|} \tag{3.23}$$

Now, by using equations (3.21) and (3.22), then $\zeta(1.5), \zeta(2), \zeta(2.5), \zeta(3)$ can be approximated by using different values of n. Fig. 3.28, Fig. 3.29, Fig. 3.30, and Fig. 3.31 show $\zeta(s)$ approximations and the corresponding relative error.

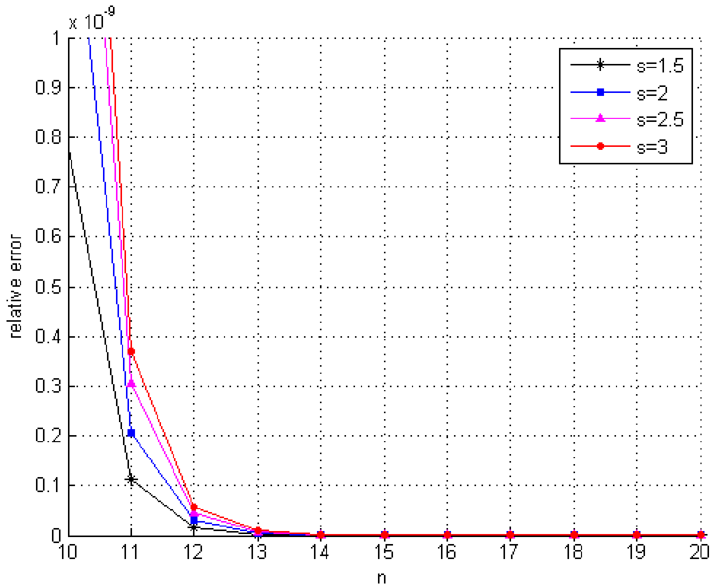
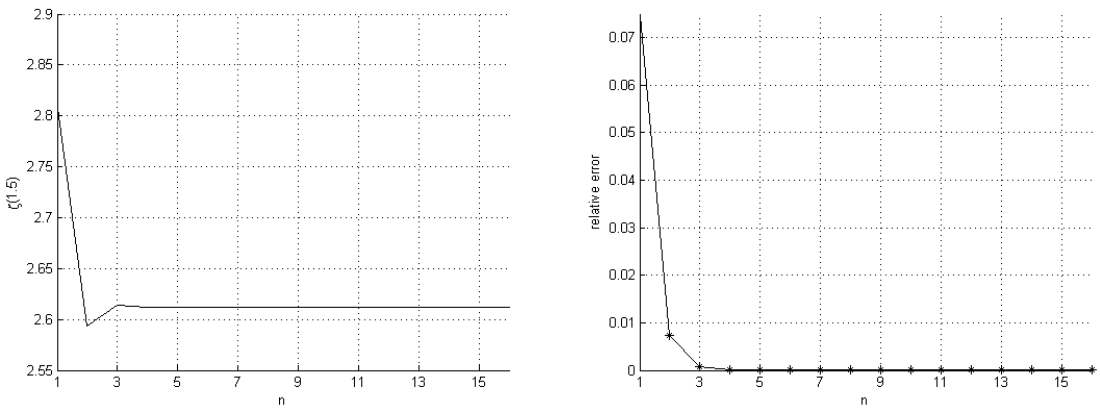


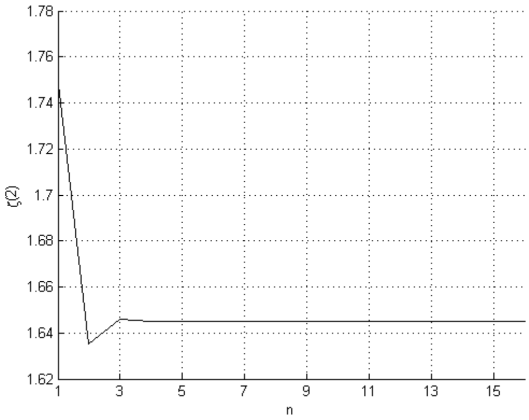
Figure 3.27: Comparison between $\zeta(s)$ approximations obtained by Borwein Algorithm case 3.4.1 for $s = 1.5, 2, 2.5, 3$.



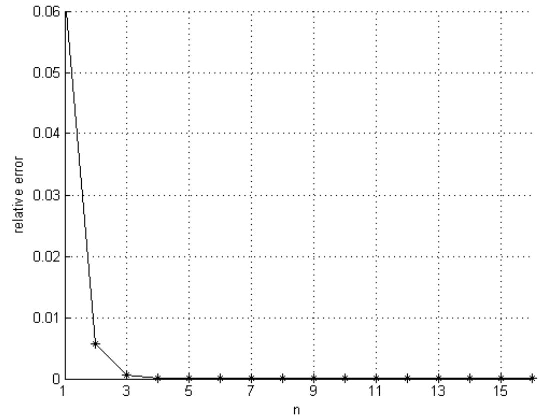
(a) $\zeta(1.5)$ approximations versus n.

(b) Relative error versus n.

Figure 3.28: Approximating $\zeta(1.5)$ by Borwein Algorithm case 3.4.2.

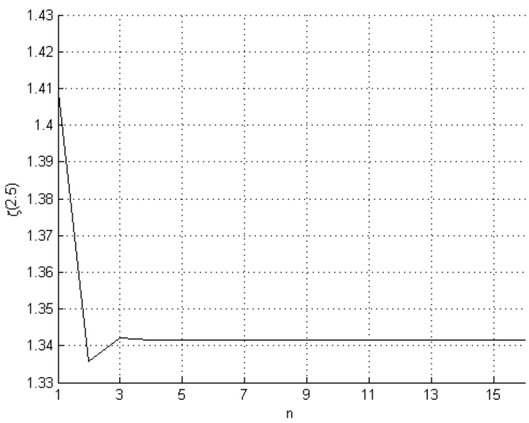


(a) $\zeta(2)$ approximations versus n.

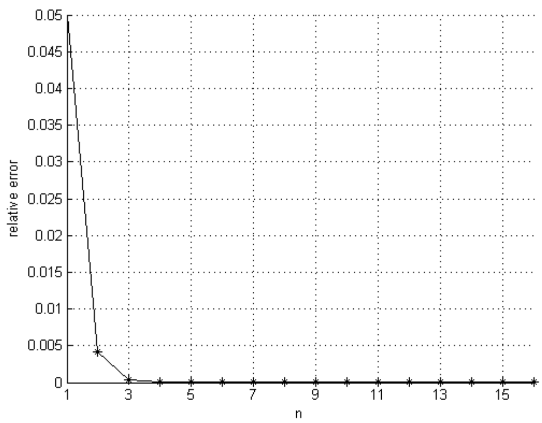


(b) Relative error versus n.

Figure 3.29: Approximating $\zeta(2)$ by Borwein Algorithm case 3.4.2.

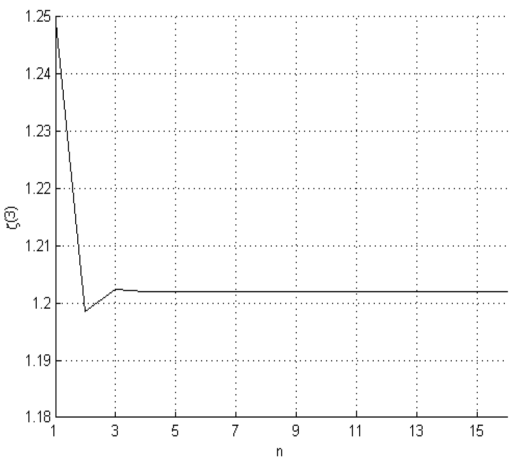


(a) $\zeta(2.5)$ approximations versus n.

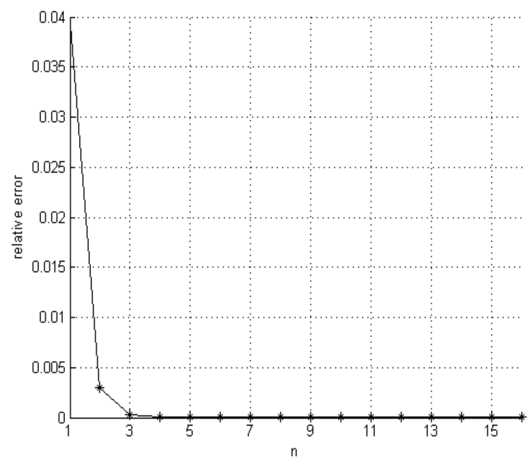


(b) Relative error versus n.

Figure 3.30: Approximating $\zeta(2.5)$ by Borwein Algorithm case 3.4.2.



(a) $\zeta(3)$ approximations versus n.



(b) relative error versus n.

Figure 3.31: Approximating $\zeta(3)$ by Borwein Algorithm case 3.4.2.

Fig. 3.32 gives a comparison between the relative error associated with $\zeta(s)$ approximations with Borwein algorithm case 3.4.2 for $s = 1.5, 2, 2.5, 3$.

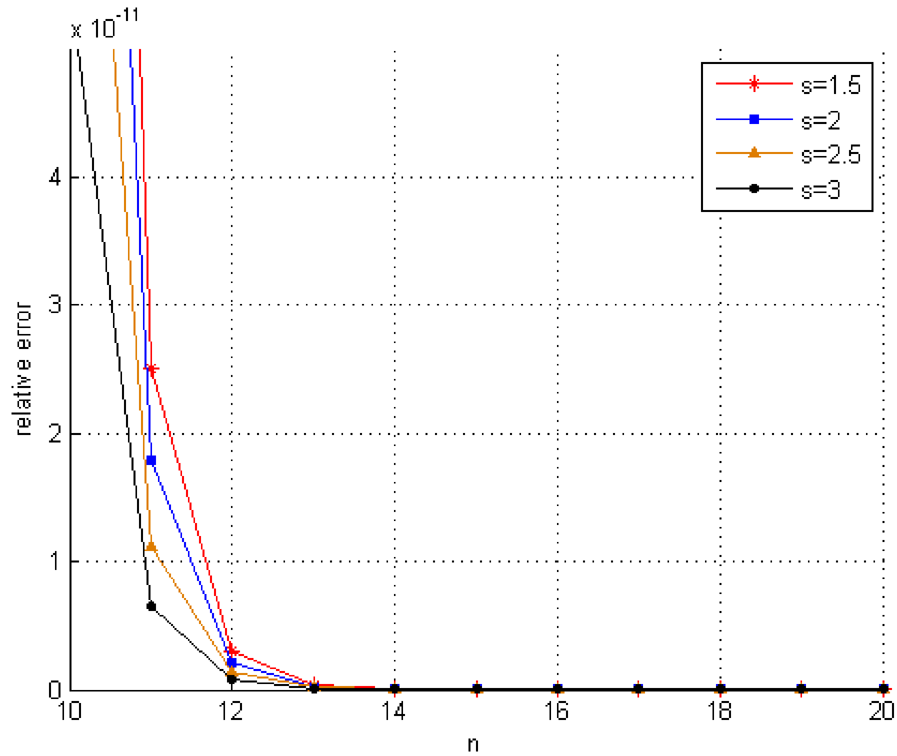


Figure 3.32: Comparison between $\zeta(s)$ approximations obtained by Borwein Algorithm case 3.4.2 for $s = 1.5, 2, 2.5, 3$ with respect to the relative error.

For $N = 20$, Table 3.8 shows the values of relative error in approximating $\zeta(s)$ by Borwein Algorithm case 3.4.2 for different values of s.

s	1.5	2	2.5	3
ε	1.40×10^{-19}	9.78×10^{-20}	6.00×10^{-20}	3.33×10^{-20}

Table 3.8: The values of relative error in approximating $\zeta(s)$ by Borwein Algorithm case 3.4.2 with $N = 20$.

Now, Table 3.9 and Table 3.10 present the suitable values of n for approximating $\zeta(s)$ with a relative error of $\varepsilon = 1 \times 10^{-3}$ and $\varepsilon = 1 \times 10^{-5}$ respectively.
Table 3.9 shows that in the case of using Borwein Algorithm Case 3.4.2, only two terms are needed for the series in (3.22), whereas three terms are needed for the series in (3.19) when

using Borwein Algorithm Case 3.4.1 to obtain an accuracy $\varepsilon = 1 \times 10^{-3}$ in approximating $\zeta(1.5)$. Comparing these results with the ones given in Table 3.2, Table 3.6, and Table 3.7, this shows that Borwein algorithms are preferable in the context of approximating $\zeta(s)$.

	$s = 1.5$	$s = 2$	$s = 2.5$	$s = 3$
Borwein Algorithm case 3.4.1	4	4	3	3
Borwein Algorithm case 3.4.2	3	3	3	3

Table 3.9: Suitable degree n for relative accuracy of roughly $\varepsilon = 1 \times 10^{-3}$

	$s = 1.5$	$s = 2$	$s = 2.5$	$s = 3$
Borwein Algorithm case 3.4.1	6	6	6	6
Borwein Algorithm case 3.4.2	5	5	5	5

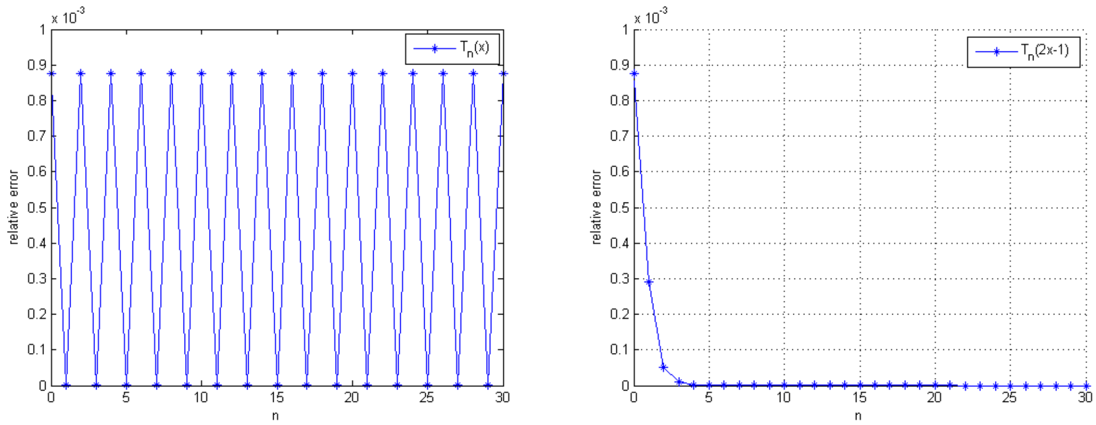
Table 3.10: Suitable degree n for relative accuracy of roughly $\varepsilon = 1 \times 10^{-5}$

In the next chapter, we shall use Borwein algorithm with different kinds of polynomials in order to find better approximations for $\zeta(s)$.

Chapter 4

NUMERICAL RESULTS

In this chapter, different kinds of polynomials will be embedded in Borwein algorithm which yield approximations with different accuracy. In particular, these polynomials include the families presented in Chapter 2, namely; classical, discrete, multiple d-orthogonal, Horadam, and traditional polynomials. During our work, Jacobi, multiple d-orthogonal, and Horadam polynomials are shifted to the interval $[0,1]$ with a suitable transformation before using them with Borwein algorithm in order to get better approximations. Throughout the coming discussion we shall symbolize the shifted polynomials with the same symbols that stand for the ordinary ones. For instance, T-Chebyshev polynomials $T_n(x)$ are used with the replacement $T_n(2x-1)$, by which new polynomials are generated with similar properties to $T_n(x)$. Fig. 4.1 illustrates the behavior of the relative error as n increases in approximating $\zeta(2)$ by Borwein algorithm in the case of $T_n(x)$ and $T_n(2x-1)$.



(a) without shifting (b) with shifting

Figure 4.1: The behavior of the relative error when approximating $\zeta(2)$ by Borwein algorithm using $T_n(x)$ and $T_n(2x-1)$.

For every set of polynomials, the relative error in calculating $\zeta(s)$ at argument s using Borwein algorithm is computed for different values of the polynomial degree n. The behavior of the relative error as n increases is observed and manifested by plotting it for different values of s. Afterwards, comparisons between the behavior of relative error associated with different kinds of polynomials at fixed value of s are presented as n increases. Typically, we'll compare most of polynomials with Chebyshev and Gegenbauer polynomials.
For classes of polynomials with free parameter, say α in the form $p_n^{\alpha}(x)$, comparisons between the approximations obtained by $p_n^{\alpha}(x)$ for different values of α are shown with respect to relative error at fixed values of n and s. Also, we'll offer, in the case of some of these polynomials, three dimensional graphs that exhibit the relative error resulted from using different values of the couple (n,α). More substantially, we look for bounds for α values by which the relative error along with $\zeta(s)$ approximations will not be better a part from this bound.
In fact, we are concern only about $\zeta(s)$ approximations when $s = 1.5, 2, 2.5, 3$. Thus, all results during our work are introduced by the means of these values. In the following sections, we implicate various polynomials with different types of orthogonality in Borwein algorithm, then we use other polynomials, some of them are from Horadam family, and the others are considered as traditional ones.

4.1 Approximating $\zeta(s)$ with Orthogonal Polynomials

This section is devoted for examining some polynomials of the classical, discrete, and multiple-d orthogonal classes by checking out the efficiency of $\zeta(s)$ approximations resulted from Borwein algorithm by the means of these polynomials. In particular, Chebyshev, Legendre, Gegenbauer, Associated Laguerre, and Bessel polynomials are included in Borwein algorithm as classical orthogonal ones. Then, we include discrete Charlier polynomials, and use Pincherle and Horadam-Pethe polynomials from Humbert family, and Kinney polynomials as a generalized case of Humbert.

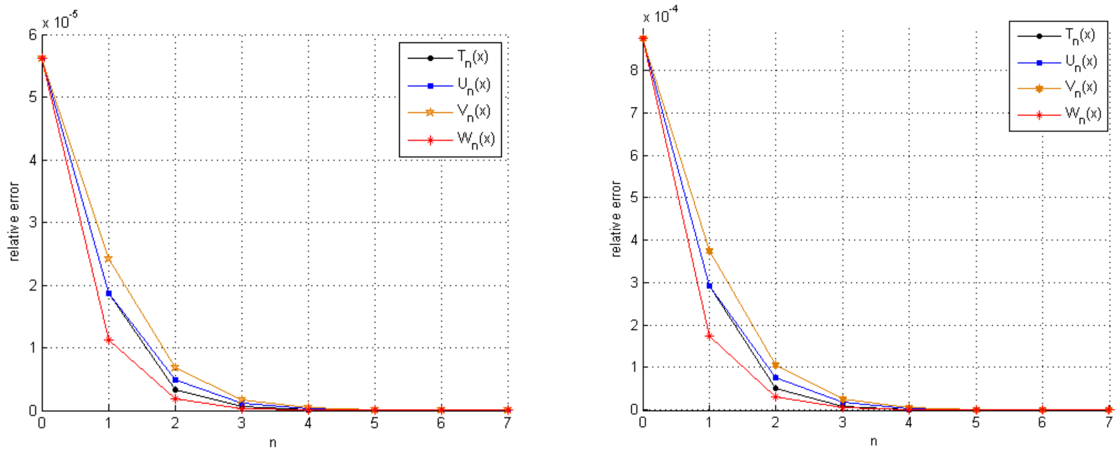
4.1.1 Approximation by Chebyshev Polynomials

Here, we are going to implicate some polynomials of Jacobi type in Borwein algorithm, those are given explicitly by equation (2.8). In particular, we now use Chebyshev polynomials which are presented in section (2.1.1) with Borwein algorithm and try to verify the efficiency of their approximations. The common four types of Chebyshev polynomials are T-Chebyshev, U-Chebyshev, V-Chebyshev, and W-Chebyshev. The explicit formulas for those polynomials are given by the equations (2.12), (2.15), (2.18), and (2.21) respectively.

The four kinds of Chebychev polynomials compete in enhancing the accuracy in approximating $\zeta(s)$. Fig. 4.2 and Fig. 4.3 show how the relative error behaves when using these different polynomials with different degree n in approximating $\zeta(s)$.

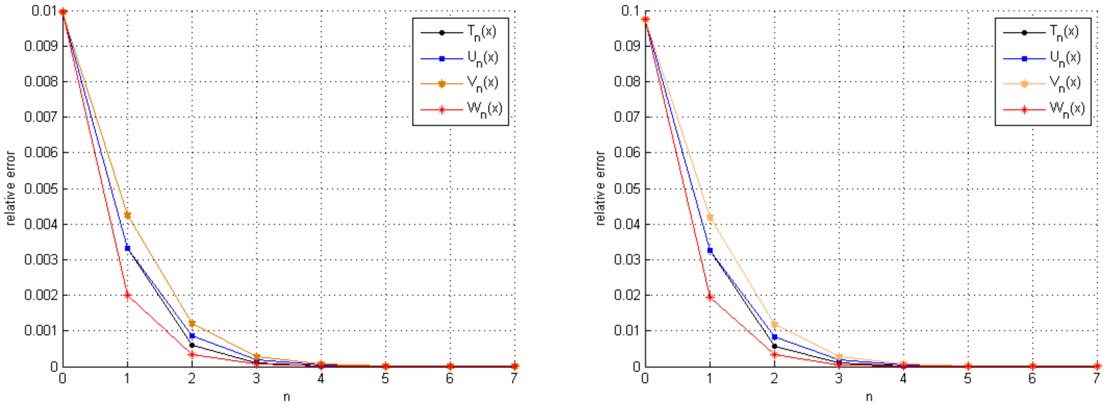
It is remarkable to observe that the relative error associated with using W-Chebychev polynomials $W_n(x)$ is less than the one associated with using T-Chebychev polynomials $T_n(x)$ at every n. Actually, $T_n(x)$ are the polynomials suggested by P. Borwein [14] and considered to be a good choice to try. In similar way, we can conclude that those polynomials are preferable in the arrangement; W-Chebyshev, T-Chebyshev, U-Chebyshev, and V-Chebychev polynomials according to the amount of relative error associated with their approximations of $\zeta(s)$ at fixed n and s.

Comparisons within the same kind of Chebyshev polynomials are made when using different values of s and this is clarified in Fig. 4.4.



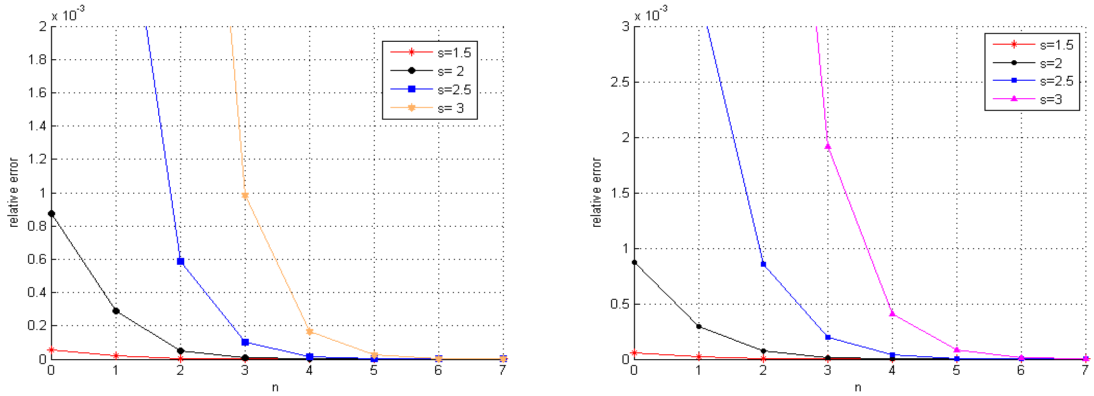
(a) $s = 1.5$ (b) $s = 2$

Figure 4.2: Comparisons between the different kinds of Chebychev polynomials in approximating $\zeta(s)$ at $s = 1.5, 2$.

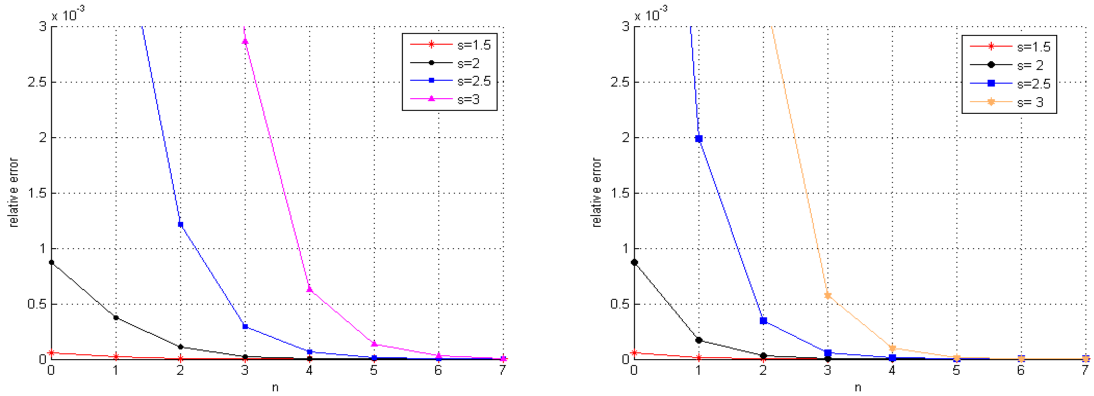


(a) $s = 2.5$ (b) $s = 3$

Figure 4.3: Comparisons between the different kinds of Chebychev polynomials in approximating $\zeta(s)$ at $s = 2.5, 3$.



(a) $T_n(x)$ (b) $U_n(x)$



(c) $V_n(x)$ (d) $W_n(x)$

Figure 4.4: The relative error in approximating $\zeta(s)$ in the case of using $T_n(x)$, $U_n(x)$, $V_n(x)$, and $W_n(x)$ polynomials.

For $n = 7$, Table 4.1 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using the different kinds of Chebyshev polynomials.

	$s = 1.5$	$s = 2$	$s = 2.5$	$s = 3$
$T_7(x)$	4.90×10^{-10}	7.60×10^{-9}	8.70×10^{-8}	8.50×10^{-7}
$U_7(x)$	1.90×10^{-9}	2.98×10^{-8}	3.40×10^{-7}	3.32×10^{-6}
$V_7(x)$	3.05×10^{-9}	4.77×10^{-8}	5.40×10^{-7}	5.31×10^{-6}
$W_7(x)$	2.88×10^{-10}	4.49×10^{-9}	5.10×10^{-8}	5.00×10^{-7}

Table 4.1: The values of relative error in approximating $\zeta(s)$ using Chebyshev polynomials.

4.1.2 Approximation by Legendre Polynomials

Legendre Polynomials $\lambda_n(x)$ which are given by (2.10), represent a particular case of Jacobi polynomials and they are accurate in approximating $\zeta(s)$. Fig. 4.5 shows the relative error when using Legendre polynomials in approximating $\zeta(s)$.

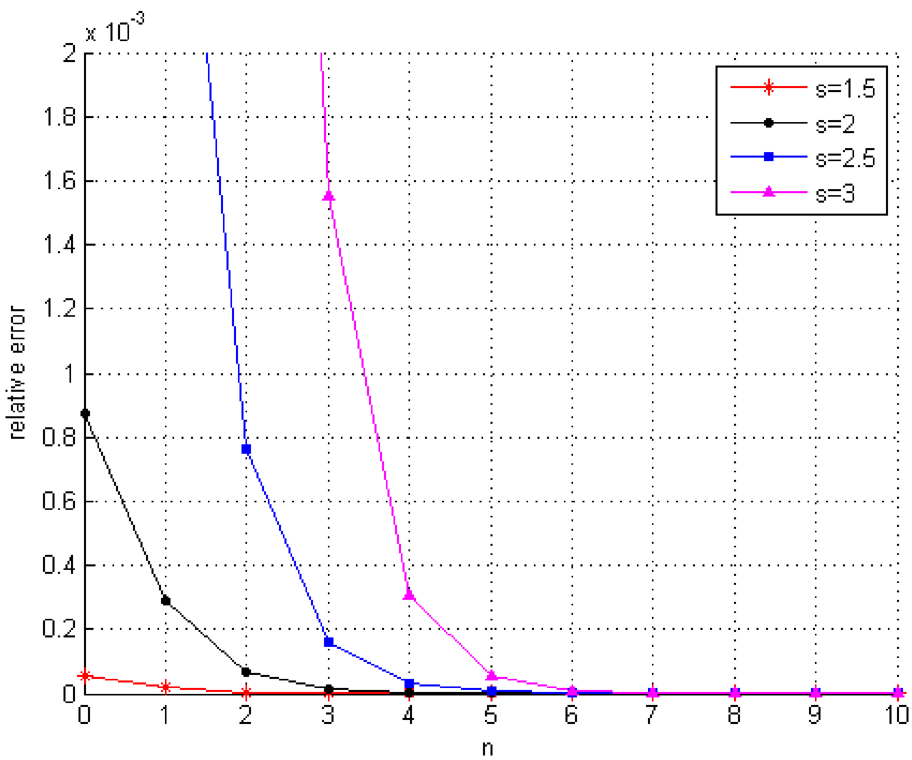
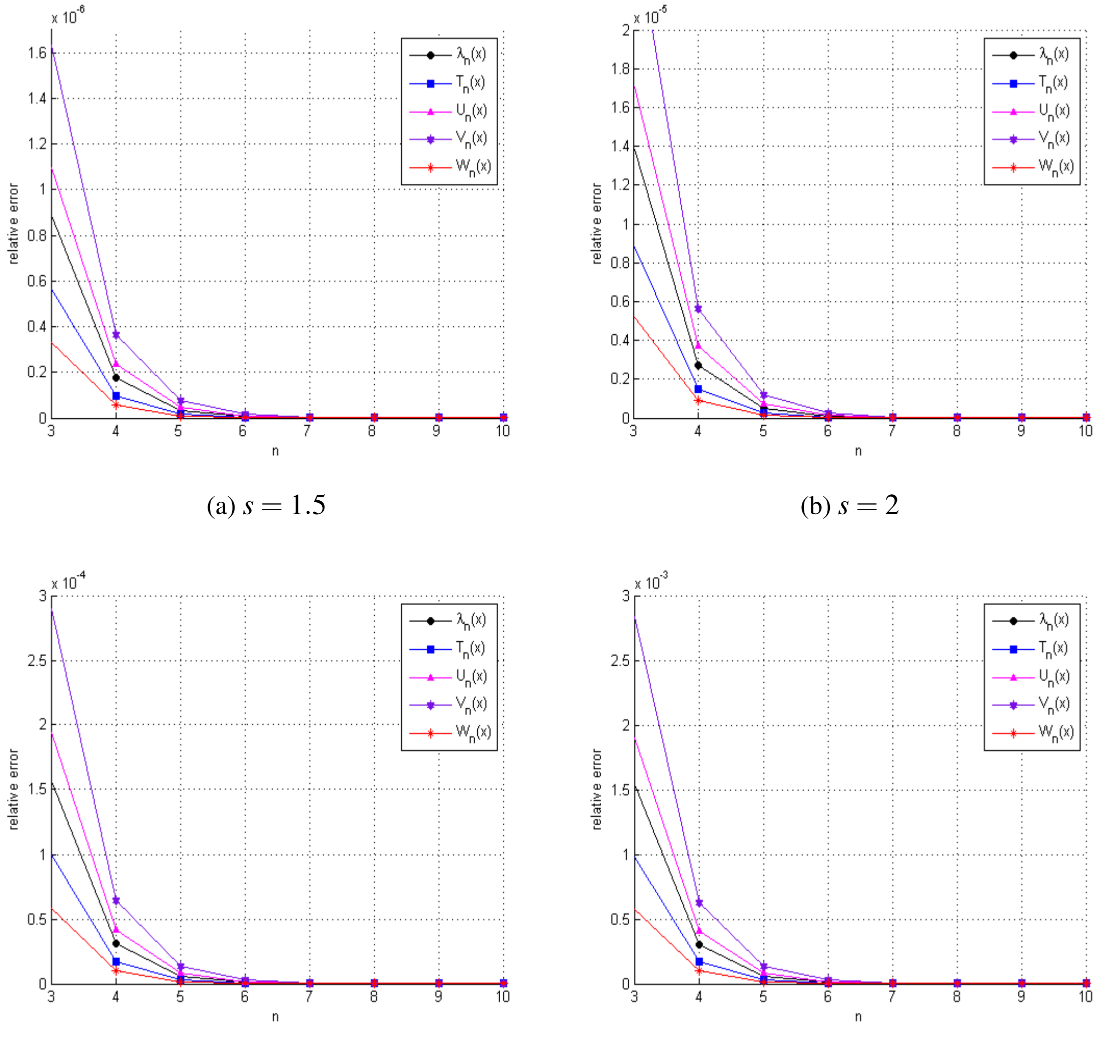


Figure 4.5: Comparisons between $\zeta(s)$ approximations obtained by using $\lambda_n(x)$ for different values of s.

The approximation value of $\zeta(s)$ using $\lambda_n(x)$ is better than the one associated with using $U_n(x)$ and $V_n(x)$, but worse than $\zeta(s)$ approximations obtained by $T_n(x)$ and $W_n(x)$. Fig. 4.6 shows comparisons between these polynomials for different values of s.

For $n = 10$, Table 4.2 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using Legendre polynomials.



(a) $s = 1.5$

(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.6: Comparisons between $\zeta(s)$ approximations obtained by using $\lambda_n(x)$, $T_n(x)$, $U_n(x)$, $V_n(x)$ and $W_n(x)$.

s	1.5	2	2.5	3
ε	6.94×10^{-12}	1.08×10^{-10}	1.23×10^{-9}	1.21×10^{-8}

Table 4.2: The values of relative error in approximating $\zeta(s)$ using $\lambda_n(x)$ with $n = 10$.

4.1.3 Approximation by Gegenbauer Polynomials

The class of Gegenbauer polynomials $C_n^\gamma(x)$ defined by equation (2.23) has a real importance in approximating $\zeta(s)$. The free parameter γ of Gegenbauer polynomials $C_n^\gamma(x)$ changes the accuracy of $\zeta(s)$ approximations in illuminating way. That is, the relative error decreases as γ does, until $\gamma = -0.5$, after this crucial value of γ, the relative error will increase by decreasing γ. Fig. 4.7 describes this variation in the case of approximating $\zeta(2)$.

The value $\gamma = -0.5$ is exactly the bound of γ values, by which the relative error associated with using $C_n^{-0.5}(x)$ in approximating $\zeta(s)$ is less than the one associated with $C_n^\gamma(x)$ for other values of γ. Boundedness is illustrated in Fig. 4.8 by taking more values of γ within the interval $(-0.6, -0.4)$.

This variation in improving $\zeta(s)$ approximations by changing γ can be described simply by the sketch in Fig. 4.9.

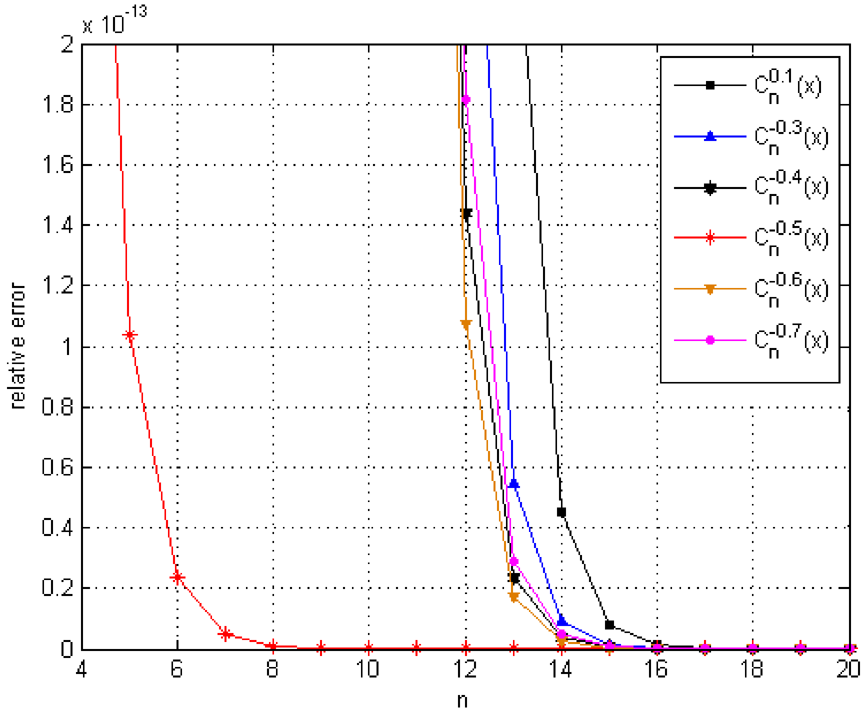


Figure 4.7: Comparison between different Gegenbauer polynomials in approximating $\zeta(2)$.

For Gegenbauer polynomials $C_n^\lambda(x)$ with $\gamma \in (-0.5, -0.4)$, these are better than the ones with bigger $\gamma \in (-0.5, -0.4)$ as discussed previously. But, they are worse than all Gegenbauer polynomials $C_n^{\lambda^*}(x)$ with $\gamma^* > -(1+\gamma)$, such that $\gamma^* \in (-0.6, -0.5)$. For example, $C_n^{-0.45}(x)$ are worse than $C_n^{-0.53}(x)$. This can be verified by Fig. 4.10 and clarified in a simple way by Diagram 4.11.

Note that $\gamma = -(1+\gamma)$ when $\gamma = -0.5$, this actually makes sense, since no better approximations can be obtained by going apart from $\gamma = -\frac{1}{2}$ neither from right nor from left.

This special value ($\gamma = -0.5$) of Gegenbauer polynomials is associated with nice polynomials $C_n^{-0.5}(x)$ which are preferable in the context of approximating $\zeta(s)$ with Borwein Algorithm. Practically, these polynomial are better than all kinds of Chebyshev polynomials.

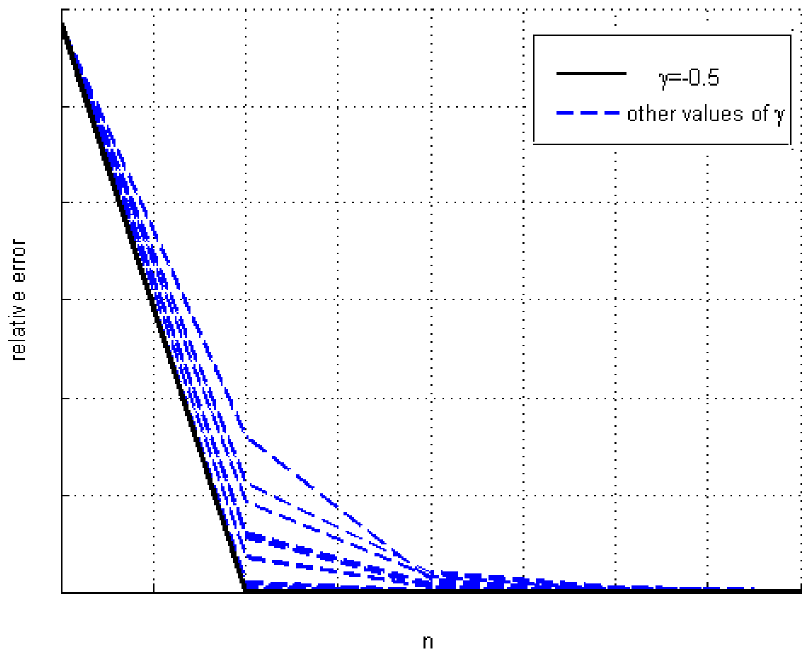


Figure 4.8: The bound value of γ values in approximating $\zeta(s)$ by $C_n^{\gamma}(x)$.

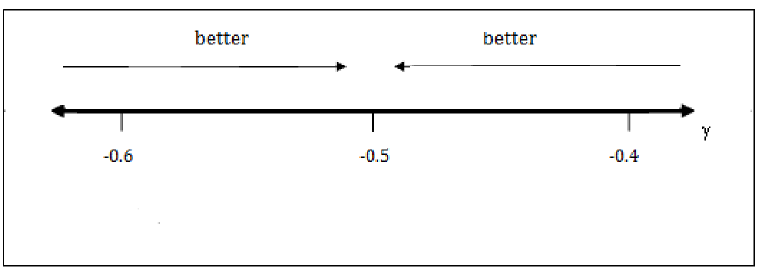
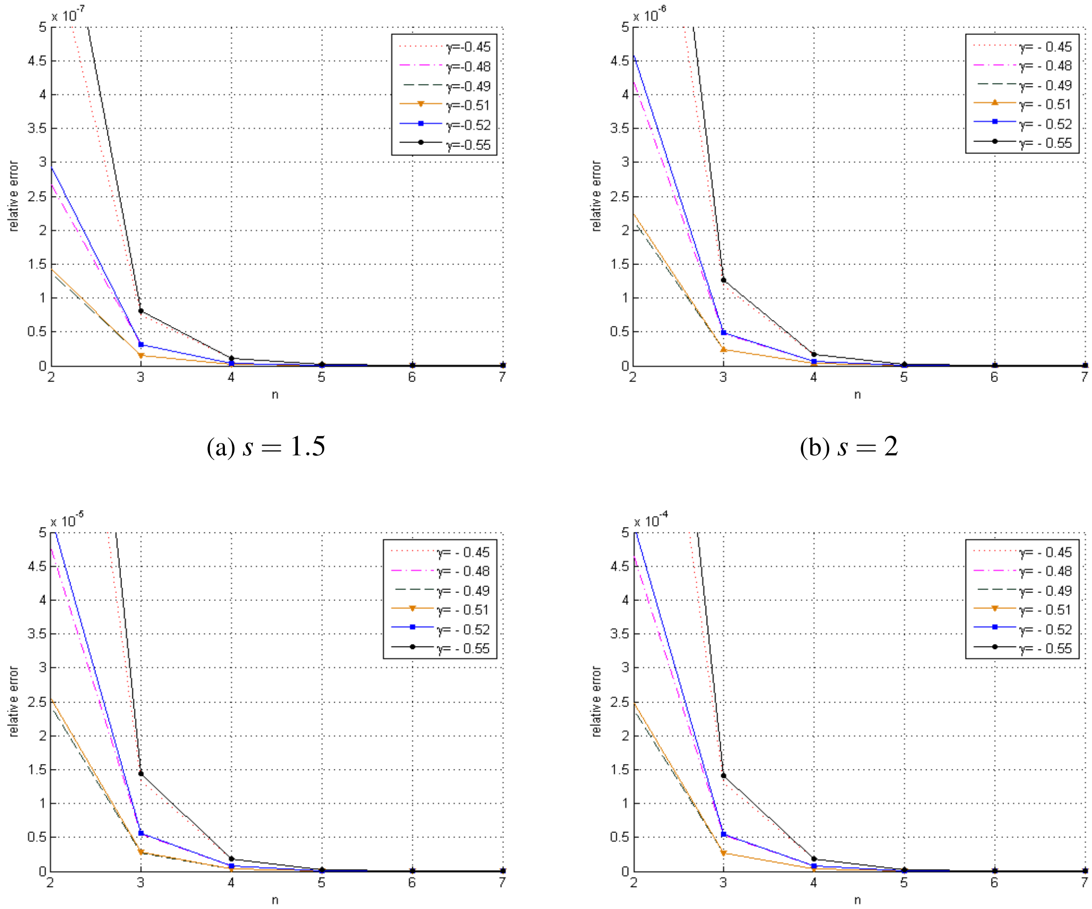


Figure 4.9: Accuracy behavior of $\zeta(s)$ approximations using $C_n^{\gamma}(x)$.

Now, to see how this bound value ($\gamma = -0.5$) leads to good approximations of $\zeta(s)$, consider Fig. 4.12, Fig. 4.13, Fig. 4.14, and Fig. 4.15. From the figures, one can conclude that there are infinite numbers of Gegenbauer polynomials that are better than Chebyshev polynomials in approximating $\zeta(s)$ with Borwein algorithm.



(a) $s = 1.5$

(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.10: The behavior of the relative error associated with $\zeta(s)$ approximations obtained by $C_n^{\lambda}(x)$ with $\gamma \in (-0.6, -0.4)$.

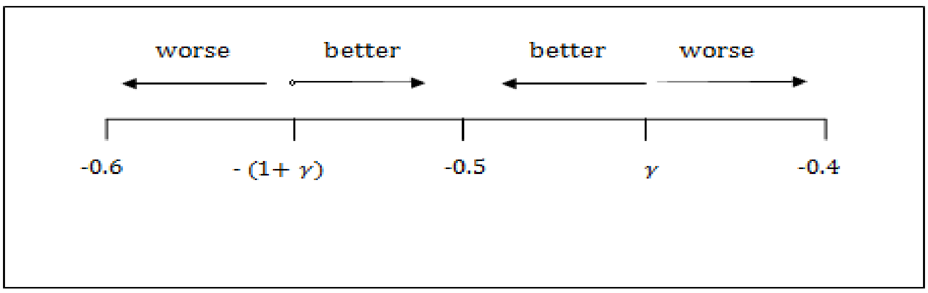


Figure 4.11: The behavior of $\zeta(s)$ approximations obtained by $C_n^{\lambda}(x)$ with $\gamma \in (-0.6, -0.4)$.

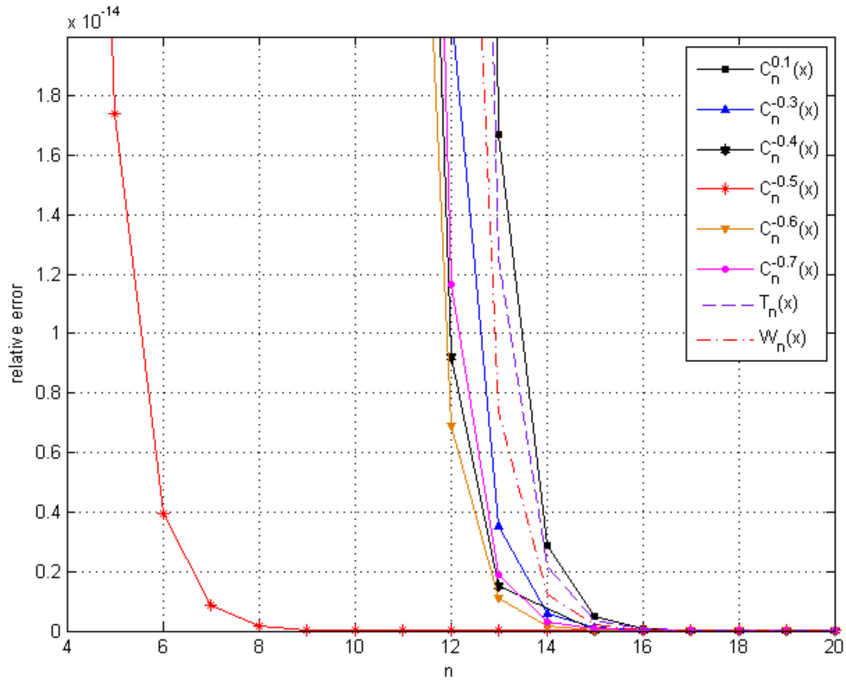


Figure 4.12: Comparison between different kinds of Gegenbauer and Chebyshev polynomials in approximating $\zeta(1.5)$.

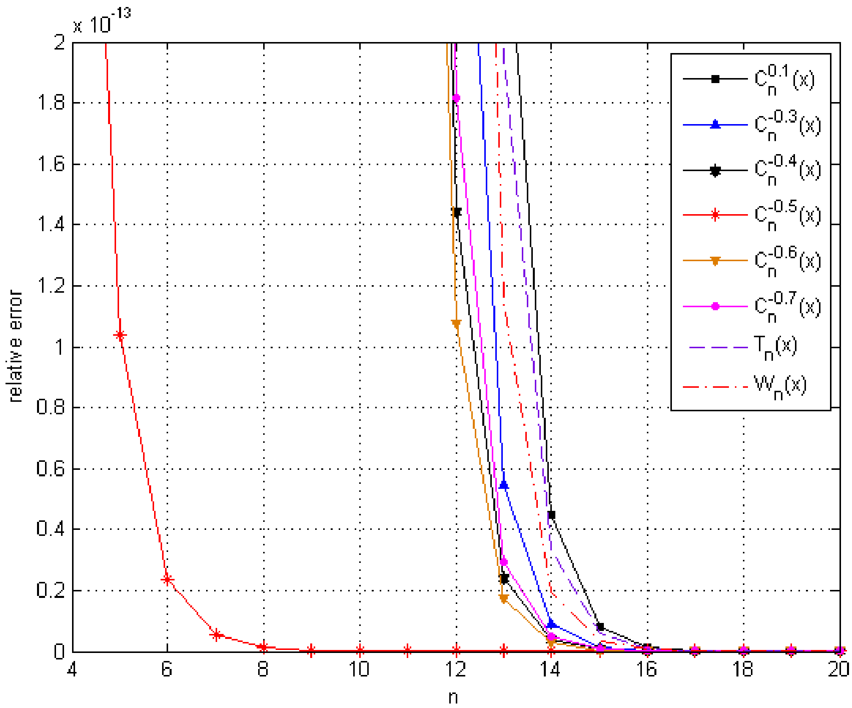


Figure 4.13: Comparison between different kinds of Gegenbauer and Chebyshev polynomials in approximating $\zeta(2)$.

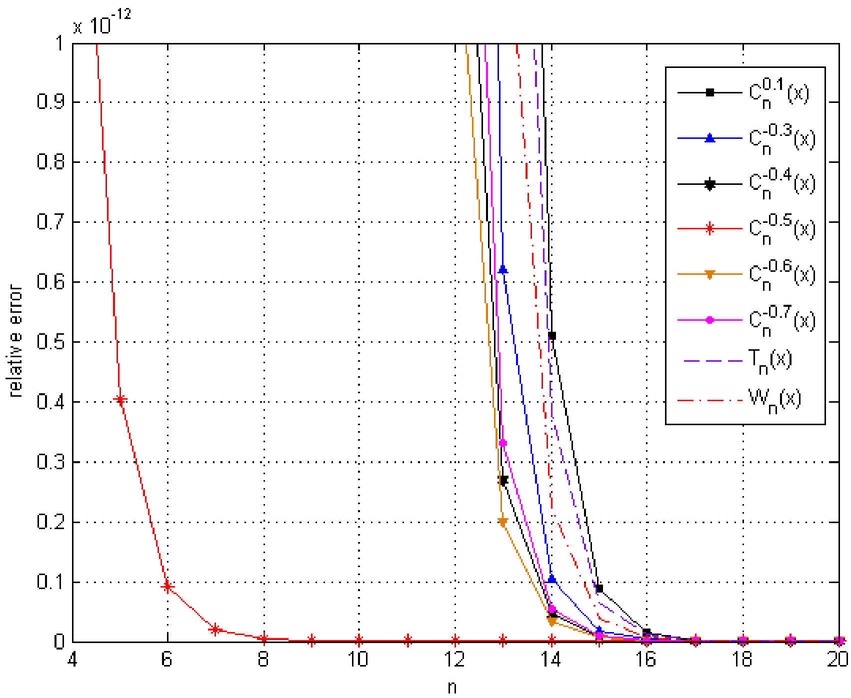


Figure 4.14: Comparison between different kinds of Gegenbauer and Chebyshev polynomials in approximating $\zeta(2.5)$.

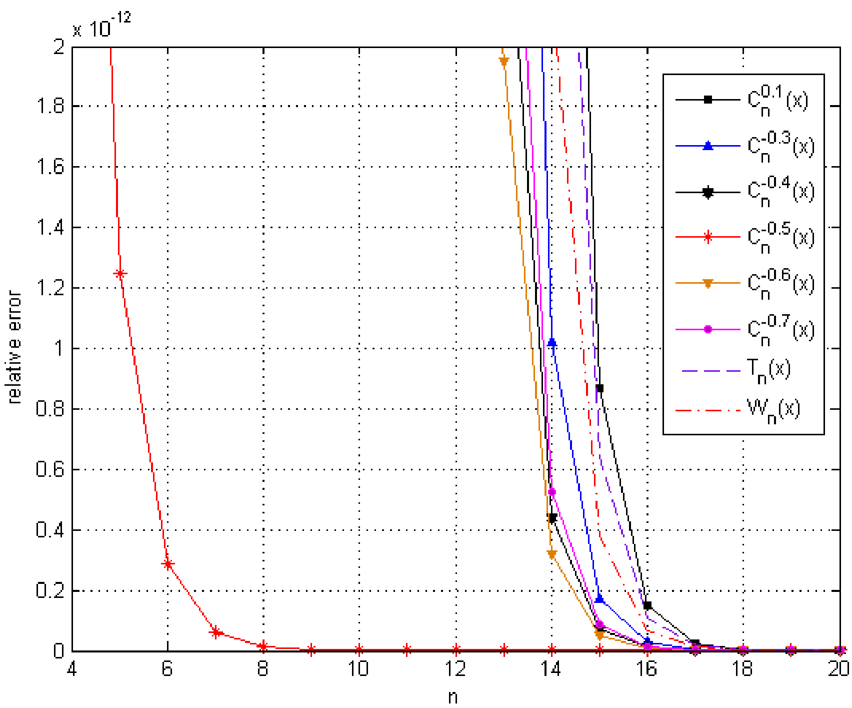


Figure 4.15: Comparison between different kinds of Gegenbauer and Chebyshev polynomials in approximating $\zeta(3)$.

As a family of polynomials with two different free parameters n and γ, one may need to determine the best couple to use in order to obtain a specific accuracy. To do that, we offer the surface of relative error resulted from changing n and γ which is exhibited in Fig. 4.16.

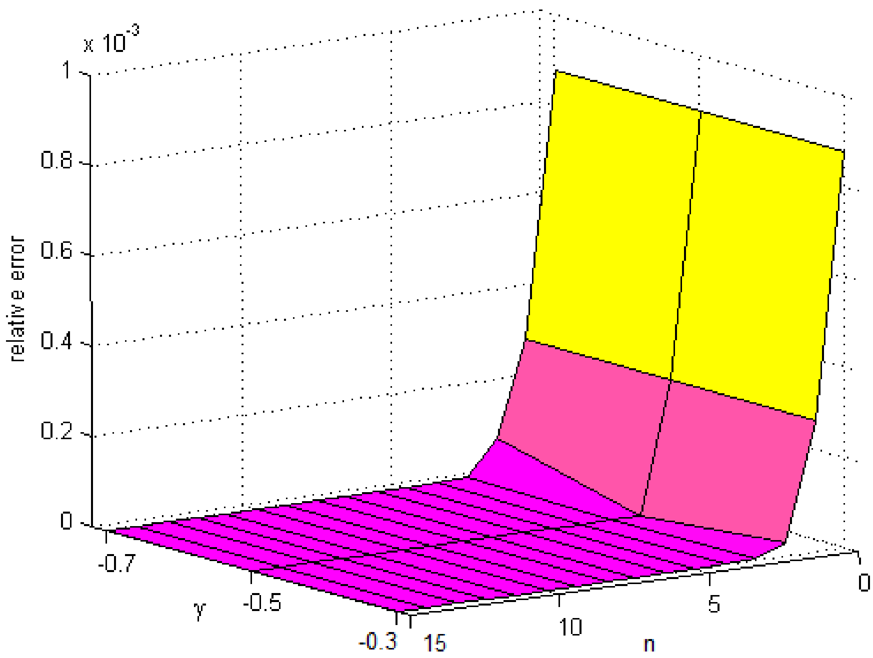


Figure 4.16: Relative error versus the values of (n,γ) in approximating $\zeta(2)$ with $C_n^{\gamma}(x)$.

4.1.4 Approximation by Associated Laguerre Polynomials

Associated or Generalized Laguerre polynomials $L_n^{\alpha}(x)$ are presented by equation (2.27) in Chapter 2. Implementing Borwein algorithm with $L_n^{\alpha}(x)$ polynomials gives different approximations of $\zeta(s)$ with different accuracy depending on the value of α. In the case of integer parameter α, $\zeta(s)$ approximations get better as the value of α decreases until some bound value of α, after which the relative error becomes constant. This is presented in Fig. 4.17 which describes the relative error associated with $L_n^{\alpha}(x)$ in approximating $\zeta(2)$ at $\alpha = 1, 0, -1, -2, -7$.

In fact, $\alpha = -7$ is the bound of α values in improving the accuracy of $\zeta(s)$ approximations, Fig. 4.18, Fig. 4.19, Fig. 4.20, and Fig. 4.21 show this boundedness.

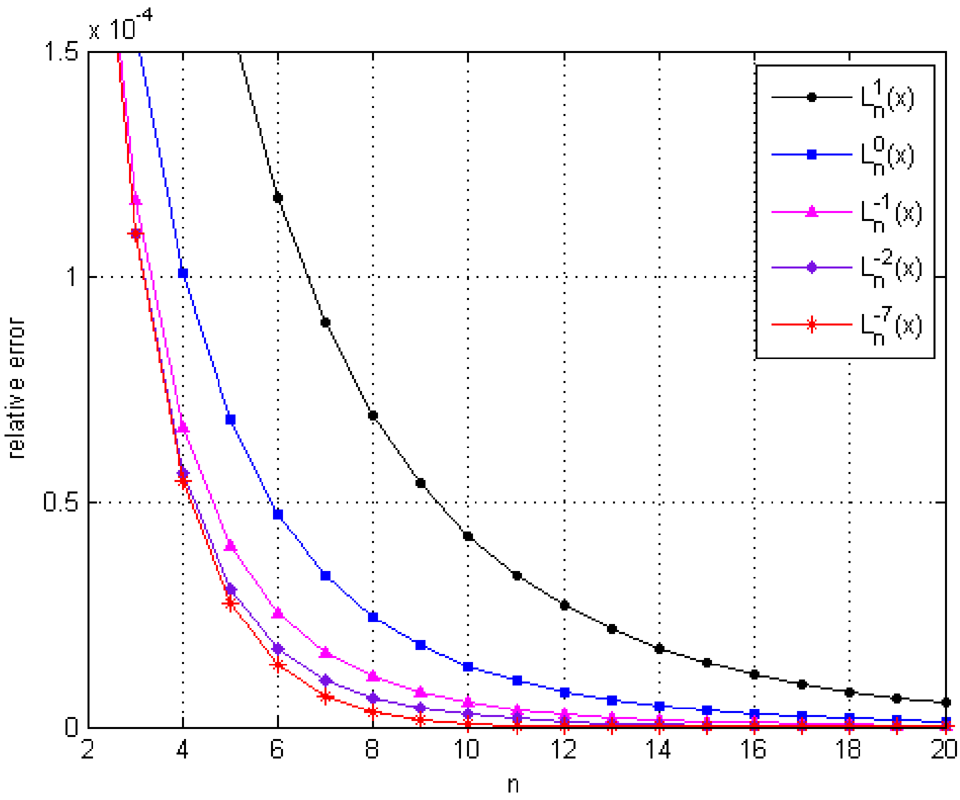


Figure 4.17: Approximating $\zeta(2)$ by $L_n^{\alpha}(x)$ polynomials.

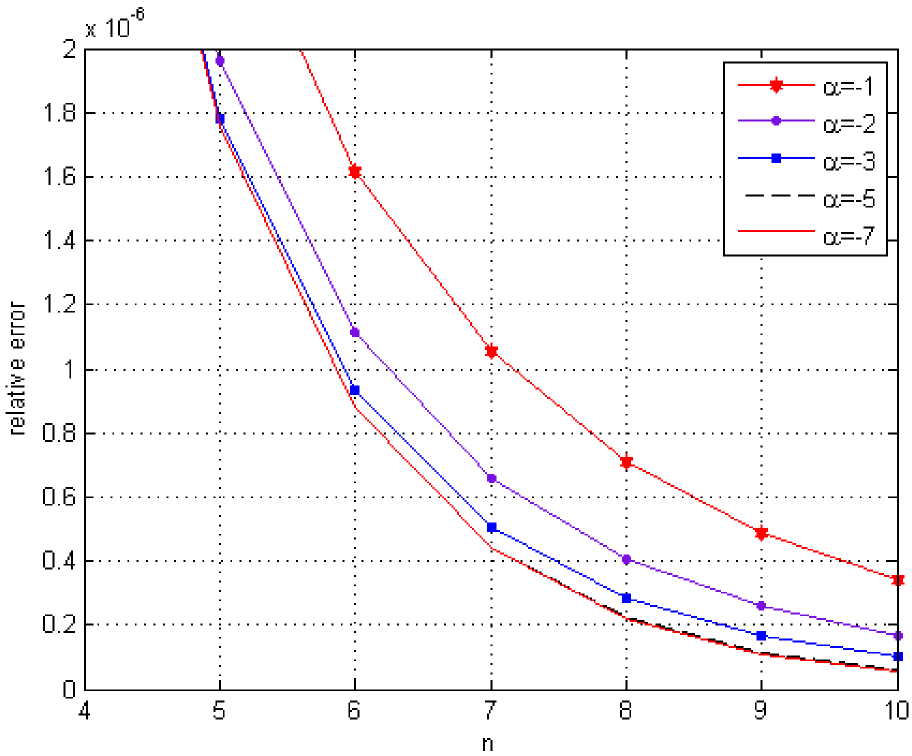


Figure 4.18: The bound of α values in approximating $\zeta(1.5)$.

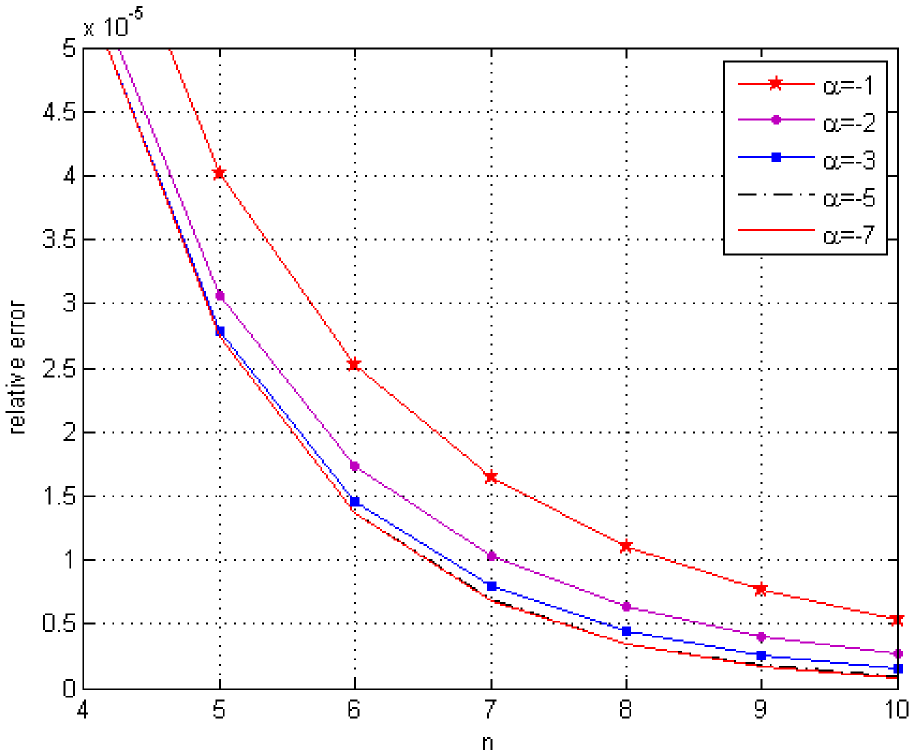


Figure 4.19: The bound of α values in approximating $\zeta(2)$.

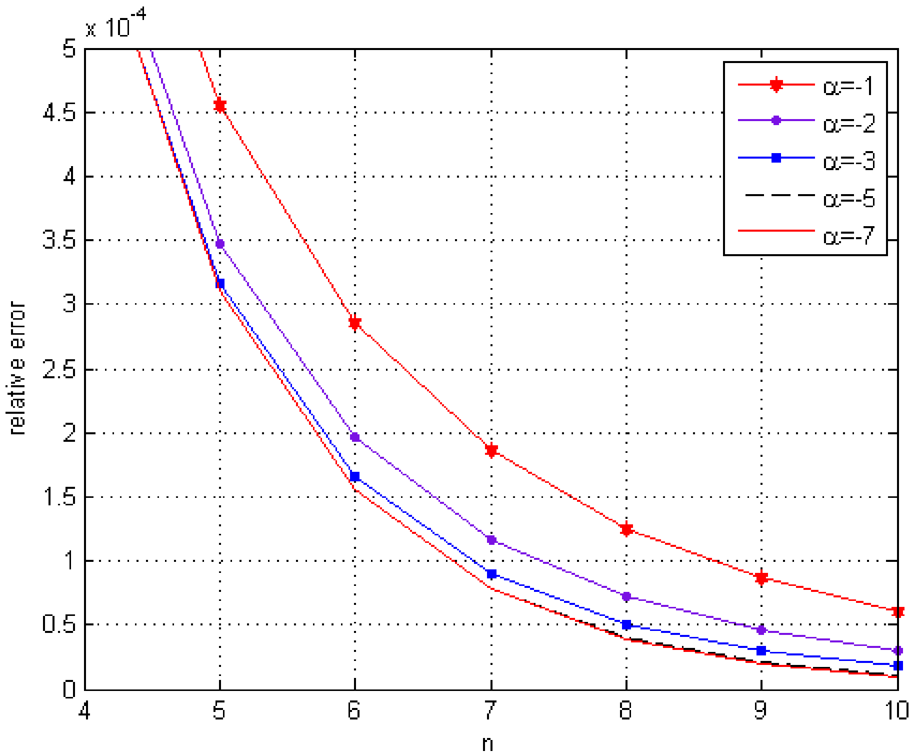


Figure 4.20: The bound of α values in approximating $\zeta(2.5)$.

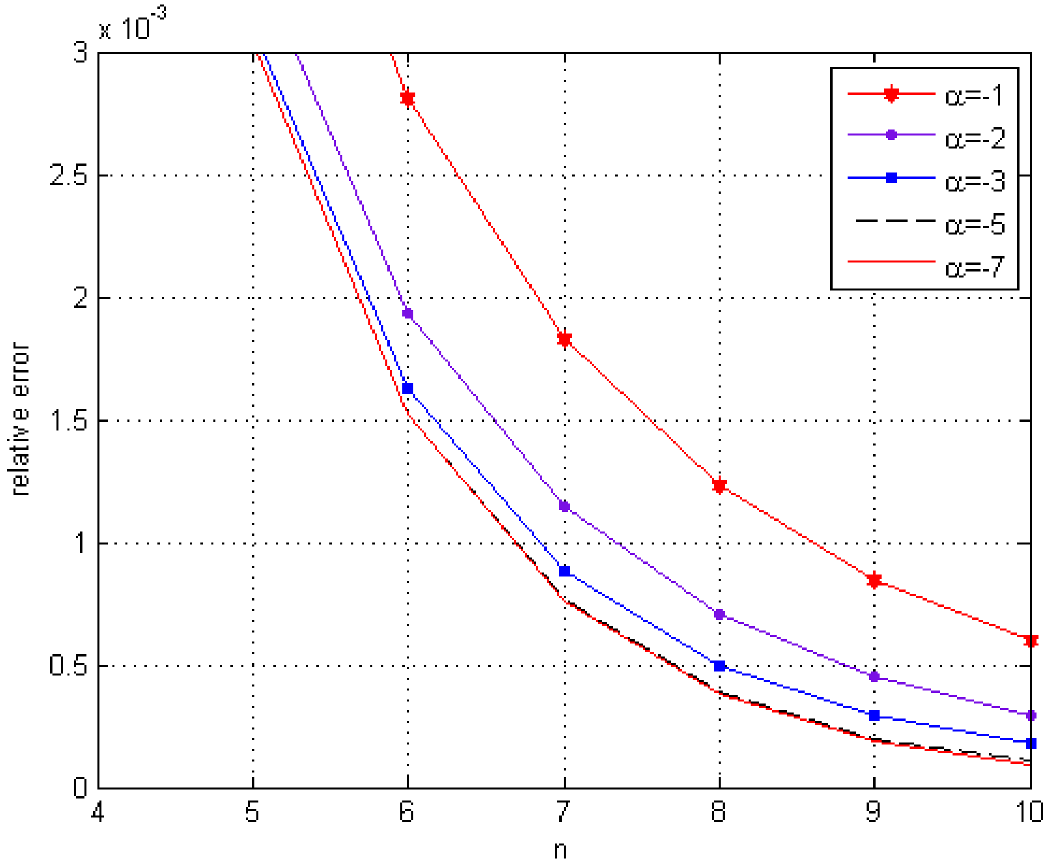
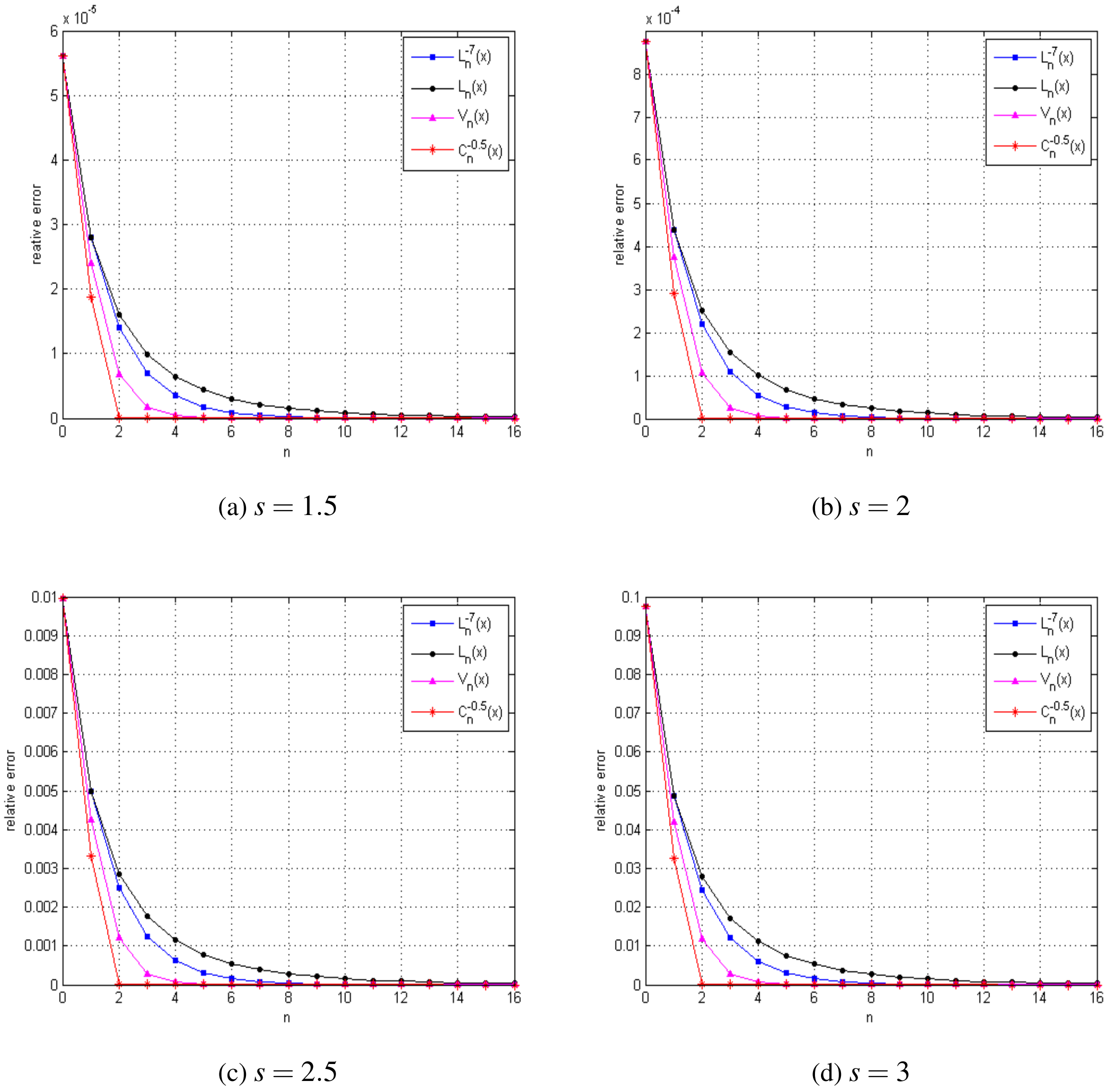


Figure 4.21: The bound of α values in approximating $\zeta(3)$.

Although $L_n^{-7}(x)$ polynomials are preferable in the family of Associated Laguerre polynomials, but on the other hand they are worse than many Jacobi polynomials. In particular, all the polynomials we have dealt with in this chapter are better than $L_n^{\alpha}(x)$. Fig. 4.22 shows comparisons between $L_n^{-7}(x)$, $T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$ in approximating $\zeta(s)$.

As it is illustrated in the figures, $L_n^{-7}(x)$ are worse than $V_n(x)$ at any n which are the third type of Chebyshev polynomials whose approximations were investigated to be the worst within Chebyshev polynomials due to their low accuracy in approximating $\zeta(s)$. At the same time, they are also worse than Legendre polynomials. So, all Associated Laguerre polynomials with different integer α are worse than all Chebyshev and so Gegenbauer polynomials.



(a) $s = 1.5$ (b) $s = 2$

(c) $s = 2.5$ (d) $s = 3$

Figure 4.22: Comparison between $L_n^{-7}(x)$, $L_n(x), V_n(x)$, and $C_n^{-0.5}(x)$ in approximating $\zeta(s)$.

If the free parameter α is not integer, then the corresponding polynomials will behave in the same way as in the integer case (Fig. 4.23, Fig. 4.24, Fig. 4.25, and Fig. 4.26). In fact, the relative error associated with $\zeta(s)$ approximations will decrease by decreasing α. This is the case until $\alpha = -1.9$, afterwards the relative error will increase.

Now, the ordinary Laguerre polynomials which attained when $\alpha = 0$, behave as the other polynomials in the class. Fig. 4.27 shows comparison between $\zeta(s)$ approximations obtained by $L_n(x)$ for different values of s, where Fig. 4.28 offers comparisons between Laguerre, T-Chebyshev, W-Chebychev, and Gegenbauer polynomials.

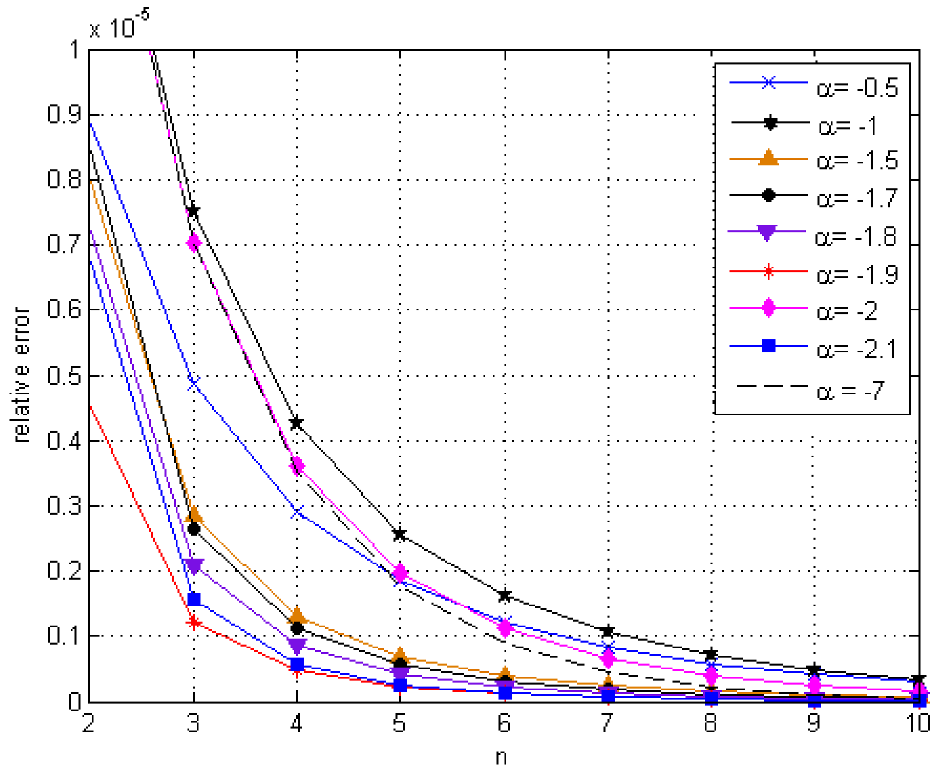


Figure 4.23: Comparison between $L_n^{\alpha}(x)$ polynomials with different non-integer values of α in approximating $\zeta(1.5)$.

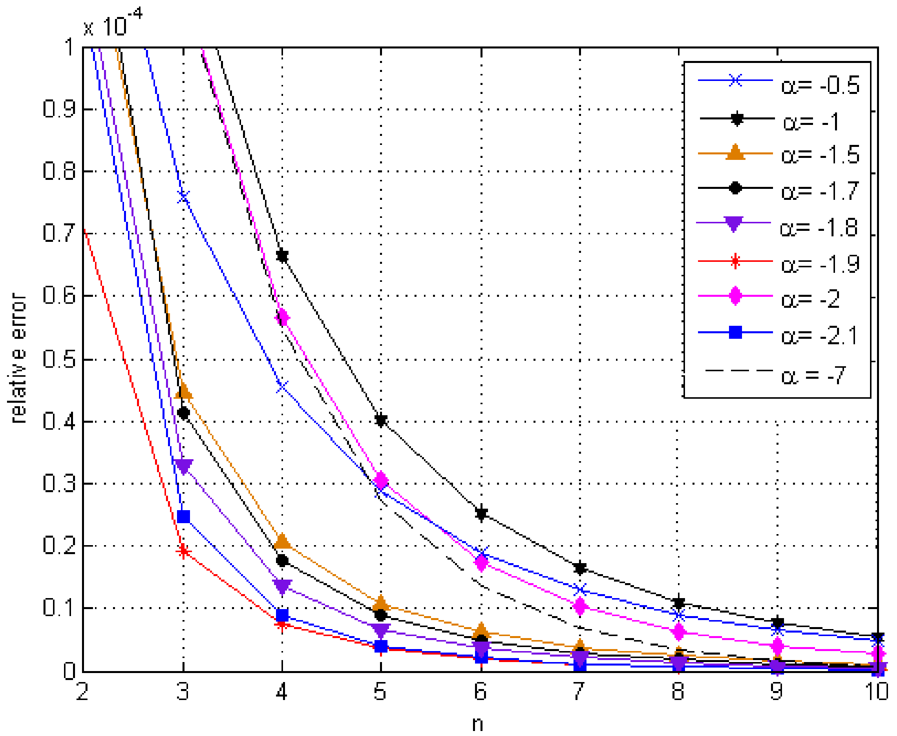


Figure 4.24: Comparison between $L_n^{\alpha}(x)$ polynomials with different non-integer values of α in approximating $\zeta(2)$.

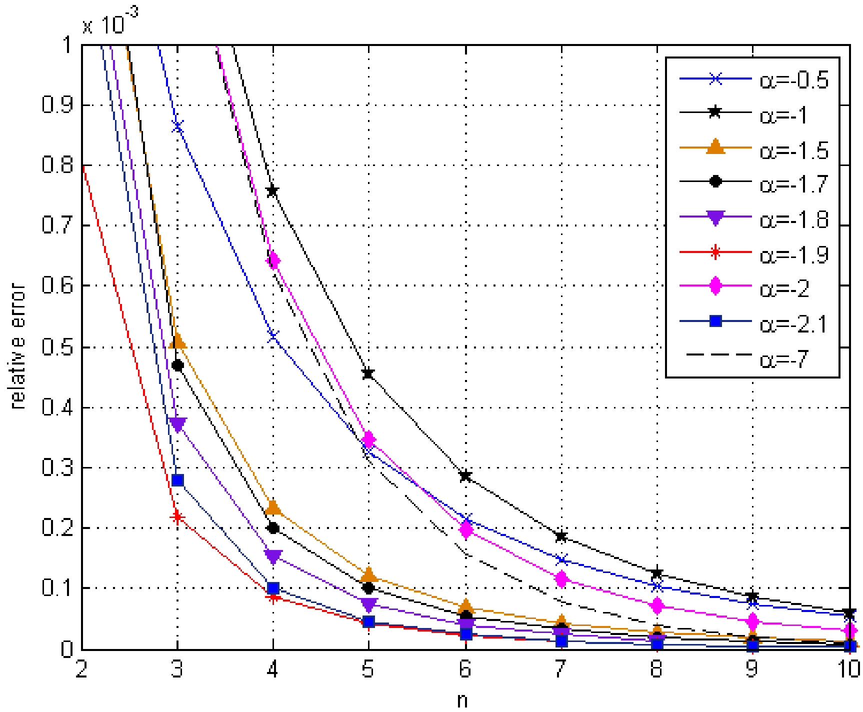


Figure 4.25: Comparison between $L_n^{\alpha}(x)$ polynomials with different non-integer values of α in approximating $\zeta(2.5)$.

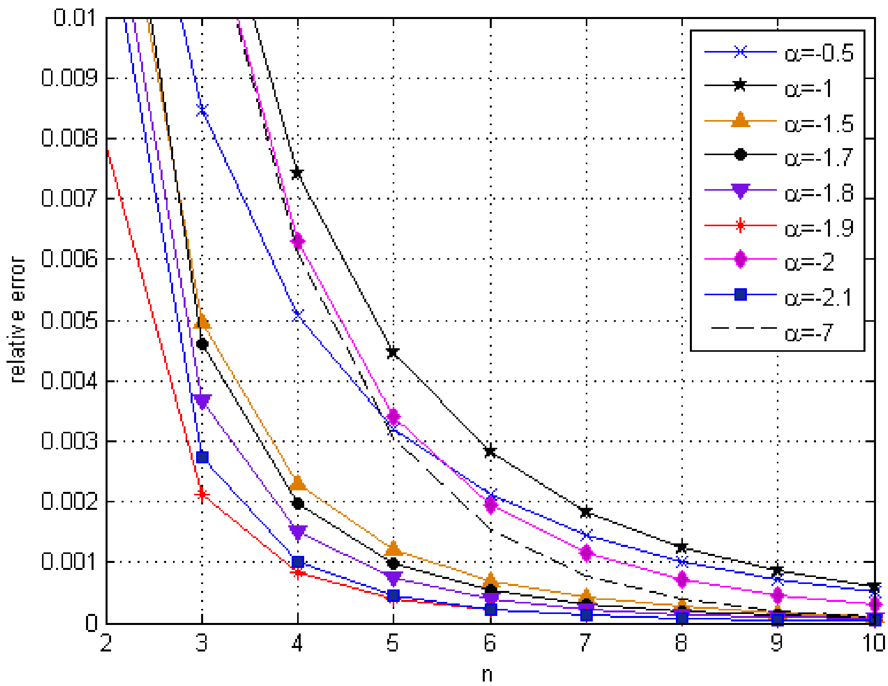


Figure 4.26: Comparison between $L_n^{\alpha}(x)$ polynomials with different non-integer values of α in approximating $\zeta(3)$.

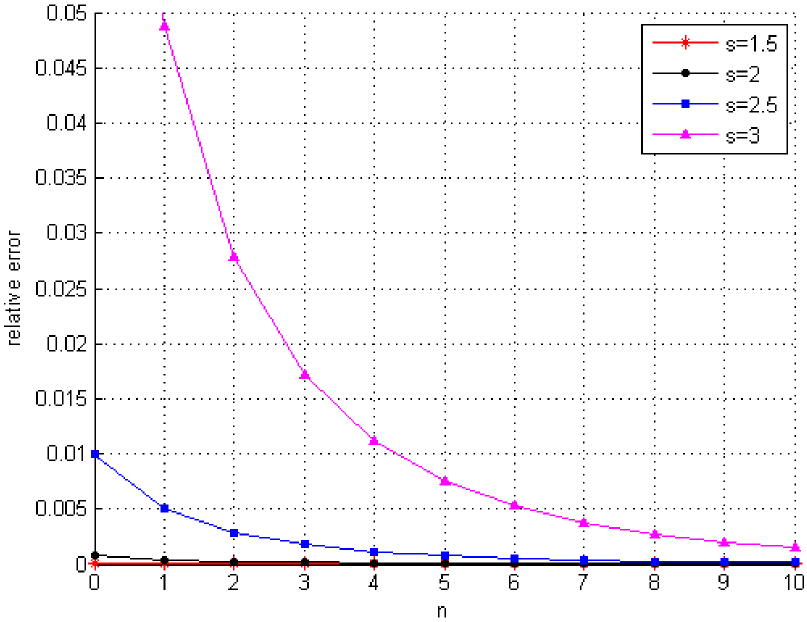
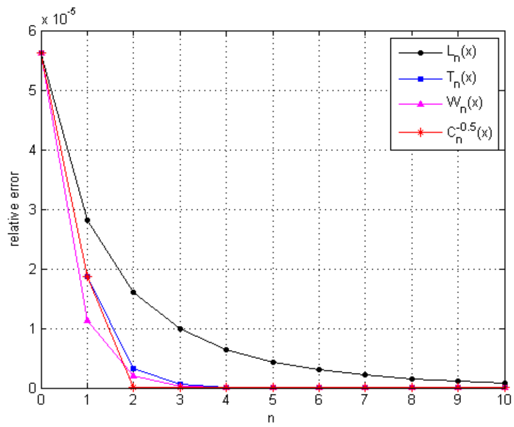
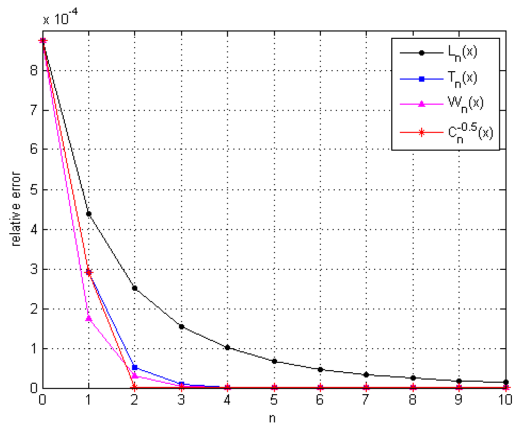


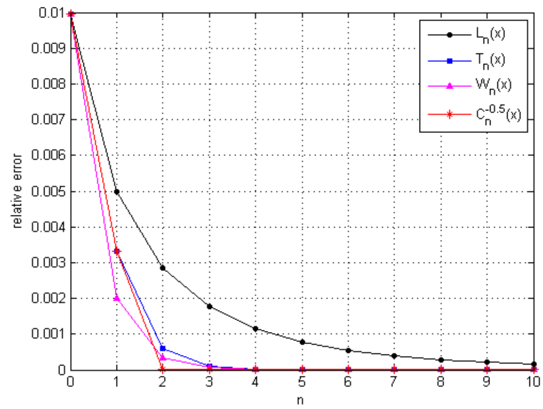
Figure 4.27: Comparison between $\zeta(s)$ approximations obtained by $L_n(x)$ for different values of s.



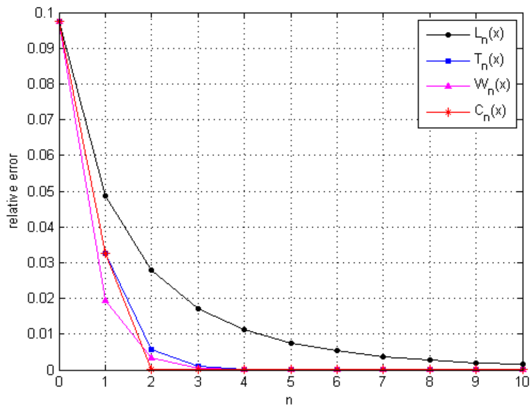
(a) $s = 1.5$



(b) $s = 2$



(c) $s = 2.5$



(d) $s = 3$

Figure 4.28: Comparison between $L_n(x), T_n(x), W_n(x)$,and $C_n^{-0.5}(x)$ polynomials in approximating $\zeta(s)$.

We can present the relative error coming from using $L_n^\alpha(x)$ polynomials in approximating $\zeta(s)$ by three dimensional graph by showing the values of the relative error when using certain values of n and α. Fig. 4.29 manifests the relative error associated with approximating $\zeta(2)$ with such couple (n,α) where α takes the values $1,-3,-7$ and n reaches 15.

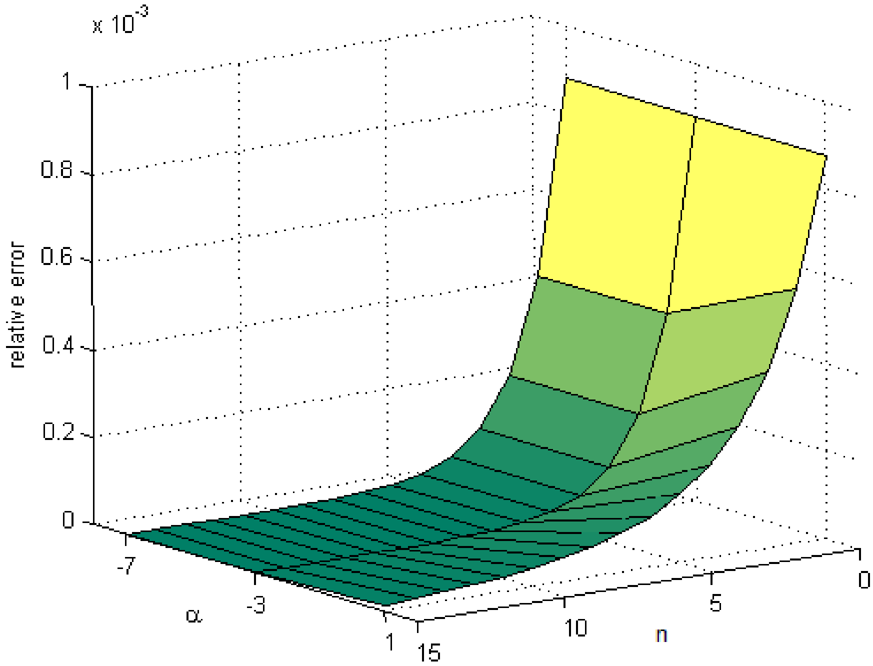


Figure 4.29: Relative error versus (n,α) values in approximating $\zeta(2)$ by $L_n^\alpha(x)$.

For $n=10$, Table 4.3 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using different kinds of Associated Laguerre polynomials.

	$s=1.5$	$s=2$	$s=2.5$	$s=3$
$L_{10}^{-1}(x)$	3.43×10^{-7}	5.36×10^{-6}	6.09×10^{-5}	5.98×10^{-4}
$L_{10}^{-3}(x)$	1.02×10^{-7}	2.61×10^{-6}	1.80×10^{-5}	1.77×10^{-9}
$L_{10}^{-5}(x)$	6.13×10^{-8}	9.57×10^{-7}	1.09×10^{-5}	1.07×10^{-4}
$L_{10}^{-6}(x)$	5.87×10^{-8}	8.83×10^{-7}	9.98×10^{-6}	9.97×10^{-5}
$L_{10}^{-7}(x)$	5.50×10^{-8}	8.60×10^{-7}	9.77×10^{-6}	9.59×10^{-5}
$L_{10}^{-8}(x)$	5.50×10^{-8}	8.60×10^{-7}	9.77×10^{-6}	9.59×10^{-5}

Table 4.3: The values of relative error in approximating $\zeta(s)$ using $L_{10}^\alpha(x)$.

4.1.5 Approximation by Bessel Polynomials

The fourth type of the classical orthogonal polynomials, namely; Bessel polynomials $\beta_n(x)$ are presented in (2.1.1) by equation (2.31) in explicit way. Bessel polynomials give approximations of $\zeta(s)$ with different accuracy when they are implicated in Borwein algorithm depending on the argument s. As the case of other classes of polynomials, the approximations obtained by $\beta_n(x)$ polynomials are better when s is small (Fig. 4.30).

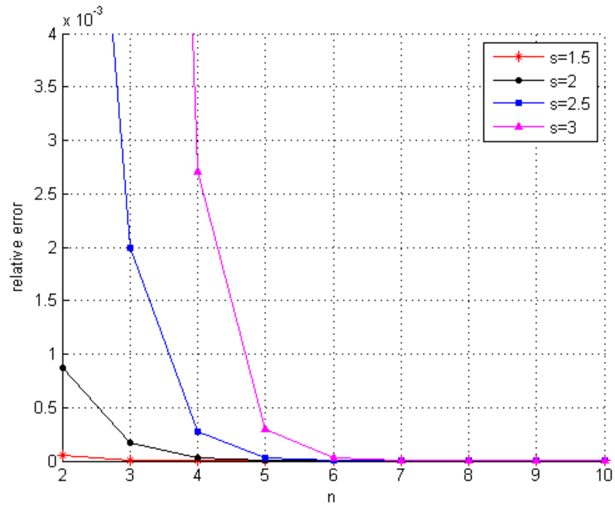
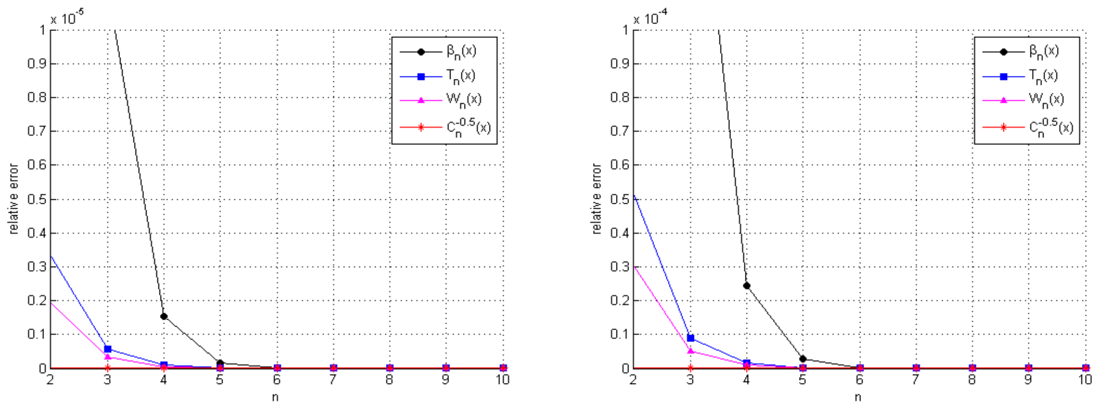


Figure 4.30: Comparison between $\zeta(s)$ approximations obtained by $\beta_n(x)$ for different values of s.

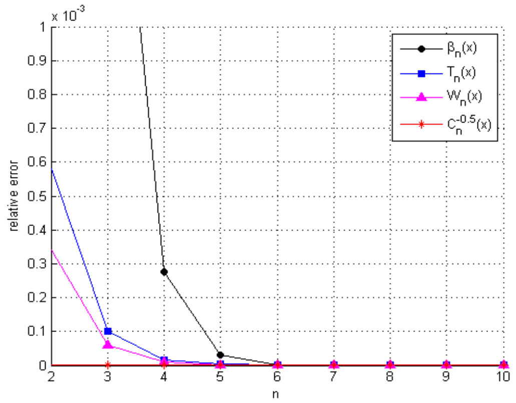
$\beta_n(x)$ polynomials are still worse than $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ as presented in Fig. 4.31 and Fig. 4.32.



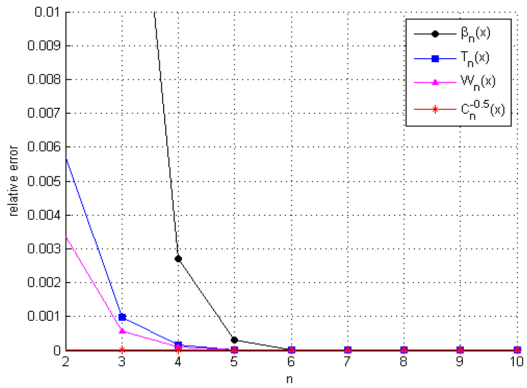
(a) $s = 1.5$

(b) $s = 2$

Figure 4.31: Comparisons between $\zeta(s)$ approximations obtained by $\beta_n(x), T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$ for $s = 1.5, 2$.



(a) $s = 2.5$



(b) $s = 3$

Figure 4.32: Comparisons between $\zeta(s)$ approximations obtained by $\beta_n(x), T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$ for $s = 2.5, 3$.

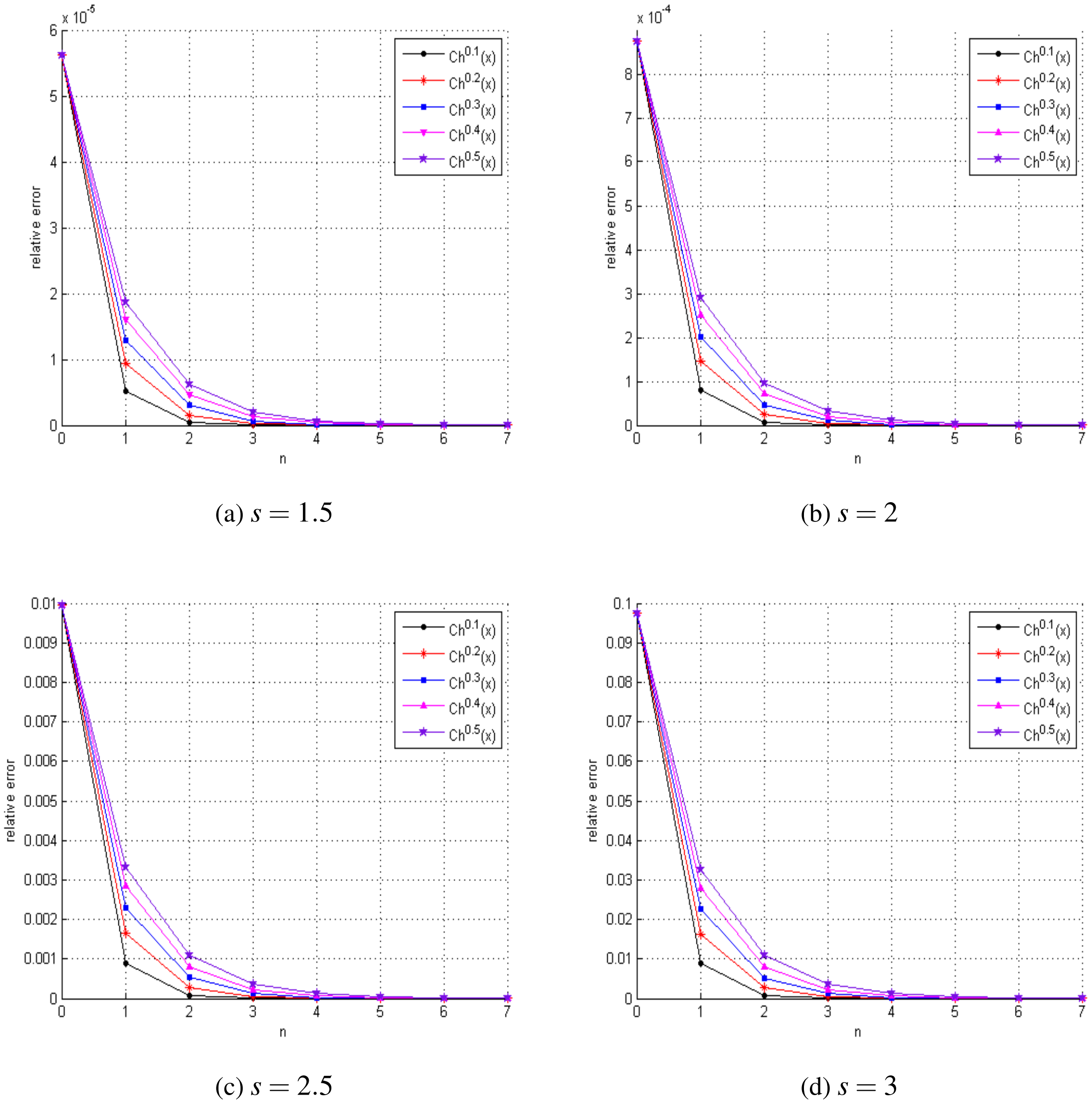
For $n = 10$, Table 4.4 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using Bessel polynomials.

s	1.5	2	2.5	3
ε	1.40×10^{-12}	3.73×10^{-12}	4.24×10^{-11}	4.16×10^{-10}

Table 4.4: The values of relative error in approximating $\zeta(s)$ using $\beta_n(x)$ with $n = 10$.

4.1.6 Approximation by Charlier Polynomials

An obvious example of the discrete orthogonal polynomials is Charlier polynomials $Ch_n^{\alpha}(x)$, which are obtained explicitly by equation (2.43). The results obtained by using these polynomials with Borwein algorithm are described by plotting the relative error corresponding to Charlier polynomials' degree at fixed values of α and s. Smaller values of α for $Ch_n^{\alpha}(x)$ polynomials are preferable for the approximation process with sufficient small values of n. Fig. 4.33 offers comparisons between the approximations obtained by using $Ch_n^{\alpha}(x)$ with different values of α.

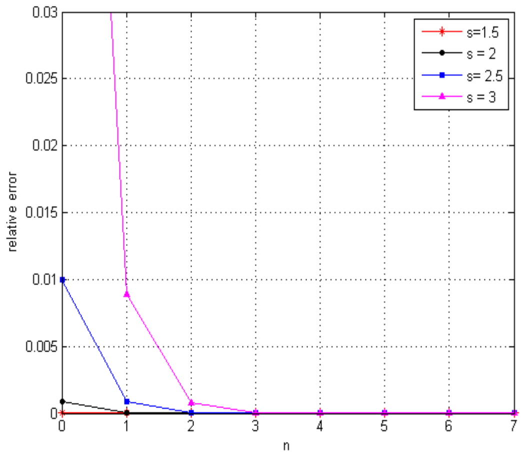


(a) $s = 1.5$ (b) $s = 2$

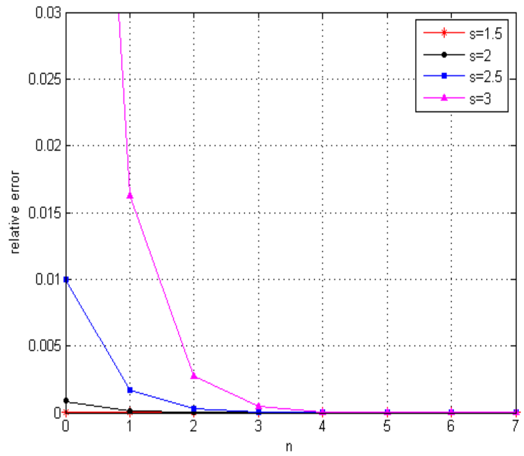
(c) $s = 2.5$ (d) $s = 3$

Figure 4.33: Comparisons between $\zeta(s)$ approximations obtained by Ch_n^α at different values of α.

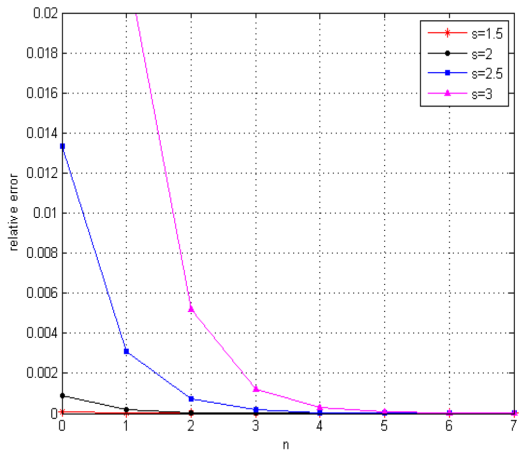
Small values of s yield small values of the relative error at fixed values of n and α. Fig. 4.34 gives comparison between the relative error obtained for different values of s at the values $\alpha = 0.1, 0.2, 0.3, 0.4, 0.5$.



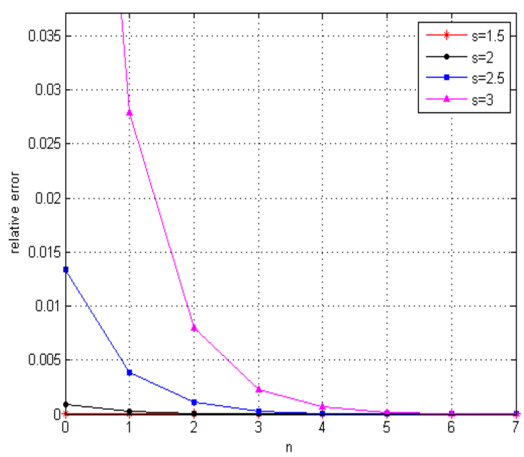
(a) $\alpha = 0.1$



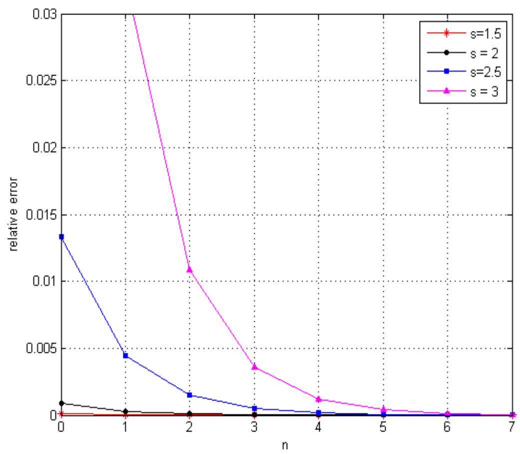
(b) $\alpha = 0.2$



(c) $\alpha = 0.3$



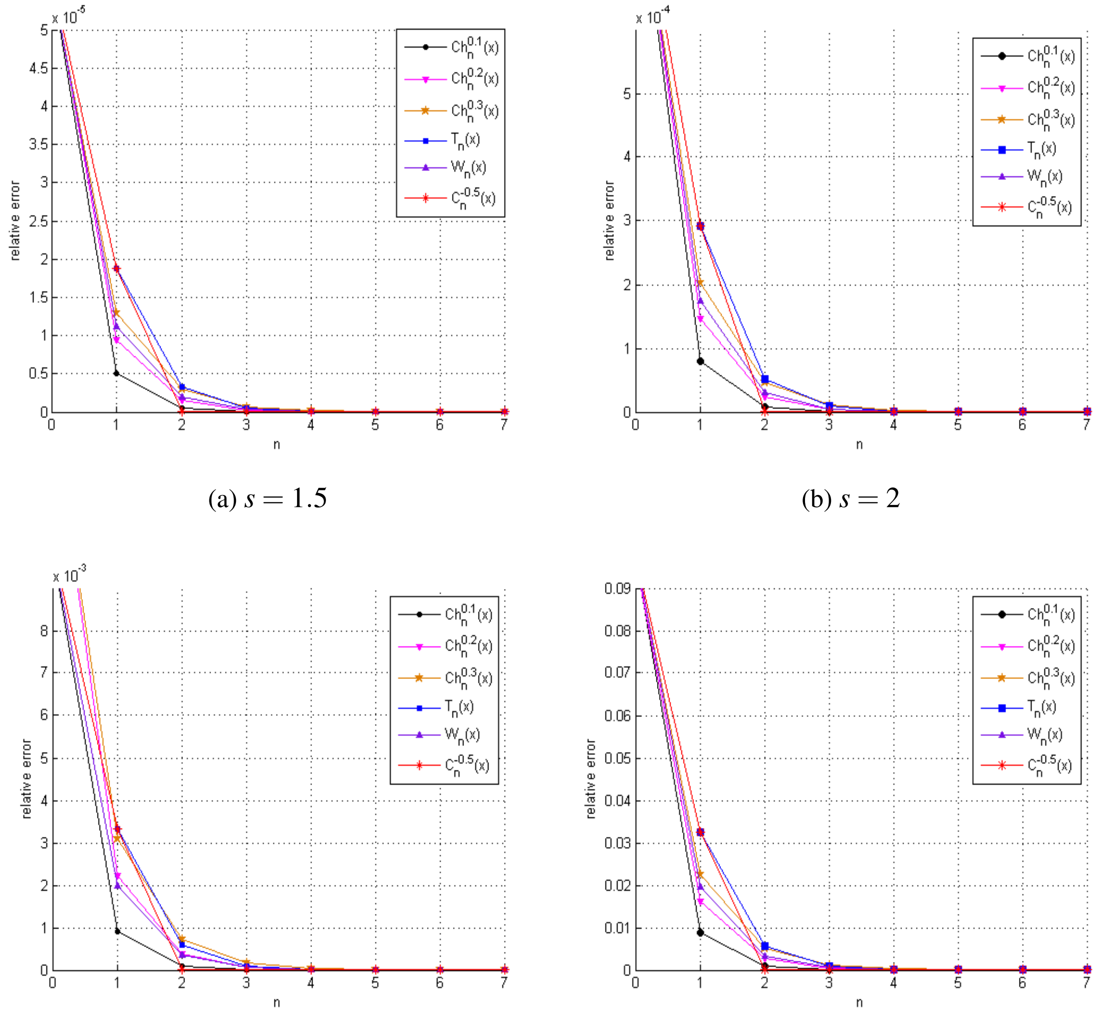
(d) $\alpha = 0.4$



(e) $\alpha = 0.5$

Figure 4.34: Comparisons between $\zeta(s)$ approximations obtained by $Ch_n^{\alpha}(x)$ at different values of α.

In fact, Charlier polynomials compete with the standard Chebychev polynomials in approximating the Riemann Zeta function by Borwein algorithm. For example, $Ch_n^{0.3}(x)$ are better than $T_n(x)$ and $C_n^{-0.5}(x)$ only for $n = 1, 2$. On the other hand they are worse than $W_n(x)$ for any degree n. But, $Ch_n^{0.2}(x)$ polynomials are better than all Chebychev polynomials for $n \leq 30$ and worse than $C_n^{-0.5}(x)$ for $n \geq 3$. Similarly, $Ch_n^{0.1}(x)$ polynomials are better than Chebychev polynomials until $n = 23$. Fig. 4.35 exhibits the behavior of relative error associated with using different Charlier polynomials, $T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$ polynomials.



(a) $s = 1.5$

(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.35: Comparisons between $T_n(x), W_n(x)$, $C_n^{-0.5}(x)$, and $Ch_n^{\alpha}(x)$, where $\alpha = 0.1, 0.2, 0.3$ in approximating $\zeta(s)$.

For $n = 7$, Table 4.35 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using different kinds of Charlier polynomials.

	$s=1.5$	$s=2$	$s=2.5$	$s=3$
$Ch_7^{0.1}(x)$	4.68×10^{-12}	4.50×10^{-11}	5.11×10^{-10}	5.01×10^{-9}
$Ch_7^{0.2}(x)$	3.11×10^{-10}	3.13×10^{-9}	3.55×10^{-8}	3.50×10^{-7}
$Ch_7^{0.3}(x)$	1.98×10^{-9}	3.05×10^{-8}	3.47×10^{-7}	3.40×10^{-6}
$Ch_7^{0.4}(x)$	8.73×10^{-9}	1.36×10^{-7}	1.55×10^{-6}	1.52×10^{-5}
$Ch_7^{0.5}(x)$	2.57×10^{-8}	4.01×10^{-7}	4.55×10^{-6}	4.47×10^{-5}

Table 4.5: The values of relative error in approximating $\zeta(s)$ using Charlier polynomials.

4.1.7 Approximation by Humbert Polynomials

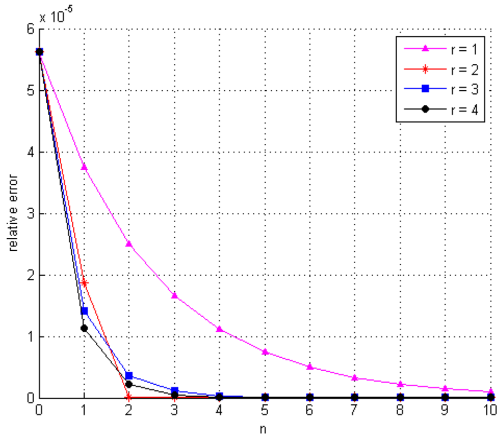
As manifested in (2.2.1), the set of Humbert polynomials $h_n^{r,\gamma}(x)$ forms an extension of Gegenbauer polynomials $C_n^{\gamma}(x)$

$$h_n^{2,\gamma}(x) = C_n^{\gamma}(x) \tag{4.1}$$

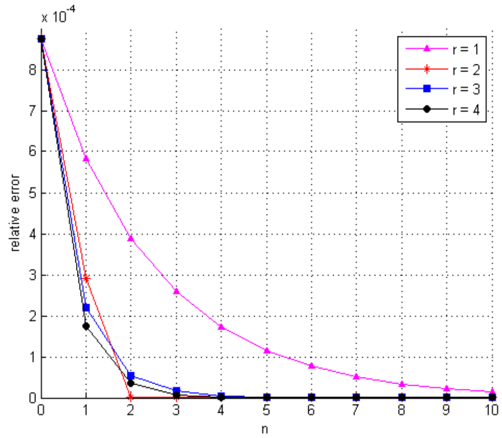
Since the particular case of theses polynomials (Gegenbauer polynomials) are investigated numerically to be great tools for evaluating $\zeta(s)$ by Borwein algorithm, then it makes sense to include Humbert polynomials in the algorithm to examine their efficiency and to find out the best choice of (r,γ) values in order to get better approximations for $\zeta(s)$. It was shown in the discussion held in (4.1.3) that $\zeta(s)$ approximations obtained by Gegenbauer polynomials $C_n^{\gamma}(x)$ take their optimal values when $\gamma=-0.5$. Thus, this value is the perfect choice of γ when $r=2$ for the Humbert polynomial.

After implicating Humbert polynomials in Borwein algorithm, the results show that by increasing r, the associated relative error decreases at fixed values of n and γ. Amazingly, $r=2$ is the bound of r values at which the relative error reaches its minimum for a reasonable polynomial degree n. Fig. 4.36, Fig. 4.37, Fig. 4.38, Fig. 4.39, and Fig. 4.40 describe this by showing comparisons between $\zeta(s)$ approximations obtained by using the different values $r=1,2,3,4$ for $h_n^{r,\gamma}(x)$ at; $\gamma=-0.5,-0.1,0.1$, and 0.5 respectively. This is investigated for different values of s.

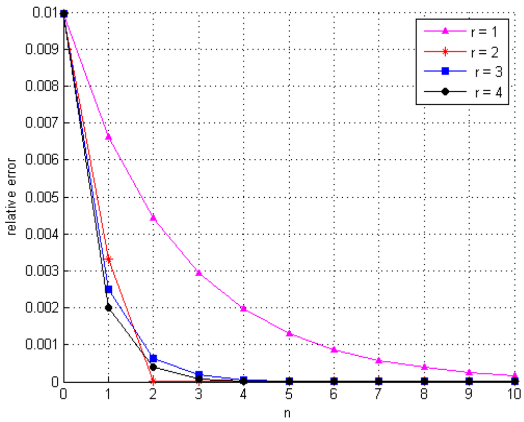
The best choice for calculating Riemann Zeta function by Borwein algorithm with Humbert polynomials is $h_n^{2,-0.5}(x)$ polynomials, which are indeed our celebrated polynomials $C_n^{-0.5}(x)$ presented in (4.1.3).



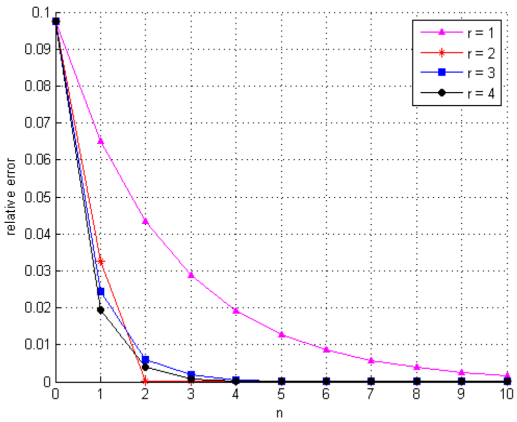
(a) $s = 1.5$



(b) $s = 2$

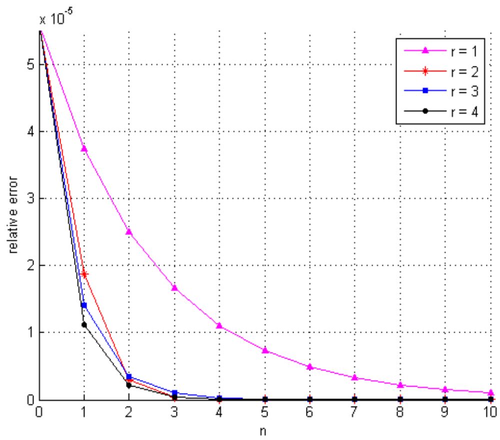


(c) $s = 2.5$

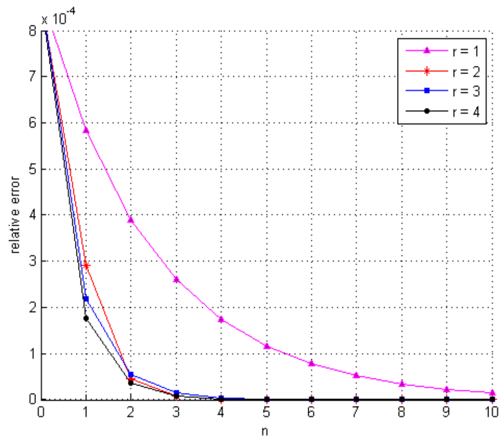


(d) $s = 3$

Figure 4.36: Comparisons between $\zeta(s)$ approximations obtained by $h_n^{r,-0.5}(x)$ at $r = 1,2,3,4$.

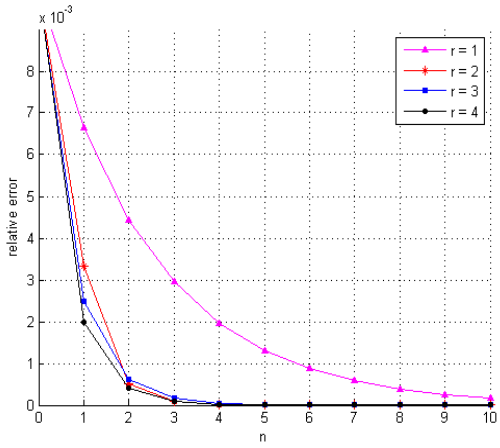


(a) $s = 1.5$

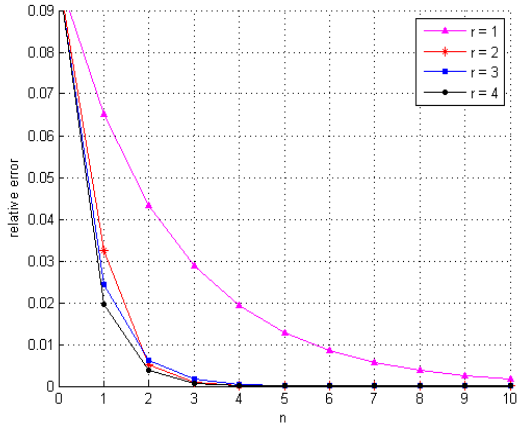


(b) $s = 2$

Figure 4.37: Comparisons between $\zeta(s)$ approximations obtained by $h_n^{r,-0.1}(x)$ at $r = 1,2,3,4$ for $s = 1.5, 2$.

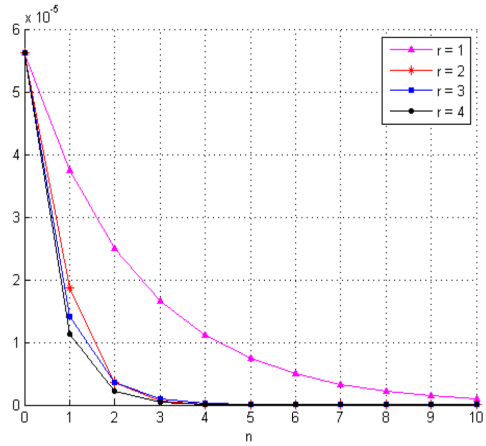


(a) $s = 2.5$

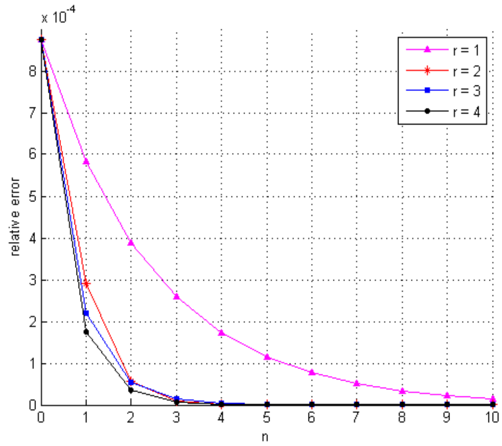


(b) $s = 3$

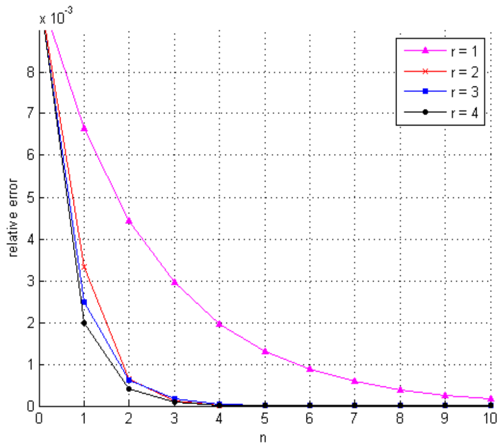
Figure 4.38: Comparisons between $\zeta(s)$ approximations obtained by $h_n^{r,-0.1}(x)$ at $r = 1,2,3,4$ for $s = 2.5, 3$.



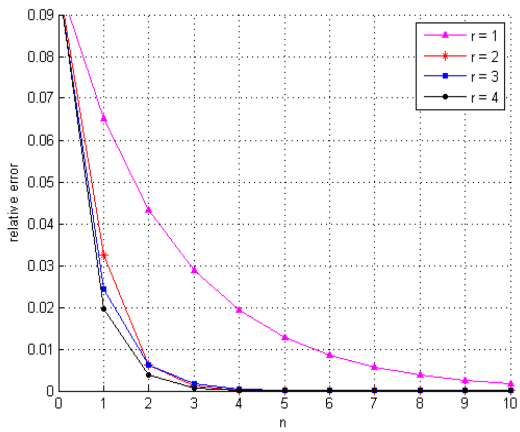
(a) $s = 1.5$



(b) $s = 2$

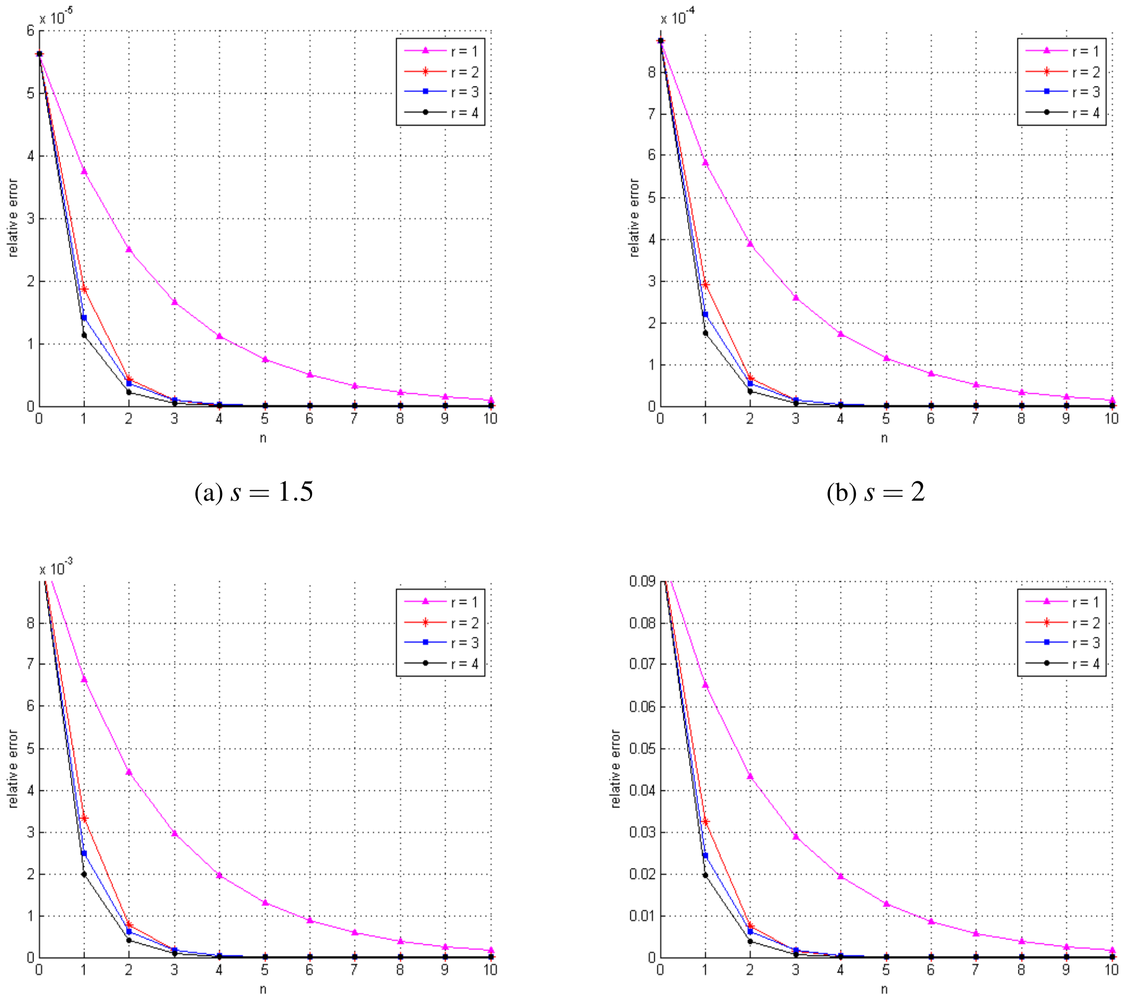


(c) $s = 2.5$



(d) $s = 3$

Figure 4.39: Comparisons between $\zeta(s)$ approximations obtained by $h_n^{r,0.1}(x)$ at $r = 1,2,3,4$ for $s = 2.5, 3$.



(a) $s = 1.5$ (b) $s = 2$

(c) $s = 2.5$ (d) $s = 3$

Figure 4.40: Comparisons between $\zeta(s)$ approximations obtained by $h_n^{r,0.5}(x)$ at $r = 1,2,3,4$

For $n = 10$, Table 4.6 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using Humbert polynomials with the values $r = 1,2,2.5,3$.

	$s = 1.5$	$s = 2$	$s = 2.5$	$s = 3$
$h_{10}^{1,-0.5}(x)$	9.74×10^{-7}	1.52×10^{-5}	1.73×10^{-4}	1.69×10^{-3}
$h_{10}^{2,-0.5}(x)$	7.57×10^{-18}	4.54×10^{-17}	1.76×10^{-16}	5.45×10^{-16}
$h_{10}^{3,-0.5}(x)$	7.62×10^{-11}	1.22×10^{-9}	1.38×10^{-8}	1.36×10^{-7}
$h_{10}^{4,-0.5}(x)$	1.56×10^{-11}	1.41×10^{-10}	1.30×10^{-9}	9.53×10^{-9}

Table 4.6: The values of relative error in approximating $\zeta(s)$ using Humbert polynomials.

The last result can be shown in three dimensions by plotting the relative error associated with $h_n^{r,\gamma}(x)$ polynomials at different values of r and γ using fixed value of n. In Fig. 4.41, we present the relative error associated with using $h_2^{r,\gamma}(x)$ in approximating $\zeta(2)$, which drops in clear way at the couple $(r = 2, \gamma = -0.5)$.

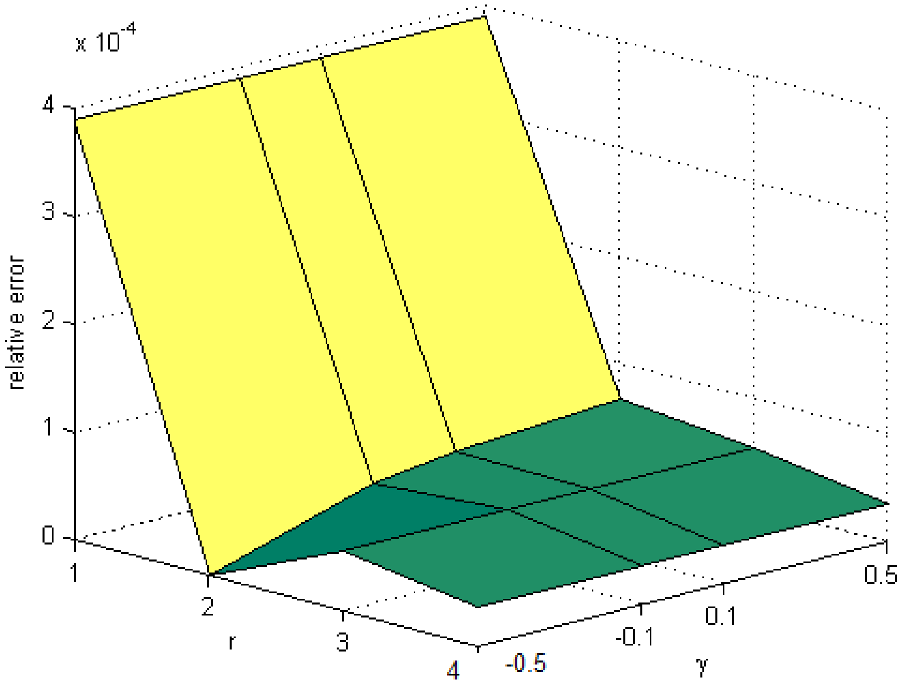


Figure 4.41: Relative error versus the values of (r,γ) using $h_2^{r,\gamma}(x)$ in approximating $\zeta(2)$.

Now, when using $h_n^{4,\gamma}(x)$ in approximating $\zeta(2)$, the resulted relative error is given for different values of (n,γ) in Fig. 4.42. In the case of $r = 2$, this is presented previously in (4.1.3) by Fig. 4.16.

For fixed values of γ, the relative error is presented for various values of n and r. For instance, Fig. 4.43 and Fig. 4.44 show the relative error associated with approximating $\zeta(2)$ using $h_n^{r,\gamma}(x)$ when $\gamma = -0.5$,0.5 respectively.

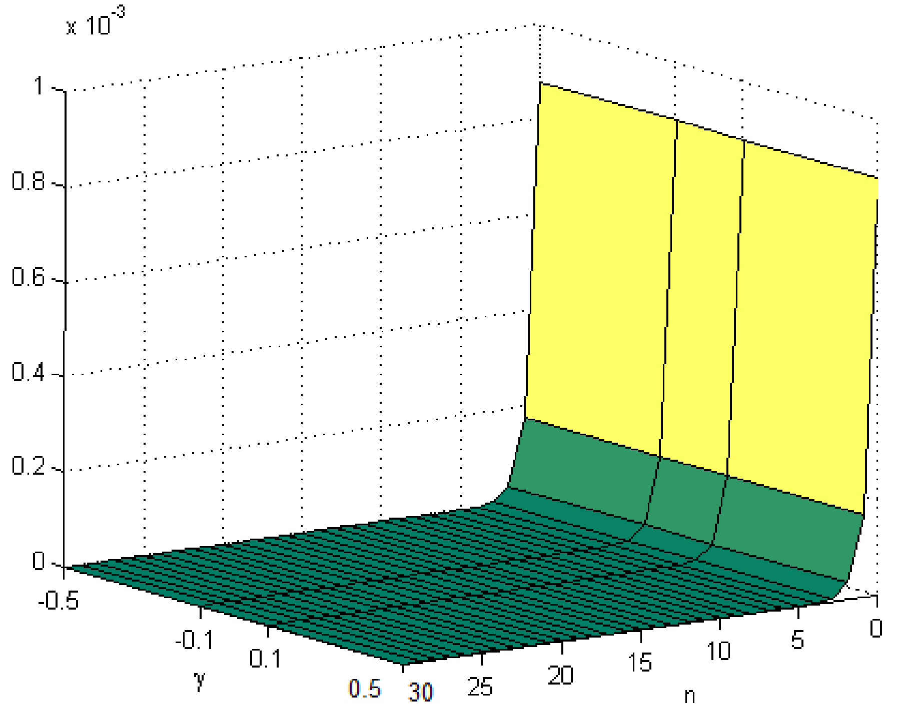


Figure 4.42: Relative error versus the values (n,γ) using $h_n^{4,\gamma}(x)$ in approximating $\zeta(2)$.

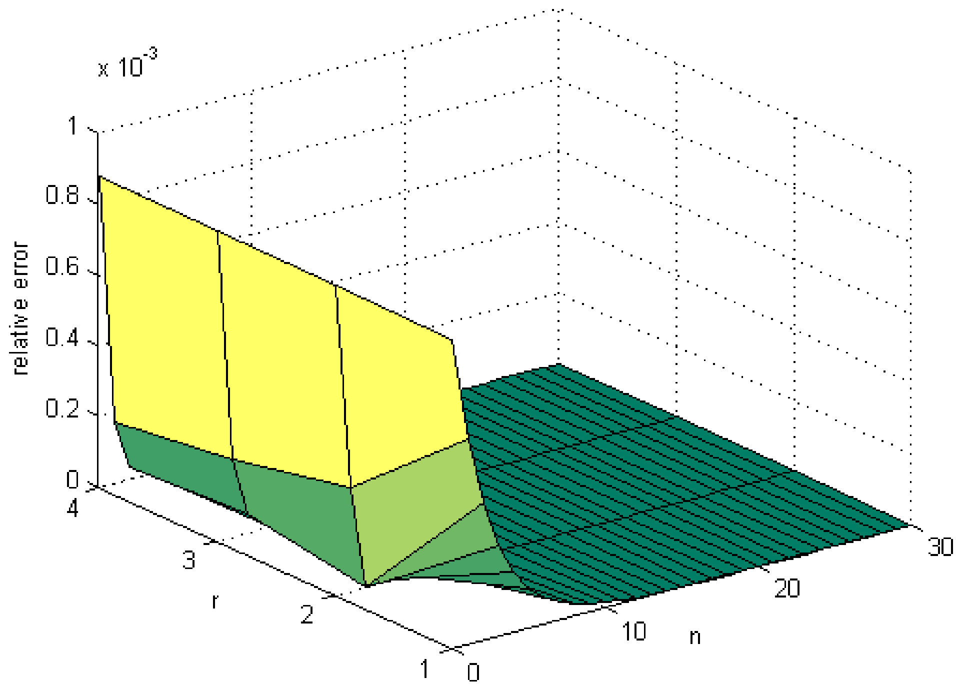


Figure 4.43: Relative error versus the values of (n,r) in approximating $\zeta(2)$ by $h_n^{r,-0.5}(x)$.

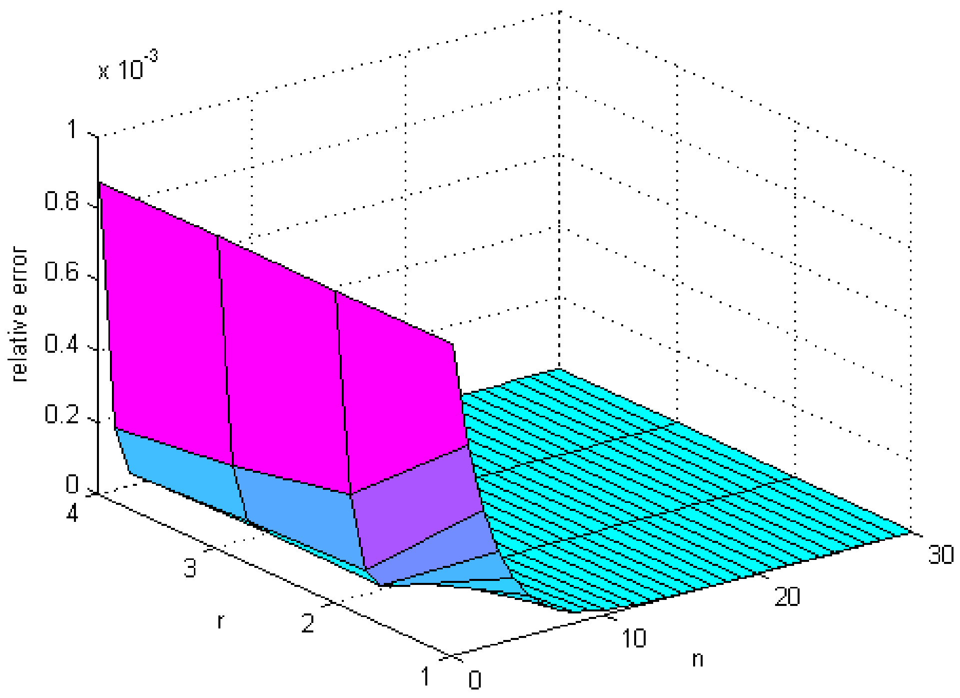


Figure 4.44: Relative error versus the values of (n,r) in approximating $\zeta(2)$ by $h_n^{r,0.5}(x)$.

In the following, particular cases of Humbert polynomials are used with Borwein algorithm.

4.1.7.1 Approximation by Pincherle Polynomials

Pincherle polynomials $\rho_n(x)$ which are obtained from Humbert polynomials by taking $r=3$ and $\gamma=-0.5$;

$$\rho_n(x) = h_n^{3,-0.5}(x)$$

give good approximations for the Riemann Zeta function when implicating them in Borwein algorithm. Fig.4.45 describes the relative error caused by using Pincherle polynomials in approximating $\zeta(s)$ for different values of s.

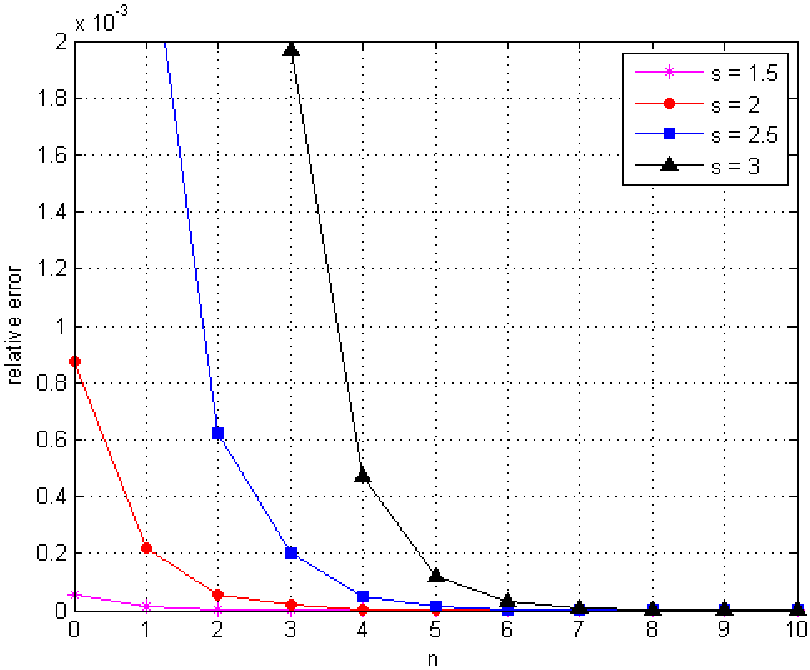
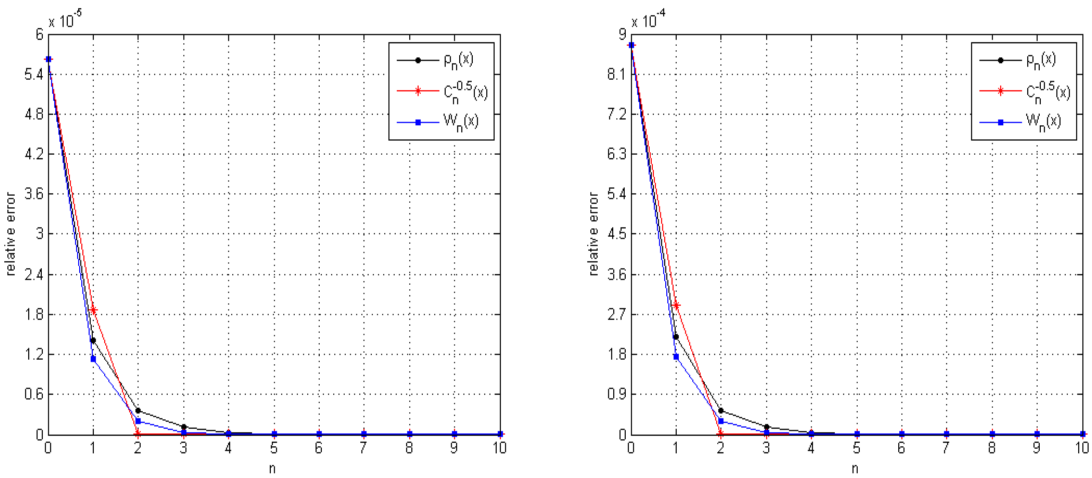


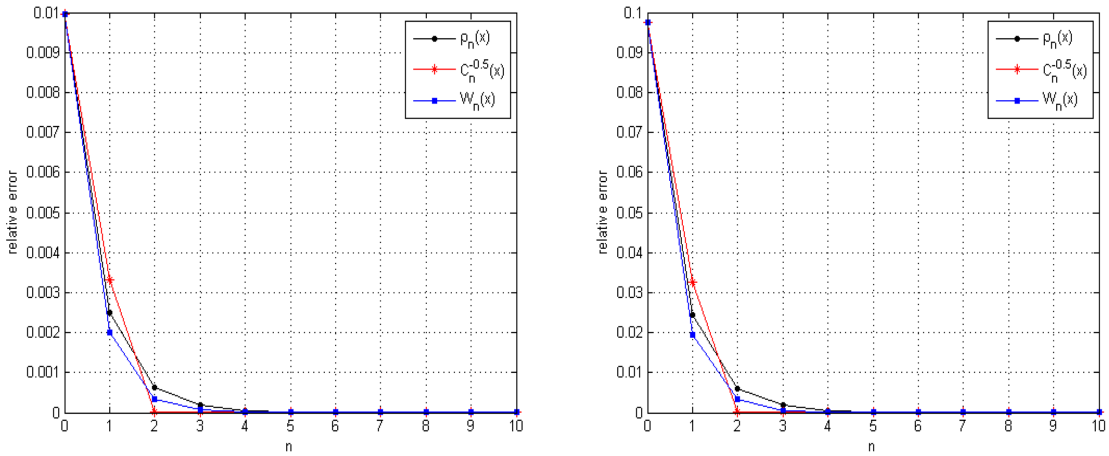
Figure 4.45: comparison between $\zeta(s)$ approximations obtained by $\rho_n(x)$ for different values of s.

Since Pincherle polynomials $\rho_n(x)$ are special cases of Humbert polynomials, then $\rho_n(x)$ are worse than $h_n^{r,-0.5}(x)$ when $r = 2$, which are our celebrated polynomials $h_n^{2,-0.5}(x)$ $(C_n^{-0.5}(x))$. In fact, W-Chebyshev polynomials are better than Pincherle polynomials as it is clarified in Fig. 4.46 and Fig. 4.47.



(a) $s = 1.5$ (b) $s = 2$

Figure 4.46: Comparisons between $\zeta(s)$ approximations obtained by $\rho_n(x)$, $C_n^{-0.5}(x)$, and $W_n(x)$ polynomials for $s = 1.5, 2$.



(a) $s = 2.5$

(b) $s = 3$

Figure 4.47: Comparisons between $\zeta(s)$ approximations obtained by $\rho_n(x)$, $C_n^{-0.5}(x)$, and $W_n(x)$ polynomials for $s = 2.5, 3$.

4.1.7.2 Approximation by Horadam-Pethe polynomials

The set of Horadam-Pethe polynomials is a special case of Milovanović Dordevic polynomials $R_{n,r}^{\lambda}(x)$. They take a part in approximating $\zeta(s)$ as Humbert type polynomials. Since,

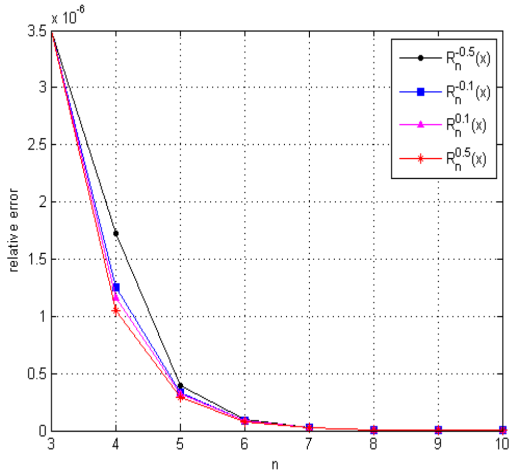
$$R_{n+1}^{\lambda}(x) = h_n^{3,\lambda}(\frac{2x}{3})$$

They are clarified and given explicitly by equation (2.49), with the restriction

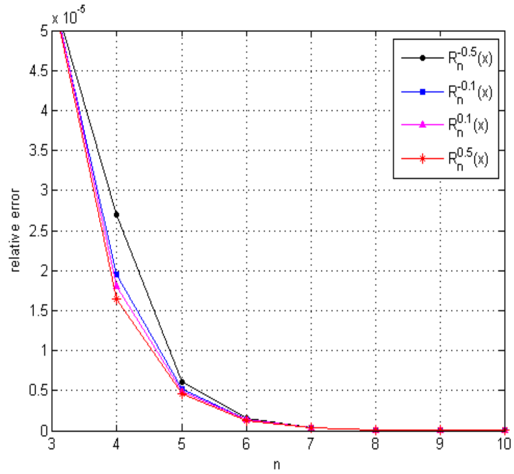
$$\lambda \geq -\frac{1}{2}$$

to hold orthogonality.

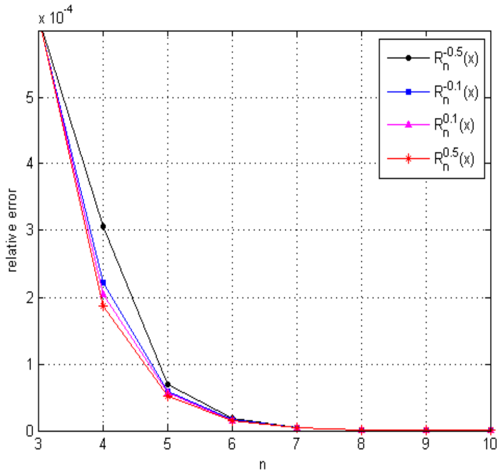
By using $R_n^{\lambda}(x)$ polynomials with Borwein algorithm and changing the values of the parameter λ, it is notable that the accuracy of $\zeta(s)$ approximations gets better as λ increases. Fig. 4.48 presents the resulting relative error when using $R_n^{\lambda}(x)$ with $\alpha = -0.5\,, -0.1\,, 0.1\,, 0.5$. It is clear that the differences between the values of the relative error are little when changing λ. Actually, the accuracy of $\zeta(s)$ approximations using $R_n^{\lambda}(x)$ is better for large values of λ.



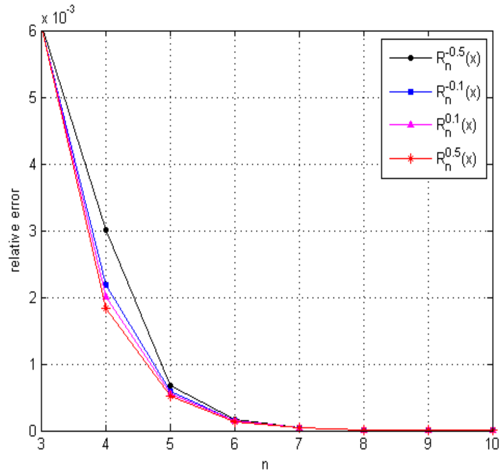
(a) $s = 1.5$



(b) $s = 2$



(c) $s = 2.5$



(d) $s = 3$

Figure 4.48: Comparisons between $\zeta(s)$ approximations obtained by $R_n^{-0.5}(x), R_n^{-0.1}(x), R_n^{0.1}(x)$, and $R_n^{0.5}(x)$ polynomials.

For $R_n^{0.5}(x)$ polynomials, we firstly provide a comparison between $\zeta(s)$ approximations obtained by the means of these polynomials for different values of s by Fig. 4.49. Then we make comparisons between $R_n^{0.5}(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials in respect to the relative error of $\zeta(s)$ approximations. This is shown by Fig. 4.50.

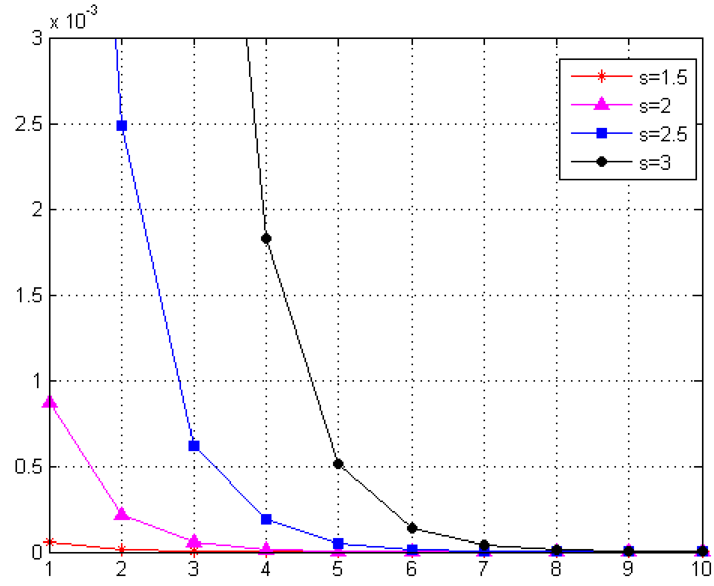
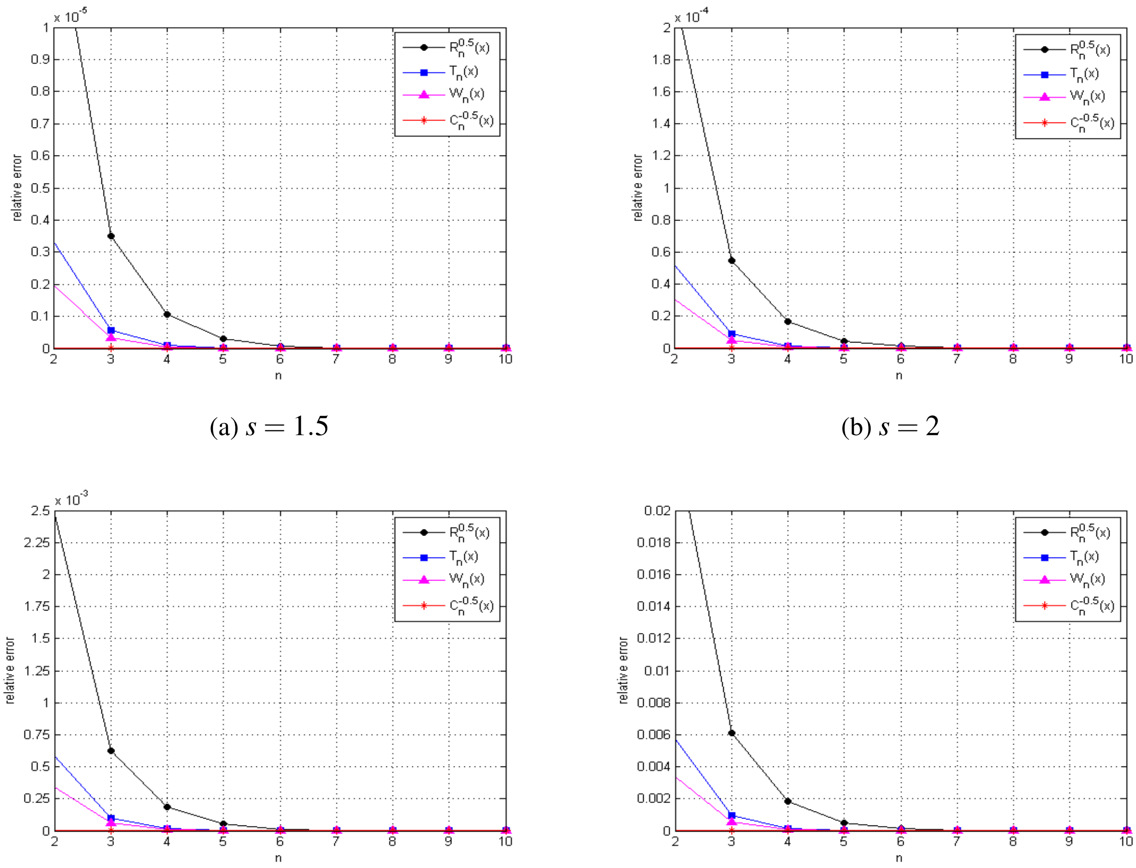


Figure 4.49: Comparison between $\zeta(s)$ approximations obtained by $R_n^{0.5}(x)$ for different values of s.



(a) $s = 1.5$

(b) $s = 2$

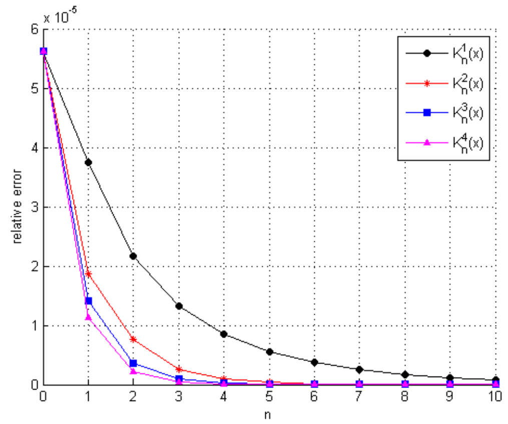
(c) $s = 2.5$

(d) $s = 3$

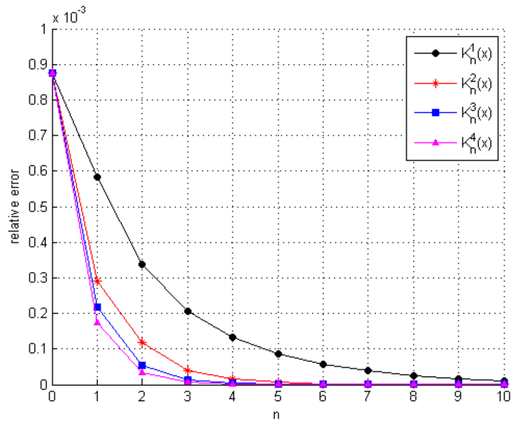
Figure 4.50: Comparisons between $\zeta(s)$ approximations obtained by $R_n^{0.5}(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials.

4.1.8 Approximation by Kinney Polynomials

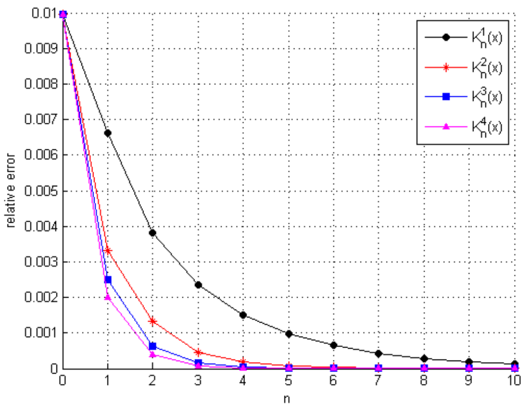
Here, we show the results follow from inserting Kinney polynomials $K_n^r(x)$ within Borwein algorithm as a representative case of the Generalized Humbert polynomials. These polynomials were given in Chapter 2 explicitly by equation (2.58). Parameter r in the case of Kinney polynomials is a positive integer, by which the relative error associated with $K_n^r(x)$ decreases as r increases for sufficient small values of n. We make a comparison between $\zeta(s)$ approximations obtained by using $K_n^r(x)$ for different values of r ($r = 1,2,3,4$). This is presented in Fig. 4.51.



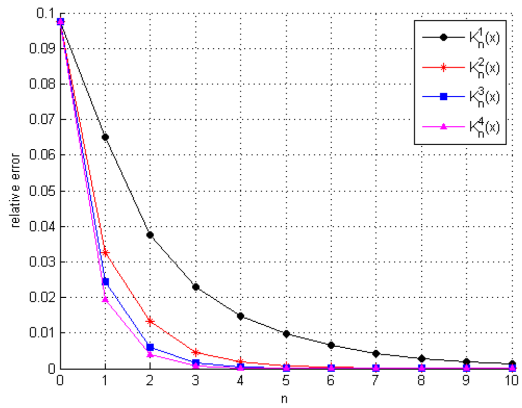
(a) $s = 1.5$



(b) $s = 2$



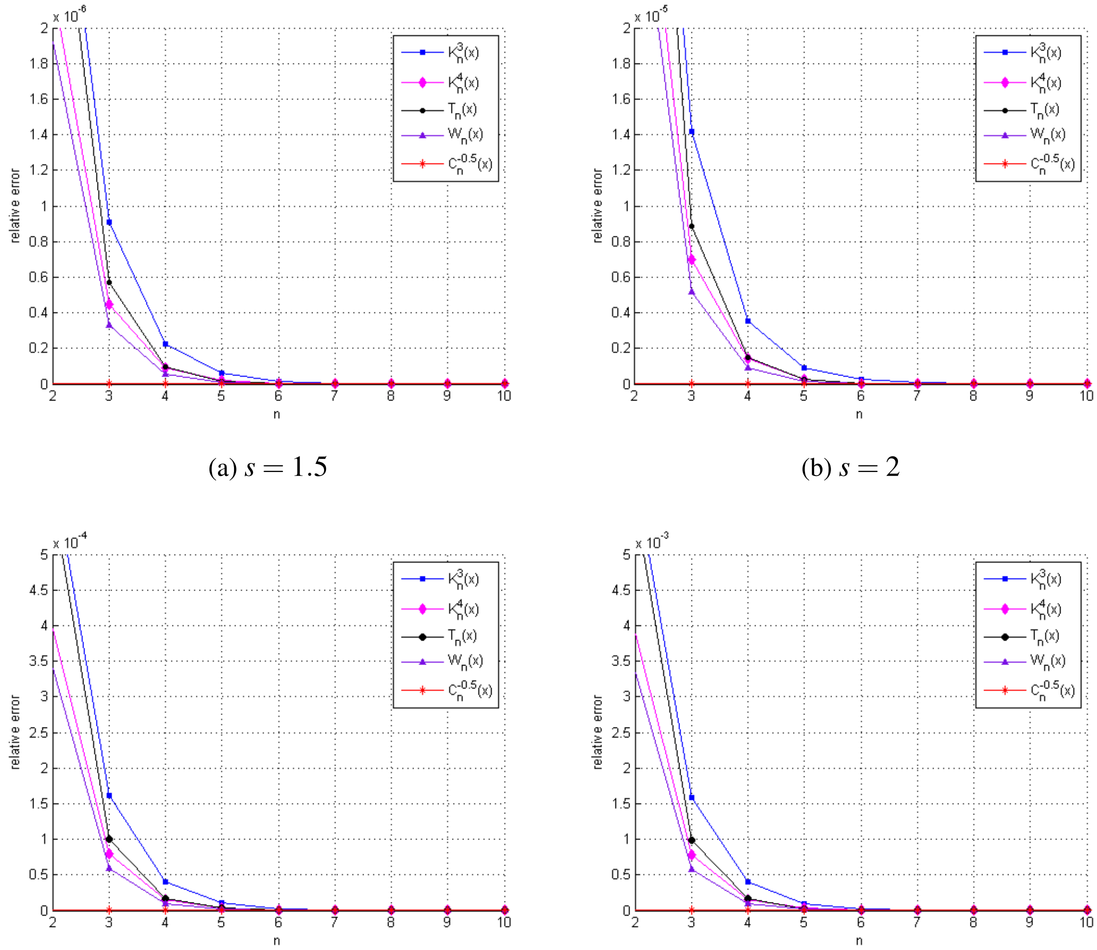
(c) $s = 2.5$



(d) $s = 3$

Figure 4.51: Comparisons between $\zeta(s)$ approximations obtained by using $K_n^r(x)$ at $r = 1,2,3,4$.

Kinney polynomials $K_n^r(x)$ with $r \geq 4$ overcome T-Chebychev polynomials in the context of $\zeta(s)$'s approximation. At the same time, the approximations obtained by $K_n^r(x)$ for $r \leq 3$ are worse than those obtained by $T_n(x)$. This is illustrated in Fig. 4.52 which shows comparisons between $\zeta(s)$ approximations obtained by $K_n^3(x)$, $K_n^4(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$.



(a) $s = 1.5$

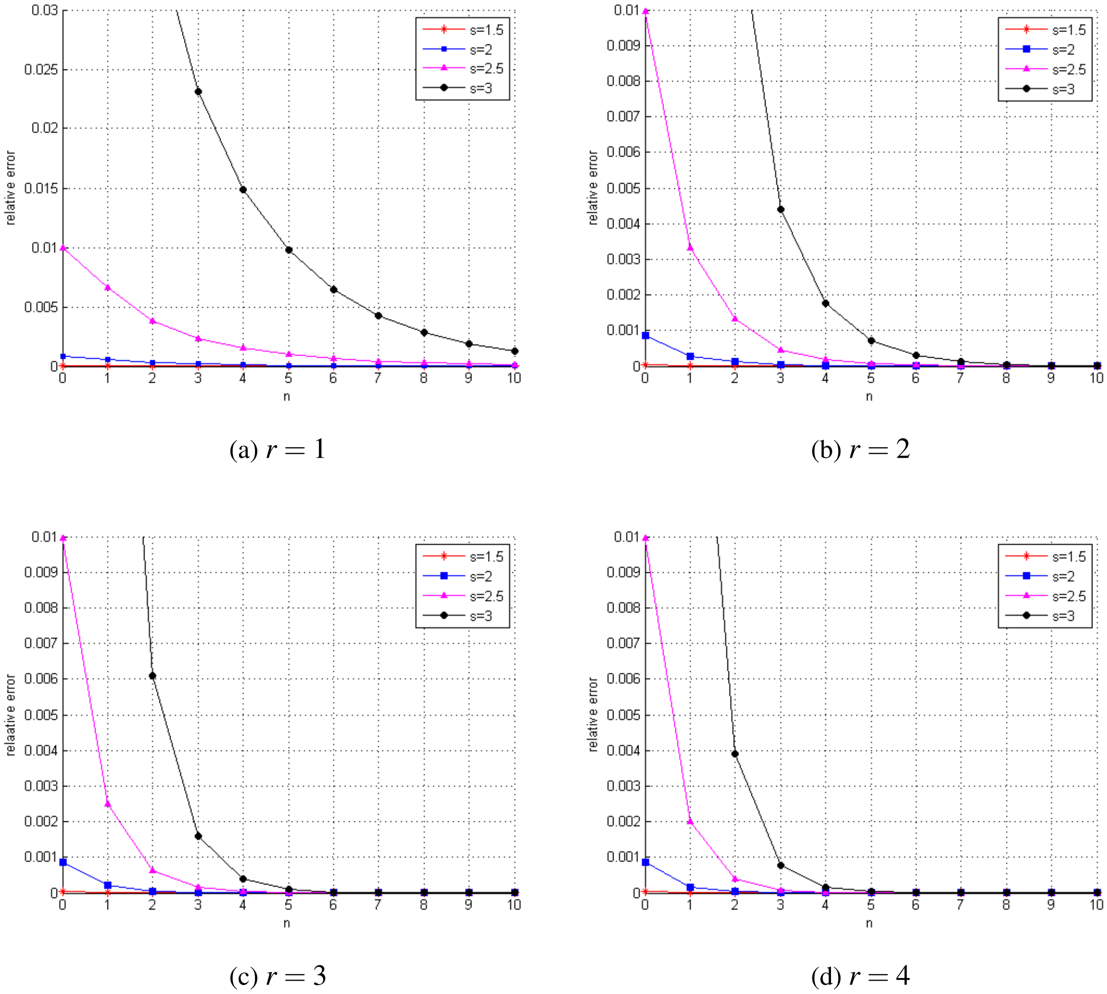
(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.52: Comparisons between $\zeta(s)$ approximations obtained by $K_n^3(x)$, $K_n^4(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials.

The approximations resulted from using $K_n^r(x)$ polynomials with Borwein algorithm get better as argument s gets smaller as it is shown in Fig. 4.53.



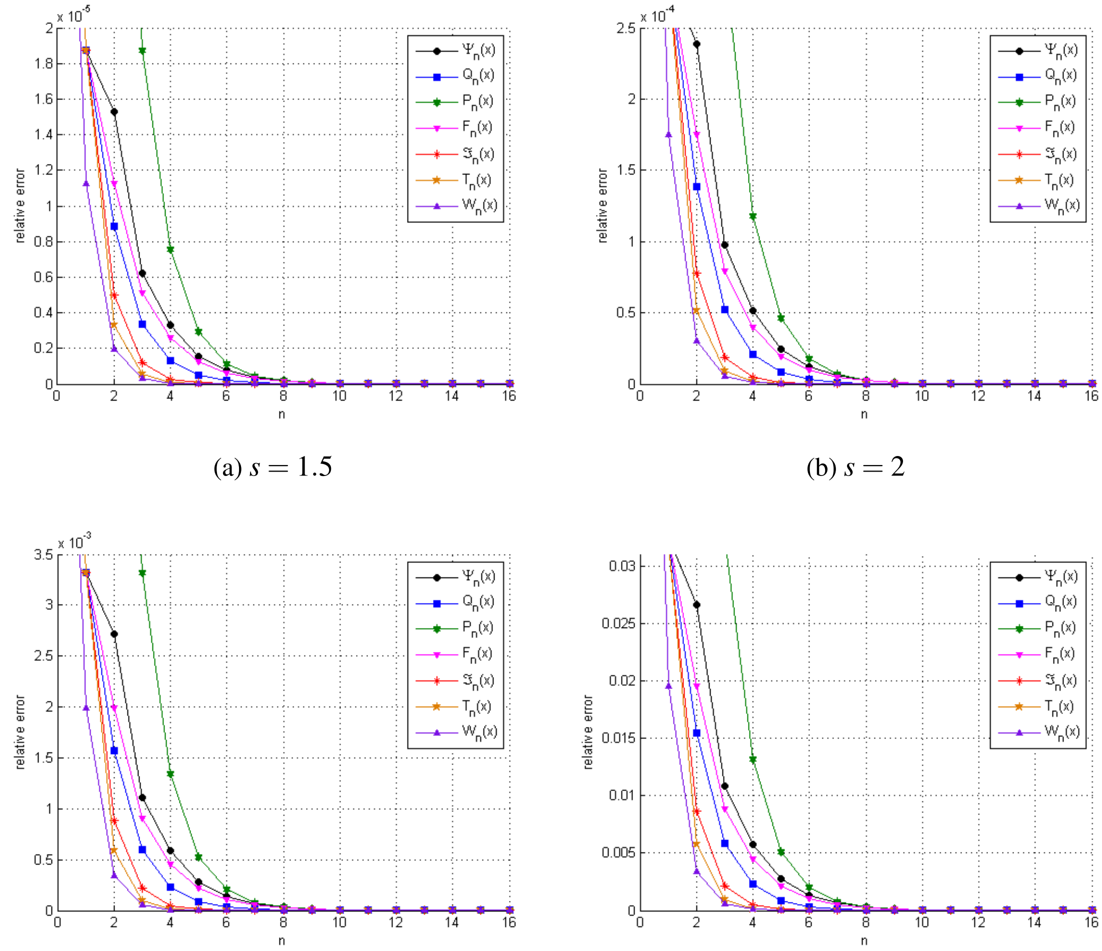
(a) $r = 1$ (b) $r = 2$

(c) $r = 3$ (d) $r = 4$

Figure 4.53: Comparisons between $\zeta(s)$ approximations obtained by $K_n^r(x)$ for different values of s in the case of $r = 1, 2, 3, 4$.

4.2 Approximating $\zeta(s)$ with Horadam Polynomials

Horadam family has a large number of polynomials, like: Fermat $\Im_n(x)$, Fibonacci $F_n(x)$, Pell $P_n(x)$, Pell-Lucas $Q_n(x)$, Lucas $\Psi_n(x)$, T-Chebyshev $T_n(x)$, and U-Chebyshev $U_n(x)$ polynomials which were given in details in section (2.3). When inserting Horadam polynomials into Borwein algorithm, the accuracy in approximating $\zeta(s)$ is expressed by introducing the relative error. Fig. 4.54 shows comparisons between $\zeta(s)$ approximations obtained by different types of Horadam polynomials with respect to the relative error.



(a) $s = 1.5$ (b) $s = 2$

(c) $s = 2.5$ (d) $s = 3$

Figure 4.54: Comparisons between $\zeta(s)$ approximations using different kinds of Horadam polynomials.

In the following discussion, we reveal and detail the accuracy of the approximations obtained by the different kinds of Horadam polynomials.

4.2.1 Approximation by Fermat Polynomials

In general, Fermat polynomials include the famous Fermat polynomials $\Im_n(x)$, Fermat polynomials of the first kind $\phi_n(x)$, and of the second kind $\theta_n(x)$ which were introduced by A. F. Horadam and presented by the equations (2.75), (2.74), and (2.73) respectively. The relative error corresponding to their approximations is presented in Fig. 4.55.

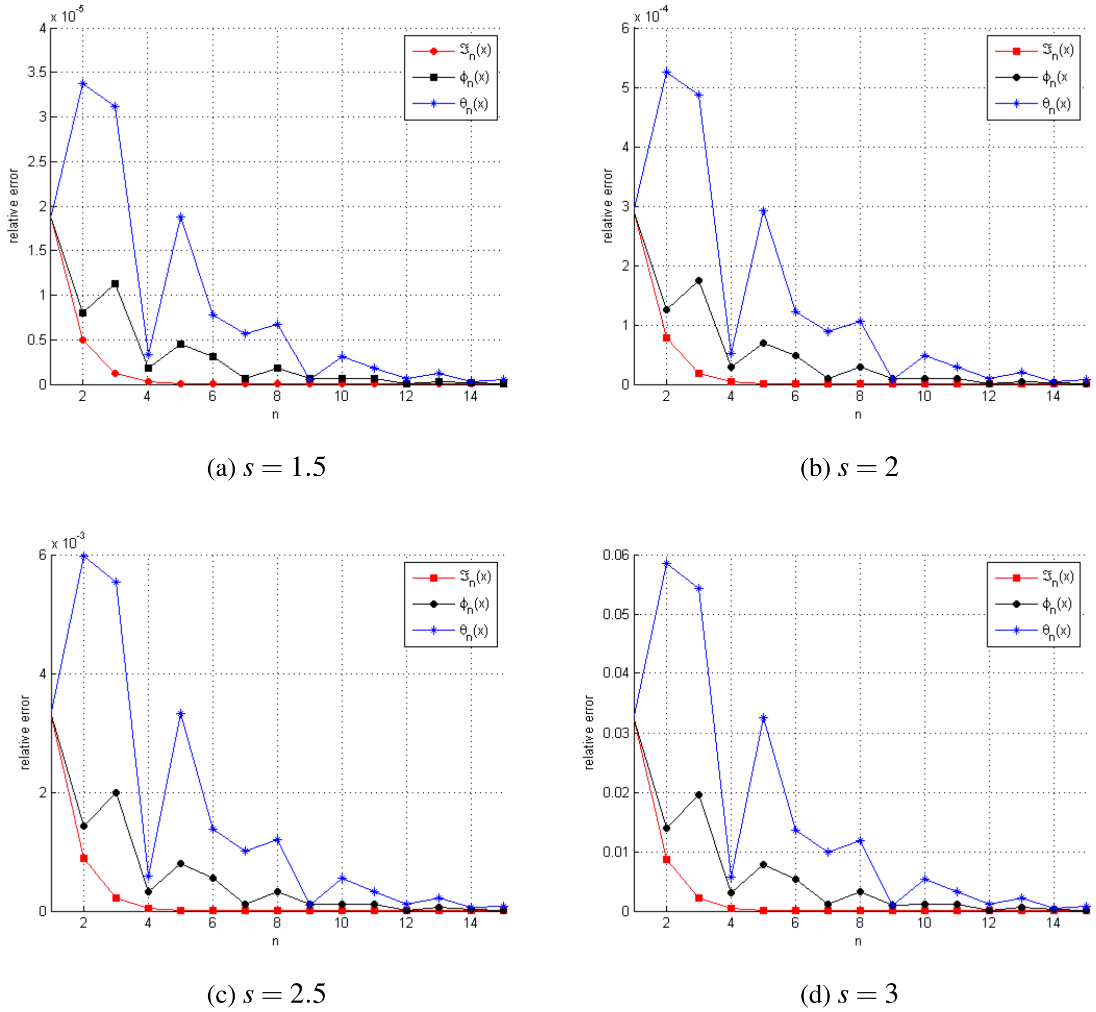


Figure 4.55: Comparisons between $\zeta(s)$ approximations obtained by the different kinds of Fermat polynomials.

Famous Fermat polynomials $\Im_n(x)$ are good in approximating $\zeta(s)$ for different values of s. Actually, $\zeta(s)$ approximations in the case of small argument s are better than those with larger s. Fig. 4.56 provides a comparison between the approximations obtained by $\Im_n(x)$ for different values of s.

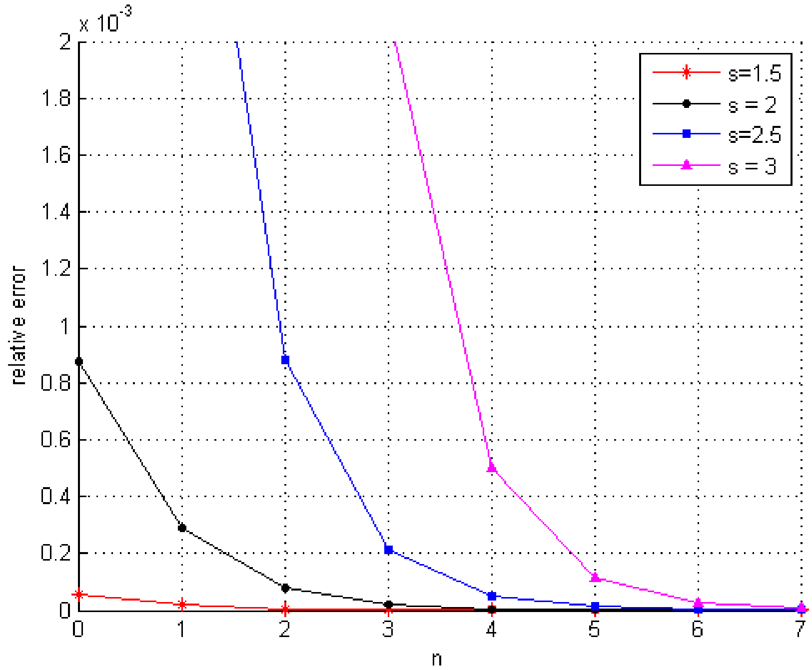
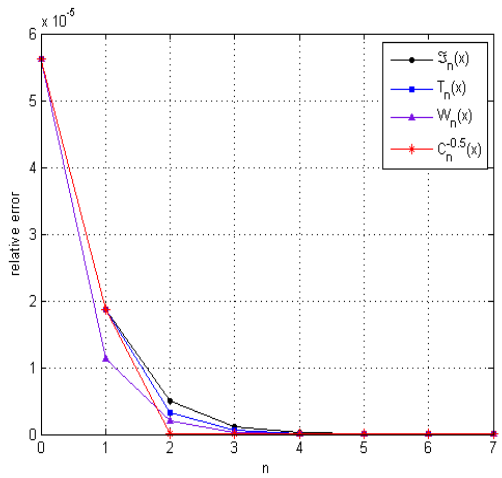
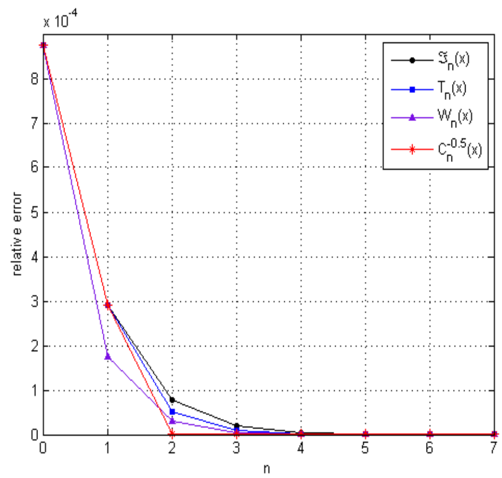


Figure 4.56: Comparison between $\zeta(s)$ approximations obtained by $\Im_n(x)$ for different values of s.

The results show that $\zeta(s)$ approximations obtained by implicating standard Chebychev polynomials and the celebrated Gegenbauer polynomials $C_n^{-0.5}(x)$ are better than those carried out with Fermat polynomials with different kinds. This is shown by plotting the relative error associated with using $\Im_n(x)$, $T_n(x)$, $W_n(x)$ and $C_n^{-0.5}(x)$ in approximating $\zeta(s)$ with Borwein algorithm as it is manifested in Fig. 4.57 and Fig. 4.58 for different values of s.

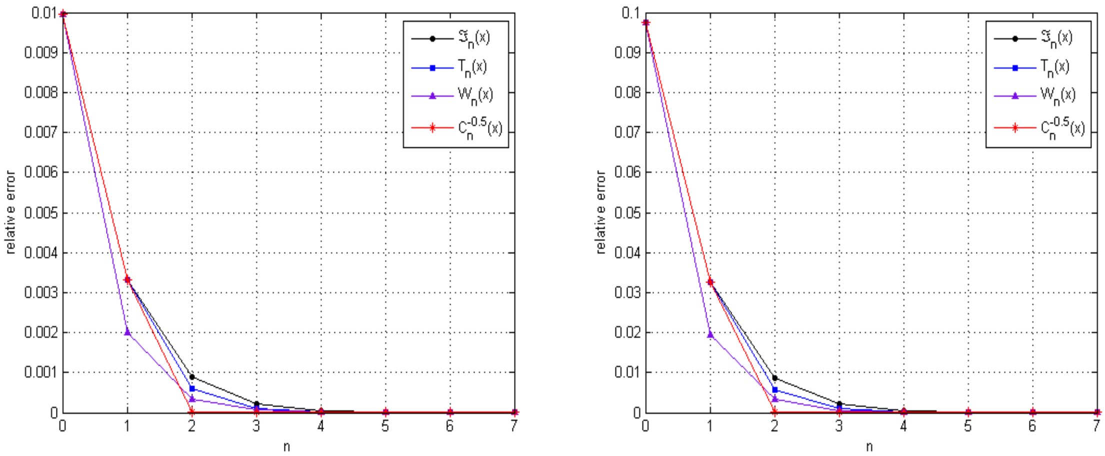


(a) $s = 1.5$



(b) $s = 2$

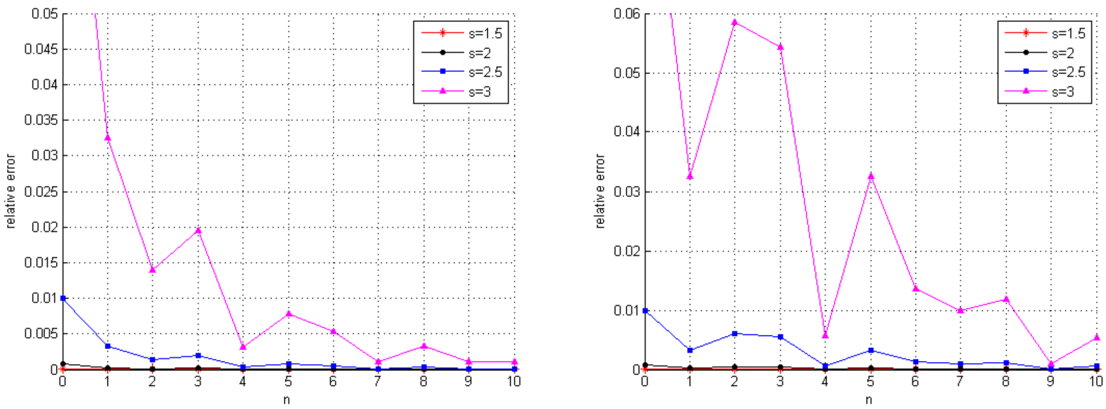
Figure 4.57: comparisons between $\zeta(s)$ approximations obtained by $\Im_n(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials for $s = 1.5, 2$.



(a) $s = 2.5$ (b) $s = 3$

Figure 4.58: comparisons between $\zeta(s)$ approximations obtained by $\Im_n(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials for $s = 2.5, 3$.

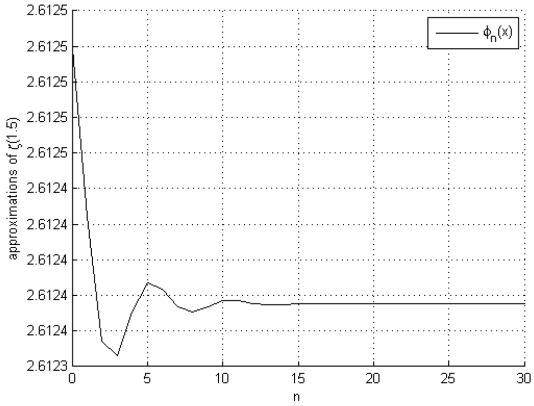
Now, comparisons between $\zeta(s)$ approximations for different values of s are presented in Fig. 4.59 in the case of using $\phi_n(x)$ and $\theta_n(x)$ polynomials.



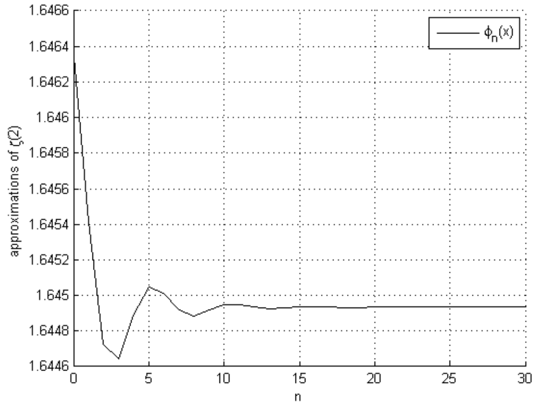
(a) $\phi_n(x)$ (b) $\theta_n(x)$

Figure 4.59: Comparisons between $\zeta(s)$ approximations obtained for different values of s by $\phi_n(x)$ and $\theta_n(x)$ polynomials.

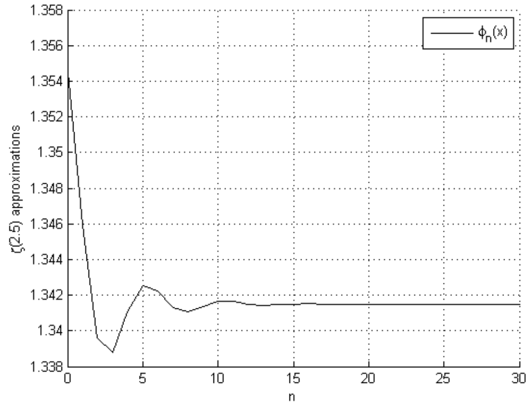
The value of $\zeta(s)$ approximations, which were obtained by $\phi_n(x)$ and $\theta_n(x)$ are presented in Fig. 4.60, Fig. 4.61, and Fig. 4.62 for different values of n and s. For every s, $\zeta(s)$ approximations converge to a limiting value as n increases which is indeed the exact value of $\zeta(s)$.



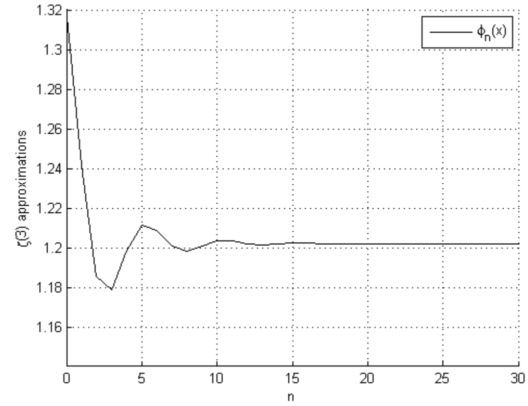
(a) $s = 1.5$



(b) $s = 2$

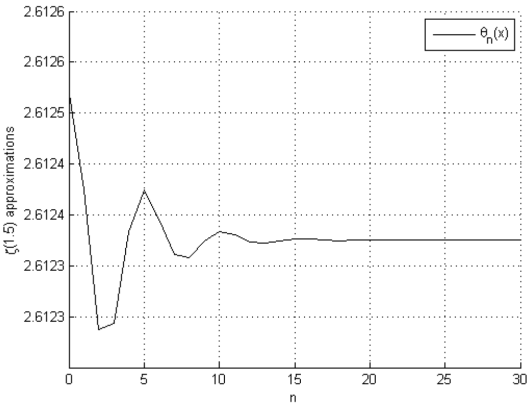


(c) $s = 2.5$

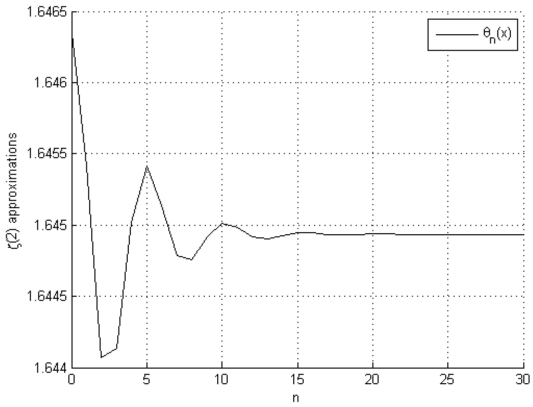


(d) $s = 3$

Figure 4.60: $\zeta(s)$ approximations using $\phi_n(x)$.

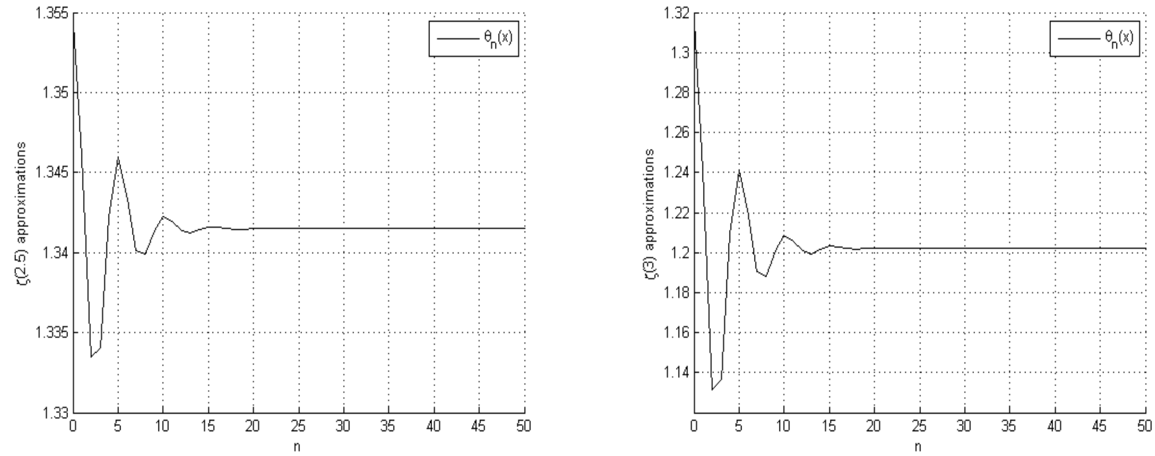


(a) $s = 1.5$



(b) $s = 2$

Figure 4.61: $\zeta(s)$ approximations using $\theta_n(x)$ for $s = 1.5, 2$.



(a) $s = 2.5$ (b) $s = 3$

Figure 4.62: $\zeta(s)$ approximations using $\theta_n(x)$ for $s = 2.5, 3$.

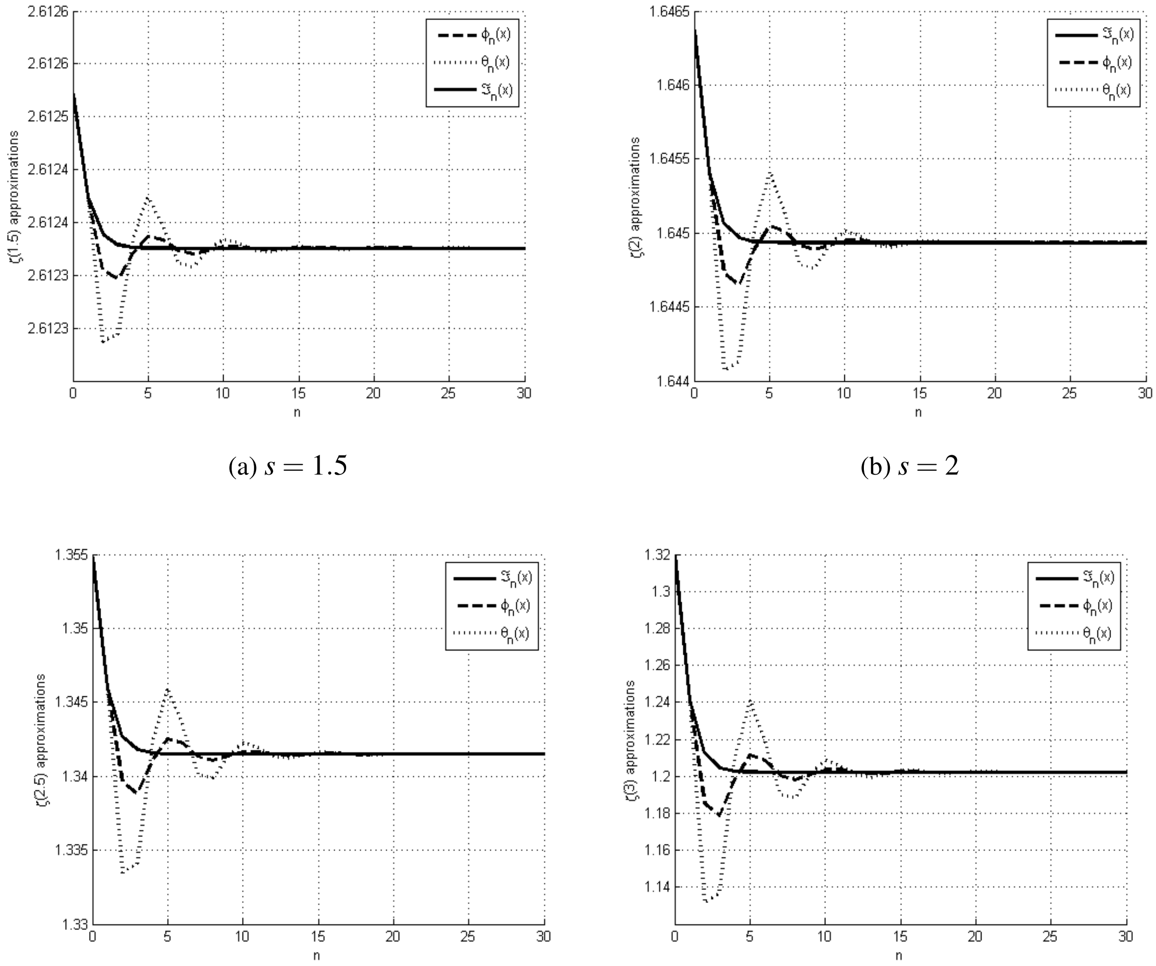
The approximations of $\zeta(s)$ obtained by any kind of Fermat polynomials are compared with the ones obtained by the other kinds. Fig. 4.63 shows these comparisons for different values of s.

4.2.2 Approximation by Fibonacci Polynomials

Fibonacci Polynomials $F_n(x)$ are considered as a generalization of the famous Fibonacci numbers and these are presented in section (2.3) in details. Fibonacci polynomials are obtained by either equation (2.76) or equation (2.77).

The behavior of the relative error associated with using $F_n(x)$ polynomials for approximating $\zeta(s)$ with Borwein algorithm is described in Fig. 4.64. Note that $\zeta(s)$ approximations are better in the case of small s.

Fibonacci polynomials, although they are good in approximating $\zeta(s)$ (since the relative error is decreasing as n increasing), but they are still worse than $T_n(x)$, $W_n(x)$, and so $C_n^{-0.5}(x)$ polynomials. Fig. 4.65 shows comparisons between these polynomials in approximating $\zeta(s)$.



(a) $s = 1.5$ (b) $s = 2$

(c) $s = 2.5$ (d) $s = 3$

Figure 4.63: Comparisons between $\zeta(s)$ approximations obtained by $\Im_n(x)$, $\phi_n(x)$, and $\theta_n(x)$.

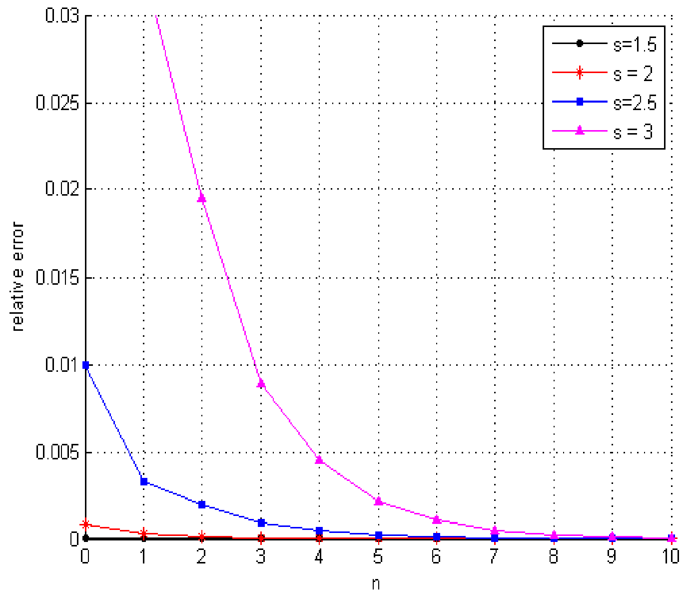
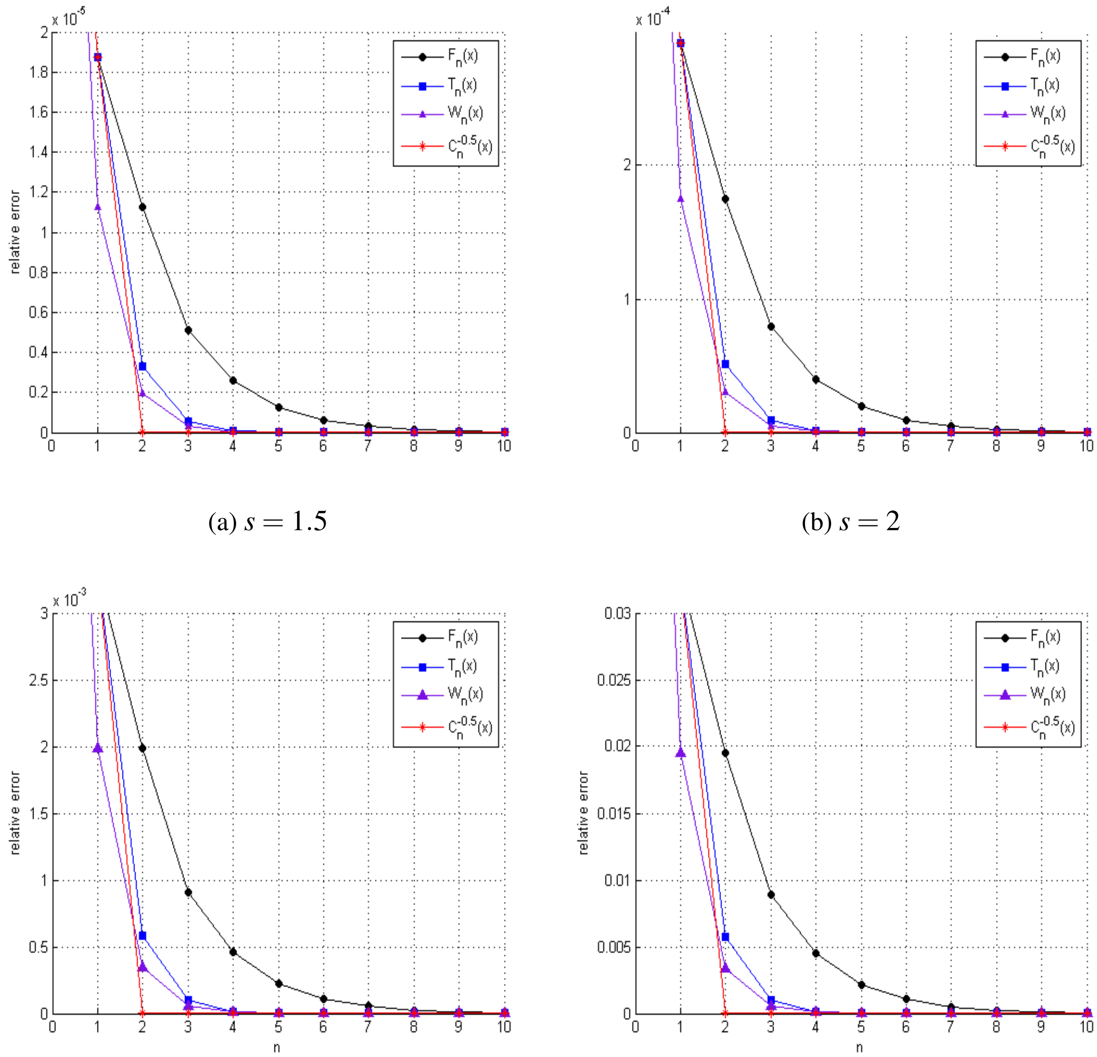


Figure 4.64: Comparison between the accuracy of $\zeta(s)$ approximations obtained by $F_n(x)$ polynomials for different values of s.



(a) $s = 1.5$

(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.65: Comparisons between the accuracy of $\zeta(s)$ approximations obtained by $F_n(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials.

4.2.3 Approximation by Pell Polynomials

Pell polynomials $P_n(x)$, where $n \geq 1$ are given by equation (4.66) and discussed elaborately in section (2.3). A comparison between the relative error associated with $\zeta(s)$ approximations using $P_n(x)$ polynomials is shown in Fig. 4.66 for different values of s .

By the same technique, we can make comparisons between $P_n(x)$ and the previously represented polynomials $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ as it is depicted in Fig. 4.67.

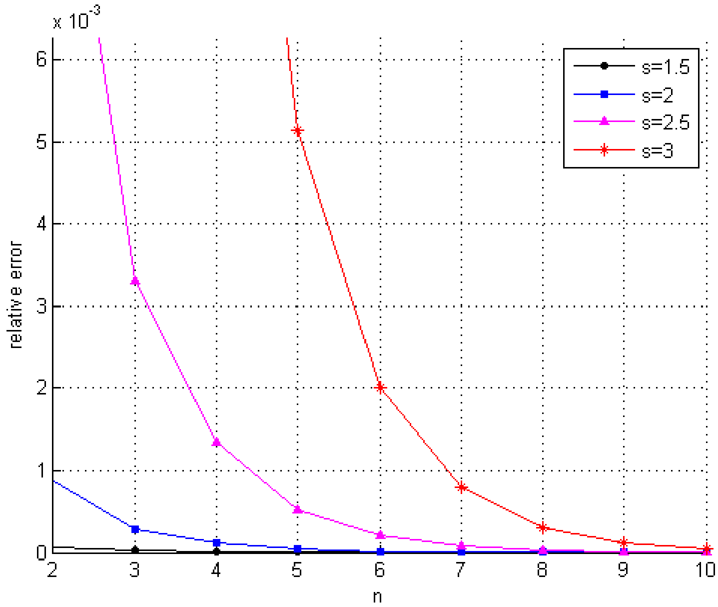
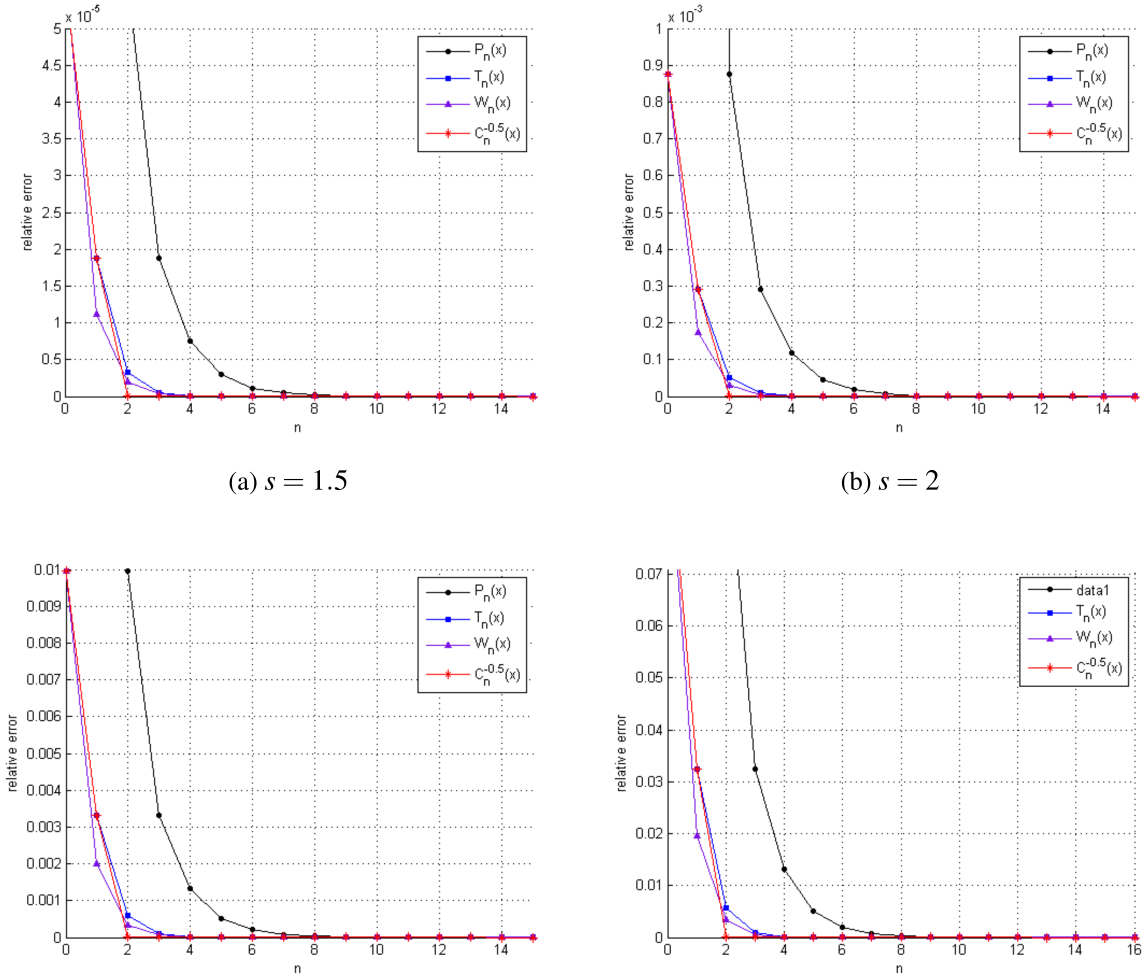


Figure 4.66: Comparison between $\zeta(s)$ approximations obtained by $P_n(x)$ for different values of s.



(a) $s = 1.5$

(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.67: Comparisons between $\zeta(s)$ approximations obtained by $P_n(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$.

4.2.4 Approximation by Pell Lucas Polynomials

Pell Lucas polynomials $Q_n(x)$ are given directly by equation (2.85), other formulas can be found in section (2.3). Like other polynomials, the accuracy of $\zeta(s)$ approximations attained from implicating $Q_n(x)$ in Borwein algorithm gets better as argument s gets smaller. This is illustrated in Fig. 4.68.

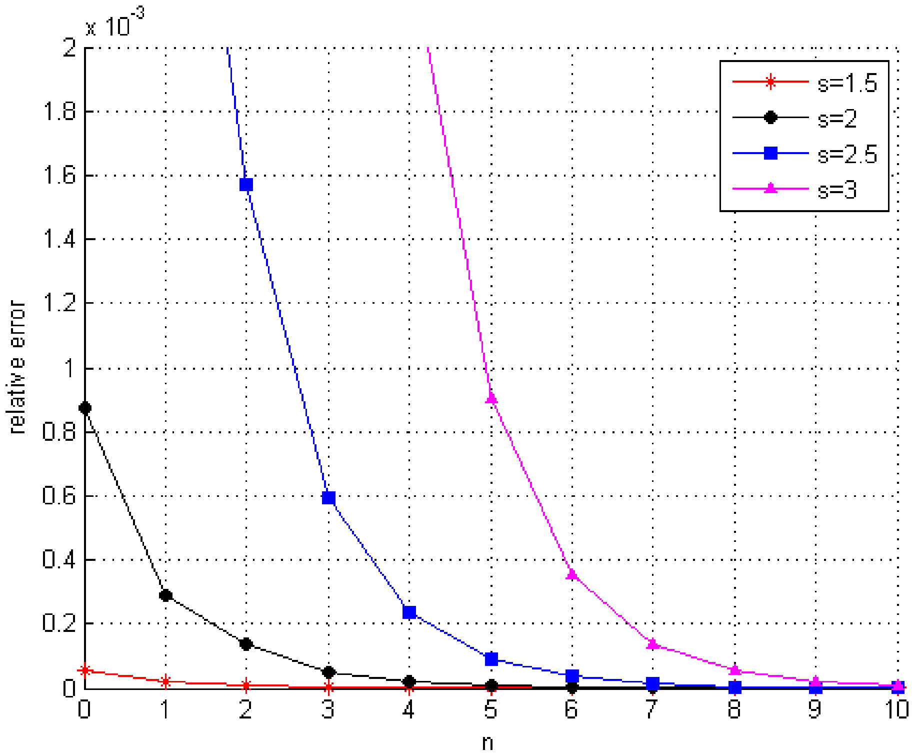
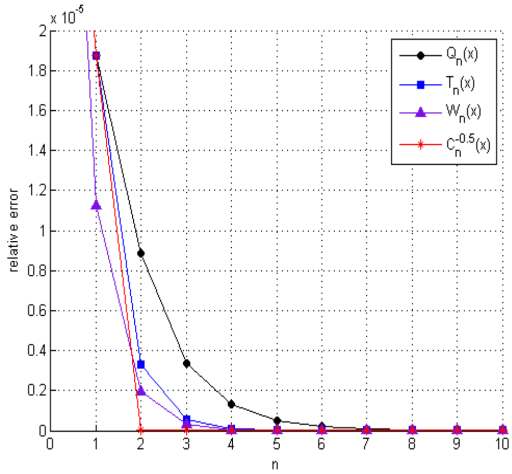
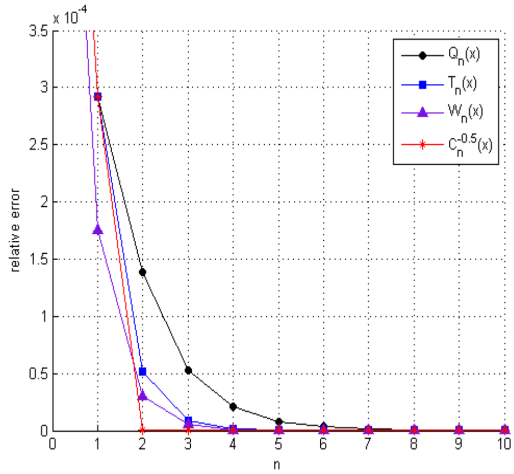


Figure 4.68: Comparison between $\zeta(s)$ approximations obtained by $Q_n(x)$ for different values of s.

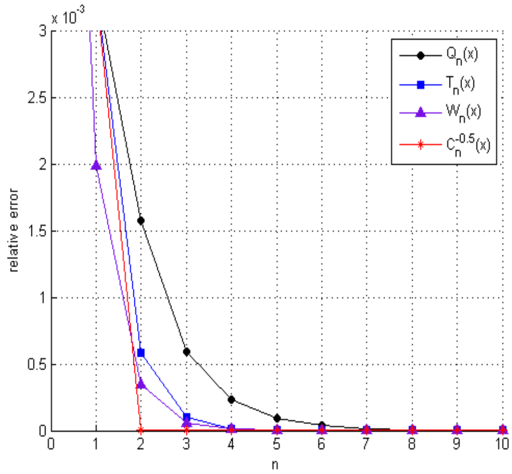
Pell Lucas polynomials are not good as the case of some other orthogonal polynomials. Indeed, $Q_n(x)$ are worse than $T_n(x)$, and thus $W_n(x)$ and $C_n^{-0.5}(x)$ polynomials. This can be shown by plotting the values of relative error associated with using those polynomials with different values of n as Fig. 4.69 exhibits.



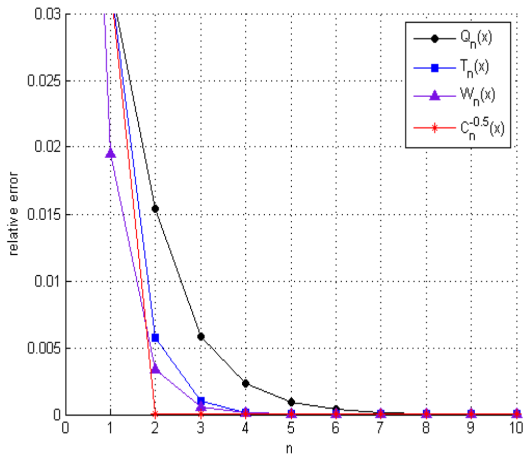
(a) $s = 1.5$



(b) $s = 2$



(c) $s = 2.5$



(d) $s = 3$

Figure 4.69: Comparisons between $Q_n(x), T_n(x), W_n(x)$ and $C_n^{-0.5}(x)$ polynomials in approximating $\zeta(s)$.

4.2.5 Approximation by Lucas Polynomials

Lucas polynomials $\Psi_n(x)$ are presented in Chapter 2 and expressed by various formulas ((2.79), (2.80), and (2.81)). If $\Psi_n(x)$ polynomials are embedded in Borwein algorithm, then they give approximations with different values of accuracy when changing the values of n or s. In fact, their approximations are better in the case of small argument s as shown in Fig. 4.70.

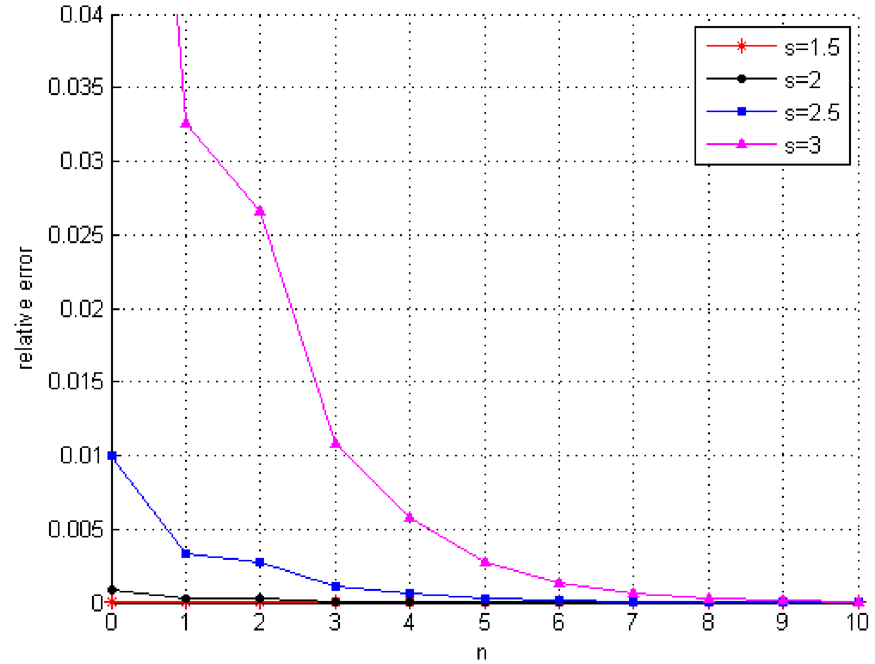
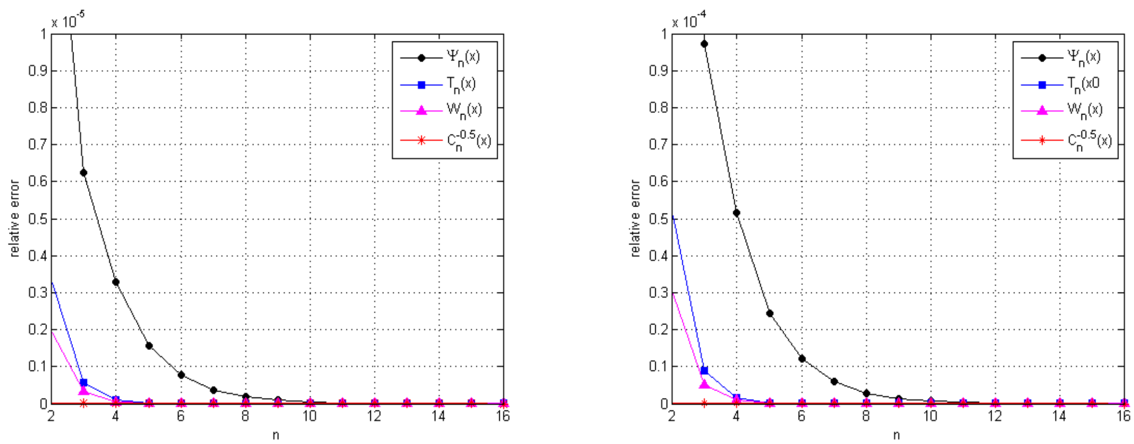


Figure 4.70: Comparison between $\zeta(s)$ approximations obtained by $Q_n(x)$ for different values of s.

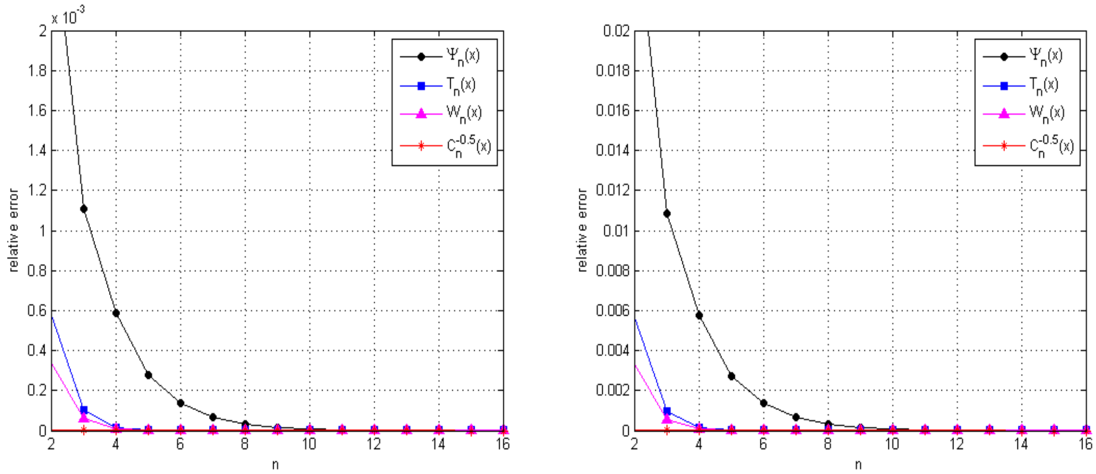
As the case of $Q_n(x)$ polynomials, $\Psi_n(x)$ are worse than $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials in approximating $\zeta(s)$, Fig. 4.71 and Fig. 4.72 describe this. In fact, the results obtained by $\Psi_n(x)$ are worse than all of the presented Horadam polynomials except of Pell polynomials as we pointed out in Fig. 4.54. A comparison between the values of the relative error when using Horadam polynomials of degree 10 for approximating $\zeta(s)$ with Borwein algorithm are presented in Table 4.7.



(a) $s = 1.5$

(b) $s = 2$

Figure 4.71: Comparisons between $\Psi_n(x), T_n(x), W_n(x)$ and $C_n^{-0.5}(x)$ polynomials in approximating $\zeta(s)$ for $s = 1.5, 2$.



(a) $s = 2.5$ (b) $s = 3$

Figure 4.72: Comparisons between $\Psi_n(x), T_n(x), W_n(x)$ and $C_n^{-0.5}(x)$ polynomials in approximating $\zeta(s)$ for $s = 2.5, 3$.

For $n = 10$, Table 4.7 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using different kinds of Horadam polynomials.

	$s = 1.5$	$s = 2$	$s = 2.5$	$s = 3$
$\Im_{10}(x)$	4.15×10^{-11}	6.48×10^{-10}	7.35×10^{-9}	7.22×10^{-8}
$Q_{10}(x)$	4.78×10^{-9}	7.46×10^{-8}	8.48×10^{-7}	8.32×10^{-6}
$P_{10}(x)$	2.73×10^{-8}	4.26×10^{-7}	4.84×10^{-6}	4.75×10^{-5}
$F_{10}(x)$	3.53×10^{-8}	5.51×10^{-7}	6.26×10^{-6}	6.14×10^{-5}
$\Psi_{10}(x)$	4.47×10^{-8}	6.98×10^{-7}	7.92×10^{-6}	7.78×10^{-5}

Table 4.7: The values of relative error in approximating $\zeta(s)$ using Horadam polynomials.

4.3 Approximating $\zeta(s)$ with Traditional Polynomials

In this section, Borwein algorithm is used to approximate $\zeta(s)$ with polynomials that do not obviously follow specific class. Some of these polynomials have familiar names like, Bernoulli, Bernstein Basis, and Newton polynomials. Some others, have no distinguishable names.

4.3.1 Approximation by polynomials of the form $x^n(1-x)^n$

Polynomials of the form

$$P^*(x) = x^n(1-x)^n \tag{4.2}$$

are with even degree always, and can be expressed by

$$P^*(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{k+n} \tag{4.3}$$

Implication of these polynomials within Borwein algorithm gives great approximations with little relative error. In fact, we can approximate $\zeta(s)$ with 50 digit of accuracy by using degree around 18. Actually, $P^*(x)$ polynomials are definitely the best polynomials were found during our work to approximate $\zeta(s)$.

The accuracy of the approximations obtained by these polynomials are better in the case of small values of s. Fig. 4.73 depicts this behavior by showing the relative error in approximating $\zeta(s)$ for different values of s.

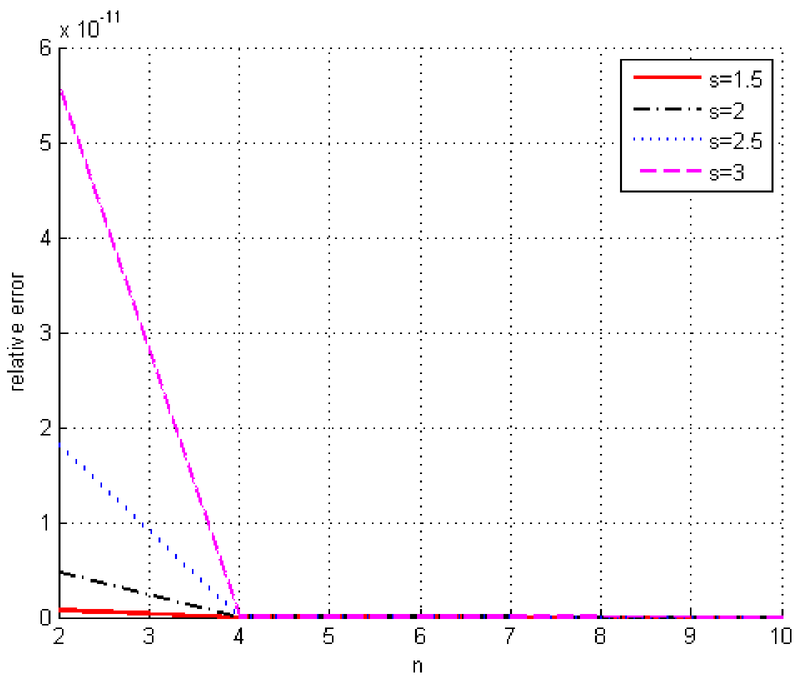
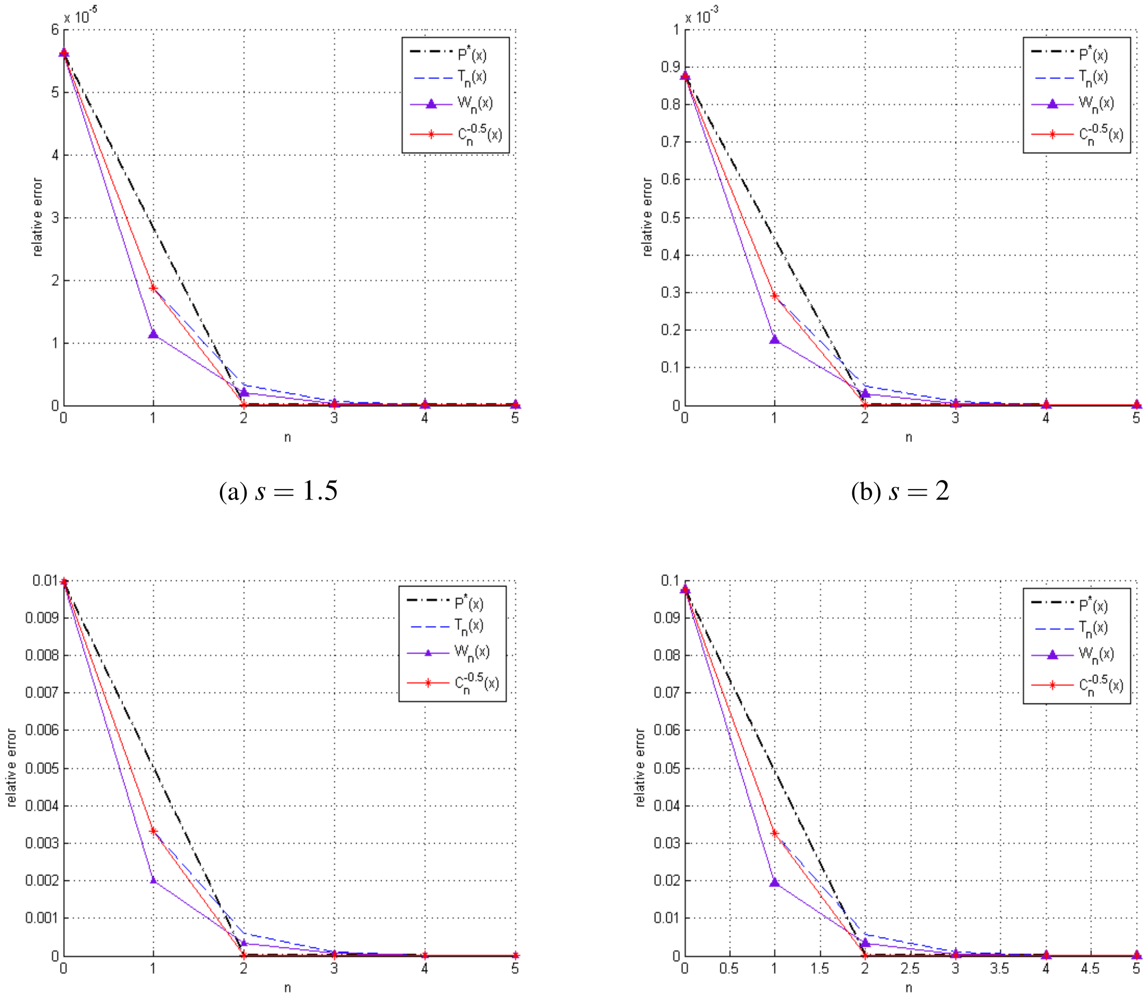


Figure 4.73: Comparison between $\zeta(s)$ approximations obtained by $P^*(x)$ for different values of s.

$P^*(x)$ polynomials are better than $W_n(x)$ and thus better than all Chebyshev polynomials in approximating $\zeta(s)$ with Borwein algorithm with degree $n \geq 2$. This is exhibited by Fig. 4.74 which reveals the relative error associated with approximating $\zeta(s)$ using $P^*(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$. Furthermore, $P^*(x)$ polynomials are better than $C_n^{-0.5}(x)$ polynomials with degree $n \geq 4$ as described in Fig. 4.75.



(a) $s = 1.5$ (b) $s = 2$

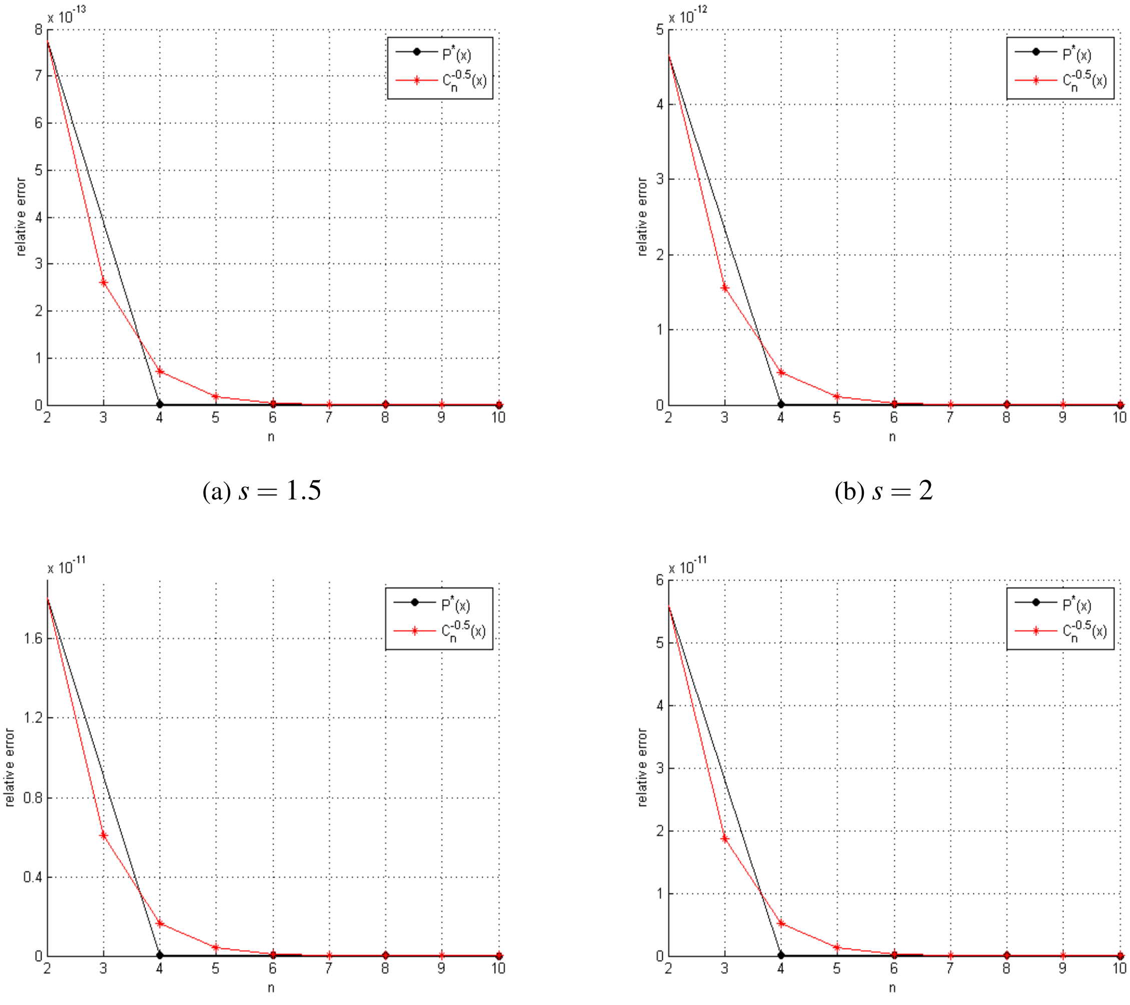
(c) $s = 2.5$ (d) $s = 3$

Figure 4.74: Comparisons between $\zeta(s)$ approximations obtained by $P^*(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$.

For $n = 10$, Table 4.8 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using $P_n^*(x)$.

s	1.5	2	2.5	3
ε	1.21×10^{-32}	3.29×10^{-32}	7.70×10^{-32}	1.61×10^{-31}

Table 4.8: The values of relative error in approximating $\zeta(s)$ using $P_n^*(x)$ with $n = 10$.



(a) $s = 1.5$ (b) $s = 2$

(c) $s = 2.5$ (d) $s = 3$

Figure 4.75: Comparisons between $\zeta(s)$ approximations obtained by $P^*(x)$ and $C_n^{-0.5}(x)$.

4.3.2 Approximation by Bernoulli Polynomials

Bernoulli polynomials $B_n(x)$ which were discussed in Chapter 2 and given explicitly by equation (2.91), are used with Borwein algorithm after some transformations for the variable x. The results show that smaller values of s are associated with little error and so better approximations for $\zeta(s)$ as it is illustrated in Fig. 4.76.

The approximations obtained by embedding Bernoulli polynomials in Borwein algorithm are not good as in the case of Chebyshev and Gegenbauer polynomials. This is manifested in Fig. 4.77 which exhibits the relative error associated with using the polynomials $B_n(x), T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$.

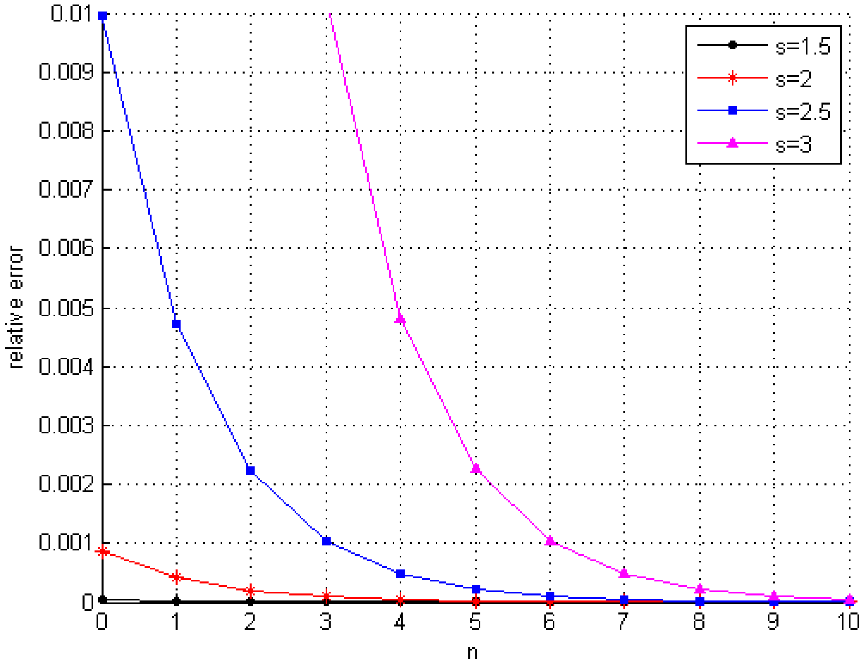
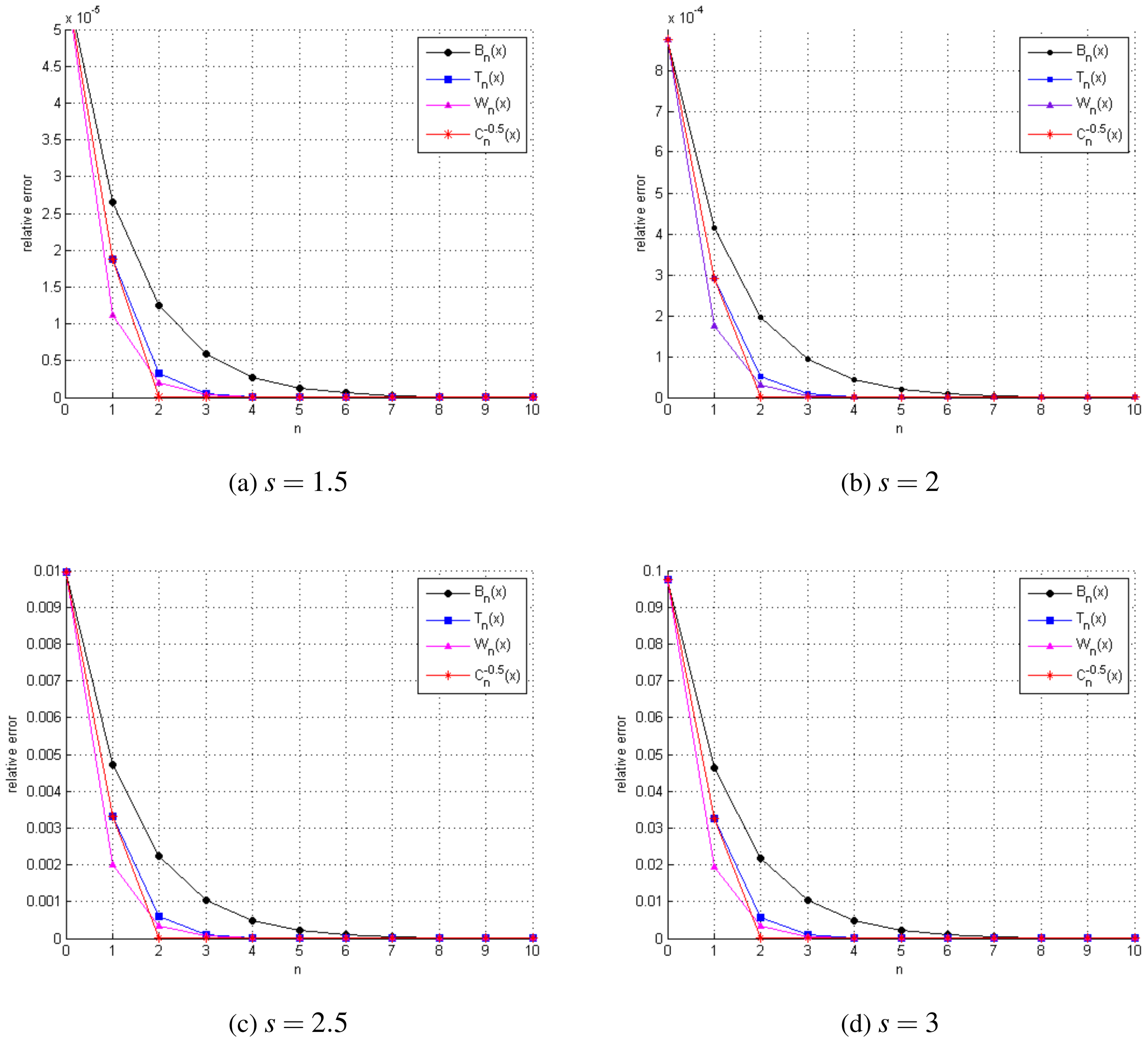


Figure 4.76: Comparison between $\zeta(s)$ approximations obtained by $B_n(x)$.



(a) $s = 1.5$

(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.77: Comparisons between $\zeta(s)$ approximations obtained by $B_n(x), T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$.

For $n = 10$, Table 4.9 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using Bernoulli polynomials.

s	1.5	2	2.5	3
ε	2.76×10^{-8}	4.31×10^{-7}	4.89×10^{-6}	4.80×10^{-5}

Table 4.9: The values of relative error in approximating $\zeta(s)$ using $B_n(x)$ with $n = 10$.

4.3.3 Approximation by Bernstein Basis Polynomials

Bernstein Basis polynomials $\Omega_n^{\alpha}(x)$ are presented by formula (2.94). These polynomials are defined with parameter α whose values range between zero and n. The relative error associated with approximating $\zeta(s)$ using $\Omega_n^0(x)$ polynomials is depicted in Fig. 4.78.

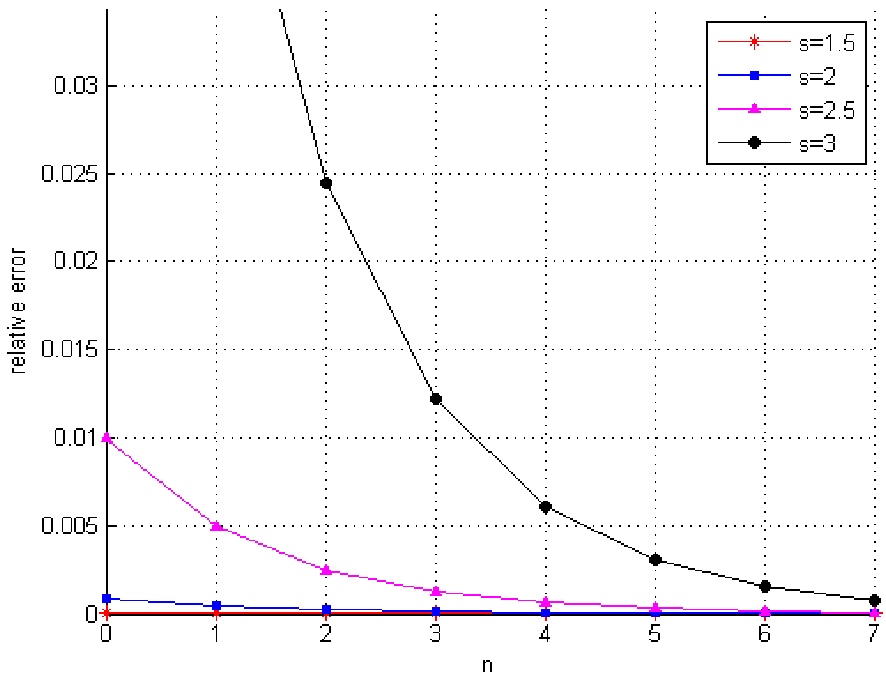
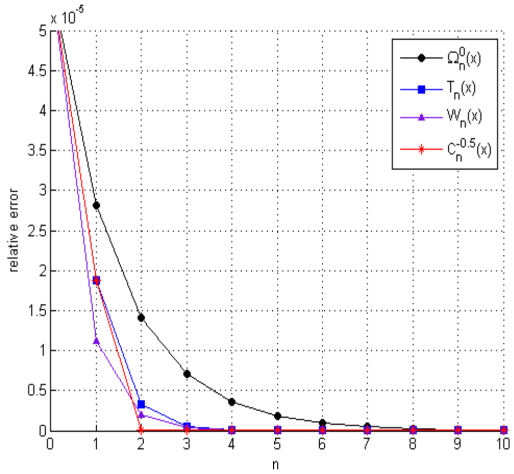


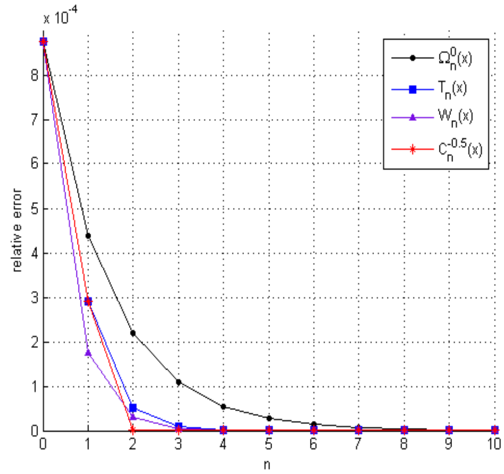
Figure 4.78: Comparison between $\zeta(s)$ approximations obtained by $\Omega_n^0(x)$ for different values of s.

$\Omega_n^0(x)$ polynomials do not give good approximations for $\zeta(s)$ as it is in the case of Chebyshev and Gegenbauer polynomials. Fig. 4.79 shows the relative error when using the polynomials $\Omega_n^0(x), T_n(x), W_n(x)$, $C_n^{-0.5}(x)$ to approximate $\zeta(s)$.
Bernstein polynomials surpass themselves in approximating $\zeta(s)$ when changing the values

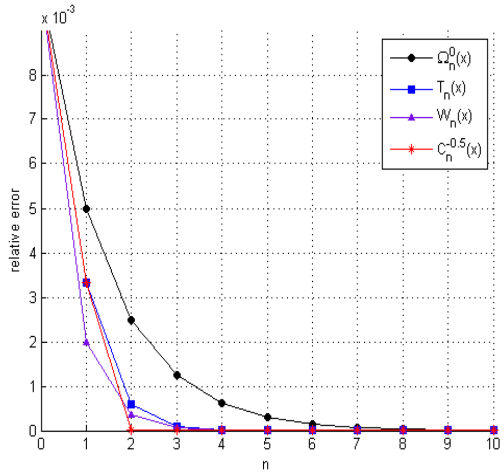
of α. Larger allowed value of α gives better approximations as well as larger values of n. Fig. 4.80, Fig. 4.81, Fig. 4.82, and Fig. 4.83 present the relative error yield when using $\Omega_n^\alpha(x)$ polynomials. It is remarkable to observe that the relative error decreases when at least one of n and α increases.



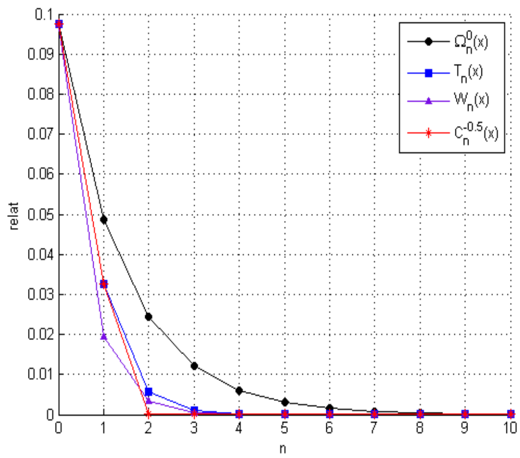
(a) $s = 1.5$



(b) $s = 2$



(c) $s = 2.5$



(d) $s = 3$

Figure 4.79: Comparisons between $\zeta(s)$ approximations obtained by $\Omega_n^0(x), T_n(x), W_n(x)$ and $C_n^{-0.5}(x)$.

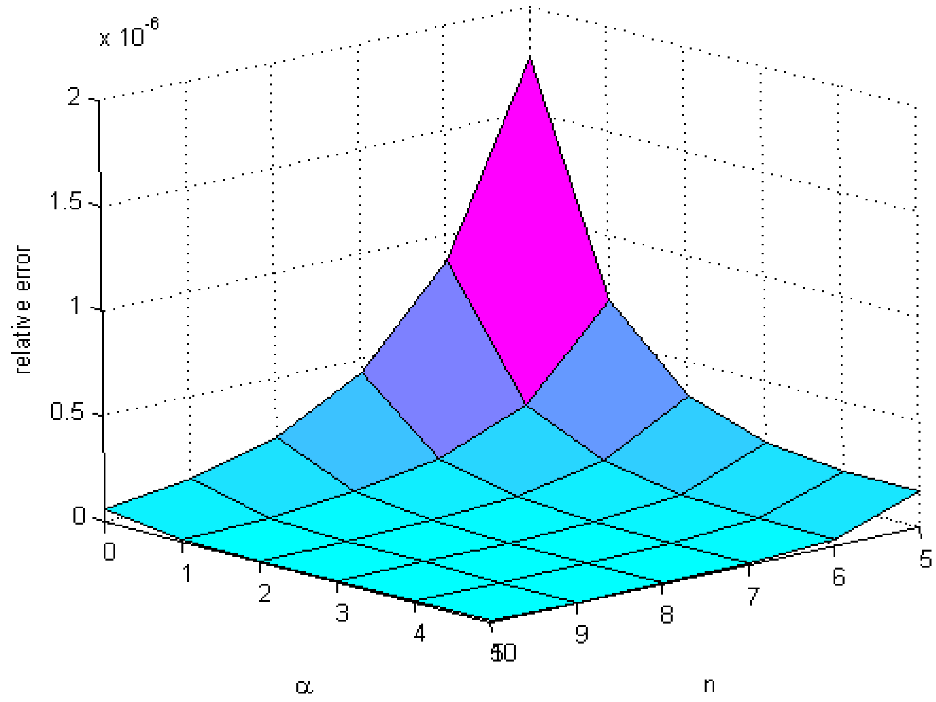


Figure 4.80: The behavior of the relative error when changing n and α in approximating $\zeta(1.5)$ by $\Omega_n^\alpha(x)$.

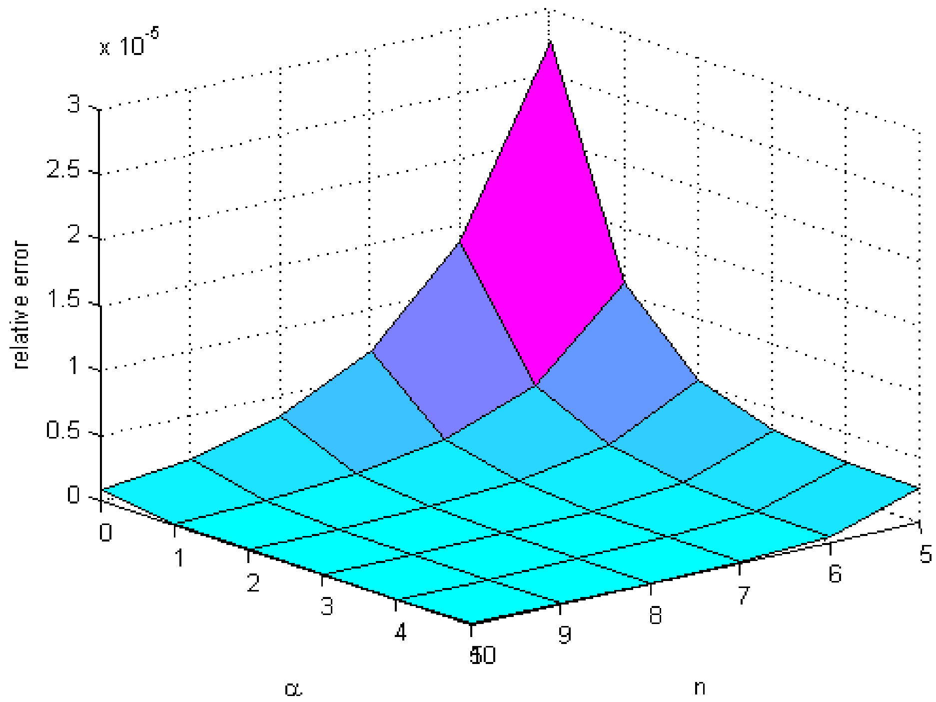


Figure 4.81: The behavior of the relative error when changing n and α in approximating $\zeta(2)$ by $\Omega_n^\alpha(x)$.

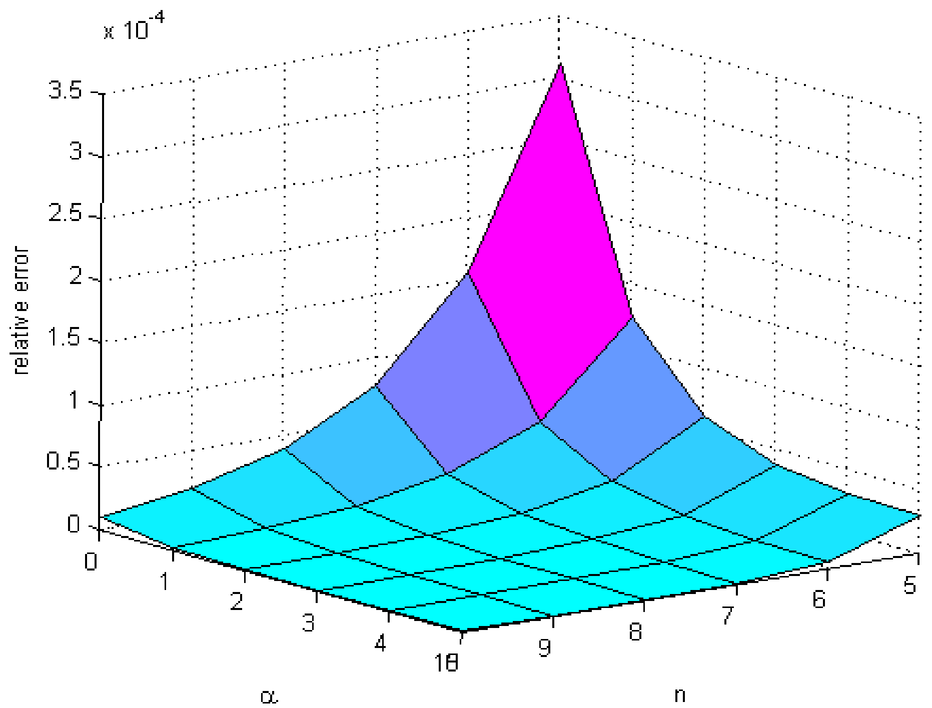


Figure 4.82: The behavior of the relative error when changing n and α in approximating $\zeta(2.5)$ by $\Omega_n^{\alpha}(x)$.

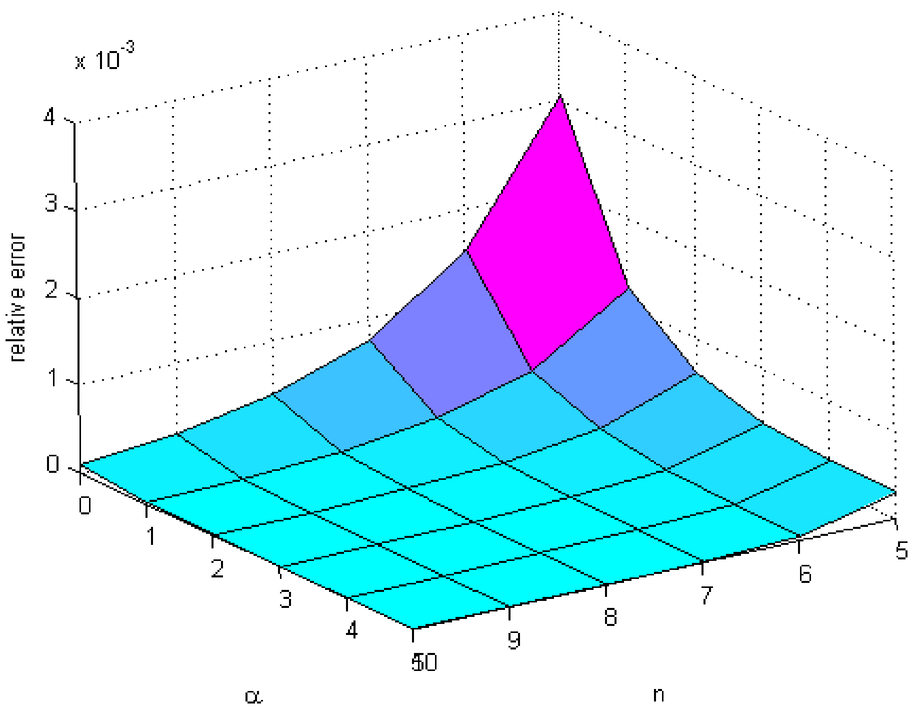


Figure 4.83: The behavior of the relative error when changing n and α in approximating $\zeta(3)$ by $\Omega_n^{\alpha}(x)$.

For $n = 7$, Table 4.10 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using $\Omega_n^0(x)$.

s	1.5	2	2.5	3
ε	4.39×10^{-7}	6.85×10^{-6}	7.77×10^{-5}	7.63×10^{-4}

Table 4.10: The values of relative error in approximating $\zeta(s)$ using $\Omega_n^0(x)$ with $n = 7$.

4.3.4 Approximation by Newton Polynomials

Newton polynomials $N_n(x)$, which are given explicitly by equation (2.92), give better approximations for $\zeta(s)$ with smaller values of s when they are embedded in Borwein algorithm. This is described in Fig. 4.84.

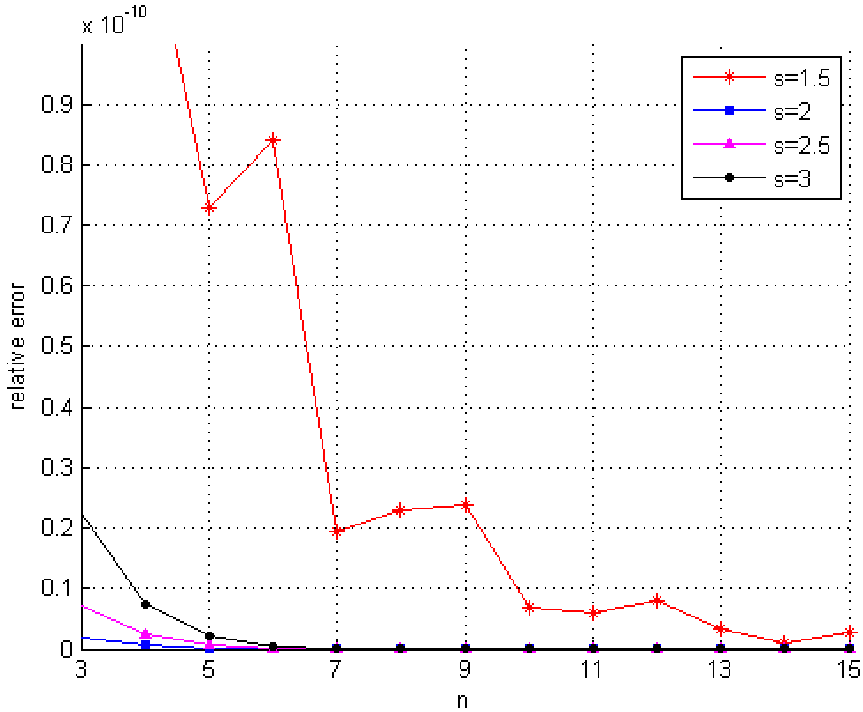
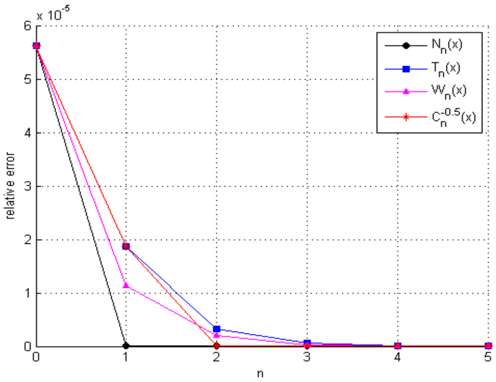
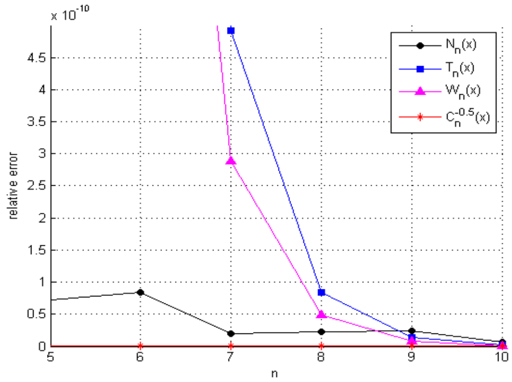


Figure 4.84: Comparison between $\zeta(s)$ approximations obtained by $N_n(x)$ for different values of s.

Newton polynomials are worse than $C_n^{-0.5}(x)$ for $n \geq 2$, but at the same time they are better than all Chebyshev polynomials when $s = 2, 2.5, 3$. When $s = 1.5$, $N_n(x)$ are better than T-Chebyshev polynomials only when $n < 9$. We clarify this in Fig. 4.85, Fig. 4.86, Fig. 4.87, and Fig. 4.88 which compare the relative error yields in the case of using $N_n(x), T_n(x), W_n(x)$, and $C_n^{-0.5}(x)$ polynomials.

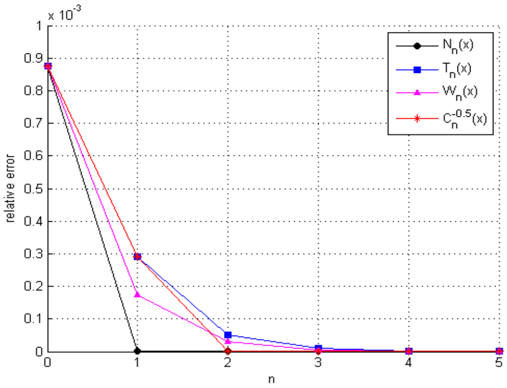


(a) $n = 0,1,2,3,4,5$

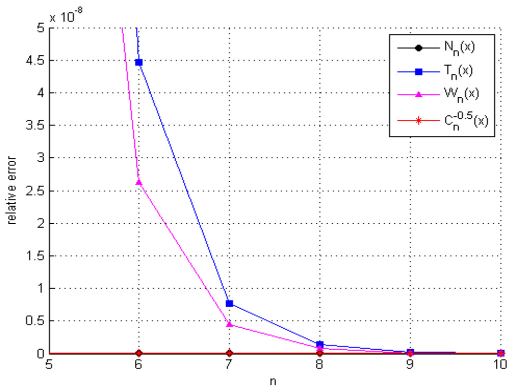


(b) $n = 5,6,7,8,9,10$

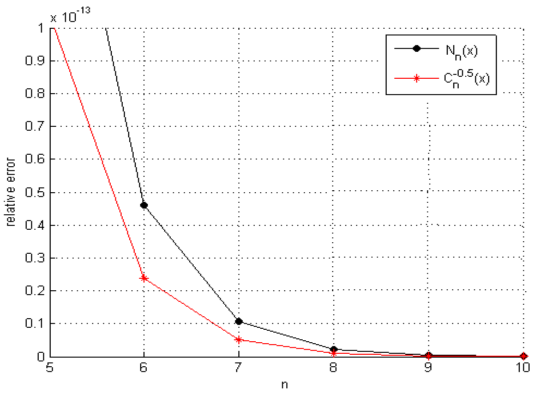
Figure 4.85: Comparisons between $\zeta(1.5)$ approximations obtained by $N_n(x), T_n(x), W_n(x)$, $C_n^{-0.5}(x)$.



(a) $n = 0,1,2,3,4,5$

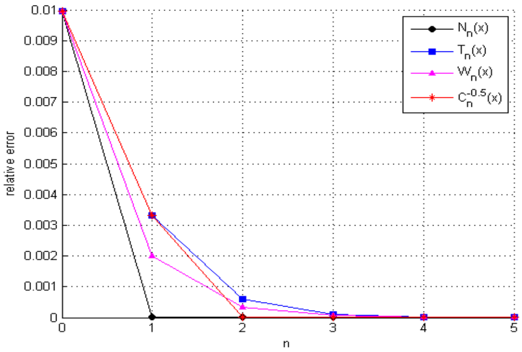


(b) $n = 5,6,7,8,9,10$

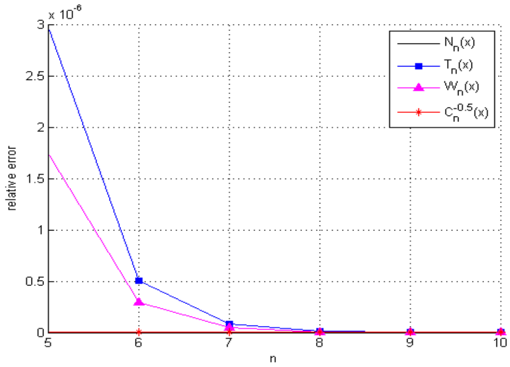


(c) $n = 5,6,7,8,9,10$

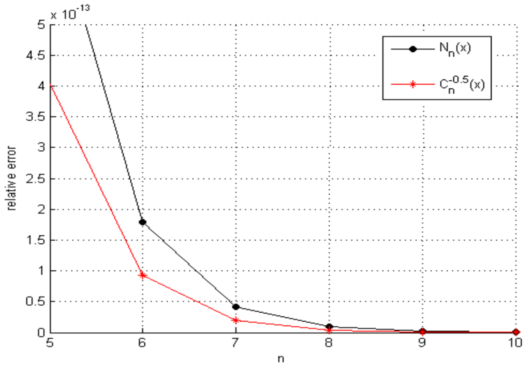
Figure 4.86: Comparisons between $\zeta(2)$ approximations obtained by $N_n(x), T_n(x), W_n(x)$, $C_n^{-0.5}(x)$.



(a) $n = 0, 1, 2, 3, 4, 5$

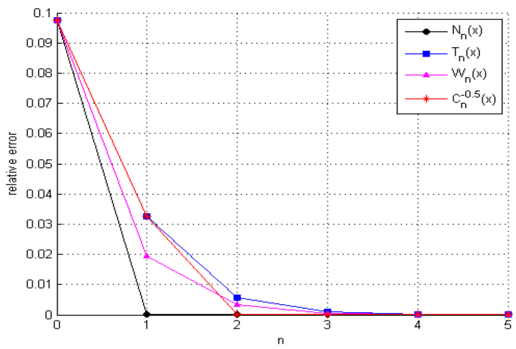


(b) $n = 5, 6, 7, 8, 9, 10$

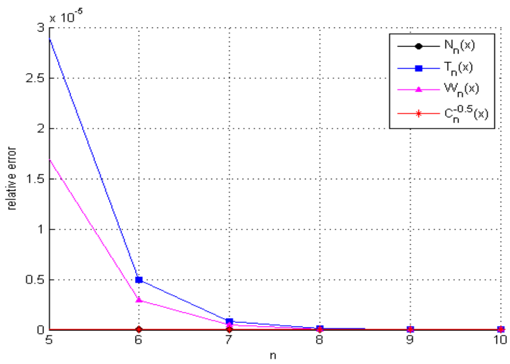


(c) $n = 5, 6, 7, 8, 9, 10$

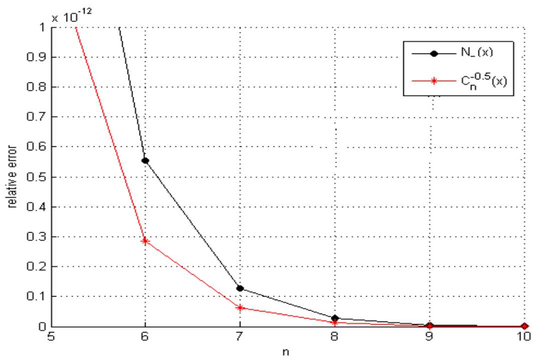
Figure 4.87: Comparisons between $\zeta(2.5)$ approximations obtained by $N_n(x), T_n(x), W_n(x), C_n^{-0.5}(x)$.



(a) $n = 0, 1, 2, 3, 4, 5$



(b) $n = 5, 6, 7, 8, 9, 10$



(c) $n = 5, 6, 7, 8, 9, 10$

Figure 4.88: Comparisons between $\zeta(3)$ approximations obtained by $N_n(x), T_n(x), W_n(x), C_n^{-0.5}(x)$.

For $n = 10$, Table 4.11 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using $N_n(x)$.

s	1.5	2	2.5	3
ε	6.74×10^{-12}	8.10×10^{-17}	3.21×10^{-16}	9.71×10^{-16}

Table 4.11: The values of relative error in approximating $\zeta(s)$ using $N_n(x)$ with $n = 10$.

4.3.5 Approximation by Pell-Simulated Polynomials

Pell polynomials $P_n(x)$ which are involved in section (2.3), are modified to obtain other polynomials $\tilde{P}_n(x)$ as it is shown in Table 4.12. We call the new polynomials; Simulated-Pell polynomials.

n	$P_n(x)$	$\tilde{P}_n(x)$
0	0	0
1	1	$x+1$
2	$2x$	$2x^2+2x+2$
3	$4x^2+1$	$4x^3+4x^2+4x+1$
4	$8x^3+4x$	$8x^4+8x^3+8x^2+4x+4$
5	$16x^4+12x^2+1$	$16x^5+16x^4+16x^3+12x^2+12x+1$

Table 4.12: $P_n(x)$ and $\tilde{P}_n(x)$ polynomials.

Note that $\tilde{P}_1(x) = x+1$, and thus $\tilde{p}_1(-1) = 0$, which means that we can not use this polynomial for approximating $\zeta(s)$ for any argument s at all. This because the division by zero is not allowed in Borwein algorithm 3.4.1.

If $\tilde{P}_n(x)$ polynomials are included inwards Borwein algorithm, then the resulting relative error decreases as argument s decreases. This is clarified in Fig. 4.89. Those polynomials have a property that the relative error associated with their $\zeta(s)$ approximations fluctuates as n increases, but its value is decreased in the long-range of n. Fig. 4.90 shows this behavior for different values of s and also shows the behavior of the ones correspond to $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ polynomials, so it makes comparisons between these polynomials in approximating $\zeta(s)$.

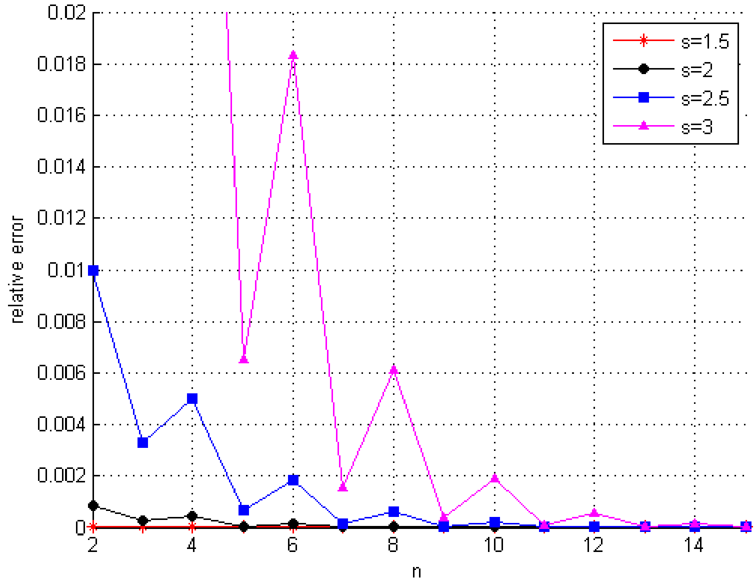
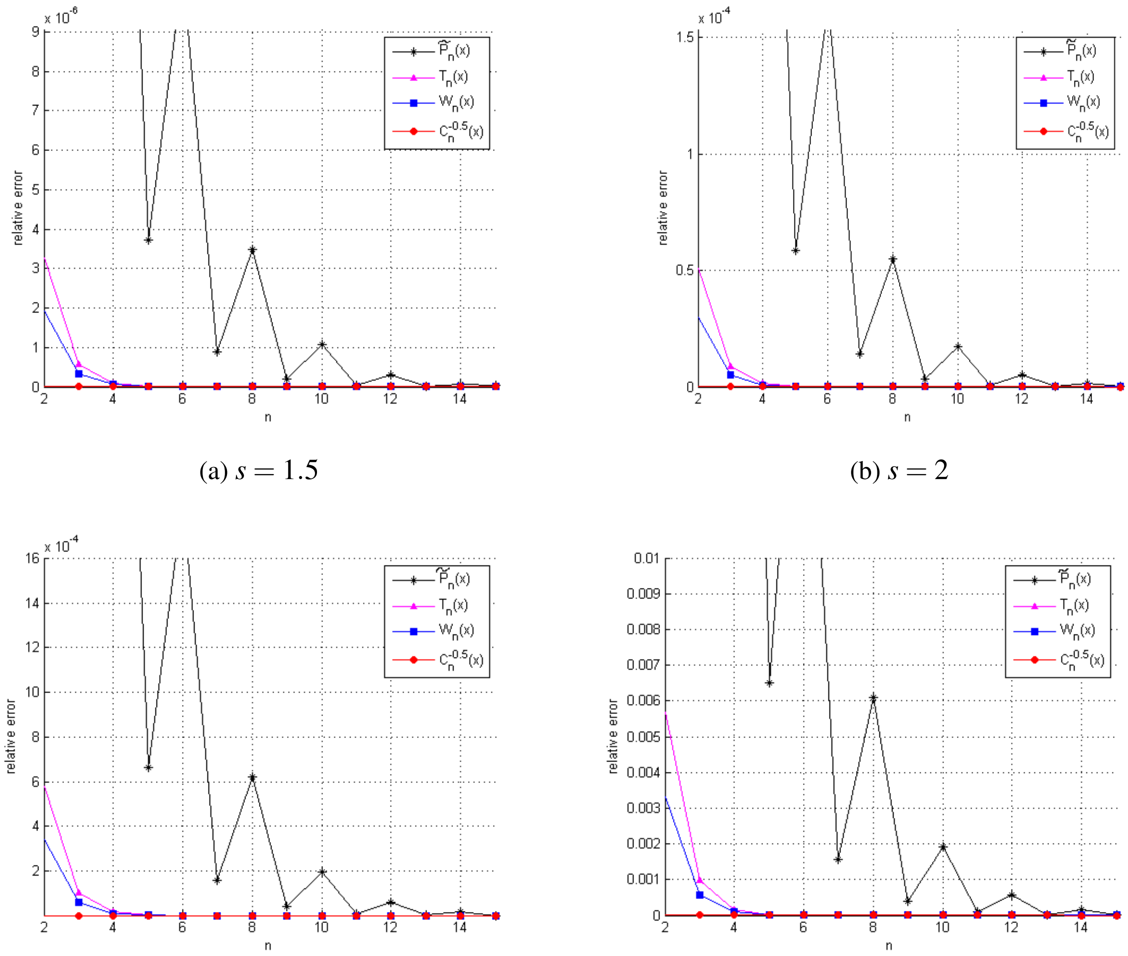


Figure 4.89: Comparison between $\zeta(s)$ approximations obtained by $\tilde{P}_n(x)$ for different values of s.



(a) $s = 1.5$

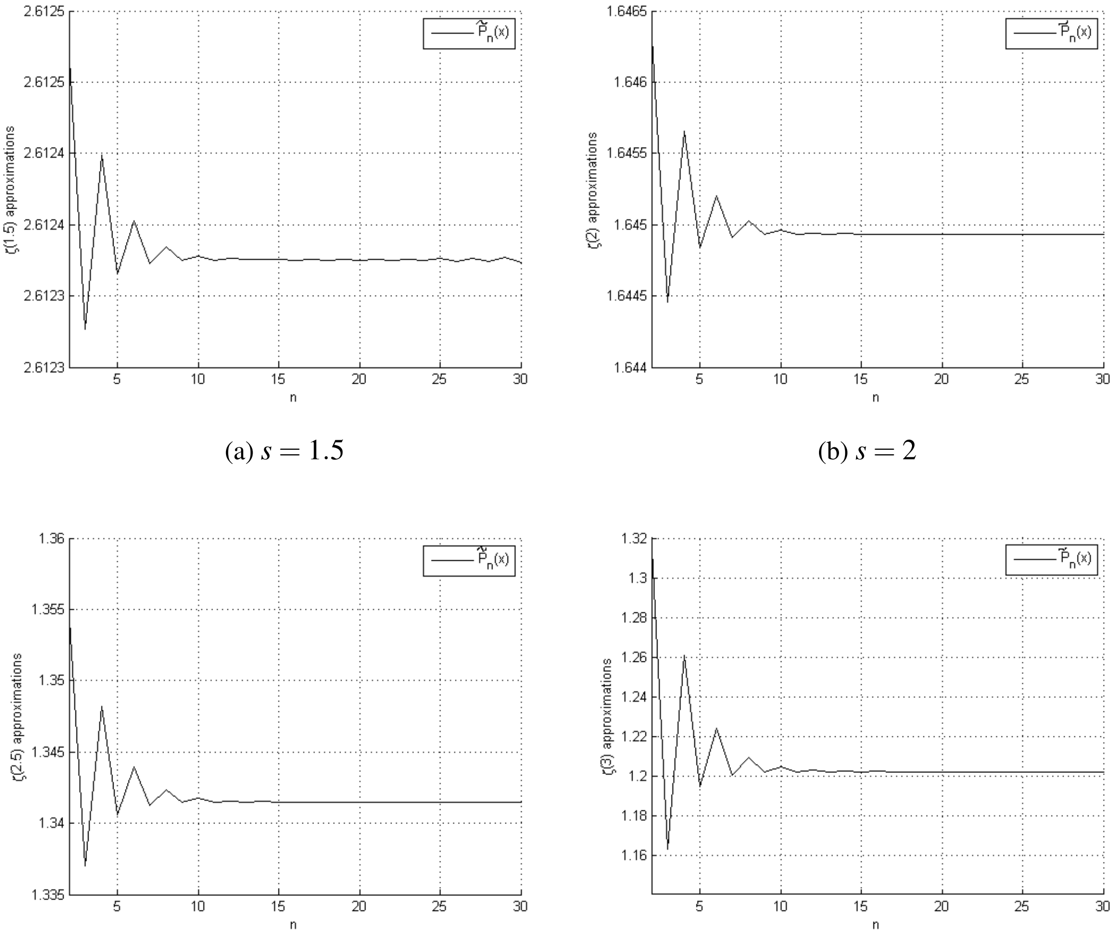
(b) $s = 2$

(c) $s = 2.5$

(d) $s = 3$

Figure 4.90: Comparisons between $\zeta(s)$ approximations obtained by $\tilde{P}_n(x)$, $T_n(x)$, $W_n(x)$, and $C_n^{-0.5}(x)$ for different values of s.

The values of $\zeta(s)$ approximations oscillate as the degree of $\tilde{P}_n(x)$ polynomials n increases. This is illustrated in Fig. 4.91 for $\zeta(1.5)$, $\zeta(2)$, $\zeta(2.5)$, and $\zeta(3)$ approximations.



(a) $s = 1.5$ (b) $s = 2$

(c) $s = 2.5$ (d) $s = 3$

Figure 4.91: $\zeta(s)$ approximations obtained by $\tilde{P}_n(x)$ polynomials.

For $n = 10$, Table 4.13 shows the values of relative error resulting from approximating $\zeta(s)$ for different values of s using $\tilde{P}_n(x)$.

s	1.5	2	2.5	3
ε	1.08×10^{-6}	1.71×10^{-5}	1.94×10^{-4}	1.91×10^{-3}

Table 4.13: The values of relative error in approximating $\zeta(s)$ using $\tilde{P}_n(x)$ with $n = 10$.

In Table 4.14 and Table 4.15, we give some approximations of $\zeta(1.5)$ as well as the corresponding relative error ε when using $P_n^*(x)$ and $C_n^{-0.5}(x)$ with Borwein algorithm respectively.

n	$\zeta(1.5)$ approximations	ε
2	2.6123753486875247906389964955578	7.795×10^{-13}
4	2.6123753486854883491410602028899	2.217×10^{-18}
6	2.6123753486854883433491070433376	2.065×10^{-22}
8	2.6123753486854883433485675676 56	1.02×10^{-28}

Table 4.14: $\zeta(1.5)$ approximations and the corresponding relative error using $P_n^*(x)$ polynomials.

n	$\zeta(1.5)$ approximations	ε
1	2.6124242586121044127743894488 33	1.87×10^{-5}
3	2.6123753486861700248195302204 81	2.609×10^{-13}
5	2.6123753486855337906024187595 98	1.740×10^{-14}
7	2.6123753486854906083047570743 80	8.67×10^{-16}

Table 4.15: $\zeta(1.5)$ approximations and the corresponding relative error using $C_n^{-0.5}(x)$ polynomials.

A comparison between the methods used in our work is given by Table 4.16 in the case of approximating $\zeta(2)$.

Method	parameters' values	relative error	CPU time
Direct	$N = 55266$	$\varepsilon = 1.00 \times 10^{-5}$	0.265 sec
EMS	$M = 3, N = 3$	$\varepsilon = 8.35 \times 10^{-7}$	0.250 sec
Borwein Algorithm with $C_n^{-0.5}(x)$	$n = 2$	$\varepsilon = 4.68 \times 10^{-12}$	0.109 sec
Borwein Algorithm with $P_n^*(x)$	$n = 2$	$\varepsilon = 4.68 \times 10^{-12}$	0.062 sec

Table 4.16: Approximating $\zeta(2)$.

CONCLUSIONS

Fast and accurate approximation methods for calculating Riemann Zeta function $\zeta(s)$ are presented and numerically verified. These methods include; direct approach by which an approximation is obtained by cutting off Euler series by finite number of terms. This method is reasonable for sufficient large s, but it is too bad for small s. In particular, results show that very large number of terms are needed to approximate $\zeta(1.5)$ with relative error $\varepsilon = 1.5 \times 10^{-3}$.

In order to enhance the approximations obtained by direct approach, some rapid series were developed to accelerate the convergence in the case of some few integer values of s ($s = 2,3,4,5,7$). Actually, these series are good alternative for Euler series when s is small. Our work shows that until $s = 5$, these fast series are better (when $s = 5$, the differences between the two methods are negligible). But, when $s \geq 7$, direct approach is better.

Euler Maclaurin Summation is an effective method for calculating $\zeta(s)$, but its problems condensed in including Bernoulli numbers which need special tools to be calculated.

Furthermore, Borwein algorithms are presented, which are considered as the best methods for calculating $\zeta(s)$. These algorithms are preferable with small values of s. Refinements for these algorithms are fulfilled by inserting polynomials of the form $p(x) = x^n(1-x)^n$, by which an approximation for $\zeta(s)$ is obtained with the least relative error through out our work. In fact, $p(x) = x^9(1-x)^9$ are associated with relative error around $\varepsilon = 1 \times 10^{-50}$.

The next better approximation of $\zeta(s)$ is obtained by Gegenbauer polynomials $C_n^{\gamma}(x)$. In fact, these polynomials take their minimal relative error in approximating $\zeta(s)$ when $\gamma = -\frac{1}{2}$. $C_{56}^{-\frac{1}{2}}(x)$ are associated with relative error around $\varepsilon = 1 \times 10^{-50}$. Also, as an extension of Gegenbauer polynomials, Humbert polynomials $h_n^{r,\gamma}(x)$ are implicated in Borwein algorithm giving approximation with the least relative error when $(r = 2, \gamma = -\frac{1}{2})$.

Chebyshev polynomials, which include the four types $T_n(x), U_n(x), V_n(x)$, and $W_n(x)$ are included in Borwein algorithm. In fact, $T_n(x)$ polynomials are suggested by P. Borwein and used to construct independent algorithm, but $W_n(x)$ are the best among Chebyshev poly-

nomials. As discrete type, Charlier polynomials $Ch_n^{\alpha}(x)$ are used with Borwein algorithm giving approximations better than those obtained by $W_n(x)$ in the case of sufficient small values of α.
Other polynomials, include Legendre, associated Laguerre, Bessel, Kinny, Horadam, Bernoulli, Newton, and Bernstein Basis polynomials are used with Borwein algorithm giving efficient approximations for $\zeta(s)$.

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C++ PROGRAMS

The following codes are used to approximate $\zeta(s)$ with Borwein Algorithm by the means of Gegenbauer polynomials.

```
#include<iostream>
#include<vector>
#include <boost/math/special_functions/log1p.hpp>
#include <boost/math/special_functions/gamma.hpp>
#include <boost/math/special_functions/pow.hpp>
#include<iomanip>
#include<fstream>
#include<limits>
using namespace std;
long double evaluation(long double x,vector<long double>& polyn, int n );
long double coefficients(int j, vector<long double>& polyn,long double minus);
long double substitution(long double x, vector<long double>& polyn,int n,
long double s);
long double integration(vector<long double>& f, int N);
int factorial( int n);
long double polynomial(int m,int n, long double alpha);
int main(){
cout<<fixed;
int k,N=100000;
long double s;
cout<<"enter s"<<endl;
cin>>s;
long double alpha;
```

```
cout<<"enter alpha"<<endl;
cin>>alpha;
int n;
cout<<"enter Gegenbauer polynomial degree"<<endl;
cin>>n;
long double cc=0;
long double ff=0;
long double coeff=0;
vector<long double> polyn(0);
vector<long double> f(0);
vector<long double> c(0);
for(int i=0;i<=n;i++){
cc=polynomial(i,n,alpha);
polyn.push_back(cc);}
cout<<"polynomial:\n";
for(int ii=0;ii<=n;ii++)
cout<<polyn[ii]<<endl;
long double minus =evaluation(-1,polyn,n);
ff=substitution(0.000000000000001, polyn,n,s);
f.push_back(ff);
for(k=1;k<=N-1;k++){
long double k2(k);
ff=substitution(k2/N,polyn,n,s);
f.push_back(ff);}
ff=substitution(0.9999999999999999, polyn,n,s);
f.push_back(ff);
long double I=integration(f,N);
long double tws=minus*(1-pow(2,1-s));
long double eta =I/(tws*boost::math::tgamma(s));
long double summ=0;
for (int j=0;j<=n-1;j++){
coeff =coefficients(j,polyn,minus);
```

```
c.push_back(coeff);
summ=summ+(c[j]/(pow(1+j,s)));}
long double zeta=(summ/(-1*tws))+ eta ;
cout<<"\n zeta( "<<s<<")="<<zeta<<endl;
return 0;}
long double evaluation(long double x,vector<long double>& polyn,int n ){
long double val=0;
for(int i=0;i<=n ;i++){
val+= polyn[i]*pow(x,i);}
return val;}
long double coefficients(int j, vector<long double>& polyn,long double minus){
long double sum=0;
for(int k=0;k<=j;k++){
sum+=pow(-1,k)*polyn[k];}
long double coeff =pow(-1,j)*(sum - minus);
return coeff;}
long double substitution(long double x, vector<long double>& polyn,int n,
long double s){
long double result;
long double sum=0;
for (int k=0;k<=n;k++){
sum+=polyn[k]*pow(x,k);}
result = (sum *pow(-1*boost::math::log1p(x-1),s-1))/(1+x);
return result;}
long double integration(vector<long double>& f, int N){
int k;
for(k=1;k<N;k++){
if(k%2==0)
f[k]*=2;
else
f[k]*=4;}
long double sum=0;
```

```
for(k=1;k<N;k++){
sum+=f[k];}
sum+=f[0]+f[N];
sum=(sum/(3*N));
return sum ;}
int factorial( int n){
int fact=1, j;
for(j=1;j<=n;j++){
fact*=j;}
return fact;}
long double polynomial(int m,int n, long double alpha){
long double result=0;
int limit;
if(n==0)      result=1;
else{
if (n%2==0) limit=n/2;
else        limit=(n-1)/2;
for (int k=0;k<=limit;k++){
if(m<=n-2*k){
long double nn1=( factorial(k));
long double nn2=( factorial(m) );
long double nn3=( factorial(n-2*k-m) );
result+=pow(-1,n+k-m)*pow(2,n-2*k+m)*boost::math::tgamma(n-k+alpha)/nn1/nn2/nn3
/boost::math::tgamma(alpha);}
else result+=0;}}
return result;}
```

Now, we show the codes of embedding polynomials of form $x^n(1-x)^n$ into Borwein Algorithm:

```
#include<iostream>
#include<vector>
#include <boost/math/special_functions/log1p.hpp>
#include <boost/math/special_functions/gamma.hpp>
#include <boost/math/special_functions/pow.hpp>
```

```
#include<iomanip>
#include<fstream>
#include<limits>
#include<ctime>
using namespace std;
long double evaluation(long double x,vector<long double>& polyn, int n );
long double coefficients(int j, vector<long double>& polyn,long double minus);
long double substitution(long double x, vector<long double>& polyn,int n,
long double s);
long double integration(vector<long double>& f, int N);
int factorial( int n);
long double polynomial(int k,int n);
int main()
{
cout<<fixed;
int k,N=100000;
long double s;
cout<<"enter s"<<endl;
cin>>s;
int n;
cout<<"enter an even polynomial degree"<<endl;
cin>>n;
long double cc=0;
long double ff=0;
long double coeff=0;
vector<long double> polyn(0);
vector<long double> f(0);
vector<long double> c(0);
for(int i=0;i<=n;i++){
cc=polynomial(i,n);
polyn.push_back(cc);}
cout<<"polynomial:\n";
```

```
for(int ii=0;ii<=n;ii++)
cout<<polyn[ii]<<endl;
long double minus =evaluation(-1,polyn,n);
ff=substitution(0.0000000000000001, polyn,n,s);
f.push_back(ff);
for(k=1;k<=N-1;k++){
long double k2(k);
ff=substitution(k2/N,polyn,n,s);
f.push_back(ff);}
ff=substitution(0.9999999999999999, polyn,n,s);
f.push_back(ff);
long double I=integration(f,N);
long double tws=minus*(1-pow(2,1-s));
long double eta =I/(tws*boost::math::tgamma(s));
long double summ=0;
for (int j=0;j<=n-1;j++){
coeff =coefficients(j,polyn,minus);
c.push_back(coeff);
summ=summ+(c[j]/(pow(1+j,s)));}
long double zeta=(summ/(-1*tws))+ eta ;
cout<<"\n zeta("<<s<<")= "<<zeta<<endl;
return 0;}
long double evaluation(long double x,vector<long double>& polyn,int n ){
long double val=0;
for(int i=0;i<=n ;i++){
val+= polyn[i]*pow(x,i);}
return val;}
long double coefficients(int j, vector<long double>& polyn,long double minus){
long double sum=0;
for(int k=0;k<=j;k++){
sum+=pow(-1,k)*polyn[k];}
long double coeff =pow(-1,j)*(sum - minus);
```

```
return coeff;}
long double substitution(long double x, vector<long double>& polyn,int n,
long double s){
long double result;
long double sum=0;
for (int k=0;k<=n;k++){
sum+=polyn[k]*pow(x,k);}
result = (sum *pow(-1*boost::math::log1p(x-1),s-1))/(1+x);
return result;}
long double integration(vector<long double>& f, int N){
int k;
for(k=1;k<N;k++){
if(k%2==0)
f[k]*=2;
else
f[k]*=4;}
long double sum=0;
for(k=1;k<N;k++){
sum+=f[k];}
sum+=f[0]+f[N];
sum=(sum/(3*N));
return sum ;}
int factorial( int n){
int fact=1, j;
for(j=1;j<=n;j++){
fact*=j;}
return fact;}
long double polynomial(int k,int n){
long double result;
if(n==0)
result=1;
else{
```

```
int m=n/2;
int y=(k-m);
if (k<=m-1)     result=0;
else{
long double dd=factorial(m) ;
long double nn1=( factorial(y) );
long double nn2=( factorial(m-y) );
result=pow(-1,y)*dd/nn1/nn2;}}
return result;}
```

الملخص

طرق سريعة و دقيقة لحساب اقتران ريمان زيتا

يُعتبر اقتران ريمان زيتا من أهم الأدوات في العديد من فروع الرياضيات و الفيزياء. نظراً لندرة الصيغ الرياضية الصريحة لحساب هذا الإقتران، فإن هناك الكثير من الطرق الدقيقة و السريعة قد تم تطويرها و إثبات جدارتها لتقريب اقتران ريمان زيتا. النتائج تُظهر أن تقليل قيمة المتغير الذي يحسب الإقتران عنده يزيد من الخطأ لعدد ثابت من الحدود. لتجنب عائق بطء التقارب للقيمة الصحيحة في حالة القيم الصغيرة للمتغير s ، تم الاستعانة ببعض الخوارزميات السريعة و مقارنتها بالطريقة المباشرة. خوارزميات بوروين استُخدمت بشكل مكثَّف و موسع في هذه الاطروحة نظراً لدقتها العالية. أنواع متعددة من كثيرات الحدود دُمجت مع خورازميات بوروين. النتائج تبين أن تلك التي تتخذ الشكل $x^n(1-x)^n$ تُعطي أفضل التقريبات.