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Square and Non-Square Fully Fuzzy Linear Systems with Trapezoidal and Hexagonal Fuzzy Numbers

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Dedication

Thanks and praise to Allah who gave me life
To the messenger of knowledge and holy religion –prophet Mohammed

To my dear lovely country,Palestine .

To whom holds me, cares about my tiny inner feelings, let me satisfied at

any time, my mother .

To whom struggles, leaves his rest time to afford every need for me and my sisters and brothers, my father.

To my dearest person who leads me through the valley of darkness with the light of hope and support my fiance. I'm truly thankful for having you in my life (Mahmoud Abuhamad).

To my dear sisters and brothers who help me always remember to hold my hand towards the right way to future.

To whom taught me the best way to increase my knowledge and get me the highest point of education and light my dear teachers and lectures.

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Abstract

Square and Non-Square Fully Fuzzy Linear Systems with Trapezoidal and Hexagonal Fuzzy Numbers

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In this thesis, the definition of trapezoidal fuzzy number using right and left spread is introduced. Then, in a similar way, we defined hexagonal fuzzy number. Also, we gave a compact formulas for the operations on such fuzzy numbers using Alpha cut arithmetic. Square and non-square fully fuzzy linear systems of trapezoidal and hexagonal types using the new definitions are extensively studied. More precisely, we were able to formulate certain conditions in order to have a fuzzy solution at first and then a positive solution. Further, plenty of illustrative examples have been discussed.

Contents

Chapter 1

Introduction

Systems of linear algebraic equations play an important role in many branches of science and engineering [30]. In many applications involving linear system, part of (may be the whole) the crisp system can be replaced by fuzzy numbers. Therefore, many authors have deeply investigated such types of systems in order to develop mathematical tools and numerical procedures to obtain their solutions.

M. Paripoura et al presented a numerical algorithm for solving fuzzy systems of linear equations based on homotopy perturbation method [26]. Chandrasekaran solved fuzzy linear system by singular value decomposition method [5]. V. Vijaylakshmi et al introduced ST decomposition procedure to solve fully fuzzy linear systems [32]. Nayak et al presented a new representation of interval arithmetic; They used it to develop algorithms to solve fuzzy linear system with both triangular and trapezoidal type of fully fuzzy numbers [24]. Radhakrishnan et al solved the fully fuzzy linear systems consisting of positive fuzzy numbers using QR decomposition method [30]. Abdolmaleki proposed a semi-iterative method to find a solution of the fully fuzzy linear systems [2]. Ezzati et al used the arithmetic operations on fuzzy numbers that introduced by Kaffman and found a positive fuzzy solution for the fully fuzzy linear system of equations [10].

Abbasbandy et al used the implicit Gauss– Cholesky algorithm of ABS class (algorithm based on the propositions of Abaffy, Broyden and Spedicato introduced in 1984 for solving determined linear systems) [1].

Friedman et al numerically solved the system $AX = \hat{b}$ where A is $n \times n$ crisp matrix and $\hat{b}$ is an arbitrary fuzzy vector. They solved Fuzzy linear systems by employing the embedding approach. The original system was replaced by $2n \times 2n$ system then the solution is obtained [11, 19].

Dehghan et al introduced a number of methods for solving Fully Fuzzy linear systems which are comparable to the well-known methods such as : Cramer rule, Gaussain elimination, LU decomposition method, Richard son ,Jacobi, Jacobi over relaxation (JOR), Gauss –seidel successive over relaxation ...etc. He also shared a new method for employing linear programming for solving square and non-square fuzzy systems [6, 8]. Nasseri and Zehmakkesh proposed Huang method for computing a nonnegative solution of the fully fuzzy linear system of equations [23]. Kumar et al gave a new approach for solving fully fuzzy linear systems based on the principles of linear programming in solving a fully fuzzy linear system with arbitrary coefficients. At the same time, a solution of fully fuzzy linear system with arbitrary coefficient was introduced by them [14,15]. Several methods used for solving fully fuzzy linear systems with trianglar fuzzy numbers have been introduced by many authors [37] [16] [4] [21].

Ahmad and Ibrahim proposed new matrix method for solving positive fully fuzzy linear system; a necessary and sufficient condition were derived to have positive solution of the left –Right fuzzy linear system [20]. A method for solving fully fuzzy linear system with trapezoidal fuzzy numbers was introduced by kumar and other in [17]. Karthik et al solved the fully fuzzy linear system by partitioning the coefficient matrix into sub-matrices with trapezoidal fuzzy number matrices [30] [12]. They proposed a method to solve fully fuzzy linear system with trapezoidal fuzzy number by converting n x n fully fuzzy linear system into 4n x 4n crisp linear system of equations and solve by classical methods. The solution of Non-square $m \times n$ fully fuzzy linear system where $m > n$ was introduced by Ezzati and Yousezed [10]. They used the least square method to approximate non-negative fuzzy solution.

In this work, we will discuss fully fuzzy linear system with trapezoidal and hexagonal fuzzy numbers. In chapter two, some preliminary topics from linear algebra are introduced, such as determinant of block matrix, the generalized inverse method, the Moore-Penrose inverse and the least square method. Chapter three presents the main concepts of fuzzy sets and numbers. A new definition of the trapezoidal and hexagonal fuzzy number is given, and the addition, multiplication and scalar multiplication using the prescribed definitions are also defined in chapter three. In chapter four and five we solve fully fuzzy linear systems with new trapezoidal and hexagonal fuzzy numbers. also, we solve the system when the coefficients matrix is square invertible, singular and non-square. Moreover, we give the necessary and sufficient condition in order to have a positive solution when the fully fuzzy linear system is square and non-square.

Chapter 2

System of Linear Equations

In elementary linear algebra, many techniques can be used to discuss the solvability of homogeneous and non-homogeneous systems of linear equations. In this chapter,a formula of the determinant of block matrices in general case will be proved via induction and then the solution of square system of linear equations with singular coefficients matrix will be discussed using two methods, the Generalized and the Moore-Penrose inverse; moreover the solution non-square system of linear equations will be discussed using least square method.

2.1 Determinants of 2×2 Block matrices

Theorem 2.1.1 *[28] Let A, B, C and D be four matrices of sizes $k \times k$, $k \times (n-k)$, $(n-k) \times k$ and $(n-k) \times (n-k)$ respectively, such that D is nonsingular. The determinant of the matrix* $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *is given by* $|S| = |A - BD^{-1}C|\,|D|$.

If D is singular and A is nonsingular, then $|S| = |D - CA^{-1}B|\,|A|$. the general case is discussed in [28]. When B or C is the zero matrix, we have the following theorem.

Corollary 2.1.1 *If S is an $(m+n) \times (m+n)$ matrix* $S = \begin{bmatrix} H_{mm} & O_{mn} \\ L_{nm} & K_{nn} \end{bmatrix}$ *where O is $m \times n$ zero matrix. Then* $|S| = |H|\,|K|$.

Proof: By induction on m.

step(1) :For $m = 1$, $H_{11} = \begin{bmatrix} h_{11} \end{bmatrix}$ and $S = \begin{bmatrix} h_{11} & O_{1n} \\ L_{n1} & K_{nn} \end{bmatrix}$. Then $|S| = h_{11} \det K = |H|\,|K|$.

step(2): Assume that for $m = p$, the statement $|S| = |H|\,|K|$ is true, that is $S = \begin{bmatrix} H_{pp} & O_{pn} \\ L_{np} & K_{nn} \end{bmatrix}$. and $|S| = |H_{pp}|\,|K|$ is true.

step(3): For m=p+1, $S = \begin{bmatrix} H_{p+1p+1} & O_{p+1n} \\ L_{np+1} & K_{nn} \end{bmatrix}$, then $|S| = \sum_{i=1}^{p+1} (-1)^{i+1} H_{1i} S_p^i$, where S_p^i is the $p \times p$ matrix obtained by removing column i and row 1 of S. Then, from the second step, we have $|S_i^p| = |H_i^p|\,|K|$, where H_i^p is $p \times p$ matrix obtained by removing the first row and i^{th} column of H_{P+1P+1}. Therefore,
$|S| = \sum_{i=1}^{p+1} -1^{i+1} h_{1i}\,|H_i^p|\,|K| = \sum_{i=1}^{p+1} -1^{i+1} h_{1i}\,|H_i^p|\,(|K|) = |H_{P+1P+1}|$. □

2.2 Generalized inverse method

The generalized inverse is one of the techniques that is used to solve linear systems. The author in [3] gave an algorithm to find the generalized inverse.

Definition 2.2.1 *[3] If A is an $m \times n$ matrix, and G is an $n \times m$ matrix then G is a generalized inverse of A if it satisfies the property $AGA = A$.*

When A is a square invertible matrix then $G = A^{-1}$.

Theorem 2.2.1 *[3] Let A be an $m \times n$ matrix and assume that G is a generalized inverse of A then for any fixed $b \in R^n$:*

(i) the system $Ax = b, x \in R^n$ has a solution if and only if $AGb = b$.

(ii) if $Ax = b$ has any solution then x is a solution if and only if $x = Gb + (I - GA)z$ for some $z \in R^n$.

The following steps describe the process of finding the Generalized inverse of a non-square matrix A.

1. Choose any non-singular sub-matrix H of size k.

2. Calculate $(H^{-1})^T$.

3. In A, replace the elements of sub-matrix H by the elements of $(H^{-1})^T$ and the rest entries by zeros to get new matrix $\tilde{A}$.

4. The generalized inverse $G = (\tilde{A})^T$.

   From the above algorithm, the generalized inverse is not unique since it depends on the choice of the sub matrix H.

Example 2.2.1 *Let* $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$, *take* $H = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, *then* $(H^{-1})^T = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$.

Therefore, $G = (\tilde{A})^T$. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

2.3 Moore-Penrose inverse

The Moore -Ponrose inverse is a generalization of the inverse of the matrix. The Moore -Ponrose is defined to any matrix of arbitrary size.

Definition 2.3.1 *[27] Let* $A \in R^{m\times n}$. *A matrix* $G = A^\dagger$ *satisfies the following four equations:* $AGA = A, GAG = G, (AG)^T = AG$ *and* $(GA)^T = GA$ *is called the Moore-Penrose inverse of the matrix* A.

Theorem 2.3.1 *[27] The Moore-Penrose inverse* $A^\dagger$ *exists and unique for any matrix* A.

properties of the moore-penrose:

1-$A^{\dagger} = (A^T A)^{\dagger} A^T = A^T (A^T A)^{\dagger}$

2-$(A^T)^{\dagger} = (A^{\dagger})^T$

3-$(A^{\dagger})^{\dagger} = A$

4-$A^T A^{\dagger} = A^{\dagger} (A^T)^{\dagger}$

.

Theorem 2.3.2 *[27] The moore-penrose inverse is a generalized inverse.*

Proof: let $G = A^{\dagger}$, from defintion(2.3.1) $AGA = A$, then $A^{\dagger}$ is generalized inverse □

Example 2.3.1 *the Moore-Ponrose inverse of the matrix* $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ *using the matlab we have :*

is $A^{\dagger} = \begin{bmatrix} 0.0308 & 0.0615 \\ 0.0462 & 0.0923 \end{bmatrix}$

Example 2.3.2 *solve the system :*

$5x_1 + 4x_2 = 16$

$10x_1 + 8x_2 = -1$

$A = \begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $b = \begin{bmatrix} 16 \\ -1 \end{bmatrix}$ *A is singular so we use the Moore-penrose inverse to solve the system :* $X = A^{\dagger} b = \begin{bmatrix} 0.0244 & 0.0488 \\ 0.0195 & 0.0390 \end{bmatrix} \begin{bmatrix} 16 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.3416 \\ 0.273 \end{bmatrix}$

2.4 Least Square Method

The least squares method is a technique for solving an over-determined system of linear equations, i.e when A is a rectangular matrix of size $m \times n$, $m \geq n$. In this section, we give some facts related to the required method.

Consider the system:

$Ax = b$, where A is $m \times n$ with $m \geq n, x$ is an $n \times 1$ vector and b is an $m \times 1$ vector.

Definition 2.4.1 *[25] A least square solution to a linear system of equations $Ax = b$, is a vector x that minimizes the Euclidean norm $\|Ax - b\|$.*

Remark 2.4.1 *[25] The vector $x = (A^TA)^{-1}A^Tb$ is called the least squares solution to $Ax = b$.*

Remark 2.4.2 *[25] If the system actualy has solution, then it is automatically the least square solution. The concept of the least square is considered only when the system does not have a solution, i.e b does not lie in the range of A.*

Theorem 2.4.1 *[25] The least square solution is uniqe if and only if $rank(A) = n$.*

Example 2.4.1 *Consider the following system*

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 - 3x_2 &= -8 \\ -x_1 - x_2 &= -3. \end{aligned}$$

If $A = \begin{bmatrix} 1 & 2 \\ 2 & -3 \\ -1 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and* $b = \begin{bmatrix} 3 \\ -8 \\ -3 \end{bmatrix}$ *then* $(A^TA)^{-1}A^T = \begin{bmatrix} 0.2667 & 0.2533 & -0.2267 \\ 0.2000 & -0.1600 & -0.1200 \end{bmatrix}$

,so, the least square solution is $X = (A^TA)^{-1}A^Tb = \begin{bmatrix} -1.6672 \\ 1.76 \end{bmatrix}$

Chapter 3

Fuzzy Set and Fuzzy Number

This chapter consist of two sections. In section 1, we discuss fuzzy sets and some important defintions, Operations on Fuzzy Sets and Fuzzy Relations, alpha cuts, fuzzy numbers and operation on fuzzy numbesrs. In section 2, we define a new fuzzy number and the operations on it.

3.1 Fuzzy Set

Fuzzy set theory was firstly introduced by Zadeh in 1965. Fuzzy set is considered to be a generalization of the concept of the set. [34, 35].

3.1.1 Definitions and Notations

Definition 3.1.1 *[33] A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one.*

Definition 3.1.2 *[18] (Membership function): Let X be a universal set and A subset* *of X. The membership function is $\mu_A : X \longrightarrow [0,1]$ such that*

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases} \tag{3.1}$$

Definition 3.1.3 *[18] Let X be a fuzzy set, the membership function of X is $\mu_A : X \to [0,1]$, where the value of $\mu_A(x)$ at x shows the grade of membership of x in A.*

Definition 3.1.4 *[18] If X is discrete, then the membership function can be written $A = \{(x, \mu_A(x)) : x \in X\}$ or $A = \sum\limits_i \frac{\mu_A(x_i)}{x_i}$. The symbol '$\sum$' does not mean addition, but it means the usual union between sets. On the other hand, if X is continuous, then the set A can be written $A = \int \frac{\mu_A(x)}{x}$.*

Example 3.1.1 *X is discrete and take the value $1, 2$ for example, $A = (1, 0.3), (2, 0.5)$ or it can be written $A = 0.3/1 + 0.5/2$*

Example 3.1.2 *A fuzzy set A which is a real number closed to 1 can be defined by it's membership function $A = \int \frac{\mu_A(x)}{x}$ where, $\mu_A(x) = \dfrac{1}{1 + (x-1)^2}$*

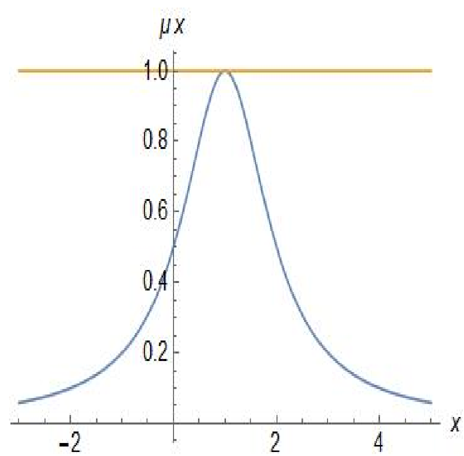

Fig. 3.1: Membership function of A

3.1.2 Operations on Fuzzy Sets

In this section the main operations on fuzzy sets together with some formulas concerning crisp sets are given.

Definition 3.1.5 *[18] A fuzzy set A is empty if and only if $\mu_A(x) = 0, \forall x \in X$.*

Definition 3.1.6 *[18] Two fuzzy sets A and B are equivalent,denoted by $A = B$ if and only if $\mu_A(x) = \mu_B(x), \forall x \in X$. And if $\mu_A(x) \neq \mu_B(x), \forall x \in X$ then $A \neq B$.*

Definition 3.1.7 *[18] $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x), \forall x \in X$.*

Definition 3.1.8 *[18] The complement set of A is denoted by $\bar{A}$ and has a membership function $\mu_{\bar{A}}(x) = 1- \ \mu_A(x), \forall x \in X$.*

Definition 3.1.9 *[18] The union of two fuzzy sets A and B is $A \cup B$ which is a fuzzy set whose membership function is defined by $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}.\forall x \in X$.*

Definition 3.1.10 *[18] The intersection of two fuzzy sets A and B is $A \cap B$ which is a fuzzy set whose membership function is defined by $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X$.*

It so trivial to extend many of the rules which are hold in crisp set to fuzzy one by using operations of union, complement, and intersection:

(1) Involution : $\bar{\bar{A}} = A$.

(2) Commutatively: $A \cup B = B \cup A$. and $A \cap B = B \cap A$

(3) Associativity: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$

(4) Distributivity: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(5)De Morgan's law :$\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

3.1.3 Alpha Cuts

Definition 3.1.11 *[36] The set* $A_\alpha = \{x : \forall x \in X, \mu_A(x) \geq \alpha\}$*is called the* $\alpha-$*cut which is a crisp set.*

Example 3.1.3 *Let* $A = \{(2, 0.5), (3, 1), (5, 0.7), (7, 0.4), (9, 0.2)\}$ *then :* $A_0 = \{2, 3, 5, 7, 9\}$. $A_{0.2} = \{2, 3, 5, 7, 9\},\ A_{0.4} = \{2, 3, 5, 7\}, A_{0.5} = \{2, 3, 5\},\ A_{0.7} = \{3, 5\},\ A_1 = \{3\}$.

Definition 3.1.12 *[36] A fuzzy set A is convex if*

$$\mu_A(\lambda x_1 + (1 - \lambda)x_2) \geq \min, \{\mu_A(x_1), \mu_A(x_2)\}$$

$x_1, x_2 \in X,$ *and* $\lambda \in [0, 1]$

Alternatively, a fuzzy set is convex if and only if all α- cuts are convex.

3.2 Fuzzy Number

In this section, the definition and basic operations on fuzzy numbers are presented. Also, the definitions of trapezoidal and hexagonal fuzzy numbers are stated. As in the definition of trapezoidal fuzzy numbers we give a new definition for hexagonal fuzzy number.

3.2.1 Definition and Notations

Definition 3.2.1 *[36] A fuzzy number A is a fuzzy set satisfies the following conditions:*

1. A is convex fuzzy set.

2.A is normalized fuzzy set (i.e $\exists x \in R,\ \mu_A(x) = 1$*)*

3. The membership function of A is piecewise continuous.

4. The membership function of A is defined on the real number.

Definition 3.2.2 *[18] : A fuzzy number A is called positive (negative) denoted by $A > 0(A < 0)$ if it's membership function $\mu_A(X)$ satisfies $\mu_A(X) = 0, \forall x \leq 0(\forall x \geq 0)$.*

3.2.2 Operations on Fuzzy Numbers

In this section, we give some operations on fuzzy numbers(note that:$\vee$ denotes maximum, $\wedge$ denotes minimum)

Definition 3.2.3 *[36] The maximum of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A\vee B}(x) = \bigvee\limits_{z=x\vee y}(\mu_A(x) \wedge \mu_B(x)), \forall x \in X$.*

Definition 3.2.4 *[36] The minimum of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A\vee B}(x) = \bigvee\limits_{z=x\wedge y}(\mu_A(x) \wedge \mu_B(x)), \forall x \in X$.*

Definition 3.2.5 *[36] The addition of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A\vee B}(x) = \bigvee\limits_{z=x+y}(\mu_A(x) \wedge \mu_B(x)), \forall x \in X$.*

Definition 3.2.6 *[36] The subtraction of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A\vee B}(x) = \bigvee\limits_{z=x-y}(\mu_A(x) \wedge \mu_B(x)), \forall x \in X$.*

Definition 3.2.7 *[36] The multiplication of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A\vee B}(x) = \bigvee\limits_{z=x*y}(\mu_A(x) \wedge \mu_B(x)), \forall x \in X$.*

Definition 3.2.8 *[36] The division of two fuzzy numbers A and B is a fuzzy set and the membership function is $\mu_{A\vee B}(x) = \bigvee\limits_{z=x/y}(\mu_A(x) \wedge \mu_B(x)), \forall x \in X$.*

3.2.3 Trapeziodal Fuzzy Number

Definition 3.2.9 *[18] Trapezoidal fuzzy number is a fuzzy number represented by four real numbers $a_1 \leq a_2 \leq a_3 \leq a_4$ denoted by $\tilde{A} = (a_1, a_2, a_3, a_4)$ and whose membership function is*

$$\mu_A(x) = \begin{cases} 0, & x \leq a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases} \tag{3.2}$$

Definition 3.2.10 *[18] If $A = (a_1, a_2, a_3, a_4)$, $B = (b_1, b_2, b_3, b_4)$ are trapezoidal fuzzy numbers then:*

1)Addition: $A \oplus B = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)$

2)Symmetric image: $-A = (-a_4, -a_3, -a_2, -a_1)$

3)multiplication: $A \otimes B = (a_1b_1, a_2b_2, a_3b_3, a_4b_4)$

Definition 3.2.11 *[18] A trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$ is said to be a zero trapezoidal fuzzy number if and only if $a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$.*

Definition 3.2.12 *[18] Two fuzzy number$A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ are equal if and only if $a_1 = b_1, a_2 = b_2, a_3 = b_3, a_4 = b_4$.*

Definition 3.2.13 *[17] Let $m \leq n$ be real numbers and γ, β are positive numbers. A non-zero fuzzy number is a Trapezoidal Fuzzy Number denoted by $\hat{A} = (m, n, \alpha, \beta)$ is a fuzzy number whose membership function is given by*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x \leq m - \gamma \\ 1 - \frac{m-x}{\gamma} & m - \gamma \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{x-n}{\beta} & n \leq x \leq n + \beta \\ 0 & x \geq n + \beta \end{cases} \tag{3.3}$$

Using definition (3.2.13) and α-cuts we can define the addition and multiplication of two trapezoidal fuzzy numbers as follows: Let $\hat{A} = (m, n, \alpha_1, \beta_1)$ be a trapezoidal fuzzy number, then for $\alpha \in [0, 1]$

$$A_\alpha = [p_1(\alpha), p_2(\alpha)] = [\alpha_1(\alpha - 1) + m, \beta_1(1 - \alpha) + n]. \tag{3.4}$$

Now, let $\hat{A} = (m, n, \alpha_1, \beta_1)$, $\hat{B} = (p, q, \alpha_2, \beta_2)$ then $A_\alpha = [\alpha_1(\alpha-1)+m, \beta_1(1-\alpha)+n]$, and $B_\alpha = [\alpha_2(\alpha - 1) + p, \beta_2(1 - \alpha) + q]$.

When $\alpha = 0, A_0 = [m - \alpha_1, n + \beta_1]$ and $B_0 = [p - \alpha_2, q + \beta_2]$. So
$A_0 + B_0 = [(m - \alpha_{1)+(} p - \alpha_2), (n + \beta_1) + (q + \beta_2)] = [m + p - (\alpha_1 + \alpha_2), n + q + (\beta_1 + \beta_2)]$
When $\alpha = 1$, $A_1 = [m, n], B_1 = [p, q]$ and $A_1 + B_1 = [m + p, n + q]$. Therefore,

$$\hat{A} \oplus \hat{B} = (m + p, n + q, \alpha_1 + \alpha_2, \beta_1 + \beta_2) \tag{3.5}$$

Similarly, for $\alpha = 0$,

$$\begin{aligned} A_0 \times B_0 &= [(m - \alpha_1)(p - \alpha_2), (n + \beta_1)(q + \beta_2)] \\ &= [(mp - m\alpha_2 - \alpha_1 p + \alpha_1\alpha_2), (nq + n\beta_2 + \beta_1 q + \beta_1\beta_2)] \\ &\simeq [(mp - (m\alpha_2 + \alpha_1 p)), (nq + (n\beta_2 + \beta_1 q))] \end{aligned}$$

and when $\alpha = 1, A_1 \times B_1 = [mp, nq]$. Hence, the multiplication of two fuzzy numbers can be

$$\hat{A} \otimes \hat{B} \simeq (mp, nq, m\alpha_2 + \alpha_1 p, n\beta_2 + \beta_1 q) \tag{3.6}$$

Definition 3.2.14 *Let $\hat{A} = (m, n, \alpha_1, \beta_1)$ be a trapezoidal fuzzy number, the scalar multiplication is defined as follows*

$$c \times \hat{A} = \begin{cases} (cm, cn, c\alpha, c\beta), & c \geq 0 \\ (cn, cm, -c\beta, -c\alpha), & c < 0 \end{cases} \tag{3.7}$$

for any scaler c.

Definition 3.2.15 *[17] A trapezoidal fuzzy number* $\hat{A} = (m, n, \alpha_1, \beta_1)$ *is identically zero if* $m = n = \alpha_1 = \beta_1 = 0$

Definition 3.2.16 *[17]* $\hat{A} = (m, n, \gamma, \beta)$ *is positive if and only if* $m - \gamma \geq 0$.

Definition 3.2.17 *[17] Let* $\hat{A} = (m, n, \alpha_1, \beta_1), B = (p, q, \alpha_2, \beta_2)$ *be two trapezoidal fuzzy number if* $\hat{A}$ *is identically equal to* B *if only if* $m = p$, $n = q, \alpha_1 = \alpha_2, \beta_1 = \beta_2$.

Example 3.2.1 *Let* $\hat{A} = (3, 5, 1, 2)$ $\hat{B} = (6, 8, 4, 5)$. *Then*

$\hat{A} \oplus \hat{B} = (3 + 6, 5 + 8, 1 + 4, 2 + 5) = (9, 13, 5, 7)$

$\hat{A} \otimes \hat{B} \simeq (3 \times 6, 5 \times 8, 3 \times 4 + 1 \times 6, 5 \times 5 + 2 \times 8) = (18, 45, 18, 41)$

3.2.4 Hexagonal Fuzzy Numbers

Definition 3.2.18 *[31] A hexagonal fuzzy number* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$,*where* $a_1, a_2, a_3, a_4, a_5, a_6$ *are real numbers whose membership function* $\mu_{\hat{A}}(x)$ *is*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < a_1 \\ \frac{1}{2}(\frac{x-a_1}{a_2-a_1}) & a_1 \leq x \leq a_2 \\ \frac{1}{2} + \frac{1}{2}(\frac{x-a_2}{a_3-a_2}) & a_2 \leq x \leq a_3 \\ 1 & a_3 \leq x \leq a_4 \\ 1 - \frac{1}{2}(\frac{x-a_4}{a_5-a_4}) & a_4 \leq x \leq a_5 \\ \frac{1}{2}(\frac{a_6-x}{a_6-a_5}) & a_5 \leq x \leq a_6 \\ 0 & x > a_6 \end{cases} \tag{3.8}$$

Definition 3.2.19 *[31] A hexagonal fuzzy number* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ *is positive (negative) if* $a_i \geq 0$ $(a_i < 0)$ *for* $i = 1, 2, 3, 4, 5, 6$.

Definition 3.2.20 *[31] Let* $\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ *,*$\hat{B} = (b_1, b_2, b_3, b_4, b_5, b_6)$ *be two hexagonal fuzzy number,* $\hat{A}= \hat{B}$ *if only if* $a_i = b_i$ *for* $i = 1, 2, 3, 4, 5, 6$*.*

Definition 3.2.21 *[31] : If*$\hat{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ *is hexagonal fuzzy number then* $-\hat{A} = (-a_6, -a_5, -a_4, -a_3, -a_2, -a_1)$ *which is the symmetric image of*$\hat{A}$ *is also a hexagonal fuzzy number.*

Definition 3.2.22 *[31](Operations of Hexagonal Fuzzy numbers) Let* $\hat{A} = \hat{B}$ *are two hexagonal fuzzy number then :*

Addition: $\hat{A} \oplus \hat{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5, a_6 + b_6)$

Multiplication: $\hat{A} \otimes \hat{B} = (a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_6b_6)$

Using α- cuts arithmetic and the left-right spreads, we give a new definition to hexagonal fuzzy numbers as follows.

Definition 3.2.23 *Let* $m \leq n$*,* $\alpha_1, \beta_1, \alpha_2,$ *and* β_2 *such that* $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2$*. A non-zero hexagonal fuzzy number denoted by* $\hat{A} = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *is a fuzzy number whose membership function is given by*

$$\mu_{\hat{A}}(x) = \begin{cases} 0 & x < m - \alpha_1 \\ \frac{1}{2}(\frac{x-m+\alpha_1}{\alpha_1-\alpha_2}) & m - \alpha_1 \leq x \leq m - \alpha_2 \\ 1 + \frac{1}{2}(\frac{x-m}{\alpha_2}) & m - \alpha_2 \leq x \leq m \\ 1 & m \leq x \leq n \\ 1 - \frac{1}{2}(\frac{x-n}{\beta_2}) & n \leq x \leq n + \beta_2 \\ -\frac{1}{2}(\frac{x-n-\beta_1}{\beta_1-\beta_2}) & n + \beta_2 \leq x \leq n + \beta_1 \\ 0 & x > n + \beta_1 \end{cases} \tag{3.9}$$

Definition 3.2.24 *A hexagonal fuzzy number* $\hat{A} = (m, n, \alpha_1, \beta_1, \alpha_2, \beta_2)$ *is positive if* $m - \alpha_1 \geq 0$*.*

Definition 3.2.25 *The hexagonal fuzzy numbers* $\hat{A} = (m, n, \alpha_{11}, \beta_{11}, \alpha_{12}, \beta_{12})$*, and*

$\hat{B} = (p, q, \alpha_{21}, \beta_{21}, \alpha_{22}, \beta_{22})$ *are equal if* $m = p$, $n = q$, $\alpha_{11} = \alpha_{21}, \beta_{11} = \beta_{21}, \alpha_{12} = \alpha_{22}, \beta_{12} = \beta_{22}$.

Remark 3.2.1 *Hexagonal fuzzy number* $\hat{A}_H$ *is the order quadruple* $P_1(u)$, $Q_1(v)$, $Q_2(v)$, $P_2(u)$ *for* $u \in [0, 0.5]$, $v \in [0.5, 1]$ *such that* $P_1(u) = \frac{1}{2}\frac{u-m+\alpha_1}{\alpha_1-\alpha_2}$, $P_2(u) = -\frac{1}{2}\frac{u-n-\beta_1}{\beta_1-\beta_2}$, $Q_1(v) = 1 + \frac{1}{2}\frac{v-m}{\alpha_2}$ *and* $Q_2(v) = 1 - \frac{1}{2}(\frac{v-n}{\beta_2})$. *In fact, if*
$P_1(x) = \frac{1}{2}\frac{x-m+\alpha_1}{\alpha_1-\alpha_2} = \alpha$ *then* $x = 2\alpha(\alpha_1 - \alpha_2) + m - \alpha_1$,
$P_2(x) = -\frac{1}{2}\frac{x-n-\beta_1}{\beta_1-\beta_2} = \alpha$ *then* $x = -2\alpha(\beta_1 - \beta_2) + n + \beta_1$,
$Q_1(x) = 1 + \frac{1}{2}\frac{x-m}{\alpha_2} = \alpha$ *then* $x = 2\alpha_2(\alpha - 1) + m$
$Q_2(x) = 1 - \frac{1}{2}(\frac{x-n}{\beta_2}) = \alpha$ *then* $x = 2\beta_2(1 - \alpha) + n$.

Using α-cut arithmetics and the above remark, we conclude that the α-cut of hexagonal fuzzy numbers $\hat{A}$ and $\hat{B}$ are

$$A_\alpha = \begin{cases} [2\alpha(\alpha_{11} - \alpha_{12}) + m - \alpha_{11}, -2\alpha(\beta_{11} - \beta_{12}) + n + \beta_{11}] & \alpha \in [0, 0.5) \\ [2\alpha_{12}(\alpha - 1) + m, 2\beta_{12}(1 - \alpha) + n] & \alpha \in [0.5, 1] \end{cases}$$

$$B_\alpha = \begin{cases} [2\alpha(\alpha_{21} - \alpha_{22}) + p - \alpha_{21}, -2\alpha(\beta_{21} - \beta_{22}) + q + \beta_{21}] & \alpha \in [0, 0.5) \\ [2\alpha_{22}(\alpha - 1) + m, 2\beta_{22}(1 - \alpha) + n] & \alpha \in [0.5, 1] \end{cases}$$

So, for $\alpha = 0$,

$$\begin{aligned} A_0 + B_0 &= [(m - \alpha_{11}) + (p - \alpha_{21}), (n + \beta_{11}) + (q + \beta_{21})] \\ &= [(m + p - (\alpha_{11} + \alpha_{21})), (n + q + (\beta_{11} + \beta_{21}))], \end{aligned}$$

for $\alpha = 0.5$,

$$\begin{aligned} A_{0.5} + B_{0.5} &= [(m - \alpha_{12}) + (p - \alpha_{22}), (n + \beta_{12}) + (q + \beta_{22})] \\ &= [(m + p - (\alpha_{12} + \alpha_{22})), (n + q + (\beta_{12} + \beta_{22}))], \end{aligned}$$

and for $\alpha = 1$, $A_1 + B_1 = [m + p, n + q]$. So, we get the sum of two hexagonal fuzzy numbers

$$\hat{A} \oplus \hat{B} = [m + p, n + q, \alpha_{11} + \alpha_{21}, \beta_{11} + \beta_{21}, \alpha_{12} + \alpha_{22}, \beta_{12} + \beta_{22}]. \tag{3.10}$$

Example 3.2.2 *Let* $\hat{A} = (4, 6, 2, 3, 1, 2)$, $\hat{B} = (8, 10, 5, 7, 3, 6)$. *Then*

$$A_\alpha = \begin{cases} [2\alpha + 2, -2\alpha + 9] & \alpha \in [0, 0.5) \\ [2\alpha + 2, 10 - 4\alpha] & \alpha \in [0.5, 1] \end{cases}$$

$$B_\alpha = \begin{cases} [4\alpha + 3, -2\alpha + 17] & \alpha \in [0, 0.5) \\ [6\alpha + 2, 22 - 12\alpha] & \alpha \in [0.5, 1] \end{cases}$$

$$A_\alpha + B_\alpha = \begin{cases} [6\alpha + 5, -4\alpha + 26] & \alpha \in [0, 0.5) \\ [8\alpha + 4, 32 - 16\alpha] & \alpha \in [0.5, 1]. \end{cases}$$

Then, $\hat{A} \oplus \hat{B} = (12, 16, 7, 10, 4, 8)$

Similarly, we can define the multiplication of two hexagonal fuzzy numbers as follows: the α- cuts for $\alpha = 0$ and $\alpha = 0.5$ are

$$\begin{aligned} A_0 \times B_0 &= [(m - \alpha_{11}) \times (p - \alpha_{21}), (n + \beta_{11}) \times (q + \beta_{21})] \\ &= [(mp - m\alpha_{21} - \alpha_{11}p + \alpha_{11}\alpha_{21}), (nq + n\beta_{21} + \beta_{11}q + \beta_{11}\beta_{21})] \\ &\simeq [(mp - (m\alpha_{21} + \alpha_{11}p)), (nq + (n\beta_{21} + \beta_{11}q))], \end{aligned}$$

$$\begin{aligned} A_{0.5} \times B_{0.5} &= [(m - \alpha_{12}) \times (p - \alpha_{22}), (n + \beta_{12}) \times (q + \beta_{22})] \\ &= [(mp - m\alpha_{22} - \alpha_{12}p + \alpha_{12}\alpha_{22}), (nq + n\beta_{22} + \beta_{12}q + \beta_{12}\beta_{22})] \\ &\simeq [(mp - (m\alpha_{22} + \alpha_{12}p)), (nq + (n\beta_{22} + \beta_{12}q))], \end{aligned}$$

and for $\alpha = 1$, $A_1 \times B_1 = [mp, nq]$. Therefore,

$$\hat{A} \otimes \hat{B} \simeq (mp, nq, (m\alpha_{21} + \alpha_{11}p), (n\beta_{21} + \beta_{11}q), (m\alpha_{22} + \alpha_{12}p), (n\beta_{22} + \beta_{12}q)). \quad (3.11)$$

Example 3.2.3 *let* $A_H = (4, 6, 2, 3, 1, 2)$,$B_H = (8, 10, 5, 7, 3, 6)$ *find* $A_H \times B_H$

$\hat{A} \times \hat{B} \simeq (4 \times 8, 6 \times 10, 2 \times 8 + 4 \times 5, 3 \times 10 + 6 \times 7, 1 \times 8 + 4 \times 3, 2 \times 10 + 6 \times 68) =$
$(32, 60, 36, 72, 20, 56)$

Chapter 4

Fully Fuzzy Linear system with Trapezoidal Fuzzy Numbers

Fuzzy square and non-square systems with trapezoidal fuzzy numbers will be considered in the present chapter. Precisely, we deal with square invertible, singular and non-square coefficients matrices. Also, a condition for a positive solution will be derived.

4.1 Fuzzy Square Systems

In the current section, we deal with fully fuzzy linear systems where the associated linear systems have a square invertible matrix.

Consider the fuzzy linear system $\hat{A} \otimes \hat{X} = \hat{B}$ such that each entry of $\hat{A} = (\hat{a}_{ij})_{4n \times 4n}$ and $\hat{B} = (\hat{b}_1, ..., \hat{b}_n)^T$ is a trapezoidal fuzzy number, and the unknown $X = (\hat{x}_1, ..., \hat{x}_n)^T$. If $\hat{a}_{ij} = (a_{ij}; b_{ij}, \alpha_{ij}, \beta_{ij})^T$ and $\hat{x}_j = (x_j, y_j, z_j, w_j)^T$, then from definition (3.3), the equation

$$(\hat{a}_{ij} \otimes \hat{x}_j) + ... + (\hat{a}_{in} \otimes \hat{x}_n) = \hat{b}_i$$

can be written as:

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}x_j &= b_j \\ \sum_{j=1}^{n} b_{ij}y_j &= g_j \\ \sum_{j=1}^{n}(a_{ij}z_j + \alpha_{ij}x_j) &= h_j \\ \sum_{j=1}^{n}(b_{ij}w_j + \beta_{ij}y_j) &= k_j. \qquad (4.1)\end{aligned}$$

This will lead to the following algebraic systems of equations

$$\begin{aligned} Ax &= b \\ By &= g \\ Az + Mx &= h \\ Bw + Ny &= k \qquad (4.2)\end{aligned}$$

where $A = (a_{ij}), B = (b_{ij}), M = (\alpha_{ij}), N = (\beta_{ij}),\ x, y, z, w, b, g, h$ and k are coulumn vectors of size n.

The block representation of equation (4.2) is

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A & 0 \\ 0 & N & 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \\ k \end{bmatrix}$$

Definition 4.1.1 *The linear system $SX = C$ where $S = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A & 0 \\ 0 & N & 0 & B \end{bmatrix}$, $X = (x, y, z, w)^T$ and $C = (b, h, g, k)^T$ is called associated linear system of the fuzzy linear system $\hat{A} \otimes \hat{X} = \hat{B}$.*

The associated linear system $SX = C$ and the fuzzy system $\hat{A} \otimes \hat{X} = \hat{B}$ are equivalent. In other words, to solve the fuzzy linear system, it is enough to solve the associated linear system.

4.1.1 Fuzzy System with Invertible Coefficients Matrix

Theorem 4.1.1 *The block matrix S is invertible if and only if the matrices A and B in (2) are invertible.*

Proof: Let $S = \begin{bmatrix} H & 0 \\ L & H \end{bmatrix}$, where $H = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$. From theorem (2.1.1) $|S| = |H|\,|H|$, but since $H = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $|H| = |A|\,|B|$

So, $|S| = |A|\,|B|\,|A|\,|B| = |A|^2\,|B|^2$,there for $|S| \neq 0$ if and only if $|A| \neq 0\,|B| \neq 0$. $\square$

Theorem 4.1.2 *If A and B are invertible matrices, then the unique solution of $SX = C$ is given by:*

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ A^{-1}(h - MA^{-1}b) \\ B^{-1}(k - NB^{-1}g) \end{bmatrix} \tag{4.3}$$

Proof: Since A and B are invertible, the inverse of the matrix S can be obtained by elementary row operations as follows:

$$S^{-1} = \begin{bmatrix} A^{-1} & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 \\ -A^{-1}MA^{-1} & 0 & A^{-1} & 0 \\ 0 & -B^{-1}NB^{-1} & 0 & B^{-1} \end{bmatrix}$$

and the solution of the associated system follows directly. □

Remark 4.1.1 *Let A be a vector matrix, the notation $A \geq 0$ means that each entry of A is greater than or equal to zero.*

Theorem 4.1.3 *A trapezoidal fuzzy linear square system in Defintion 4.1.1 has a trapezoidal fuzzy solution if the following conditions are satisfied*

1. $B^{-1}g \geq A^{-1}b$

2. $A^{-1}h \geq A^{-1}(MA^{-1})b$

3. $B^{-1}k \geq B^{-1}(NB^{-1})g$.

Moreover, if $A^{-1}(I + MA^{-1})b \geq A^{-1}h$, then the solution is positive.

Proof: From equation (4.3), the entries of $A^{-1}b,\ B^{-1}g,\ A^{-1}(h - MA^{-1}b)$ and $B^{-1}(k - NB^{-1}g)$ determine respectively the first, second, third and fourth entries of $\hat{x}_i$. So, from definition (3.2.13) the result follows directly. Also, from definition (3.2.16), to have a positive solution we must have $A^{-1}b - A^{-1}(h - MA^{-1}b) \geq 0$ which is equivalent to $A^{-1}(I + MA^{-1})b \geq A^{-1}h$. □

Example 4.1.1 *Consider the following fuzzy linear system:*

$$\begin{aligned} (3,6,2,7) \otimes (x_1, y_1, z_1, w_1) \oplus (4,6,1,8) \otimes (x_2, y_2, z_2, w_2) &= (27,66,26,68) \\ (4,5,1,6) \otimes (x_1, y_1, z_1, w_1) \oplus (5,8,1,9) \otimes (x_2, y_2, z_2, w_2) &= (35,70,25,71). \end{aligned}$$

Then $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 6 \\ 5 & 8 \end{bmatrix}$, $M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 7 & 8 \\ 6 & 9 \end{bmatrix}$, $b = \begin{bmatrix} 27 \\ 35 \end{bmatrix}$, $g = \begin{bmatrix} 66 \\ 70 \end{bmatrix}$

$h = \begin{bmatrix} 26 \\ 25 \end{bmatrix}$, $k = \begin{bmatrix} 68 \\ 71 \end{bmatrix}$. *From theorem (4.1.2), we get* $\hat{x}_1 = (5, 6, 3, -2.888)$ *which is not a fuzzy number and* $\hat{x}_2 = (3, 5, 1, 0.5553)$.

Example 4.1.2 *consider the following trapezoidal fuzzy system:*

$$\begin{bmatrix} (5,9,4,3) & (8,10,1,5) \\ (8,12,6,11) & (7,14,2,8) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (14,15,5,26) \\ (5,18,4,31) \end{bmatrix}.$$

From theorem (4.1.2), $x = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $z = \begin{bmatrix} 0.3448 \\ 1.0345 \end{bmatrix}$, *and* $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore, $\begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (-2, -1, 0.3448, 1) \\ (3, 3, 1.0345, 1) \end{bmatrix}$.

Example 4.1.3 *Consider the system:*

$$\begin{bmatrix} (4,5,2,6) & (2,3,1,4) \\ (6,8,4,10) & (2,4,1,5) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (18,37,15,91) \\ (24,56,21,138) \end{bmatrix}.$$

Then :$A = \begin{bmatrix} 4 & 2 \\ 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 8 & 4 \end{bmatrix}$, $M = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 6 & 4 \\ 10 & 5 \end{bmatrix}$, $b = \begin{bmatrix} 18 \\ 24 \end{bmatrix}$,

$g = \begin{bmatrix} 37 \\ 56 \end{bmatrix}$, $h = \begin{bmatrix} 15 \\ 21 \end{bmatrix}$, $k = \begin{bmatrix} 91 \\ 138 \end{bmatrix}$. *From (4.1.2)* $x = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ $y = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $z = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ $w = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$

and $\begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (3,5,0,6) \\ (3,4,3,5) \end{bmatrix}$.

Example 4.1.4 *Consider the system:*

$$\begin{bmatrix} (10,12,6,7) & (8,9,4,7) & (8,10,5,8) \\ (12,1510,11) & (5,7,3,3) & (7,8,4,6) \\ (9,10,8,8) & (14,15,9,12) & (11,16,7,10) \end{bmatrix} \begin{bmatrix} (x_1,y_1,z_1,w_1) \\ (x_2,y_2,z_2,w_2) \\ (x_3,y_3,z_3,w_3) \end{bmatrix} = \begin{bmatrix} (152,260,123,291) \\ (147,277,146,294) \\ (189,342,182,382) \end{bmatrix}.$$

Using theorem (4.1.2), we get $x = \begin{bmatrix} 8 \\ 6 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 11 \\ 8 \\ 7 \end{bmatrix}$, $z = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ *and* $w = \begin{bmatrix} 4.5135 \\ 4.2703 \\ 1.1757 \end{bmatrix}$.

So, $\hat{x}_1 = (8,11,2,4.5135), \hat{x}_2 = (8,1,6,4.2703)$, *and* $\hat{x}_3 = (3,7,1,1.1757)$ *which are positive trapezoidal fuzzy numbers.*

4.1.2 Fuzzy System with singular Coefficients Matrix

In this subsection, we approximate the solution of singular associated linear system using Moore -Penrose and a generalized inverse.

From equations (4.2) and the Moore-Penrose inverse of A and B, we have

$$\begin{aligned} x &= A^{+}b \\ y &= B^{+}g \\ z &= A^{+}(h - MA^{+}b) \\ w &= B^{+}(k - NB^{+}g), \end{aligned} \tag{4.4}$$

and by the generalized inverse technique,

$$\begin{aligned} x &= G_1 b \\ y &= G_2 g \\ z &= G_1(h - MG_1 b) \\ w &= G_2(k - NG_2 g), \end{aligned} \tag{4.5}$$

where G_1 and G_2 are the generalized inverses of A and B respectively.

Example 4.1.5 *Consider the system*

$$\begin{bmatrix} (2,3,1,4) & (3,4,2,6) \\ (4,6,2,9) & (6,7,5,8) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (15,39,13,123) \\ (24,47,21,138) \end{bmatrix}.$$

Then $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 6 & 7 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$, $N = \begin{bmatrix} 4 & 6 \\ 9 & 8 \end{bmatrix}$, $b = \begin{bmatrix} 15 \\ 24 \end{bmatrix}$,

$g = \begin{bmatrix} 39 \\ 47 \end{bmatrix}$, $h = \begin{bmatrix} 13 \\ 21 \end{bmatrix}$, $k = \begin{bmatrix} 123 \\ 138 \end{bmatrix}$.

Using (4.4), the solution can be derived as follows :

$x = A^{+}b = \begin{bmatrix} 0.0308 & 0.0615 \\ 0.0462 & 0.0923 \end{bmatrix} \begin{bmatrix} 15 \\ 24 \end{bmatrix} = \begin{bmatrix} 1.9385 \\ 2.9077 \end{bmatrix}$, $z = A^{+}(h - Mx) = \begin{bmatrix} 0.3205 \\ 0.4807 \end{bmatrix}$,

$y = B^{+}g = \begin{bmatrix} -2.3333 & 2 \\ 1.3333 & -1 \end{bmatrix} \begin{bmatrix} 39 \\ 47 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ *and* $w = B^{+}(k - Ny) = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

So, $\begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (1.9385, 3, 0.3205, 6) \\ (2.9077, 5, 0.4807, 8) \end{bmatrix}$.

On the other hand, from(4.5), we have $x = Gb = \begin{bmatrix} 7.5 \\ 0 \end{bmatrix}$ *where* $G = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$ *is the* *generalized inverse of A. Since B is non-singular,* $y = \begin{bmatrix} -2.3333 & 2 \\ 1.3333 & -1 \end{bmatrix} \begin{bmatrix} 39 \\ 47 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$,

$z = G(h - Mx) = \begin{bmatrix} 2.7 \\ 0 \end{bmatrix}$ *and* $w = B^{-1}(k - Ny) = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$.

So, $\begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (7.5, 3, 2.7, 6) \\ (0, 5, 0, 8) \end{bmatrix}$.

Example 4.1.6 *Consider the system*

$$\begin{bmatrix} (8,9,5,6) & (10,11,3,9) \\ (13,18,20,10) & (7,22,6,6) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (63,155,15,95) \\ (29,290,29,133) \end{bmatrix}.$$

Then $x = \begin{bmatrix} 8 & 10 \\ 13 & 17 \end{bmatrix}^{-1} \begin{bmatrix} 63 \\ 29 \end{bmatrix} = \begin{bmatrix} -2.04.5 \\ 7.9324 \end{bmatrix}$, $y = \begin{bmatrix} 9 & 11 \\ 18 & 22 \end{bmatrix}^{+} \begin{bmatrix} 155 \\ 290 \end{bmatrix} = \begin{bmatrix} 6.5495 \\ 8.005 \end{bmatrix}$,

$z = \begin{bmatrix} 0.119 \\ 0.0511 \end{bmatrix}$ *and* $w = \begin{bmatrix} 0.2015 \\ 0.2402 \end{bmatrix}$.

So, $\begin{bmatrix} (x_1, y_1, z_1, w_1) \\ (x_2, y_2, z_2, w_2) \end{bmatrix} = \begin{bmatrix} (-2.045, 6.5495, 0.119, 0.2015) \\ (7.9324, 8.005, 0.0511, 0.2402) \end{bmatrix}$.

4.2 Fuzzy System with Non-square Coefficients Matrix

In this section, fuzzy linear system whose associated coefficients matrices A, B, M and N are of size $m \times n$ where $m > n$ will be studied.

Consider the fuzzy linear system $\tilde{A} \otimes \hat{X} = \tilde{B}$, then similar to the case of invertible coefficient matrix, we have the following associated system

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ M & 0 & A & 0 \\ 0 & N & 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \\ k \end{bmatrix}. \tag{4.6}$$

Elementary row operations reduce system (4.6) to the following system:

$$\begin{bmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} b \\ g \\ h - M(A^TA)^{-1}A^Tb \\ k - N(B^TB)^{-1}B^Tg \end{bmatrix}. \tag{4.7}$$

Denote system (4.7) by $SX = C$, where $S = diag(A, B, A, B)$.

Corollary 4.2.1 *The systems* $SX = C$ *and* $\tilde{A} \otimes \hat{X} = \tilde{B}$ *are equivalent.*

Proof: If $(A^TA)^{-1}$ and $(B^TB)^{-1}$ exist, then the least square solution is given by

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (A^TA)^{-1}A^Tb \\ (B^TB)^{-1}B^Tg \\ (A^TA)^{-1}A^T(h - M(A^TA)^{-1}A^Tb) \\ (B^TB)^{-1}B^T(k - N(B^TB)^{-1}B^Tg) \end{bmatrix}. \tag{4.8}$$

Otherwise, if at least one of the matrices A^TA, B^TB is singular, then using Moore-Penrose method, the least square solution is give by

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} (A^TA)^{+}A^Tb \\ (B^TB)^{+}B^Tg \\ (A^TA)^{+}A^T(h - M(A^TA)^{+}A^Tb) \\ (B^TB)^{+}B^T(k - N(B^TB)^{+}B^Tg) \end{bmatrix}. \tag{4.9}$$

$\square$

Corollary 4.2.2 *Assume A^TA and B^TB are nonsingular. A trapezoidal fuzzy linear non-square system has a trapezoidal fuzzy solution if the following conditions are satisfied*

1. $(B^TB)^{-1}B^Tg \geq (A^TA)^{-1}A^Tb$

2. $(A^TA)^{-1}A^Th \geq (A^TA)^{-1}A^TM(A^TA)^{-1}A^Tb$

3. $(B^TB)^{-1}B^Tk \geq (B^TB)^{-1}B^TN(B^TB)^{-1}B^Tg.$

Moreover, if $(A^TA)^{-1}A^T(I + M(A^TA)^{-1}A^Tb) \geq (A^TA)^{-1}$ then the trapezoidal fuzzy non-square system is fully.

In the following examples, we apply the above theorem to determine whether the solution of the given system is fuzzy number with or without solving the system. Further if the system has a fuzzy solution is it positive or not.

Example 4.2.1 *Consider the following system:*

$$\begin{bmatrix} (6,8,3,4) & (7,9,2,5) \\ (5,8,1,3) & (9,11,5,7) \\ (6,9,2,4) & (5,7,1,2) \end{bmatrix} \begin{bmatrix} (x_1,y_1,z_1,w_1) \\ (x_2,y_2,z_2,w_2) \end{bmatrix} = \begin{bmatrix} (41,48,30,32) \\ (43,45,43,42) \\ (59,60,50,28) \end{bmatrix}.$$

Then $A = \begin{bmatrix} 6 & 7 \\ 5 & 9 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 9 \\ 8 & 11 \\ 9 & 7 \end{bmatrix}$, $M = \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 2 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 4 & 5 \\ 3 & 7 \\ 4 & 2 \end{bmatrix}$, $b = \begin{bmatrix} 41 \\ 43 \\ 59 \end{bmatrix}$, $g = \begin{bmatrix} 48 \\ 45 \\ 60 \end{bmatrix}$,

$h = \begin{bmatrix} 30 \\ 43 \\ 50 \end{bmatrix}$ and $k = \begin{bmatrix} 32 \\ 42 \\ 28 \end{bmatrix}$.

Since A^TA and B^TB are nonsingular, then using (4.8) we get:

$x = \begin{bmatrix} 9.6226 \\ -1.0119 \end{bmatrix}$, $y = \begin{bmatrix} 8.0231 \\ -1.7615 \end{bmatrix}$, $z = \begin{bmatrix} 0.4639 \\ 3.0489 \end{bmatrix}$ and $w = \begin{bmatrix} -5.0751 \\ 6.1318 \end{bmatrix}$.

Therefore, $\hat{x}_1 = (9.6226, 8.0231, 0.4639, -5.0751)$ and $\hat{x}_2 = (-1.0119, -1.7615, 3.0489, 6.1318)$ which are not fuzzy since in x_1 $\beta = -5.0751 \leq 0$, in x_2 $-1.0119 \geq -1.7615$ which mean $m \geq n$ and this since the conditions 1 and 3 in corollary (4.2.2) are not satisfied .

Example 4.2.2 *In the system:*

$$\begin{bmatrix} (1,5,0.5,1) & (7,9,5,3) \\ (2,7,1.25,1) & (3,5,2.5,1) \\ (6,10,4,1) & (5,11,3,1) \end{bmatrix} \begin{bmatrix} (x_1,y_1,z_1,w_1) \\ (x_2,y_2,z_2,w_2) \end{bmatrix} = \begin{bmatrix} (20,200,18,95) \\ (15,160,14,56) \\ (13,106,12,166) \end{bmatrix}.$$

$A = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 6 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 9 \\ 7 & 5 \\ 10 & 11 \end{bmatrix}$, $M = \begin{bmatrix} 0.5 & 5 \\ 1.25 & 2.5 \\ 4 & 3 \end{bmatrix}$, $N = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 20 \\ 15 \\ 13 \end{bmatrix}$, $g = \begin{bmatrix} 200 \\ 160 \\ 106 \end{bmatrix}$, $h = \begin{bmatrix} 18 \\ 14 \\ 12 \end{bmatrix}$ and $k = \begin{bmatrix} 95 \\ 56 \\ 66 \end{bmatrix}$. Since the matrices A^TA and B^TB are invertible, from corollary (4.2.1) we have $\hat{x}_1 = (-0.0811, 5.0383, 0.2001, 1.6639)$ and $\hat{x}_2 = (3.0541, 15.2808, 0.3111, 3.3561)$ which are trapezoidal fuzzy numbers since the first three conditions in corollary (4.2.2) are satisfy.

Example 4.2.3 *If we consider the following matrices:* $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 4 & 4 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0.3 \\ 0.25 & 0.75 \\ 0.5 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 3 & \\ 43 & 2 \\ 1 & 8 \end{bmatrix}$, $b = \begin{bmatrix} 12 \\ 18 \\ 14 \end{bmatrix}$, $g = \begin{bmatrix} 20 \\ 32 \\ 44 \end{bmatrix}$, $h = \begin{bmatrix} 11 \\ 17 \\ 13 \end{bmatrix}$ *and* $k = \begin{bmatrix} 23 \\ 26 \\ 44 \end{bmatrix}$.

Then the solution is $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (3.6667, 4, 2.2556, 0.0417) \\ (3.6667, 4, 2.2556, 0.0417) \end{bmatrix}$, *which is a positive solution.*

Chapter 5

Fully Fuzzy Linear System with Hexagonal Fuzzy Numbers

The aim of the current chapter is to extend the theory introduced in the previous chapter to fuzzy linear systems with hexagonal Fuzzy numbers defined by (3.9).

5.1 Fuzzy System with Square Coefficients Matrix

Consider the fully fuzzy linear system $\hat{A} \otimes \hat{X} = \hat{B}$ such that each entry of $\hat{A} = (\hat{a}_{ij})$, $\hat{B} = (\hat{b}_j)$ are hexagonal, where $\hat{a}_{ij} = (a_{ij}; b_{ij}, \alpha_{ij}, \beta_{ij}\gamma_{ij}, \zeta_{ij})^T$, $\hat{x}_j = (x_j, y_j, z_j, w_j, u_j, v_j)^T$, and $\hat{b}_j = (b_j, g_j, h_j, k_j, l_j, t_j)^T$. Then, using equation (3.11) leads to:

$$\hat{a}_{ij} \otimes \hat{x}_j = (a_{ij}x_j, b_{ij}y_j, a_{ij}z_j + \alpha_{ij}x_j, b_{ij}w_j + \beta_{ij}y_j, a_{ij}u_j + \gamma_{ij}x_j, b_{ij}v_j + \zeta_{ij}y_j)$$

So, the ith row of the system $\hat{A} \otimes \hat{X} = \hat{B}$ is $(\hat{a}_{ij} \otimes \hat{x}_j) + ... + (\hat{a}_{in} \otimes \hat{x}_n) = \hat{b}_i$ can be written as :

$$
\begin{aligned}
\sum_{j=1}^{n} a_{ij}x_j &= b_j \\
\sum_{j=1}^{n} b_{ij}y_j &= g_j \\
\sum_{j=1}^{n} (a_{ij}z_j + \alpha_{ij}x_j) &= h_j \\
\sum_{j=1}^{n} (b_{ij}w_j + \beta y_j) &= k_j \\
\sum_{j=1}^{n} (a_{ij}u_j + \gamma x_j) &= l_j \\
\sum_{j=1}^{n} (b_{ij}v_j + \zeta y_j) &= t_j \qquad (5.1)
\end{aligned}
$$

Hence, we get the following algebraic linear systems:

$$
\begin{aligned}
Ax &= b \\
By &= g \\
Az + Mx &= h \\
Bw + Ny &= k \\
Au + Tx &= l \\
Bv + Uy &= t \qquad (5.2)
\end{aligned}
$$

where $A = (a_{ij}), B = (b_{ij}), M = (\alpha_{ij}), N = (\beta_{ij}), T = (\gamma_{ij}), U = (\zeta_{ij}), b, g, h, k, l, t$ are vector of size n. The matrices A, B, M, N, T and U are of size $n \times n$. The associated linear system of the hexagonal fuzzy linear system is

$$SX = C \qquad (5.3)$$

where $S = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ M & 0 & A & 0 & 0 & 0 \\ 0 & N & 0 & B & 0 & 0 \\ T & 0 & 0 & 0 & A & 0 \\ 0 & U & 0 & 0 & 0 & B \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \\ w \\ u \\ v \end{bmatrix}$ and $C = \begin{bmatrix} b \\ g \\ h \\ k \\ l \\ t \end{bmatrix}$.

5.1.1 Fuzzy System with Invertible Coefficients Matrix

In the following theorems, we prove that the solution of the fuzzy and associated systems are equivalent, and then, some applications will be introduced.

Theorem 5.1.1 *The matrix S is invertible if and only if A and B are invertible.*

Proof: let $S = \begin{bmatrix} H & O \\ L & K \end{bmatrix}$, where $H = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ M & 0 & A \end{bmatrix}$ L= $\begin{bmatrix} 0 & B & 0 \\ T & 0 & 0 \\ 0 & U & 0 \end{bmatrix}$ $K = \begin{bmatrix} B & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{bmatrix}$.
Then, from theorem (2.1.1), we have $|S| = |H|\,|k|$, Also, H and K can be partitioned as $H = \begin{bmatrix} A & O \\ L_1 & K_1 \end{bmatrix}$, and $K = \begin{bmatrix} H & O \\ 0 & K_2 \end{bmatrix}$ hence , $|S| = |H|\,|k|\,|A|\,|K_1|\,|B|\,|K_2|$, Thus $|S| = |A|^3\,|B|^3$. Therefore, S is invertible if and only if A and B are invertible. □

Theorem 5.1.2 *If S is invertible, then the uniqe solution of SX=C is given by*

$$X = \begin{bmatrix} A^{-1}b \\ B^{-1}g \\ A^{-1}(h - MA^{-1}b) \\ B^{-1}(k - NB^{-1}g) \\ A^{-1}(l - TA^{-1}b) \\ B^{-1}(t - UB^{-1}g) \end{bmatrix} \tag{5.4}$$

Proof: Apply elementary row operations on the augmented matrix

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ M & 0 & A & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & N & 0 & B & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ T & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & U & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

to get the inverse of S:

$$S^{-1} = \begin{bmatrix} A^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & B^{-1} & 0 & 0 & 0 & 0 \\ -A^{-1}MA^{-1} & 0 & A^{-1} & 0 & 0 & 0 \\ 0 & -B^{-1}NB^{-1} & 0 & B^{-1} & 0 & 0 \\ -A^{-1}TA^{-1} & 0 & 0 & 0 & A^{-1} & 0 \\ 0 & B^{-1}UB^{-1} & 0 & 0 & 0 & B^{-1} \end{bmatrix}$$

□

Example 5.1.1 *Consider the system:*

$$\begin{bmatrix} (8,12,5,4,3,1) & (9,10,7,5,5,3) \\ (12,16,10,9,6,4) & (13,15,10,7,8,4) \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (22,56,65,139,35,88) \\ (30,88,83,197,49,117) \end{bmatrix}.$$

Then $A = \begin{bmatrix} 8 & 9 \\ 12 & 13 \end{bmatrix}$, $B = \begin{bmatrix} 12 & 10 \\ 16 & 15 \end{bmatrix}$, $M = \begin{bmatrix} 5 & 7 \\ 10 & 10 \end{bmatrix}$, $N = \begin{bmatrix} 4 & 5 \\ 9 & 7 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 3 \\ 4 & 4 \end{bmatrix}$ *and* $U = \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$. *The constant matrices are* $b = \begin{bmatrix} 22 \\ 30 \end{bmatrix}$, $g = \begin{bmatrix} 56 \\ 88 \end{bmatrix}$, $h = \begin{bmatrix} 65 \\ 83 \end{bmatrix}$,

$k = \begin{bmatrix} 139 \\ 197 \end{bmatrix}$, $l = \begin{bmatrix} 35 \\ 49 \end{bmatrix}$ *and* $t = \begin{bmatrix} 88 \\ 117 \end{bmatrix}$. *Apply theorem (5.1.2) on the associated system to get* $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (-4,-2,2,5,1,3) \\ (6,8,3,6.25,1,3) \end{bmatrix}$.

Corollary 5.1.1 *A hexagonal fuzzy linear system (5.3) has a hexagonal fuzzy solution if the following conditions are satisfied*

1. $B^{-1}g \geq A^{-1}b$

2. $A^{-1}l \geq A^{-1}TA^{-1}b$

3. $B^{-1}t \geq B^{-1}UB^{-1}h$

4. $A^{-1}(h-l) \geq A^{-1}(M-T)A^{-1}b$

5. $B^{-1}(k-t) \geq B^{-1}(N-U)B^{-1}g$.

Moreover, if $A^{-1}(I + MA^{-1})b \geq A^{-1}h$ *then the solution is positive.*

Proof: The proof of the corollary followed directly from definition (3.2.23), equation (5.3) and the fact that the solution of subsystems (5.2) are exactly the components of the desired solution. □

Example 5.1.2 *If*

$$\begin{aligned} (8,9,6,7,5,6) \otimes \hat{x}_1 \oplus (4,7,3,2,2,1) \otimes \hat{x}_2 &= (56,102,45,137,33.5,36) \\ (5,10,4,3,3,2) \otimes \hat{x}_1 \oplus (3,4,2,4,1,3) \otimes \hat{x}_2 &= (38,78,30,83,19,70). \end{aligned}$$

Then, $A = \begin{bmatrix} 8 & 4 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 9 & 7 \\ 10 & 4 \end{bmatrix}$, $M = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$, $N = \begin{bmatrix} 7 & 2 \\ 3 & 4 \end{bmatrix}$, $T = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$ *and* $U = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$. *The constant matrices are* $b = \begin{bmatrix} 56 \\ 38 \end{bmatrix}$, $g = \begin{bmatrix} 102 \\ 78 \end{bmatrix}$, $h = \begin{bmatrix} 45 \\ 30 \end{bmatrix}$,

$k = \begin{bmatrix} 137 \\ 83 \end{bmatrix}$, $l = \begin{bmatrix} 33.5 \\ 19 \end{bmatrix}$ *and* $t = \begin{bmatrix} 36 \\ 70 \end{bmatrix}$. *Apply theorem (5.1.2) on the associated system to get* $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (4, 8, 0.25, 5, 0.125, 1) \\ (6, 3, 0.25, 3, 0.125, 1) \end{bmatrix}$. *Note that* $\hat{x}_2$ *is non-fuzzy since and* $B^{-1}g = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ *which is not greater or equal to* $A^{-1}b = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

Example 5.1.3 *Consider the following system*

$$\begin{bmatrix} (10, 12, 5, 4, 4, 3) & (7, 8, 3, 5, 2, 1) \\ (6, 9, 3, 3, 1, 2) & (9, 10, 4, 3, 3, 2) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1, u_1, v_1) \\ (x_2, y_2, z_2, w_2, u_2, v_2) \end{bmatrix} = \begin{bmatrix} (122, 180, 112, 189, 71, 116) \\ (102, 171, 90, 159, 47, 112) \end{bmatrix}.$$

Then $A = \begin{bmatrix} 10 & 7 \\ 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 12 & 8 \\ 9 & 10 \end{bmatrix}$, $M = \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}$, $N = \begin{bmatrix} 4 & 5 \\ 3 & 3 \end{bmatrix}$, $T = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ *and* $U = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$. *The constant matrices are* $b = \begin{bmatrix} 122 \\ 102 \end{bmatrix}$, $g = \begin{bmatrix} 180 \\ 171 \end{bmatrix}$, $h = \begin{bmatrix} 112 \\ 90 \end{bmatrix}$, $k = \begin{bmatrix} 189 \\ 159 \end{bmatrix}$, $l = \begin{bmatrix} 71 \\ 47 \end{bmatrix}$ *and* $t = \begin{bmatrix} 116 \\ 112 \end{bmatrix}$. *Apply theorem (5.1.2) on the associated system to get* $\begin{bmatrix} (x_1, y_1, z_1, w_1, u_1, v_1) \\ (x_2, y_2, z_2, w_2, u_2, v_2) \end{bmatrix} = \begin{bmatrix} (8, 9, 4, 5, 2, 4) \\ (6, 9, 2, 6, 1, 4) \end{bmatrix}$ *which are two positive fuzzy numbers* .

Example 5.1.4 *The solution of the following system*

$$\begin{bmatrix} (5, 8, 0, 0, 0, 0) & (0, 0, 0, 0, 0, 0) \\ (0, 0, 0, 0, 0, 0) & (7, 10, 0, 0, 0, 0) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1, \alpha_1, \beta_1) \\ (x_2, y_2, z_2, w_2, \alpha_2, \beta_2) \end{bmatrix} = \begin{bmatrix} (45, 64, 36, 49, 35, 27) \\ (77, 130, 63, 65, 56, 54) \end{bmatrix}$$

is $\begin{bmatrix} (x_1, y_1, z_1, w_1, \alpha_1, \beta_1) \\ (x_2, y_2, z_2, w_2, \alpha_2, \beta_2) \end{bmatrix} = \begin{bmatrix} (9, 10, 7.25, 6.225, 7, 3.375) \\ (11, 13, 9, 6.5, 8, 5.4) \end{bmatrix}$.

Example 5.1.5 *The solution of the following system*

$$\begin{bmatrix} (8,12,5,4,3,1) & (9,10,7,5,5,3) \\ (12,16,10,9,6,4) & (13,15,10,7,8,4) \end{bmatrix} \begin{bmatrix} (x_1,y_1,z_1,w_1,\alpha_1,\beta_1) \\ (x_2,y_2,z_2,w_2,\alpha_2,\beta_2) \end{bmatrix} = \begin{bmatrix} (22,56,65,139,35,88) \\ (30,88,83,197,49,117) \end{bmatrix}$$

is $\begin{bmatrix} (x_1,y_1,z_1,w_1,\alpha_1,\beta_1) \\ (x_2,y_2,z_2,w_2,\alpha_2,\beta_2) \end{bmatrix} = \begin{bmatrix} (-4,-2,2,6.25,1,3) \\ (6,8,3,3,1,3) \end{bmatrix}$.

5.1.2 Fuzzy System with Singular Coefficients Matrix

The previous section dealt with invertible matrices. Using either the generalized or the Moore-Penrose inverse, the solution of associated system will be investigated, namely, when at least one of the matrices A and B is singular.

Remark 5.1.1 *For the algebraic linear systems (5.2) when A or B is singular, the approximate solution can be found by both Moore -Penrose and General inverses.*

1. Moore -Penrose inverse using the following equations

$$\begin{aligned} x &= A^\dagger b \\ y &= B^\dagger g \\ z &= A^\dagger(h - MA^\dagger b) \\ w &= B^\dagger(k - NB^\dagger g) \\ u &= A^\dagger(l - TA^\dagger b) \\ v &= B^\dagger(t - UB^\dagger g). \end{aligned}$$

2. The generalized inverse using the following equations:

$$\begin{aligned} x &= G_1 b \\ y &= G_2 g \\ z &= G_1(h - MG_1 b) \\ w &= G_2(k - NG_2 g) \\ u &= G_1(l - TG_1 b) \\ v &= G_2(t - UG_2 g). \end{aligned}$$

Corollary 5.1.2 *A hexagonal fuzzy linear system 5.3 when A or B is singular,has a hexagonal fuzzy solution if the following conditions are satisfied*

1. $B^\dagger g \geq A^\dagger b$

2. $A^\dagger l \geq A^\dagger T A^\dagger b$

3. $B^\dagger t \geq B^\dagger U B^\dagger h$

4. $A^\dagger(h - l) \geq A^\dagger(M - T)A^\dagger b$

5. $B^\dagger(k - t) \geq B^\dagger(N - U)B^\dagger g.$

Moreover, if $A^\dagger(I + MA^\dagger)b \geq A^\dagger h$*, then the solution is positive.*

Example 5.1.6 *Consider the following system*

$$\begin{bmatrix} (2,5,1,3,1,2) & (2,3,0,0,0,0) \\ (6,10,3,4,2,3) & (6,8,5,5,3,3) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1, \alpha_1, \beta_1) \\ (x_2, y_2, z_2, w_2, \alpha_2, \beta_2) \end{bmatrix} = \begin{bmatrix} (24,40,20,30,18,26) \\ (36,40,25,35,23,30) \end{bmatrix}.$$

Then $A = \begin{bmatrix} 2 & 2 \\ 6 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 \\ 10 & 8 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$, $N = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$,

$U = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 24 \\ 36 \end{bmatrix}$, $g = \begin{bmatrix} 40 \\ 40 \end{bmatrix}$, $h = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$, $k = \begin{bmatrix} 30 \\ 35 \end{bmatrix}$, $l = \begin{bmatrix} 18 \\ 23 \end{bmatrix}$ *and* $t = \begin{bmatrix} 26 \\ 30 \end{bmatrix}$.

Note that the matrices A and B is singular, then

$$x = A^\dagger b = \begin{bmatrix} 0.0250 & 0.0750 \\ 0,0250 & 0,0750 \end{bmatrix} \begin{bmatrix} 24 \\ 36 \end{bmatrix} = \begin{bmatrix} 3.3000 \\ 3.3000 \end{bmatrix},$$

$$y = B^{\dagger}g = \begin{bmatrix} 0.0244 & 0.0488 \\ 0.0195 & 0.0390 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 2.9268 \\ 2.3415 \end{bmatrix}$$

$$z = A^{\dagger}(g - MA^{\dagger}b) = \begin{bmatrix} 0.3125 \\ 0.3125 \end{bmatrix} \; w = B^{\dagger}(k - NB^{\dagger}h) = \begin{bmatrix} 1.0827 \\ 0.8662 \end{bmatrix}$$

$$\alpha = A^{\dagger}(l - TA^{\dagger}b) = \begin{bmatrix} 0.8550 \\ 0.8550 \end{bmatrix} \; \beta = B^{\dagger}(O - UB^{\dagger}h) \; = \begin{bmatrix} 1.1838 \\ 0.9471 \end{bmatrix}.$$

So, we get the two non-fuzzy numbers

$\hat{x}_1 = (3.3000, 2.9268, 0.3125, 1, 0872, 0.8550, 1, 1838)$

$\hat{x}_2 = (3.3000, 2.3415, 0.3125, 0.8662, 0.8550, 0.9471).$

Secondly, by the generalized inverse

$$x = G_1 b = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 24 \\ 36 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}, \; y = G_2 g = \begin{bmatrix} .2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \end{bmatrix},$$

$$z = G_1(h - nG_1 b) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \; w = G_2(k - NG_2 g) = \begin{bmatrix} 1.2000 \\ 0 \end{bmatrix}, \alpha = G_1(l - TG_1 b) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

and $\beta = G_2(o - TG_2 g) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$

SO, $\hat{x}_1 = (12, 8, 4, 1.2, 3, 2)$*, and* $\hat{x}_2 = (0, 0, 0, 0, 0, 0)$*. Note that the solution using Moore -Penrose general inverse is more sufficient than the Generalized inverse of Matrix*

Example 5.1.7 *By the Moore-Penrose inverse, the solution of the following system*

$$\begin{bmatrix} (4,5,3,5,2,4) & (6,7,4,5,3,3) \\ (8,10,5,3,2,2) & (9,14,7,8,5,5) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1, \alpha_1, \beta_1) \\ (x_2, y_2, z_2, w_2, \alpha_2, \beta_2) \end{bmatrix} = \begin{bmatrix} (35, 65, 33, 60, 24.1, 32) \\ (60, 115, 53, 54, 34.6, 39) \end{bmatrix}$$

is $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} (3.75, 3.9865, 0.3542, 0, 0995, 0.2667, 0.0751) \\ (3.3333, 5.5811, 1.0278, 0.1393, 0.9222, 0.1051) \end{bmatrix}$ *which is a hexagonal fuzzy solution.*

5.2 Fuzzy System with Non-Square Coefficients Matrix

let $\tilde{A} = (\hat{a}_{ij})$ and $\tilde{B} = (\hat{b}_j)_{j=1}^n$ be two fuzzy matrices with hexagonal fuzzy numbers entries. Following the same procedure in section 5.1, we get a system similar to the system (5.2) where A, B, M, N, U, T are $m \times n$ matrices and such that $(A^T A)^{-1}$ and $(B^T B)^{-1}$ are exist. Then the associated system $SX = C$ is of size $6m \times 6n$ and has the following form:

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ M & 0 & A & 0 & 0 & 0 \\ 0 & N & 0 & B & 0 & 0 \\ T & 0 & 0 & 0 & A & 0 \\ 0 & U & 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b \\ g \\ h \\ k \\ l \\ o \end{bmatrix}$$

The above system can be reduced by elementary row operations to

$$\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 & 0 \\ 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 \\ 0 & 0 & 0 & 0 & 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ u \\ v \end{bmatrix} = \begin{bmatrix} b \\ g \\ h - M(A^T A)^{-1} A^T b \\ k - N(B^T B)^{-1} B^T g \\ l - T(A^T A)^{-1} A^T b \\ t - U(B^T B)^{-1} B^T g \end{bmatrix} \tag{5.5}$$

Remark 5.2.1 *The solution of $SX = C$ is of the form:*

$$\begin{bmatrix} x \\ y \\ z \\ w \\ u \\ v \end{bmatrix} = \begin{bmatrix} (A^TA)^{-1}A^Tb \\ (B^TB)^{-1}B^Tg \\ (A^TA)^{-1}A^T(h - M(A^TA)^{-1}A^Tb) \\ (B^TB)^{-1}B^T(i - N(B^TB)^{-1}B^Tg) \\ (A^TA)^{-1}A^T(j - T(A^TA)^{-1}A^Tb) \\ (B^TB)^{-1}B^T(k - U(B^TB)^{-1}B^Tg) \end{bmatrix} \tag{5.6}$$

Proof: Since A^TA and B^TB are invertible the solution $SX = C$, is $X = (S^T)^{-1}C$.

□

Corollary 5.2.1 *A hexagonal fuzzy linear non-square system has a positive solution if the following conditions are satisfied:*

1. $(B^TB)^{-1}B^Tg \geq (A^TA)^{-1}A^Tb$,
2. $(A^TA)^{-1}A^Tj \geq (A^TA)^{-1}A^TT(A^TA)^{-1}A^Tb)$,
3. $(B^TB)^{-1}B^Tt \geq (B^TB)^{-1}B^TU(B^TB)^{-1}B^Tg)$,
4. $(A^TA)^{-1}A^T(h - l) \geq (A^TA)^{-1}A^T(M - T)(A^TA)^{-1}A^Tb$,
5. $(B^TB)^{-1}B^T(k - t) \geq (B^TB)^{-1}B^T(N - U)(B^TB)^{-1}B^Tg$,
6. $(A^TA)^{-1}A^T[I + M(A^TA)^{-1}A^T]b \geq (A^TA)^{-1}A^Th$

Proof: From (1), we can see $y \geq x$ from (ii) and (iii) we see α and$\beta \geq 0$, and when we Rearrange equations $(iv), (v)$ we have $z - \alpha \geq 0, w - \beta \geq 0$. □

Example 5.2.1 *Condiser the system*

$$\begin{bmatrix} (6,9,3,5,2,3) & (5,7,4,6,3,4) \\ (7,8,5,8,4,5) & (9,10,7,7,6,5) \\ (4,6,2,3,1,2) & (8,11,6,4,5,1) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1, \alpha_1, \beta_1) \\ (x_2, y_2, z_2, w_2, \alpha_2, \beta_2) \end{bmatrix} = \begin{bmatrix} (25,51,23,96,15,50) \\ (33,68,32,144,30,61) \\ (45,72,36,189,28,31) \end{bmatrix}$$

The coefficients matrices are:

$$A=\begin{bmatrix}6&5\\7&9\\4&8\end{bmatrix},\ B=\begin{bmatrix}9&7\\8&10\\6&11\end{bmatrix},\ M=\begin{bmatrix}3&4\\5&7\\2&6\end{bmatrix},\ N=\begin{bmatrix}5&6\\8&7\\3&4\end{bmatrix},\ T=\begin{bmatrix}3&4\\5&5\\2&1\end{bmatrix}\ U=\begin{bmatrix}2&3\\4&6\\1&5\end{bmatrix},\text{ and the}$$

$$\text{constant matrices } b=\begin{bmatrix}25\\33\\45\end{bmatrix}\ h=\begin{bmatrix}23\\32\\36\end{bmatrix},\ g=\begin{bmatrix}51\\68\\72\end{bmatrix},\ k=\begin{bmatrix}96\\144\\189\end{bmatrix},\ l=\begin{bmatrix}15\\30\\28\end{bmatrix}\ t=\begin{bmatrix}50\\61\\31\end{bmatrix}.$$

Therefore,

$(x_1, y_1, z_1, w_1, \alpha_1, \beta_1) = (-1.5405, 0.210, 3, 0.1183, 1.6273)$

$(x_2, y_2, z_2, w_2, \alpha_2, \beta_2) = (5.7325, 6, 0.1807, 4, 0.0737, 1.2367)$.

Example 5.2.2 *Consider the system*

$$\begin{aligned}(4,8,2,6,1,1)\otimes\hat{x}_1\oplus(8,10,6,7,4,5)\otimes\hat{x}_2 &= (47,70,46,60,28,35)\\ (5,6,4,3,3,2)\otimes\hat{x}_1 &= (25,50,24,50,18,28)\\ (7,9,5,5,3,3)\otimes\hat{x}_2 &= (36,66,35,55,20,30)\end{aligned}$$

$$\textit{Solution: } A=\begin{bmatrix}5&0\\0&7\\4&8\end{bmatrix},\ B=\begin{bmatrix}6&0\\0&9\\8&10\end{bmatrix},\ M=\begin{bmatrix}4&0\\0&5\\2&6\end{bmatrix},\ N=\begin{bmatrix}3&0\\0&5\\6&7\end{bmatrix},\ T=\begin{bmatrix}3&0\\0&3\\1&4\end{bmatrix},$$

$$U=\begin{bmatrix}2&0\\0&3\\1&5\end{bmatrix},\ b=\begin{bmatrix}25\\36\\47\end{bmatrix},\ g=\begin{bmatrix}50\\66\\70\end{bmatrix},\ h=\begin{bmatrix}24\\35\\46\end{bmatrix},\ k=\begin{bmatrix}50\\55\\60\end{bmatrix},\ l=\begin{bmatrix}18\\20\\28\end{bmatrix},\ t=\begin{bmatrix}28\\30\\35\end{bmatrix},$$

$$x=(A^TA)^{-1}A^Tb=\begin{bmatrix}4.2319\\4.3591\end{bmatrix},\ y=(B^TB)^{-1}B^Tg=\begin{bmatrix}4.4564\\5.1795\end{bmatrix},$$

$$z = (A^TA)^{-1}A^T(h - M(A^TA)^{-1}A^Tb) = \begin{bmatrix} 0.8514 \\ 1.3118 \end{bmatrix},$$

$$w = (B^TB)^{-1}B^T(i - N(B^TB)^{-1}B^Tg) = \begin{bmatrix} 1.4433 \\ 0.6437 \end{bmatrix},$$

$$\alpha = (A^TA)^{-1}A^T(j - T(A^TA)^{-1}A^Tb) = \begin{bmatrix} 0.7446 \\ 0.6662 \end{bmatrix},$$

$$\beta = (B^TB)^{-1}B^T(k - U(B^TB)^{-1}B^Tg) = \begin{bmatrix} 1.1391 \\ 0.4723 \end{bmatrix}$$

so, $\hat{x}_1 = (4.2319, 4.4564, 0.8514, 1.4433, 0.7446, 1.1391)$

$\hat{x}_2 = (4.3591, 5.1795, 1.3118, 0.6437, 0.6662, 0.4723)$ *which are two positive fuzzy number since all conditions of corollary* (5.2.1) *are satisfied .*

Example 5.2.3 *Consider the system*

$$\begin{bmatrix} (3,4,0,0,0,0) & (6,8,1,3,2,2) \\ (2,5,1,1,1,1) & (4,6,3,2,2,1) \\ (5,7,4,4,3,2) & (0,0,0,0,0,0) \\ (0,0,0,0,0,0) & (9,10,6,7,5,4) \end{bmatrix} \begin{bmatrix} (x_1, y_1, z_1, w_1, \alpha_1, \beta_1) \\ (x_2, y_2, z_2, w_2, \alpha_2, \beta_2) \end{bmatrix} = \begin{bmatrix} (51,92,49,80,40,70) \\ (34,87,33,80,27,60) \\ (35,63,33,60,29,50) \\ (45,70,40,70,31,60) \end{bmatrix}$$

The coefficients matrices are:

$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \\ 5 & 0 \\ 0 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 8 \\ 5 & 6 \\ 7 & 0 \\ 0 & 10 \end{bmatrix}$, $M = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 4 & 0 \\ 0 & 6 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 3 \\ 1 & 2 \\ 4 & 0 \\ 0 & 7 \end{bmatrix}$, $T = \begin{bmatrix} 0 & 2 \\ 1 & 2 \\ 3 & 0 \\ 0 & 5 \end{bmatrix}$ $U = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}$, *and the constant matrices* $b = \begin{bmatrix} 51 \\ 34 \\ 35 \\ 45 \end{bmatrix}$ $h = \begin{bmatrix} 92 \\ 87 \\ 63 \\ 70 \end{bmatrix}$, $g = \begin{bmatrix} 49 \\ 33 \\ 33 \\ 4 \end{bmatrix}$, $k = \begin{bmatrix} 90 \\ 80 \\ 60 \\ 70 \end{bmatrix}$, $l = \begin{bmatrix} 40 \\ 27 \\ 29 \\ 31 \end{bmatrix}$ $t = \begin{bmatrix} 70 \\ 60 \\ 50 \\ 60 \end{bmatrix}$.

Therefore,

$(x_1, y_1, z_1, w_1, \alpha_1, \beta_1) = (7, 9, 3.0742, 5.4643, 2.9296, 4, 9178)$

$(x_2, y_2, z_2, w_2, \alpha_2, \beta_2) = (5, 7, 2.3915, 3.8261, 1.4874, 3, 6355)$ *which are two positive fuzzy number since all conditions of corollary* $(5.2.1)$ *are satisfied .*

Conclusions

In the present work, the solution of fully fuzzy linear systems with trapezoidal and hexagonal fuzzy numbers are derived by transforming the given system into a crisp system with block coefficients matrix. Different methodologies have been used to reach the solution according to the core matrices whether they are square, non-square, invertible and singular. After that we study the conditions to have positive solutions in each states. In literature, almost all researchers approximated the multiplication of two Trapezoidal fuzzy numbers when the arithmetic alpha-cuts are used . As a future work, we will discuss the developed theory presented in the current work with no restriction on the multiplication.

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ملخص

الأنظمة الخطية الضبابية الشبه منحرفة والسداسية الكاملة

أسيل صالح محمد قاروط

في هذه الأطروحة، نحن نقدم تعريفاً جديداً للأعداد الضبابية الشبه منحرفة بأستخدام الانتشار الأيمن والأيسر كما وعرفنا الاعداد الضبابية السداسية بنفس الطريقة، ثم قمنا بتعريف العمليات على هذه الاعداد بأستخدام حساب الافا كت . وبعد ذلك درسنا الانظمة الضبابية الشبه منحرفة والسداسية الكاملة المربعة وغير المربعة باستخدام التعريفات الجديدة. وتوصلنا الى الشروط الازمه للحصول على حل ضبابي في البداية ثم ان يكون هذا الحل موجب .وقد نوقش الكثير من الأمثلة التوضيحية.