The Arab American University

Faculty of Graduate Studies

The Bivariate Shifted Legendre Functions for Nonlinear Volterra Integral Equation

by

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This Thesis was Submitted

in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in

Applied Mathematics

August, 2017

Committee Decision

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Dedication

To my dear father and mother who supported me in my educational career and encouraged me to continue my studies and help me in all aspects, including moral and material.

To my dear wife for my heart that helped me and tolerated me to be busy with her and my daughter because of the study.

To my brothers and sisters and friends who have continued with me and helped me.

Acknowledgements

First of all, the great continuous thank to Allah who gave me the ability and patience to handle the years I spent during my study.

I would like to express my deepest appreciation to my supervisor Dr. Abdelhalim Ziqan for the generosity, guidance and helpful suggestions throughout the preparation of this work.

My sincere thanks to the examination committee for their careful reading to my thesis and valuable feedback.

I will never forget my family, my father, my mother, my wife, my brothers and my sisters for their infinite support, also my daughter Nada.

Finally, my great gratitude to Mathematics Department at The Arab American University

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Abstract

The Bivariate Shifted Legendre Functions for Nonlinear Volterra Integral Equation

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In this thesis, the Bivariate Shifted Legendre Method for solving one, two and three-dimensional nonlinear Volterra Integral Equation are introduced and analyzed. More specific, three-dimensional Bivariate Shifted Legendre has been investigated in details and some formulas related to three-dimensional Volterra integral equation are deduced. Further, in order to find the approximated solution, the Volterra integral Equation of the first kind is converted to a second kind by using Leibniz integral formula and then the obtained integral equation is reduced to a system of linear algebraic equations using the bivariate shifted Legendre functions operational matrices. Finally, many numerical examples were provided to demonstrate the applicability and the accuracy of the presented method.

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Chapter 1

Introduction

An integral equation is the equation in which the unknown function $u(x)$ appears under an integral sign . In 1884, Volterra started working on integral equations, but his serious study began in 1896. The name "integral equation" was given by Du Bios Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908 [53].

Volterra integral equations arise in many scientific areas such as population dynamics, biology, engineering, spread of epdemics, and semi- conductor devices.

The destinction between Fredhom and Volterra integral equations is analogous to the distinction between boundary and initial value problems in ordinary differential equations [29].

Volterra integral equations can be considered as a generalization of initial value problems. In practice, Volterra equations frequently occur in connection with time- dependent or evolutionary systems [11], [26].

The most well known formula of integral equations is given by:

$$h(x)u(x) = g(x)f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \tag{1.1}$$

where $\alpha(x)$ and $\beta(x)$ are the limits of integration such that either both of them are functions or both of them constants or at least one of them is constant , λ is a constant, $K(x,t)$ is a known function of two variables x, and t which is called the kernel of the integral equation, while the function $u(x)$ is the solution of integral equation. [28].

Basing on the limits of integrations, equation (1.1) can be classified into two known types as follows

1. If at least one limit of integration in eqation (1.1) is variable, then it is called a Volterra integral equation given and it has the form:

$$h(x)u(x) = g(x)f(x) + \lambda \int_a^x k(x,t)u(t)dt \tag{1.2}$$

   If $h(x) = 0$, eqation (1.1) is called a Volterra integral equation of the first kind.

   If $h(x) = 1$, eqation (1.1) is called a Volterra integral equation of the second kind.

   The function $g(x)$ determines the homogeneity of the equation.

2. If the limits of integration in equation 1.1 are constants, eqation (1.1) is called Fredholm integral equation and given by the formula:

$$h(x)u(x) = g(x)f(x) + \lambda \int_a^b k(x,t)u(t)dt. \tag{1.3}$$

   Where a and b are constants.

   Also, it can be characterized in a similar manner as in Volterra integral equation.

It is interesting to mention that any equation includes both derivatives and integrals of the unknown function is called integro-differential equation, as for example the following equation

$$h(x)\frac{d^k u}{dx^k}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt \tag{1.4}$$

Moreover, [46] the integral equation is called nonlinear if the kernel function is a nonlinear function in $u(x)$

$$h(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t,u(t))dt \tag{1.5}$$

or it has other form which is :

$$h(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)F(u(t))dt \tag{1.6}$$

where F is a nonlinear function of u

For the n-independent variables $x = (x_1, x_2, ..., x_n)$, the n-dimensional integral equation is:

$$h(x)u(x) = f(x) + \int_G K(x,y)u(y)dy \tag{1.7}$$

where $x, y \in \mathbb{R}^n, G \subseteq \mathbb{R}^n$.

Many analytical and numerical methods has been used to solve integral equations. In [5,6,12,20–22,48,51,53] the author gathered many analytical and numerical methods to solve different types of one and two-dimensional integral equations like the Adomian decomposition method, modified Adomian decomposition method, Variational iteration method, Laplace transform method, successive approximations method, series solution method, rationalized haar functions, triangular functions, collocation and iterated collocation, and differential transform method . It was noted that little has been done to solve the first kind cases. The numerical solution of the equation of the first kind has been considered in [8, 9, 34]. Malik Negad et.al [34] studied the numerical solution of the first kind using block- pulse functions.

In this thesis, the method that introduced in [40] was extended to solve the nonlinear Volterra integral equation of the first kind with some conditions. The considered problem is solved using the Bivariate shifted Legendre orthogonal functions. The main idea of this technique is to reduce the equation to a systems of nonlinear algebraic equations [40]. In addition to the current chapter, the thesis is arranged as follows: in chapter two, general concepts like inner product, and Newton's method have been introduced. In chapters three four and five, the method of the Bivariate shifted Legendre functions, operational matrices of integration, and the product of operational matrices are introduced in details and the approximated solution of the one, two, and three dimensional Volterra integral equations discussed. Numerical examples as will as figures obtained from the simulation that done using Matlab to illustrate the accuracy of the presented method.

Chapter 2

Preliminaries

In this chapter, and to be familiar with the notations that will be used throughout this work, some basic terminologies are needed. Mainly, the chapter consists of the definition of inner product spaces and the method of Newton for systems of nonlinear equations.

2.1 Inner Product

Recall that if u, and v are two vectors in 3-space, then the inner product $< u, v >$ is a function satisfying the following conditions: $[1, 56]$:

1. $< u, v >=< v, u >$
2. $< Ku, v >= K < u, v >$, K is scaler
3. $< u, u >= 0$ if $u = 0$, and $< u, u >> 0$ if $u \neq 0$
4. $< u + v, w >=< u, w > + < v, w >$, where w is a vector.

Throughout this work, we will deal with real valued functions defined on a finite interval $[a, b]$.

Definition 2.1 *[56](Inner product of function) The inner product of two functions f and g on an interval $[a, b]$ is the number $< f, g >$.*

$$< f, g >= \int_a^b f(x)g(x)dx$$

2.2 Orthogonal Functions

Motivated by the fact that two geometric vectors u and v are orthogonal whenever their inner product is zero. Define orthogonal functions in the same manner.

Definition 2.2 *[38, 56](Orthogonal Functions) Two fnctions f and g are orthogonal on an interval $[a, b]$ if*

$$< f, g >= \int_a^b f(x)g(x)dx = 0 \tag{2.1}$$

2.2.1 Orthogonal sets

We are primarily interested in infinite sets of orthogonal functions.

Definition 2.3 *[56](Weight Function) A set of real- valued functions $\{\psi_0(x),\ \psi_1(x), ...\}$ is said to be orthogonal with respect to a nonnegative function $w(x)$ on an interval $[a, b]$ if*

$$< \psi_m(x), \psi_n(x) >= \int_a^b w(x)\psi_m(x), \psi_m(x)dx = 0, m \neq n$$

where $w(x)$ is called the weight function. The norm of $\psi_m(x)$ is defined as

$$< \psi_m(x), \psi_m(x) >= ||\psi_m(x)||^2 = \int_a^b w(x)|\psi_m(x)|^2 dx$$

The inner product space of all continuous real valued functions on the interval$[-1, 1]$ with the inner product defined in (2.1) can be completed to give a Hilbert space. This space consists of a special type of polynomials which are called Legendre's Polynomials.

Definition 2.4 *[27](Legendre Functions) The Legendre polynomials are defined by Rodrigues formula*

$$L_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \ , \ n = 0, 1, 2, ... \tag{2.2}$$

For arbitrary real or complex values of the variable x.

The first three Legendre polynomials are: $L_0(x) = 1, \ L_1(x) = x, \ L_2(x) = \frac{1}{2}(3x^2-1)$. The general expression for the n^{th} Legendre polynomials is obtined from 2.2 using Binomial expansion

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k}$$

which implies that:

$$L_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (2n-2k)!}{2^k k!(n-k)!(n-2k)!} x^{n-2k}$$

where the symbol $[\frac{n}{2}]$ denotes the largest integer less than or equal $[\frac{n}{2}]$.

These specific n^{th} degree polynomials are called Legendre polynomials and are denoted $L_n(x)$. [4]

Recurrence relations that relate Legendre polynomials of different degrees are also important in some aspects of their applications.

We state, without proof, the three-term recurrence relation.

The well known Legendre polynomials of degree n are defined on the interval $[-1, 1]$ as follows:

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x), \quad n = 1, 2, 3, ...,$$

where $L_0(x) = 1, \quad L_1(x) = x$.

Although, we have assumed that the parameter n in Legendre's differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dL_n(x)}{dx}\right]+n(n+1)L_n(x)=0,$$

$n = 0, 1, 2, ...$ Represent a non negative integer. for more general settings n can be any real number. Any solution of legendre's equation is called a Legendre function.

The orthogonality condition is:

$$\int_{-1}^{1} L_n(x)L_m(x)dx = \begin{cases} \frac{2}{2n+1} & \text{if } m = n \\ 0 & otherwise \end{cases}$$

For one to use these polynomials on the interval $[0, l]$, we defined the so called Shifted Legendre polynomials using the change of variables $x = \frac{2}{l}t-1$. Let the Shifted Legendre polynomials denoted by $P_n(t)$, which can be obtained as follows:

$$P_{n+1}(t) = \frac{(2n+1)}{(n+1)}(\frac{2}{l}t-1)P_n(t) - \frac{n}{n+1}P_{n-1}(t), \quad n = 1, 2, ...$$

where $P_0(t) = 1, \quad P_1(t) = \frac{2}{l}t - 1.$

The analytical form [13] of the Shifted Legendre polynomials $P_n(t)$ of degree n is given by:

$$P_n(t) = \sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!t^k}{(n-k)!(k!)^2}$$

The orthogonality condition is:

$$\int_{0}^{l} P_n(t)P_m(t)dt = \begin{cases} \frac{l}{2n+1} & \text{if } m = n \\ 0 & otherwise \end{cases}$$

2.2.2 The Kronecker Product

Definition 2.5 *[50, 54] Let $A = [a_{ij}]$ be an $m \times n$ matrix, and let B be a $p \times q$, then the Kronecker product of A and B is that $(mp) \times (nq)$ matrix defined by:*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{bmatrix}$$

Remarks:

1. Sometimes the kronecker product is also called direct product or tensor product.

2. Let I_n be the $n \times n$ identity matrix, and let I_m be the $m \times m$ identity matrix. Then $I_n \otimes I_m$ is the $nm \times mn$ identity matrix. Obviously $I_m \otimes I_n = I_n \otimes I_m = I_{mn}$.

3. Let A_n be an arbitrary $n \times n$ matrix, and let O_m be the $m \times m$ zero matrix. Then $A_n \otimes O_m = O_{mn}$.

4. The kronecker product is satisfying the Distributivity and the Associativity properties, i.e:

$$\begin{aligned} A \otimes (B + C) &= A \otimes B + A \otimes C, \quad (\textit{Distributivity}), \\ (A \otimes B) \otimes C &= A \otimes (B \otimes C), \quad (\textit{Associativity}). \end{aligned}$$

2.3 Newton's Method for System of nonlinear equations

Newton Raphson method is an iterative method for solving nonlinear system of algebraic equations. [55]

The system of n- nonlinear equations in n unknowns is given by:

$$\begin{aligned}
f_1(x_1, x_2, ..., x_n) &= 0 \\
f_2(x_1, x_2, ..., x_n) &= 0 \\
&\vdots \\
f_n(x_1, x_2, ..., x_n) &= 0
\end{aligned}$$

This system can be written using a single form as:

$$F(X) = O$$

where the vector X contains the variabls $(x_1, x_2, ..., x_n)$, the vector F is the vector of functions f_i, and O is the zero vecor.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}. \tag{2.3}$$

As in newton method of one variable, we need to start with an initial guess X_0. In theory, the more variables one has, the harder it is to find the initial guess. In practice, that must be overcome by reasonable assumptions about the possible values of the solution. Once X_0 is chosen, let $\Delta X = X_1 - X_0$, then it can be approximated around the vector X_0 using Taylor expansion as follows [49, 52]:

$$F(X_0 + \Delta X) \approx F(X_0) + J(F(X_0))\Delta X, \tag{2.4}$$

where

$$
\begin{aligned}
J(F(X_0)) &= \frac{\partial(f_1, f_2, ..., f_n)}{\partial(x_1, x_2, ..., x_n)}(X_0) \\
J(F(X_0) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(X_0) & \frac{\partial f_1}{\partial x_2}(X_0) & ... & \frac{\partial f_1}{\partial x_n}(X_0) \\ \frac{\partial f_2}{\partial x_1}(X_0) & \frac{\partial f_2}{\partial x_2}(X_0) & ... & \frac{\partial f_2}{\partial x_n}(X_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(X_0) & \frac{\partial f_n}{\partial x_2}(X_0) & ... & \frac{\partial f_n}{\partial x_n}(X_0) \end{bmatrix}
\end{aligned}
$$

,

J is called the Jacobian.

Newton's method is based on constructing of a sequence of vectors that converges to X, such that $F(X) = 0$. Let F be a continuously differentiable function at X_0, and the target is to find X that makes $F(X)$ equal to the zero vector. For that purpose, choose X_1 as follows [16]:

$$F(X_0) = J(F(X_0))(X_1 - X_0) = 0$$

Since $J(F(X_0))$ is a a square matrix, we can solve this equation by:

$$X_1 = X_0 - J^{-1}(F(X_0))F(X_0)$$

In general

$$X_{n+1} = X_n - J^{-1}(F(X_n))F(X_n)$$

provided that the inverse of the Jacobian matrix is exist. However, in practice, we never use the inverse of matrix for computations. Rather, we can do the following. First, solve the equation

$$J(F(X_0))\Delta X = -F(X_0)$$

$J(F(X_0)),\quad F(X_0)$ are knows, so it is a linear system of equations, which can be solved efficiently and accurately. Once we have the solution of the vector ΔX, we can obtain the improved estimate X_1 which is

$$X_1 = X_0 + \Delta X$$

For subsequent steps, we have the following process:

$$J(F(X_n))\Delta X = -F(X_n)$$

where $X_{n+1} = \Delta X + X_n$.

A convergence criterion of the solution of the system of nonlinear equations could be ,for example, that the maximum of the absolute values of the function $f_i(X_n)$ is smaller that a certain tolerance ϵ,

$$Max_i|f_i(X_n)| < \epsilon$$

Another possibility for convergence is that the magnitude of the vector $F(X_n)$ be smaller than the tolerance

$$|F(X_n)| < \epsilon$$

We can also use the difference consecutive values of the solution

$$Max_i|(X_i)_{n+1} - (X_i)_n| < \epsilon$$

or $\Delta X = X_{n+1} - X_n < \epsilon$.

Chapter 3

One-Dimensional Nonlinear Volterra Integral Equation

In this chapter, we present a numerical methods for the solution of nonlinear one-dimensional Volterra integral equation **1D-VIE** of the form

$$\int_0^x K(x,t)u^p(t)dt = f(x), x \in [0,l], \quad t \in [0,l]. \tag{3.1}$$

where $u(x)$ is an unknown function called the solution of the integral equation, p is a positive integer number, K is the kernel function and f is a smooth function. Also, the functions f and K are required to satisfy the conditions $f(0) = 0$ and $K(x,x) \neq 0, \forall x \in [0,l]$. Many problems in mathematics, physics and engineering could be reduced to integral equations of the first kind are inherently ill-posed problems meaning that the solution is generally unstable, and small changes to the problem can make very large changes to the solutions. [47]

As a result of this type of problems, the numerical solution becoms very difficult to reach. In deed, a small error could lead to an unbounded error. To overcome the ill-posdenss, we transform nonlinear **1D-VIE** of the first kind with the conditions $f(0) = 0$, $K(x,x) \neq 0$, $\forall x \in [0,l]$ to a nonlinear **1D- VIE** of the second kind. [43] The nonlinear **1D-VIE** of the second kind can be obtained by differentiating equation (3.1) with respect to x using Leibniz's Integral Rule.

Theorem 3.1 *[23][Leibniz's Integral Rule] Suppose that the function g has a uniformly continuous partial derivative $\frac{\partial g}{\partial x}$ and let*

$$f(x) = \int_{a(x)}^{b(x)} g(x, y)\, dy$$

Where a and b are continuously differentiable funtions defined on(x_0, x_1). Then, for $x \in (x_0, x_1)$ we have

$$\frac{\partial f}{\partial x} = \int_{a(x)}^{b(x)} \frac{\partial g(x, y)}{\partial x}\, dy + b'(x)g(x, b(x)) - a'(x)g(x, a(x)) \tag{3.2}$$

Thus, using (3.2), the integral equation(3.1) can be reduced to :

$$\frac{\partial f}{\partial x} = \int_{0}^{x} \frac{\partial K(x, t)}{\partial x} u^p(t)dt + K(x, x)u^p(x) \tag{3.3}$$

Equation (3.3) then will have the following shape:

$$u^p(x) \;\; = \;\; \int_{0}^{x} K_1(x, t)u^p(t)dt + F(x) \tag{3.4}$$

where $K_1(x, t) = -\frac{\partial K(x,t)}{\partial x}/K(x, x)$, and $F(x) = \frac{\partial f(x)}{\partial x}/K(x, x)$

As a conclusion, in order to solve equation (3.1), it is enough to solve equation (3.4).

Let us first intorduce the shifted lengedre polynomial as follows

Definition 3.1 *[2, 15] The well-known Legendre polynomials of degree $n = 0, 1, \ldots$ are defined on the interval [-1,1] and can be determined with the aid of the following* *recurrence formula*

$$L_{n+1}(t) = \left(\frac{2n+1}{n+1}\right) t L_n(t) - \left(\frac{n}{n+1}\right) L_{n-1}(t), \quad n = 1, 2, \dots \tag{3.5}$$

where $L_0(t) = 1$ *and* $L_1(t) = t$.

For one to use these shifted polynomials on the interval $[a, b] = [0, l]$, we define the so called shifted legnedre polynomials by using the change of variable

$$t = \frac{2}{b-a}[x - \frac{b+a}{2}] \implies t = \frac{2}{l}(x - \frac{l}{2}) = \frac{2}{l}x - 1.$$

Now, let the shifted legendre polynomials $L_n(\frac{2}{l}x - 1)$ be deonted by $P_n(x)$, then $P_n(x)$ can be obtained as follows

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right)\left(\frac{2}{l}x - 1\right) P_n(x) - \left(\frac{n}{n+1}\right) P_{n-1}(x), \quad n = 1, 2, \dots \tag{3.6}$$

where $P_0(x) = 1$ and $P_1(x) = \frac{2}{l}x - 1$. Let $P_n(x)$ and $P_m(x)$ be two functions defined on some interval $[0, l]$. These functions are called with respect to a weight function $w(x) = 1$ on $[0, l]$ s.t.

$$\int_0^l w(x) P_n(x) P_m(x) dx = \begin{cases} \frac{l}{2n+1} & \text{if } m = n \\ 0 & \textit{otherwise} \end{cases}$$

The bivariate shifted Legendre functions are defined on $x \in [0, l]$ as follows

$$\psi_n(x) = P_n(x) = L_n(\frac{2}{l}x - 1), \quad n = 0, 1, 2, \dots \tag{3.7}$$

and are orthgonal with respect to the weight function $w(x) = 1$ such that

$$\int_0^l w(x)\psi_n(x)\psi_m(x)dx = \begin{cases} \frac{l}{2n+1} & \text{if m = n} \\ 0 & o.w \end{cases} \tag{3.8}$$

3.1 Approximation with shifted legendre polynomials

A function $f(x) \in L^2[0, l]$ may be expanded by in terms of Legendre polynomials as follows

$$f(x) = \sum_{n=0}^{\infty} C_n\psi_n(x) \tag{3.9}$$

where C_n is constant given by

$$C_n = \frac{\langle f(x), \psi_n(x)\rangle}{\langle \psi_n(x), \psi_n(x)\rangle}. \tag{3.10}$$

If the infinite series in equation(3.9) is truncated up to term N, [30, 47] then

$$f(x) \simeq \sum_{n=0}^{N} C_n\psi_n(x) = C^T\psi(x) \tag{3.11}$$

Where C and ψ are $(N + 1) \times 1$ vectors given by

$$C = \begin{bmatrix} C_0 & C_1 \cdots C_N \end{bmatrix}^T \text{ and } \psi = \begin{bmatrix} \psi_0 & \psi_1 \cdots \psi_N \end{bmatrix}^T$$

Now,

$$\langle f(x), \psi_n(x)\rangle = \left\langle \sum_{j=0}^{\infty} C_j\psi_j(x), \psi_n(x) \right\rangle \tag{3.12}$$

$$= \langle C_n\psi_n(x), \psi_n(x)\rangle$$

$$= C_n\langle \psi_n(x), \psi_n(x)\rangle$$

Therefore, from equation(3.8) the constant C_n can be computed by the following formula [10, 32]

$$C_n = \frac{\langle f(x), \psi_n(x)\rangle}{\| \psi_n \|_2^2} = \frac{2n+1}{l} \int_0^l f(x)\psi_n(x)dx \tag{3.13}$$

3.2 Approximating the kernel function

Let $K(x,t) \in L^2\big([0,l] \times [0,l]\big)$, we have

$$K(x,t) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \psi_r(x) K_{rs} \psi_s(t) \tag{3.14}$$

where

$$K_{rs} = \frac{\langle \psi_r(x), \langle K(x,t), \psi_s(t)\rangle\rangle}{\langle \psi_r(x), \psi_r(x)\rangle \langle \psi_s(t), \psi_s(t)\rangle} \tag{3.15}$$

And can be accessed through the previous equation [45]

$$\langle \psi_r(x), \langle K(x,t), \psi_s(t)\rangle\rangle = \left\langle \psi_r(x), \langle \sum_{r,s=0}^{\infty} \psi_r(x) K_{rs} \psi_s(t), \psi_s(t)\rangle \right\rangle$$

$$= \langle \psi_r(x), K_{rs} \| \psi_s(t) \|_2^2 \psi_r(x)\rangle = K_{rs} \| \psi_r(x) \|_2^2 \| \psi_s(t) \|_2^2$$

$$K_{rs} = \frac{\langle \psi_r(x), \langle K(x,t), \psi_s(t)\rangle\rangle}{\| \psi_r(x) \|_2^2 \| \psi_s(t) \|_2^2}$$

Therefore,

$$K_{rs} = \frac{(2r+1)}{l} \frac{(2s+1)}{l} \int_0^l \int_0^l \psi_r(x) K(x,t) \psi_s(t)\, dt\, dx \tag{3.16}$$

If the infinite series in equation(3.14) is truncated, then it can be written as

$$K(x,t) \simeq \sum_{r=0}^{N} \sum_{s=0}^{N} \psi_r(x) K_{rs} \psi_s(t) = \psi^T(x) K \psi(t) \tag{3.17}$$

$$K(x,t) \simeq \psi^T(x) K \psi(t), \quad r = s = 0, 1, \cdots, N \tag{3.18}$$

Similarly, any function $K^1(x,t) \in L^2\big([0,l]\times[0,l]\big)$ can be expanded in terms of bivariate shifted legendre functions as [32, 44]

$$K_1(x,t) \simeq \psi^T(x) K^1 \psi(t) \tag{3.19}$$

where K^1 is a $(N+1)\times(N+1)$ block matrix has the form

$$K^1 = [k_r]_{r=0}^N = \begin{bmatrix} k_0 \\ k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix}$$

and $k_i = \begin{bmatrix} k_{i0} & k_{i1} & \cdots & k_{iN} \end{bmatrix}, \quad i = 0, 1, \ldots, N$

Therefore, $\quad K^1 = \begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix}.$

3.3 Integration opertational matrix

The ordinary differential recurrence relations of the orthogonal Legendre polynomials with respect to x is given by

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{3.20}$$

After differentiating with respect to x and rearranging the terms we get the following

$$(2n+1)P_n(x) = (n+1)\dot{P}_{n+1}(x) - (2n+1)x\dot{P}_n(x) + n\dot{P}_{n-1}(x) \tag{3.21}$$

On the other hand, we have [3]

$$x\dot{P}_n(x) = nP_n(x) + \dot{P}_{n-1}(x) \tag{3.22}$$

Now, [15] Substitute equation(3.22) in equation(3.21) to get

$$P_n(x) = \frac{(n+1)}{(2n+1)}\dot{P}_{n+1}(x) - \frac{(2n+1)}{(2n+1)}\Bigg[nP_n(x) + \dot{P}_{n-1}(x)\Bigg] + \frac{n}{(2n+1)}\dot{P}_{n-1}(x)$$

$$P_n(x) + nP_n(x) = \frac{(n+1)}{(2n+1)}\dot{P}_{n+1}(x) - \dot{P}_{n-1}(x) + \frac{n}{(2n+1)}\dot{P}_{n-1}(x)$$

$$(2n+1)P_n = \dot{P}_{n+1}(x) - \dot{P}_{n-1}(x) \tag{3.23}$$

Equation(3.23) can be written in the form

$$\psi_n(x) = A_n\dot{\psi}_{n+1}(x) + B_n\dot{\psi}_n(x) + C_n\dot{\psi}_{n-1}(x) \tag{3.24}$$

Where $\psi_n(x)$ are the shifted legendre polynomials on $[0, l]$ and the coefficients A_n, B_n and C_n, $\quad n = 0, 1, 2, \cdots$ are constants. The coeficients A_n, B_n and C_n can be evaluated by integraing the recurrence relation(3.24) from 0 to x as follows

$$\int_0^x \psi_n(x)\,dx = A_n\psi_{n+1}(x) + B_n\psi_n(x) + C_n\psi_{n-1}(x) + D_n\psi_0(x) \tag{3.25}$$

Then Substitute $n = 0, 1, \ldots$ in equation(3.25)to get

$$A_0 = \frac{l}{2}, \quad B_0 = \frac{l}{2}, \quad C_0 = 0, \quad D_0 = 0$$

$$A_1 = \frac{l}{6}, \quad B_1 = 0, \quad C_1 = \frac{-l}{6}, \quad D_1 = 0$$

$$A_2 = \frac{l}{10}, \quad B_2 = 0, \quad C_2 = \frac{-l}{10}, \quad D_2 = 0$$

$$A_n = \frac{l}{2(2n+1)}, \quad B_n = 0, \quad C_n = \frac{-l}{2(2n+1)}, \quad D_n = 0, \quad n \geq 1$$

For simplicity, we may write equation(3.25) in the form [17, 44]

$$\int_0^x \psi(t)\, dt \simeq P\psi(x) \tag{3.26}$$

Where ψ has the form

$$\psi(x) = \Big[\psi_0, \psi_1, \psi_2, \cdots, \psi_N\Big]^T. \tag{3.27}$$

The matrix P is an $(N+1)\times(N+1)$ matrix which is called the integration operational matrix of one-dimensional shifted Legendre polynomials and has the following form [38, 44, 45]

$$P = \begin{bmatrix} B_o & A_o & 0 & 0 & ... & 0 & 0 & 0 \\ D_1 + C_1 & B_1 & A_1 & 0 & ... & 0 & 0 & 0 \\ D_2 & C_2 & B_2 & A_2 & ... & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ D_{n-2} & 0 & 0 & 0 & ... & C_{n-2} & B_{n-2} & A_{n-2} \\ D_{n-1} & 0 & 0 & 0 & ... & 0 & C_{n-1} & B_{n-1} \end{bmatrix}$$

$$= \frac{l}{2} \begin{bmatrix} 1 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & ... & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & ... & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & ... & \frac{-1}{2N-1} & 0 & \frac{1}{2N-1} \\ 0 & 0 & 0 & 0 & ... & 0 & \frac{-1}{2N+1} & 0 \end{bmatrix}$$

Thus, it is shown that any arbitrary signal can be approximated as a series of orhogonal functions. If the number of terms in the series is finite, then the integral of the square of the error is a minimum in the approximation. The polynomial $\psi_n(x)$ of degree n in any system of orthogonal polynomials can be generated by solving a differential equation of order $(2m+1)$. Fortunately, a three-term recurrence formula is available by menas of which the orthogonal polynomials in any system can be generated recursively, or consequently, one should not go for the solution of the differential equation. By choosing the weighting-function $w(x)$ and the interval $[0, l]$, any syestem of orthogonal polynomials, e.g, Legendre, Laguerre, Hermite can be generated. Each of these orthgonoal polynomials is shown to satisfy a differential equation and an ordinary differential recurrence relation, the latter is found to be useful in the derivation of the integration operational matrix. [14]

3.4 The product operational matrix

The aim of this section is to compute the product of $\psi(x)$ and $\psi^T(x)$, and put the result in a compact form. This result will be needed for further computations in this work. Now, let $C = \Big[C_0, C_1, C_2, \cdots, C_N\Big]^T$ and from equation (3.27) we have [24, 42]

$$\begin{aligned}
\psi(x)\psi^T(x)C &= \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} \begin{bmatrix} \psi_0 & \psi_1 & \cdots & \psi_N \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix} \\
&= \begin{bmatrix} \psi_0\psi_0 & \psi_0\psi_1 & \cdots & \psi_0\psi_N \\ \psi_1\psi_0 & \psi_1\psi_1 & \cdots & \psi_1\psi_N \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N\psi_0 & \psi_N\psi_1 & \cdots & \psi_N\psi_N \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_N \end{bmatrix}.
\end{aligned}$$

Simply, the last matrix can be written as follows:

$$\psi(x)\psi^T(x)C = \begin{bmatrix} \sum_{j=0}^{N} C_j\psi_j\psi_0 \\ \sum_{j=0}^{N} C_j\psi_j\psi_1 \\ \vdots \\ \sum_{j=0}^{N} C_j\psi_j\psi_N \end{bmatrix} \tag{3.28}$$

Now, let

$$\psi_i(x)\psi_j(x) = \sum_{k=0}^{i+j} a_{ijk}\psi_k(x). \tag{3.29}$$

From the orthogonality property, the coefficients a_{ijk} can be computed using the formula below.

$$a_{ijn} = \left(\frac{2n+1}{l}\right)\int_0^l \psi_i(x)\psi_j(x)\psi_n(x)\,dx, \quad i,j,n = 0,1,\cdots,N \tag{3.30}$$

Also, we let $w_{ijn} = \int_0^l \psi_i(x)\psi_j(x)\psi_n(x)\,dx$, then $a_{ijn} = \frac{2n+1}{l}w_{ijn}$ Therefore, equation(3.29) becomes [38]

$$\psi_i(x)\psi_j(x) \simeq \sum_{n=0}^{N}\left(\frac{2n+1}{l}\right)w_{ijn}\psi_n(x) \tag{3.31}$$

Now, plug equation(3.31) into equation(3.28) to get:

$$\psi(x)\psi^T(x)C \simeq \begin{bmatrix} \sum_{j=0}^{N}\sum_{n=0}^{N} C_j\left(\frac{2n+1}{l}\right)w_{ojn}\psi_n(x) \\ \sum_{j=0}^{N}\sum_{n=0}^{N} C_j\left(\frac{2n+1}{l}\right)w_{1jn}\psi_n(x) \\ \vdots \\ \sum_{j=0}^{N}\sum_{n=0}^{N} C_j\left(\frac{2n+1}{l}\right)w_{Njn}\psi_n(x) \end{bmatrix}$$

Note that $w_{inj} = w_{ijn}$. Replace n by j, and j by n and gathering similar terms to reach the desired form:

$$\psi(x)\psi^T(x)C \simeq \tilde{C}\psi(x) \tag{3.32}$$

, where $\tilde{C}$ is a $(N+1)\times(N+1)$ product operational matrix

$$\tilde{C} = \begin{bmatrix} \tilde{C}_{00} & \tilde{C}_{01} & \tilde{C}_{02} & \cdots \tilde{C}_{0N} \\ \tilde{C}_{10} & \tilde{C}_{11} & \tilde{C}_{12} & \cdots \tilde{C}_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_{N0} & \tilde{C}_{N1} & \tilde{C}_{N2} & \cdots \tilde{C}_{NN} \end{bmatrix}$$

And, each element in $\tilde{C}$ can be obtained as follows [32]

$$\tilde{C}_{i,j} = \left(\frac{2j+1}{l}\right) \sum_{n=0}^{N} w_{ijn} C_n, \quad i, j, n = 0, 1, \cdots, N \tag{3.33}$$

Finally, before moving to the method of solution, let us introduce the following result.

Lemma 3.2 *For an* $(N+1) \times (N+1)$ *matrix* B, *we have*

$$\psi^T(x) B \psi(x) \simeq \widehat{B} \psi(x), \tag{3.34}$$

where

$$B = \begin{bmatrix} b_{00} & b_{01} & b_{02} & \cdots & b_{0N} \\ b_{10} & b_{11} & b_{12} & \cdots & b_{1N} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{N0} & b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}$$

And $\widehat{B}$ *is a* $1 \times (N+1)$ *vector defined as [42, 45]* $\widehat{B} = \begin{bmatrix} B_0 & B_1 & \cdots & B_N \end{bmatrix}$ *with*

$$B_n = \frac{2n+1}{l} \sum_{i=0}^{N} \sum_{r=0}^{N} w_{irn} b_{ir}$$

Proof:

$$\psi^T(x)B\psi(x) = \sum_{j=0}^{N} b_{0j}\psi_0\psi_j + \sum_{j=0}^{N} b_{1j}\psi_1\psi_j + ... + \sum_{j=0}^{N} b_{Nj}\psi_N\psi_j$$

According to equation (3.31), we get:

$$\begin{aligned} \psi^T(x)B\psi(x) &\simeq \sum_{n=0}^{N}\sum_{j=0}^{N}(\frac{2n+1}{l})b_{0j}w_{0jn}\psi_n(x) + \sum_{n=0}^{N}\sum_{j=0}^{N}(\frac{2n+1}{l})b_{1j}w_{1jn}\psi_n(x) \\ &+ ... + \sum_{n=0}^{N}\sum_{j=0}^{N}(\frac{2n+1}{l})b_{Nj}w_{Njn}\psi_n(x) \end{aligned}$$

When removing these totals and gathering some similar terms to some properties, we get the following figure:

$$\begin{aligned} \psi^T(x)B\psi(x) &\simeq \sum_{j=0}^{N}(\frac{2(0)+1}{l})b_{0j}w_{0j0}\psi_0(x) + \sum_{j=0}^{N}(\frac{2(1)+1}{l})b_{0j}w_{0j1}\psi_1(x) \\ &+ ... + \sum_{j=0}^{N}(\frac{2(N)+1}{l})b_{0j}w_{0jN}\psi_N(x) + \sum_{j=0}^{N}(\frac{2(0)+1}{l})b_{1j}w_{1j0}\psi_0(x) \\ &+ \sum_{j=0}^{N}(\frac{2(1)+1}{l})b_{1j}w_{1j1}\psi_1(x) + ... + \sum_{j=0}^{N}(\frac{2(N)+1}{l})b_{1j}w_{1jN}\psi_N(x) \\ &+ ... + \sum_{j=0}^{N}(\frac{2(0)+1}{l})b_{Nj}w_{Nj0}\psi_0(x) + \sum_{j=0}^{N}(\frac{2(1)+1}{l})b_{Nj}w_{Nj1}\psi_1(x) \\ &+ ... + \sum_{j=0}^{N}(\frac{2(N)+1}{l})b_{Nj}w_{NjN}\psi_N(x) \end{aligned}$$

$$\begin{aligned} \psi^T(x)B\psi(x) &\simeq \sum_{i=0}^{N}\sum_{j=0}^{N}(\frac{2(0)+1}{l})b_{ij}w_{ij0}\psi_0(x) + \sum_{i=0}^{N}\sum_{j=0}^{N}(\frac{2(1)+1}{l})b_{ij}w_{ij1}\psi_1(x) \\ &+ ... + \sum_{i=0}^{N}\sum_{j=0}^{N}(\frac{2(N)+1}{l})b_{ij}w_{ijN}\psi_N(x) \\ &= B_0\psi_0(x) + B_1\psi_1(x) + ... + B_N\psi_N(x) \end{aligned}$$

Therefore, [40]

$$\psi^T(x)B\psi(x) \simeq \begin{bmatrix} B_0 & B_1 & \cdots & B_N \end{bmatrix} \begin{bmatrix} \psi_0 \\ \psi_1 \\ \vdots \\ \psi_N \end{bmatrix} = \widehat{B}\psi(x) \tag{3.35}$$

where $B_n = \left(\frac{2n+1}{l}\right) \sum_{i=0}^{N} \sum_{j=0}^{N} b_{ij} w_{ijn}$. □

3.5 Method of solution

In this section, we present a numerical method to find an approximate solution to the general problem(3.1) and conditions $f(0) = 0$, $K(x,x) \neq 0$, $\forall x \in [0,l]$ which corresponds with equation(3.4). We assume that the known functions in equation (3.1) satisfy the conditions that this equation has a unique solution. [43] Now, using the way mentioned previously, the functions $u^p(x), F(x)$, and $K_1(x,t)$ can be approximated by the bivariate shifted legendre polynomials as

$$u^p(x) \simeq C^T\psi(x) = \psi^T(x)C \tag{3.36}$$

$$F(x) \simeq F^T\psi(x) \tag{3.37}$$

Where F^T is an $1 \times (N+1)$ matrix of constants

$F^T = \begin{bmatrix} F_0 & F_1 & \dots & F_N \end{bmatrix}$ such that,

$$F_n = \frac{< F(x), \psi_n(x) >}{< \psi_n(x), \psi_n(x) >} = (\frac{2n+1}{l}) \int_0^l F(x)\psi_n(x)dx$$

$$K_1(x,t) \simeq \psi^T(x)K^1\psi(t) \tag{3.38}$$

Now, substituting equations(3.36),(3.37) and (3.38) in equation(3.4), this yiels to:

$$C^T\psi(x) = \int_0^x \psi^T(x)K^1\psi(t)\psi^T(t)C\,dt + F^T\psi(x)$$

Using equation (3.32) to get:

$$\begin{aligned} C^T\psi(x) &= \psi^T(x)K^1\int_0^x \tilde{C}\psi(t)\,dt + F^T\psi(x) \\ &= \psi^T(x)K^1\tilde{C}\int_0^x \psi(t)\,dt + F^T\psi(x) \end{aligned}$$

Depending on equation (3.26) to have:

$$\begin{aligned} C^T\psi(x) &= \psi^T(x)K^1\tilde{C}P\psi(x) + F^T\psi(x) \\ &= \psi^T(x)B\psi(x) + F^T\psi(x), \quad \text{since } B = K^1\tilde{C}P \end{aligned} \tag{3.39}$$

Then, by applying result(3.35) we may have

$$C^T\psi(x) = \left(\widehat{B} + F^T\right)\psi(x) \tag{3.40}$$

Hence we have:

$$C^T = \widehat{B} + F^T \tag{3.41}$$

Which corresponds with a system of linear algebriac equations in terms of the unknown elements of the vector C and can be solved easily using direct methods. The unknown function $u(x)$ can be approximated in terms of the bivariate shifted legendre polynomials as

$$u(x) \simeq A^T\psi(x) \tag{3.42}$$

Such that, the entries of the vector A are unknown. Using equation(3.32) and (3.42) it is easily obtained such that

$$u^p(x) \simeq A^T \tilde{A}^{p-1} \psi(x) \tag{3.43}$$

Finally, using $u^p(x) \simeq C^T \psi(x)$ and the result(3.43) we get

$$C^T = A^T \tilde{A}^{p-1} \tag{3.44}$$

Equation(3.44) forms a system of $(N+1)$ nonlinear equations which can be solved for elements of A using numerical methods such as Newton's iterative method.

The result(3.43) can be proved by induction that, $\forall k \in \mathbb{Z}^+$,

By using the result(3.32) and replace C by A we get

$$\psi(x)\psi^T(x)A \simeq \tilde{A}\psi(x) \tag{3.45}$$

Take the transpose of both sides of equation(3.45) this yields to

$$A^T \psi(x)\psi^T(x) \simeq \psi^T(x)\tilde{A}^T$$

Now, [44] apply result(3.42) with $p = 1$ we get

$$u^1(x) \simeq A^T \tilde{A}^0 \psi(x)$$

for $p = 2$,

$$\begin{aligned} u^2(x) &= u^1(x)u^1(x) \simeq A^T \psi(x)\psi^T(x)A \\ u^2(x) &\simeq A^T \tilde{A}^1 \psi(x) \end{aligned}$$

for $p = 3$,

$$\begin{aligned} u^3(x) &= u^2(x)u^1(x) \simeq A^T\tilde{A}\psi(x)(\psi^T(x)A) \simeq A^T\tilde{A}\tilde{A}\psi(x) \\ u^3(x) &\simeq A^T\tilde{A}^2\psi(x) \end{aligned}$$

for $p = 4$,

$$\begin{aligned} u^4(x) &= u^3(x)u^1(x) \simeq A^T\tilde{A}^2\psi(x)\psi^T(x)A \simeq A^T\tilde{A}^2\tilde{A}\psi(x) \\ u^4(x) &\simeq A^T\tilde{A}^3\psi(x) \end{aligned}$$

Let $k \in \mathbb{Z}^+$ be given and suppose equation(3.43) is true for $p = k$. Then

$$\begin{aligned} u^{k+1}(x) &= u^k(x)u^1(x) \simeq A^T\tilde{A}^{k-1}\psi(x)\psi^T(x)A \simeq A^T\tilde{A}^{k-1}\tilde{A}\psi(x) \\ u^{k+1}(x) &\simeq A^T\tilde{A}^k\psi(x) \end{aligned}$$

Thus, equation(3.43) holds for $p = k+1$, and the proof of the induction step is complete.

3.6 Numerical Examples

In this section, some examples are presented to show the reliability of this method. In order to show the error of the method, the following notation is introduced:

$$e_N(x) = |u(x) - u_N(x)|, \quad x \in [0, l]$$

where $u(x)$ is the exact solution, and $u_N(x)$ is the computed result with N.
To solve the examples, we consider N or and Newton's method is used for solving the nonlinear system. The initial guess in Newton's method is for these examples is considered to be $A^o = C$, but the number of iterations can be reduced by choosing a more closed A^o to the exact solution.

Example 3.1 *[53] Consider the following Volterra integral equation of the first kind:*

$$\frac{1}{12}x^4 + \frac{1}{3}x^3 = \int_0^x (x - t + 1)u^2(t)dt, \qquad x \in [0, 1]$$

With the exact solution $u(x) = \pm x$.

The following tables show the exact solution, the approximate solution, and the absolute error at $N = 1, 3, 5$ at some selected grid points.

Tab. 3.1: Exact and numerical solutions at $N = 1$

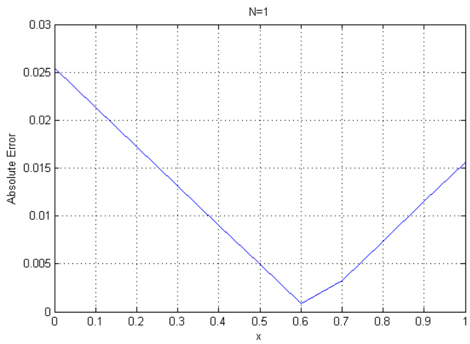
x	Exact	Approximation	Error
0	0	0.025419692516970	$2.541969251697000 \times 10^{-2}$
0.1	0.1	0.121318940774885	$2.131894077488500 \times 10^{-2}$
0.2	0.2	0.217218189032801	$1.721818903280100 \times 10^{-2}$
0.3	0.3	0.313117443729071	$1.311744372907100 \times 10^{-2}$
0.4	0.4	0.409016685548632	$9.016685548632000 \times 10^{-3}$
0.5	0.5	0.504915933806547	$4.915933806546980 \times 10^{-3}$
0.6	0.6	0.600815182064462	$8.151820644620720 \times 10^{-4}$
0.7	0.7	0.696714430322378	$3.285569677621950 \times 10^{-3}$
0.8	0.8	0.792613678580294	$7.386321419706080 \times 10^{-3}$
0.9	0.9	0.888512926838209	$1.148707316179100 \times 10^{-2}$
1	1	0.984412175096124	$1.558782490387600 \times 10^{-2}$

Tab. 3.2: Exact and numerical solutions at $N = 3$

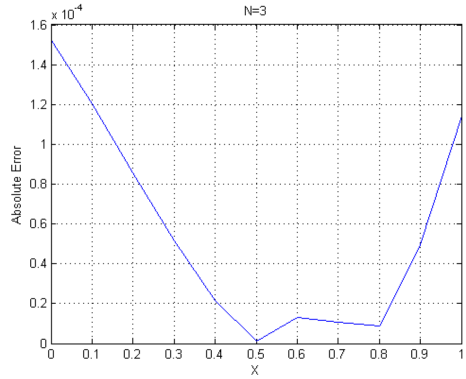
x	Exact	Approximation	Error
0	0	$1.524833089100301 \times 10^{-2}$	1.524833×10^{-4}
0.1	0.1	0.100120404160429	1.204042×10^{-4}
0.2	0.2	0.200085631630599	8.563163×10^{-5}
0.3	0.3	0.300051530302498	5.153030×10^{-5}
0.4	0.4	0.400021464759206	2.146476×10^{-5}
0.5	0.5	0.499998799583798	1.200416×10^{-6}
0.6	0.6	0.599986899359354	1.310064×10^{-5}
0.7	0.7	0.699989128668951	1.087133×10^{-5}
0.8	0.8	0.800008852095667	8.852096×10^{-6}
0.9	0.9	0.900049434222581	4.943422×10^{-5}
1	1	1.000114239632770	1.142396×10^{-4}

Tab. 3.3: Exact and numerical solutions at $N = 5$

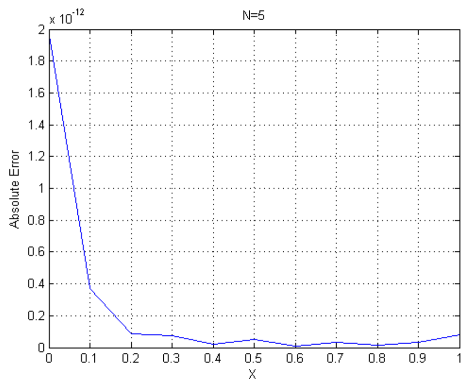
x	Exact	Approximation	Error
0	0	$1.977581646955672 \times 10^{-12}$	1.98×10^{-12}
0.1	0.1	0.100000000000369	3.68997×10^{-13}
0.2	0.2	0.199999999999915	8.50153×10^{-14}
0.3	0.3	0.299999999999992	7.99361×10^{-15}
0.4	0.4	0.400000000000021	2.09832×10^{-14}
0.5	0.5	0.500000000000049	4.89608×10^{-14}
0.6	0.6	0.600000000000008	7.99361×10^{-15}
0.7	0.7	0.699999999999966	3.39728×10^{-14}
0.8	0.8	0.799999999999984	1.60982×10^{-14}
0.9	0.9	0.900000000000034	3.39728×10^{-14}
1	1	0.999999999999922	7.80487×10^{-14}



(a) N =1



(b) N=3



(c) N =5

Fig. 3.1: The values of N versus absolute error at different x values

Example 3.2

$$\frac{1}{2}e^{3x} - \frac{1}{2}e^{2x} = \int_0^x e^{x-t}u^3(t)dt, \qquad x \in [0, 2]$$

The exact solution is $u(x) = e^x$.

The Numerical results are given in the following tables and graphs.

The following tables show the exact solution, the approximate solution, and the absolute error at $M, N = 4, 6, 8$ at some selected grid points.

Tab. 3.4: Exact and numerical solutions at $N = 4$

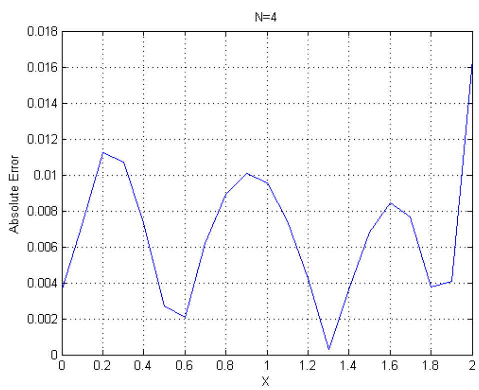
x	Exact	Approximation	Error
0	1	1.00358953	3.589530×10^{-3}
0.2	1.22140276	1.21014893	1.125383×10^{-2}
0.4	1.49182470	1.48448371	7.340990×10^{-3}
0.6	1.82211880	1.82422027	2.101470×10^{-3}
0.8	2.22554093	2.23446707	8.926140×10^{-3}
1	2.71828183	2.72781471	9.532880×10^{-3}
1.2	3.32011692	3.32433586	4.218940×10^{-3}
1.4	4.05519997	4.05158527	3.614700×10^{-3}
1.6	4.95303242	4.94459981	8.432610×10^{-3}
1.8	6.04964746	6.04589843	3.749030×10^{-3}
2	7.38905610	7.40548216	1.642606×10^{-2}

Tab. 3.5: Exact and numerical solutions at $N = 6$

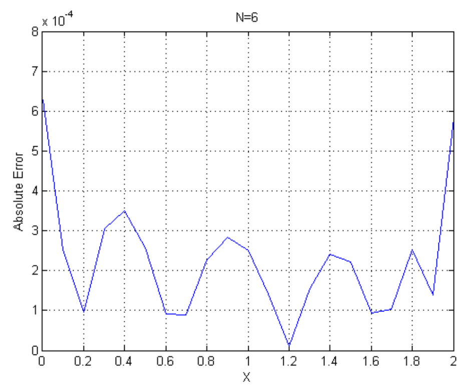
x	Exact	Approximation	Error
0	1	0.99936305	6.3695×10^{-4}
0.2	1.22140276	1.22149906	9.6300×10^{-5}
0.4	1.49182470	1.49217374	3.4904×10^{-4}
0.6	1.82211880	1.82221005	9.1250×10^{-5}
0.8	2.22554093	2.22531469	2.2624×10^{-4}
1	2.71828183	2.71803051	2.5132×10^{-4}
1.2	3.32011692	3.32012861	1.1690×10^{-5}
1.4	4.05519997	4.05544002	2.4005×10^{-4}
1.6	4.95303242	4.95312701	9.4590×10^{-5}
1.8	6.04964746	6.04939404	2.5342×10^{-4}
2	7.38905610	7.38963828	5.8218×10^{-4}

Tab. 3.6: Exact and numerical solutions at $N = 8$

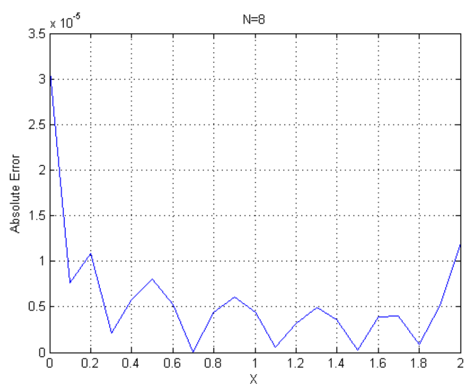
x	Exact	Approximation	Error
0	1	0.99996857	3.1430×10^{-5}
0.2	1.22140276	1.22141358	1.0820×10^{-5}
0.4	1.49182470	1.49181892	5.7800×10^{-6}
0.6	1.82211880	1.82211355	5.2500×10^{-6}
0.8	2.22554093	2.22554537	4.4400×10^{-6}
1	2.71828183	2.71828625	4.4200×10^{-6}
1.2	3.32011692	3.32011371	3.2100×10^{-6}
1.4	4.05519997	4.05519641	3.5600×10^{-6}
1.6	4.95303242	4.95303637	3.9500×10^{-6}
1.8	6.04964746	6.04964654	9.2000×10^{-7}
2	7.38905610	7.38906814	1.2040×10^{-6}



(a) N =4



(b) N=6



(c) N =8

Fig. 3.2: The values of N versus absolute error at different x values

Chapter 4

Two-Dimensional Nonlinear Volterra Integral Equation

In this chapter, a numerical method for the solution of nonlinear two- dimensional Volterra integral equation of the first kind (4.1)is presented.

$$\int_0^t \int_0^x K(x,t,y,z)u^p(y,z)dydz = f(x,t), \quad x,y \in [0,l], \quad t,z \in [0,T] \tag{4.1}$$

$(x,t) \in \Omega$, and $\Omega = [0,l] \times [0,T]$.
where $u(x,t)$ is an unlnown function called the solution of the integral equation, p is a positive integer number, K is the Kernel function, and f is a smooth function.
such that the following conditions are satisfied [40, 42]:

$$f(x,0) = 0, \forall x \in [0,l] \tag{4.2}$$

$$f(0,t) = 0, \forall t \in [0,T] \tag{4.3}$$

$$K(x,t,x,t) \neq 0, \ \forall (x,t) \in \Omega. \tag{4.4}$$

Integral equations of the first kind are in herently ill-posed problems. To overcome the ill-posedness, we transform nonlinear of the first kind with conditions to a nonlinear of the second kind by differentiating equation 4.1 with respect to t and x :

$$\frac{d}{dx}\Big[\frac{d}{dt}\big[\int_0^t\int_0^x K(x,t,y,z)u^p(y,z)dydz = f(x,t)\big]\Big].$$

Using Leibniz integral rule:

$$\begin{aligned}
\frac{d^2 f(x,t)}{dxdt} &= \frac{d}{dx}\Big[\frac{d}{dt}\int_0^t\big(\int_0^x K(x,t,y,z)u^p(y,z)dy\big)dz\Big]\\
&= \int_0^t\int_0^x \frac{d^2}{dxdt}K(x,t,y,z)u^p(y,z)dydz + \int_0^x \frac{d}{dx}K(x,t,y,t)u^p(y,t)dy\\
&+ \int_0^t \frac{d}{dt}K(x,t,x,z)u^p(x,z)dz + K(x,t,x,t)u^p(x,t) \qquad (4.5)
\end{aligned}$$

From equation 4.5 the function $u^p(x,t)$ has the form:

$$\begin{aligned}
u^p(x,t) &= \int_0^t\int_0^x K_1(x,t,y,z)u^p(y,z)dydz + \int_0^x K_2(x,t,y)u^p(y,t)dy\\
&+ \int_0^t K_3(x,t,z)u^p(x,z)dz + F(x,t) \qquad (4.6)
\end{aligned}$$

where:

$$\begin{aligned}
K_1(x,t,y,z) &= -\big[\frac{d^2}{dxdt}K(x,t,y,z)\big]/K(x,t,x,t)\\
K_2(x,t,y) &= -\big[\frac{d}{dx}K(x,t,y,t)\big]/K(x,t,x,t)\\
K_3(x,t,z) &= -\big[\frac{d}{dt}K(x,t,x,z)\big]/K(x,t,x,t)\\
F(x,t) &= \big[\frac{d^2}{dxdt}f(x,t)\big]/K(x,t,x,t)
\end{aligned}$$

The nonlinear of the second kind will be solved using the bivariate shifted Legendre functions. The obtained solution will be the solution of the nonlinear **2D- VIE** of the first kind. Therefore, we need first to introduce the shifted Legendre functions.

Definition 4.1 *[39,40,42] (Two- dimensional Shifted Legendre Functions) The shifted*

Legendre function are defined on Ω as:

$$\begin{aligned}\psi_{mn}(x,t) &= L_m(\frac{2}{l}x-1)L_n(\frac{2}{T}t-1)\\ \psi_{mn}(x,t) &= P_m(x)P_n(t)=\psi_m(x)\psi_n(t)\end{aligned}$$

where $m=0,1,...,M,\quad n=0,1,...,N.$

The two-dimensional shifted Legendre polynomials are orthogonal with respect to the weight function $w(x,t)=1$ such that:

$$\int_0^T\int_0^l w(x,t)\psi_{mn}(x,t)\psi_{ij}(x,t)dxdt=\begin{cases}\left(\frac{l}{2m+1}\right)\left(\frac{T}{2n+1}\right) & \text{if } i=m,\ \ j=n\\ 0 & otherwise\end{cases}$$

Here L_m and L_n are the well- known Legendre polynomials respectively of order m, and n which are defined on the interval [-1, 1] and satisfy the following recursive formula:

$$L_{n+1}(t)=\frac{2n+1}{n+1}tL_n(t)-\frac{n}{n+1}L_{n-1}(t),\quad n=1,2,...$$

where $L_o(t)=1$, and $L_1(t)=t$.

The shifted Legendre polynomials are defined on the interval $[0,s]$ as [15]:

$$P_{m+1}(x)=\left(\frac{2m+1}{m+1}\right)\left(\frac{2}{s}x-1\right)P_m(x)-\left(\frac{m}{m+1}\right)P_{m-1}(x)$$

where $P_0(x)=1,\ P_1(x)=\frac{2}{s}x-1,\ P_m(x)=L_m\left(\frac{2}{s}x-1\right)$, and $m=1,2,....$

4.1 Function Approximation with Shifted Legendre Function

A function $f(x,t) \in L^2(\Omega)$ can be expanded by the shifted Legendre functions series as follows [33, 39, 41]:

$$f(x,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\psi_{mn}(x,t) \tag{4.7}$$

where C_{mn} are constants given by:

$$C_{mn} = \frac{< f(x,t), \psi_{mn}(x,t) >}{< \psi_{mn}(x,t), \psi_{mn}(x,t) >}$$

The inner product in the space $L^2(\Omega)$ is defined by:

$$< f(x,t), \psi_{mn}(x,t) >= \int_0^T \int_0^l f(x,t)\psi_{mn}(x,t)dxdt$$

and the norm is defined as follows:

$$\begin{aligned} ||\psi_{mn}(x,t)||_2 &= \Big(< \psi_{mn}(x,t), \psi_{mn}(x,t) > \Big)^{\frac{1}{2}} \\ &= \Big(\int_0^T \int_0^l |\psi_{mn}(x,t)|^2 dxdt \Big)^{\frac{1}{2}}, \qquad \forall\ \psi_{mn}(x,t) \in L^2(\Omega) \end{aligned}$$

If the infinite series in equation 4.7 is truncated up to terms N and M, then we may approximate $f(x,t)$ in 4.7 as follows [39]:

$$\begin{aligned} f_{MN}(x,t) &= \sum_{m=0}^{M}\sum_{n=0}^{N} C_{mn}\psi_{mn}(x,t) = C^T\psi(x,t) \\ f(x,t) &\simeq C^T\psi(x,t) \end{aligned} \tag{4.8}$$

where C and $\psi(x,t)$ are $(M+1)(N+1)\times 1$ vectors given by:

$$C = \Big[C_{00} \quad C_{01} \quad ... \quad C_{0N}, \quad C_{10} \quad C_{11} \quad ... \quad C_{1N}, \quad ...., \quad C_{M0} \quad C_{M1} \quad ...C_{MN} \Big]^T$$

$$\psi(x,t) = \Big[\psi_{00} \quad \psi_{01} \quad ... \quad \psi_{0N}, \quad \psi_{10} \quad \psi_{11} \quad ... \quad \psi_{1N}, \quad ...., \quad \psi_{M0} \quad \psi_{M1} \quad ...\psi_{MN}\Big]^T \tag{4.9}$$

Now,

$$\begin{aligned} < f(x,t), \psi_{mn}(x,t) > &= < \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij}\psi_{ij}(x,t), \psi_{mn}(x,t) > \\ &= C_{mn} < \psi_{mn}(x,t), \psi_{mn}(x,t) > \end{aligned}$$

Therefore:

$$C_{mn} = \big(\frac{2m+1}{l}\big)\big(\frac{2n+1}{T}\big)\int_0^T \int_0^l f(x,t)\psi_{mn}(x,t)dxdt$$

4.2 approximating the Kernel Function with Shifted Legendre functions

The Kernel $K(x,t,y,z)$ [41, 42] can be approximated as:

$$K(x,t,y,z) \simeq \psi^T(x,t)K\psi(y,z)$$

where $K(x,t,y,z)$ is an $L^2(\Omega \times \Omega)$ function, and K is
an $(M+1)(N+1) \times (M+1)(N+1)$ block matrices of the form $K = [K^{(i,m)}]_{i,m=0}^{M}$, and
$K^{(i,m)} = [K_{ijmn}]_{j,n=0}^{N}, \;\; i,m = 0,1,...,M$

$$\begin{aligned} K_{ijmn} &= H\int_0^T \int_0^l \Big[\int_0^T \int_0^l K(x,t,y,z)\psi_{mn}(y,z)dydz\Big]\psi_{ij}(x,t)dxdt \\ H &= \frac{(2i+1)(2j+1)(2m+1)(2n+1)}{l^2T^2} \end{aligned}$$

The derivation of K_{ijmn} can be obtained for every $K(x,t,y,z) \in L^2(\Omega \times \Omega)$ as follows:

$$K(x,t,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \psi_{ij}(x,t) K_{ijmn} \psi_{mn}(y,z) \tag{4.10}$$

where

$$K_{ijmn} = \frac{<< K(x,t,y,z), \psi_{mn}(y,z) >, \psi_{ij}(x,t) >}{< \psi_{ij}(x,t), \psi_{ij}(x,t) >< \psi_{mn}(y,z), \psi_{mn}(y,z) >}$$

K_{ijmn} can be calculated through the following expansions [24, 45] .

$$\begin{aligned}
& << K(x,t,y,z), \psi_{mn}(y,z) >, \psi_{ij}(x,t) > \\
= \; & << \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \psi_{ab}(x,t) K_{abrs} \psi_{rs}(y,z), \psi_{mn}(y,z) >, \psi_{ij}(x,t) > \\
= \; & < \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \psi_{ab}(x,t) K_{abmn} < \psi_{mn}(y,z), \psi_{mn}(y,z) >, \psi_{ij}(x,t) > \\
= \; & < \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \psi_{ab}(x,t) K_{abmn} ||\psi_{mn}(y,x)||_2^2, \psi_{ij}(x,t) > \\
= \; & K_{ijmn} ||\psi_{mn}(y,z)||_2^2 < \psi_{ij}(x,t), \psi_{ij}(x,t) > \\
= \; & K_{ijmn} ||\psi_{mn}(y,z)||_2^2 ||\psi_{ij}(x,t)||_2^2
\end{aligned}$$

Solving the previous equation for K_{ijmn} to get [39]:

$$K_{ijmn} = \frac{<< K(x,t,y,z), \psi_{mn}(y,z) >, \psi_{ij}(x,t) >}{||\psi_{mn}(y,z)||_2^2 ||\psi_{ij}(x,t)||_2^2}$$

Therefore:

$$\begin{aligned}
K_{ijmn} &= H \int_0^T \int_0^l \left[\int_0^T \int_0^l K(x,t,y,z) \psi_{mn}(y,z) dydz \right] \psi_{ij}(x,t) dxdt \\
H &= \frac{(2i+1)(2j+1)(2m+1)(2n+1)}{l^2 T^2}
\end{aligned}$$

If the infinite series 4.10 is truncated, then it can be written as:

$$K(x,t,y,z) \simeq \sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{m=0}^{M}\sum_{n=0}^{N} \psi_{ij}(x,t)K_{ijmn}\psi_{mn}(y,z) = \psi^T(x,t)K\psi(y,z)$$

where $i, m = 0, 1, ..., N$, and $j, n = 0, 1, ..., N$.

Similarly, any function K_1 in $L^2(\Omega \times \Omega)$, K_2 in $L^2(\Omega \times [0,l])$, and K_3 in $L^2(\Omega \times [0,T])$ can be expanded in terms of bivariate shifted Legendre functions respectively as:

$$\begin{aligned} K_1(x,t,y,z) &\simeq \psi^T_(x,t)K_1\psi(y,z) \\ K_2(x,t,y) &\simeq \psi^T_(x,t)K_2\psi(y,t) \\ K_3(x,t,z) &\simeq \psi^T_(x,t)K_3\psi(x,z) \end{aligned}$$

where $K_1,\ K_2$, and K_3 are $(M+1)(N+1) \times (M+1)(N+1)$ of the form $K_q = [K_q^{(i,m)}]_{i,m=0}^{M}$, $K_q^{(i,m)} = [K_{ijmn}^q]_{j,n=0}^{N}$, $i, m = 0, 1, ..., M$, and $q = 1, 2, 3$. Legendre coefficients K_{ijmn}^q, $q = 1, 2, 3$ are given by:

$$\begin{aligned} K_{ijmn}^1 &= \frac{<< K_1(x,t,y,z), \psi_{mn}(y,z) >, \psi_{ij}(x,t) >}{||\psi_{mn}(y,z)||_2^2 ||\psi_{ij}(x,t)||_2^2} \\ &= H\int_0^T\int_0^l \left[\int_0^T\int_0^l K_1(x,t,y,z)\psi_{mn}(y,z)dydz\right]\psi_{ij}(x,t)dxdt \end{aligned}$$

$$\begin{aligned} K_{ijmn}^2 &= \frac{<< K_2(x,t,y), \psi_{mn}(y,t) >, \psi_{ij}(x,t) >}{||\psi_{mn}(y,t)||_2^2 ||\psi_{ij}(x,t)||_2^2} \\ &= H\int_0^T\int_0^l \left[\int_0^T\int_0^l K_2(x,t,y)\psi_{mn}(y,t)dydt\right]\psi_{ij}(x,t)dxdt \end{aligned}$$

$$\begin{aligned} K_{ijmn}^3 &= \frac{<< K_3(x,t,z), \psi_{mn}(x,z) >, \psi_{ij}(x,t) >}{||\psi_{mn}(x,z)||_2^2 ||\psi_{ij}(x,t)||_2^2} \\ &= H\int_0^T\int_0^l \left[\int_0^T\int_0^l K_3(x,t,z)\psi_{mn}(x,z)dxdz\right]\psi_{ij}(x,t)dxdt \end{aligned}$$

where $i, m = 0, 1, ..., M, \quad j, n = 0, 1, ..., N.$

$$K = \begin{bmatrix} K_{0000} & K_{0001} & \dots & K_{000N} & & K_{00M0} & K_{00M1} & \dots & K_{00MN} \\ K_{0100} & K_{0101} & \dots & K_{010N} & ,\dots, & K_{01M0} & K_{01M1} & \dots & K_{01MN} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ K_{0N00} & K_{0N01} & \dots & K_{0N0N} & & K_{0NM0} & K_{0NM1} & \dots & K_{0NMN} \\ & & \vdots & & \ddots & & & \vdots & \\ K_{M000} & K_{M001} & \dots & K_{M00N} & & K_{M0M0} & K_{M0M1} & \dots & K_{M0MN} \\ K_{M100} & K_{M101} & \dots & K_{M10N} & ,\dots, & K_{M1M0} & K_{M1M1} & \dots & K_{M1MN} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ K_{MN00} & K_{MN01} & \dots & K_{MN0N} & & K_{MNM0} & K_{MNM1} & \dots & K_{MNMN} \end{bmatrix}$$

The matrices $K_q, \; q = 1, 2, 3$ can be derived by Replacing the coefficients K_{ijmn} by K^q_{ijmn},

4.3 Integration Operational Matrix

The integration of the vector $\psi(x, t)$ defined by 4.9 approximately obtained as:

$$\int_0^t \int_0^x \psi(y, z) dy dz \simeq Q_1 \psi(x, t) \tag{4.11}$$

$$\int_0^x \psi(y, t) dy \simeq Q_2 \psi(x, t) \tag{4.12}$$

$$\int_0^t \psi(x, z) dz \simeq Q_3 \psi(x, t) \tag{4.13}$$

where $x \in [0, l], \quad t \in [0, T], \; Q_1, \; Q_2,$ and Q_3 are $(M+1)(N+1) \times (M+1)(N+1)$ operational matrices of integration [18] which are introduced respectively as:

$$\begin{aligned} Q_1 &= P_1 \otimes P_2 \\ Q_2 &= P_1 \otimes I_2 \\ Q_3 &= I_1 \otimes P_2 \end{aligned}$$

where $I_1,\ I_2$ are identity matrices of size $M+1,\ N+1$ respectively. P_1, and P_2 are the operational matrices of one dimension shifted Legendre polynomials defined respectively on $[0,l],[0,T]$ as follows [15,17,45]:

$$P_q = \frac{z}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \ldots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \frac{-1}{2h-1} & 0 & \frac{1}{2h-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & \frac{-1}{2h+1} & 0 \end{bmatrix}$$

$$Where\ q = 1,2$$
$$and\ z = l,T$$
$$and\ h = M,N$$

$$P_1 = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ D_1+C_1 & B_1 & A_1 & 0 & \ldots & 0 & 0 & 0 \\ D_2 & C_2 & B_2 & A_2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ D_{m-2} & 0 & 0 & 0 & \ldots & C_{m-2} & B_{m-2} & A_{m-2} \\ D_{m-1} & 0 & 0 & 0 & \ldots & 0 & C_{m-1} & B_{m-1} \end{bmatrix}$$

where,

$$
\begin{aligned}
A_0 &= \frac{l}{2},\ B_0 = \frac{l}{2},\ C_0 = 0,\ D_0 = 0 \\
A_1 &= \frac{l}{6},\ B_1 = 0,\ C_1 = \frac{-l}{6},\ D_1 = 0 \\
A_2 &= \frac{l}{10},\ B_2 = 0,\ C_2 = \frac{-l}{10},\ D_2 = 0 \\
&\vdots \\
A_m &= \frac{l}{2(2m+1)},\ B_m = 0,\ C_m = \frac{-l}{2(2m+1)},\ D_m = 0, \quad m \geq 1
\end{aligned}
$$

$$
P_2 = \begin{bmatrix}
b_0 & a_0 & 0 & 0 & ... & 0 & 0 & 0 \\
d_1 + c_1 & b_1 & a_1 & 0 & ... & 0 & 0 & 0 \\
d_2 & c_2 & b_2 & a_2 & ... & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
d_{n-2} & 0 & 0 & 0 & ... & c_{n-2} & b_{n-2} & a_{n-2} \\
d_{n-1} & 0 & 0 & 0 & ... & 0 & c_{n-1} & b_{n-1}
\end{bmatrix}
$$

where,

$$
\begin{aligned}
a_0 &= \frac{T}{2},\ b_0 = \frac{T}{2},\ c_0 = 0,\ d_0 = 0 \\
a_1 &= \frac{T}{6},\ b_1 = 0,\ c_1 = \frac{-T}{6},\ d_1 = 0 \\
a_2 &= \frac{T}{10},\ b_2 = 0,\ c_2 = \frac{-T}{10},\ d_2 = 0 \\
&\vdots \\
a_n &= \frac{T}{2(2n+1)},\ b_n = 0,\ C_n = \frac{-T}{2(2n+1)},\ d_n = 0, \quad n \geq 1
\end{aligned}
$$

Now, we are going to prove equations 4.11-4.13

Proof: Using the equations 3.24 that we reached previously in chapter three

$$\psi_m(y) = A_m\dot{\psi}_{m+1}(y) + B_m\dot{\psi}_m(y) + C_m\dot{\psi}_{m-1}(y) \tag{4.14}$$

Multiply both sides of the equation 4.14 by $\psi_n(t)$ then we get:

$$\psi_{mn}(y,t) = A_{mn}\dot{\psi}_{m+1,n}(y,t) + B_{mn}\dot{\psi}_{m,n}(y,t) + C_{mn}\dot{\psi}_{m-1,n}(y,t) \tag{4.15}$$

The coefficients $A_{mn}, B_{mn},$ and C_{mn} can be evaluated by integrating the recurrence relation 4.15 with respect to y from 0 to x

$$\begin{aligned}\int_0^x \psi_{mn}(y,t)dy &= A_{mn}\psi_{m+1,n}(x,t) + B_{mn}\psi_{m,n}(x,t) + C_{mn}\psi_{m-1,n}(x,t) \\ &+ D_{mn}\psi_{00}(x,t)\end{aligned} \tag{4.16}$$

Then, substitute $m = 0, 1..., M$, and $n = 0, 1, ..., N$ in equation 4.16 to get:

$$\begin{aligned}A_{0n} &= \frac{l}{2},\ B_{0n} = \frac{l}{2},\ C_{0n} = 0,\ D_{0n} = 0, \quad n = 0, 1, ..., N \\ A_{1n} &= \frac{l}{6},\ B_{1n} = 0,\ C_{1n} = \frac{-l}{6},\ D_{1n} = 0, \quad n = 0, 1, ..., N \\ &\vdots \\ A_{mn} &= \frac{l}{2(2m+1)},\ B_{mn} = 0,\ C_{mn} = \frac{-l}{2(2m+1)},\ D_{mn} = 0, \quad n = 0, 1, ..., N, \quad m \geq 1\end{aligned}$$

For simplicity, we may write eqution 4.16 in the form:

$$\int_0^x \psi(y,t)dy \simeq Q_2\psi(x,t)$$

Where

$$\psi(x,t) = \begin{bmatrix}\psi_{00} & \psi_{01} & ... & \psi_{0N}, & \psi_{10} & \psi_{11} & ... & \psi_{1N}, & ...., & \psi_{M0} & \psi_{M1} & ...\psi_{MN}\end{bmatrix}^T$$

and Q_2 is an $(M+1)(N+1) \times (M+1)(N+1)$ integration operational matrix of $2D$ shifted Legendre functions having the form:

$$Q_2 = \begin{bmatrix} \bar{B}_0 & \bar{A}_0 & O & O & \dots & O & O & O \\ \bar{D}_1+\bar{C}_1 & \bar{B}_1 & \bar{A}_1 & O & \dots & O & O & O \\ \bar{D}_2 & \bar{C}_2 & \bar{B}_2 & \bar{A}_2 & \dots & O & O & O \\ \bar{D}_3 & O & \bar{C}_3 & \bar{B}_3 & \bar{A}_3 & \dots & O & O \\ \bar{D}_4 & O & O & \bar{C}_4 & \bar{B}_4 & \bar{A}_4 & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \bar{D}_{m-2} & O & O & O & \dots & \bar{C}_{m-2} & \bar{B}_{m-2} & \bar{A}_{m-2} \\ \bar{D}_{m-1} & O & O & O & \dots & O & \bar{C}_{m-1} & \bar{B}_{m-1} \end{bmatrix}$$

where,

$\bar{A}_k = diag(A_{k0}, ..., A_{k,n-1}), k = 0, 1, ..., m-2$

$\bar{B}_k = diag(B_{k0}, ..., B_{k,n-1}), k = 0, 1, ..., m-1$

$\bar{C}_k = diag(C_{k0}, ..., C_{k,n-1}), k = 1, 2, ..., m-1$

$\bar{D}_k = diag(D_{k0}, ..., C_{k,n-1}), k = 1, 2, ..., m-1$

Note that the matrix Q_2 has the following form:

$$Q_2 = \begin{bmatrix} B_0 I_2 & A_0 I_2 & O & O & \dots & O & O & O \\ (D_1+C_1)I_2 & B_1 I_2 & A_1 I_2 & O & \dots & O & O & O \\ D_2 I_2 & C_2 I_2 & B_2 I_2 & A_2 I_2 & \dots & O & O & O \\ D_3 I_2 & O & C_3 I_2 & B_3 I_2 & A_3 I_2 & \dots & O & O \\ D_4 I_2 & O & O & C_4 I_2 & B_4 I_2 & A_4 I_2 & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ D_{m-2} I_2 & O & O & O & \dots & C_{m-2} I_2 & B_{m-2} I_2 & A_{m-2} I_2 \\ D_{m-1} I_2 & O & O & O & \dots & O & C_{m-1} I_2 & B_{m-1} I_2 \end{bmatrix}$$

Finally, Q_2 can be written as:

$$Q_2 = P_1 \otimes I_2$$

where I_2 is the identity matrix of order $N+1$, and O is a square zero matrix of order $N+1$.

it is clear that:

$$\begin{aligned}
A_{0n} = A_0 &= \frac{l}{2},\ B_{0n} = B_0 = \frac{l}{2},\ C_{0n} = C_0 = 0,\ D_{0n} = D_0 = 0, \quad n = 0,1,...,N \\
A_{1n} = A_1 &= \frac{l}{6},\ B_{1n}B_1 = 0,\ C_{1n} = C_1 = \frac{-l}{6},\ D_{1n} = D_1 = 0, \quad n = 0,1,...,N \\
&\vdots \\
A_{mn} = A_m &= \frac{l}{2(2m+1)},\ B_{mn} = B_m = 0,\ C_{mn} = C_m = \frac{-l}{2(2m+1)}, \\
D_{mn} = D_m &= 0,\ n = 0,1,...,N, \quad m \geq 1
\end{aligned}$$

to prove that $Q_3 = I_1 \otimes P_2$, use the equation 3.24 that we reached previously in chapter three, which is as follows:

$$\psi_n(z) = a_n\dot{\psi}_{n+1}(z) + b_n\dot{\psi}_n(z) + c_n\dot{\psi}_{n-1}(z) \tag{4.17}$$

Multiply both sides of the previous equation by $\psi_m(x)$ to get:

$$\psi_{mn}(x,z) = a_{mn}\dot{\psi}_{m,n+1}(x,z) + b_{mn}\dot{\psi}_{m,n}(x,z) + c_{mn}\dot{\psi}_{m,n-1}(x,z) \tag{4.18}$$

The coefficients a_{mn}, b_{mn}, and c_{mn} can be evaluated by integrating the recurence the relation 4.18 with respect to z from 0 to t as follows:

$$\begin{aligned}
\int_0^t \psi_{mn}(x,z)dz &= a_{mn}\psi_{m,n+1}(x,t) + b_{mn}\psi_{m,n}(x,t) + c_{mn}\psi_{m,n-1}(x,t) \\
&+ d_{mn}\psi_{0,0}(x,t) \qquad (4.19)
\end{aligned}$$

Then, after substituting $m = 0, 1..., M$, and $n = 0, 1, ..., N$ in equation 4.19 we can get:

$$\begin{aligned}
a_{m0} &= \frac{T}{2},\ b_{m0} = \frac{T}{2},\ c_{m0} = 0,\ d_{m0} = 0, \quad m = 0, 1, ..., M \\
a_{m1} &= \frac{T}{6},\ b_{m1} = 0,\ c_{m1} = \frac{-T}{6},\ d_{m1} = 0, \quad m = 0, 1, ..., M \\
&\vdots \\
a_{mn} &= \frac{T}{2(2n+1)},\ b_{mn} = 0,\ c_{mn} = \frac{-T}{2(2n+1)},\ d_{mn} = 0, \quad m = 0, 1, ..., M, \quad n \geq 1
\end{aligned}$$

For simplicity, we may write eqution 4.19 in the form:

$$\int_0^t \psi(x, z)dz \simeq Q_3 \psi(x, t)$$

where

$$\psi(x, t) = \Big[\psi_{00} \quad \psi_{01} \quad ... \quad \psi_{0N}, \quad \psi_{10} \quad \psi_{11} \quad ... \quad \psi_{1N}, \quad ...., \quad \psi_{M0} \quad \psi_{M1} \quad ...\psi_{MN}\Big]^T$$

and Q_3 is an $(M+1)(N+1) \times (M+1)(N+1)$ operational matrix of integration for two-dimensional shifted Legendre functions, such a matrix have the following form:

$$Q_3 = \begin{bmatrix} S_0 & 0 & \dots & 0 \\ 0 & S_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & S_{m-1} \end{bmatrix}, S_k = \begin{bmatrix} b_{k0} & a_{k0} & 0 & 0 & \dots & 0 & 0 & 0 \\ d_{k1}+c_{k1} & b_{k1} & a_{k1} & 0 & \dots & 0 & 0 & 0 \\ d_{k2} & c_{k2} & b_{k2} & a_{k2} & \dots & 0 & 0 & 0 \\ d_{k3} & 0 & c_{k3} & b_{k3} & a_{k3} & \dots & 0 & 0 \\ d_{k4} & 0 & 0 & c_{k4} & b_{k4} & a_{k4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ d_{k,n-2} & 0 & 0 & 0 & \dots & c_{k,n-2} & b_{k,n-2} & a_{k,n-2} \\ d_{k,n-1} & 0 & 0 & 0 & \dots & 0 & c_{k,n-1} & b_{k,n-1} \end{bmatrix},$$

$k = 0, 1, ..., m-1$.

We note that the entries of S_k are the same as the entries of the one dimensional operational matrix. Therefore,

$Q_3 = diag(P_2, ..., P_2) = I_1 \otimes P_2$

Where I_1 is the identity matrix of the order $M+1$, and O is a zero square matrix of order $N+1$.

To prove that $Q_1 = P_1 \otimes P_2$, multiply equation 4.14by with Equation 4.17 to have the following formula:

$$\psi_m(y)\psi_n(z) = (A_m\dot{\psi}_{m+1}(y)+B_m\dot{\psi}_m(y)+C_m\dot{\psi}_{m-1}(y))(a_n\dot{\psi}_{n+1}(z)+b_n\dot{\psi}_n(z)+c_n\dot{\psi}_{n-1}(z)) \tag{4.20}$$

After simplifying 4.20 and using the fact that $\psi_n(z)\psi_m(y) = \psi_{mn}(y,z)$, and integrating the resultant equation with respect to the variables z and y, the coefficients Aa_{mn}, Ab_{mn}, Ac_{mn}, $Ba_{mn}, Bb_{mn}, Bc_{mn}$, Ca_{mn}, Cb_{mn}, and Cc_{mn} can be calculated using the equation:

$$\begin{aligned}\int_0^t\int_0^x \psi_{mn}(y,z)dydz &= Aa_{mn}\psi_{m+1,n+1}(x,t) + Ab_{mn}\psi_{m+1,n}(x,t) + Ac_{mn}\psi_{m+1,n-1}(x,t)\\ &+ Ba_{mn}\psi_{m,n+1}(x,t) + Bb_{mn}\psi_{m,n}(x,t) + Bc_{m,n}\psi_{m,n-1}(x,t)\\ &+ Ca_{mn}\psi_{m-1,n+1}(x,t) + Cb_{mn}\psi_{m-1,n}(x,t) + Cc_{mn}\psi_{m-1,n-1}(x,t)\\ &+ (Ad_{mn} + Bd_{mn} + Cd_{mn} + Da_{mn} + Db_{mn} + Dc_{mn} + Dd_{mn})\psi_{00}(x,t)\end{aligned} \tag{4.21}$$

Then, substitute $m = 0, 1, ...M$, and $n = 0, 1, ..., N$ in equation 4.21 to get:

$$Aa_{mn} = \frac{lT}{4(2m+1)(2n+1)}, m = 0, 1, ..., M, \quad n = 0, 1, ..., N,$$

$$Ab_{mn} = \begin{cases} \frac{lT}{4(2m+1)(2n+1)} & \text{if } n = 0,\ m = 0, 1, ..., M \\ 0 & \text{if } n = 1, 2, ..., N\ m = 0, 1, ..., M \end{cases}$$

$$Ac_{mn} = \begin{cases} \frac{-lT}{4(2m+1)(2n+1)} & \text{if } n = 1, 2, ..., N,\ m = 0, 1, ..., M \\ 0 & \text{if } n = 0,\ m = 0, 1, ..., M \end{cases}$$

$$Ba_{mn} = \begin{cases} \frac{lT}{4(2m+1)(2n+1)} & \text{if } m = 0,\ n = 0, 1, ...N \\ 0 & \text{if } m = 1, 2, ..., M,\ n = 0, 1, ..., N \end{cases}$$

$$Bb_{mn} = \begin{cases} \frac{lT}{4(2m+1)(2n+1)} & \text{if } m = 0,\ n = 0 \\ 0 & \text{if } m = 1, 2, ..., M,\ or\ n = 1, 2, ..., N \end{cases}$$

$$Bc_{mn} = \begin{cases} \frac{-lT}{4(2m+1)(2n+1)} & \text{if } m = 0,\ n = 1, 2, ..., N \\ 0 & \text{if } m = 1, 2, ..., M,\ n = 0, 1, ..., N \\ 0 & \text{if } m = 0,\ n = 0 \end{cases}$$

$$Ca_{mn} = \begin{cases} \frac{-lT}{4(2m+1)(2n+1)} & \text{if } m = 1, 2, ..., M,\ n = 0, 1, ..., N \\ 0 & \text{if } m = 0,\ n = 0, 1, ..., N \end{cases}$$

$$Cb_{mn} = \begin{cases} \frac{-lT}{4(2m+1)(2n+1)} & \text{if } n = 0,\ m = 1, 2, ..., M \\ 0 & \text{if } n = 1, 2, ..., N,\ m = 0, 1, ..., M \\ 0 & \text{if } n = 0,\ m = 0 \end{cases}$$

$$Cc_{mn} = \begin{cases} \frac{lT}{4(2m+1)(2n+1)} & \text{if } m = 1, 2, ..., M,\ n = 1, 2, ..., N \\ 0 & \text{if } m = 0,\ or\ n = 0 \end{cases}$$

$Ad_{mn} = Bd_{mn} = Cd_{mn} = Da_{mn} = Db_{mn} = Dc_{mn} = Dd_{mn} = 0$ for $m = 0, 1, ..., M$, $n = 0, 1, ..., N$.

For simplicity, equation 4.21 can be written as follows:

$$\int_0^t \int_0^x \psi(y,z)dydz \simeq Q_1\psi_(x,t)$$

Where

$$\psi(x,t) = \Big[\psi_{00} \quad \psi_{01} \quad ... \quad \psi_{0N}, \quad \psi_{10} \quad \psi_{11} \quad ... \quad \psi_{1N}, \quad ...., \quad \psi_{M0} \quad \psi_{M1} \quad ...\psi_{MN}\Big]^T$$

and Q_1 is an $(M+1)(N+1) \times (M+1)(N+1)$ integration operational matrix of $2D$ shifted Legendre functions having the form:

$$Q_1 = \begin{bmatrix} \overline{Bb}_0 & \overline{Ab}_0 & O & O & \dots & O & O & O \\ \overline{Db}_1 + \overline{Cb}_1 & \overline{Bb}_1 & \overline{Ab}_1 & O & \dots & O & O & O \\ \overline{Db}_2 & \overline{Cb}_2 & \overline{Bb}_2 & \overline{Ab}_2 & \dots & O & O & O \\ \overline{Db}_3 & O & \overline{Cb}_3 & \overline{Bb}_3 & \overline{Ab}_3 & \dots & O & O \\ \overline{Db}_4 & O & O & \overline{Cb}_4 & \overline{Bb}_4 & \overline{Ab}_4 & \dots & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \overline{Db}_{m-2} & O & O & O & \dots & \overline{Cb}_{m-2} & \overline{Bb}_{m-2} & \overline{Ab}_{m-2} \\ \overline{Db}_{m-1} & O & O & O & \dots & O & \overline{Cb}_{m-1} & \overline{Bb}_{m-1} \end{bmatrix}$$

where,

$$\overline{Bb}_k = \begin{bmatrix} Bb_{k0} & Ba_{k0} & 0 & 0 & \dots & 0 & 0 & 0 \\ Bd_{k1} + Bc_{k1} & Bb_{k1} & Ba_{k1} & 0 & \dots & 0 & 0 & 0 \\ Bd_{k2} & Bc_{k2} & Bb_{k2} & Ba_{k2} & \dots & 0 & 0 & 0 \\ Bd_{k3} & 0 & Bc_{k3} & Bb_{k3} & Ba_{k3} & \dots & 0 & 0 \\ Bd_{k4} & 0 & 0 & Bc_{k4} & Bb_{k4} & Ba_{k4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ Bd_{k,n-2} & 0 & 0 & 0 & \dots & Bc_{k,n-2} & Bb_{k,n-2} & Ba_{k,n-2} \\ Bd_{k,n-1} & 0 & 0 & 0 & \dots & 0 & Bc_{k,n-1} & Bb_{k,n-1} \end{bmatrix},$$

$k = 0, 1, ..., m-1.$

$$\overline{Bb}_0 = \begin{bmatrix} \frac{lT}{4} & \frac{lT}{4} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{-lT}{12} & 0 & \frac{lT}{12} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{-lT}{20} & 0 & \frac{lT}{20} & \dots & 0 & 0 & 0 \\ 0 & 0 & \frac{-lT}{28} & 0 & \frac{lT}{28} & \dots & 0 & 0 \\ 0 & 0 & 0 & \frac{-lT}{36} & 0 & \frac{lT}{36} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{-lT}{4(2(n-2)+1)} & 0 & \frac{lT}{4(2(n-2)+1)} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{-lT}{4(2(n-1)+1)} & 0 \end{bmatrix},$$

$Bb_k = [O]_{(N+1)(N+1)}$, $\forall k = 1, 2, ..., m-1$, and $n = 0, 1, ..., N$

$$\overline{Ab}_k = \begin{bmatrix} Ab_{k0} & Aa_{k0} & 0 & 0 & \dots & 0 & 0 & 0 \\ Ad_{k1} + Ac_{k1} & Ab_{k1} & Aa_{k1} & 0 & \dots & 0 & 0 & 0 \\ Ad_{k2} & Ac_{k2} & Ab_{k2} & Aa_{k2} & \dots & 0 & 0 & 0 \\ Ad_{k3} & 0 & Ac_{k3} & Ab_{k3} & Aa_{k3} & \dots & 0 & 0 \\ Ad_{k4} & 0 & 0 & Ac_{k4} & Ab_{k4} & Aa_{k4} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ Ad_{k,n-2} & 0 & 0 & 0 & \dots & Ac_{k,n-2} & Ab_{k,n-2} & Aa_{k,n-2} \\ Ad_{k,n-1} & 0 & 0 & 0 & \dots & 0 & Ac_{k,n-1} & Ab_{k,n-1} \end{bmatrix},$$

$k = 0, 1, ..., m-2.$

$$
\overline{Ab}_k = \frac{1}{2k+1}\begin{bmatrix}
\frac{lT}{4} & \frac{lT}{4} & 0 & 0 & \dots & 0 & 0 & 0 \\
\frac{-lT}{12} & 0 & \frac{lT}{12} & 0 & \dots & 0 & 0 & 0 \\
0 & \frac{-lT}{20} & 0 & \frac{lT}{20} & \dots & 0 & 0 & 0 \\
0 & 0 & \frac{-lT}{28} & 0 & \frac{lT}{28} & \dots & 0 & 0 \\
0 & 0 & 0 & \frac{-lT}{36} & 0 & \frac{lT}{36} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & \frac{-lT}{4(2(n-2)+1)} & 0 & \frac{lT}{4(2(n-2)+1)} \\
0 & 0 & 0 & 0 & \dots & 0 & \frac{-lT}{4(2(n-1)+1)} & 0
\end{bmatrix},
$$

$k = 0, 1, ..., m-2$, and $n = 0, 1, ..., N$

$$
\overline{Cb}_k = \begin{bmatrix}
Cb_{k0} & Ca_{k0} & 0 & 0 & \dots & 0 & 0 & 0 \\
Cd_{k1} + Cc_{k1} & Cb_{k1} & Ca_{k1} & 0 & \dots & 0 & 0 & 0 \\
Cd_{k2} & Cc_{k2} & Cb_{k2} & Ca_{k2} & \dots & 0 & 0 & 0 \\
Cd_{k3} & 0 & Cc_{k3} & Cb_{k3} & Ca_{k3} & \dots & 0 & 0 \\
Cd_{k4} & 0 & 0 & Cc_{k4} & Cb_{k4} & Ca_{k4} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
Cd_{k,n-2} & 0 & 0 & 0 & \dots & Cc_{k,n-2} & Cb_{k,n-2} & Ca_{k,n-2} \\
Cd_{k,n-1} & 0 & 0 & 0 & \dots & 0 & Cc_{k,n-1} & Cb_{k,n-1}
\end{bmatrix},
$$

$$
\overline{Cb}_k = \frac{1}{2k+1}\begin{bmatrix}
\frac{-lT}{4} & \frac{-lT}{4} & 0 & 0 & \dots & 0 & 0 & 0 \\
\frac{lT}{12} & 0 & \frac{-lT}{12} & 0 & \dots & 0 & 0 & 0 \\
0 & \frac{lT}{20} & 0 & \frac{-lT}{20} & \dots & 0 & 0 & 0 \\
0 & 0 & \frac{lT}{28} & 0 & \frac{-lT}{28} & \dots & 0 & 0 \\
0 & 0 & 0 & \frac{lT}{36} & 0 & \frac{-lT}{36} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \dots & \frac{lT}{4(2(n-2)+1)} & 0 & \frac{-lT}{4(2(n-2)+1)} \\
0 & 0 & 0 & 0 & \dots & 0 & \frac{lT}{4(2(n-1)+1)} & 0
\end{bmatrix},
$$

where $k = 1, 2, ..., m-1$, and $n = 0, 1, ..., N$.

$$
\overline{Db}_k = \begin{bmatrix}
Db_{k0} & Da_{k0} & 0 & 0 & \dots & 0 & 0 & 0 \\
Dd_{k1}+Dc_{k1} & Db_{k1} & Da_{k1} & 0 & \dots & 0 & 0 & 0 \\
Dd_{k2} & Dc_{k2} & Db_{k2} & Da_{k2} & \dots & 0 & 0 & 0 \\
Dd_{k3} & 0 & Dc_{k3} & Db_{k3} & Da_{k3} & \dots & 0 & 0 \\
Dd_{k4} & 0 & 0 & Bc_{k4} & Db_{k4} & Da_{k4} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
Dd_{k,n-2} & 0 & 0 & 0 & \dots & Dc_{k,n-2} & Db_{k,n-2} & Da_{k,n-2} \\
Dd_{k,n-1} & 0 & 0 & 0 & \dots & 0 & Dc_{k,n-1} & Db_{k,n-1}
\end{bmatrix},
$$

$k = 1, ..., m-1$, $n = 0, 1, ..., N$ and $Db_k = [O]_{(N+1)(N+1)}$.

□

4.4 The Product Operational Matrix

Let

$$
C = \begin{bmatrix} C_{00} & C_{01} & ... & C_{0N}, & C_{10} & C_{11} & ... & C_{1N}, & ...., & C_{M0} & C_{M1} & ...C_{MN} \end{bmatrix}^T
$$

, and

$$\psi(x,t) = \begin{bmatrix} \psi_{00} & \psi_{01} & ... & \psi_{0N}, & \psi_{10} & \psi_{11} & ... & \psi_{1N}, & ...., & \psi_{M0} & \psi_{M1} & ...\psi_{MN} \end{bmatrix}^T$$

Then,

$$\psi(x,t)\psi^T(x,t)C = \begin{bmatrix} \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{00} \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{01} \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{0N} \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{10} \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{11} \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{1N} \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{M0} \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{M1} \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} C_{kq}\psi_{kq}\psi_{MN} \end{bmatrix} \tag{4.22}$$

Also, [38] $\psi_{ij}(x,t)\psi_{kq}(x,t)$ can be written as a linear combination of the two dimensional shifted Legendre functions as follows:

$$\psi_{ij}(x,t)\psi_{kq}(x,t) = \sum_{r=0}^{i+k}\sum_{s=0}^{j+q} a_{rs}\psi_{rs}(x,t) \tag{4.23}$$

and from the orthogonality property, the coefficients a_{mn} can be evaluated as follows

$$
\begin{aligned}
a_{mn} &= \frac{(2m+1)(2n+1)}{lT}\int_0^T\int_0^l \psi_{ij}(x,t)\psi_{kq}(x,t)\psi_{mn}(x,t)dxdt \qquad (4.24)\\
&= \frac{(2m+1)(2n+1)}{lT}\int_0^l \psi_i(x)\psi_k(x)\psi_m(x)dx\int_0^T \psi_j(t)\psi_q(t)\psi_n(t)dt.
\end{aligned}
$$

Now let

$$
w_{ikm} = \int_0^l \psi_i(x)\psi_k(x)\psi_m(x)dx
$$

and

$$
w'_{jqn} = \int_0^T \psi_j(t)\psi_q(t)\psi_n(t)dt
$$

, then from equations 4.23 and 4.24

$$
\psi_{ij}(x,t)\psi_{kq}(x,t) = \sum_{m=0}^{i+k}\sum_{n=0}^{j+q}\frac{(2m+1)(2n+1)}{lT}w_{ikm}w'_{jqn}\psi_{mn}(x,t). \qquad (4.25)
$$

Hence from equations (4.22,4.23),and replacing the indices m by k and vice versa as well as for q and n, [24, 38, 39] we have

$$
\psi(x,t)\psi^T(x,t)C \simeq \begin{bmatrix}
\sum_{m=0}^M\sum_{n=0}^N\sum_{k=0}^M\sum_{q=0}^N C_{mn}\frac{(2k+1)(2q+1)}{lT}w_{0km}w'_{0qn}\psi_{kq}(x,t)\\
\vdots\\
\sum_{m=0}^M\sum_{n=0}^N\sum_{k=0}^M\sum_{q=0}^N C_{mn}\frac{(2k+1)(2q+1)}{lT}w_{0km}w'_{Nqn}\psi_{kq}(x,t)\\
\sum_{m=0}^M\sum_{n=0}^N\sum_{k=0}^M\sum_{q=0}^N C_{mn}\frac{(2k+1)(2q+1)}{lT}w_{1km}w'_{0qn}\psi_{kq}(x,t)\\
\vdots\\
\sum_{m=0}^M\sum_{n=0}^N\sum_{k=0}^M\sum_{q=0}^N C_{mn}\frac{(2k+1)(2q+1)}{lT}w_{1km}w'_{Nqn}\psi_{kq}(x,t)\\
\vdots\\
\sum_{m=0}^M\sum_{n=0}^N\sum_{k=0}^M\sum_{q=0}^N C_{mn}\frac{(2k+1)(2q+1)}{lT}w_{Mkm}w'_{0qn}\psi_{kq}(x,t)\\
\vdots\\
\sum_{m=0}^M\sum_{n=0}^N\sum_{k=0}^M\sum_{q=0}^N C_{mn}\frac{(2k+1)(2q+1)}{lT}w_{Mkm}w'_{Nqn}\psi_{kq}(x,t)
\end{bmatrix}
$$

$$= \begin{bmatrix} C^{(0,0)} & C^{(0,1)} & ... & C^{(0,M)} \\ C^{(1,0)} & C^{(1,1)} & ... & C^{(1,M)} \\ \vdots & \ddots & \vdots & \vdots \\ C^{(M,0)} & C^{(M,1)} & ... & C^{(M,M)} \end{bmatrix} \psi(x,t) = \tilde{C}\psi(x,t)$$

Note that $w_{ikm} = w_{imk}$,and $w'_{jqn} = w'_{jnq}$. So, for $m = 0, 1, ...M$, let A_m be a square matrix whose entries are $[A_m]_{j,q=0}^{N} = \frac{2q+1}{T}\sum_{n=0}^{N} w'_{jqn} C_{mn}$ and for $i, k = 0, 1, ..., M$, let $\tilde{C} = [C^{(i,k)}]$, where

$$C^{(i,k)} = \frac{2k+1}{l}\sum_{m=0}^{M} w_{ikm} A_m.$$

The matrix $\tilde{C}$ is a square matrix of size $(M+1)(N+1)$ and called product operational matrix. [40–42], Therefore,

$$\psi(x,t)\psi^T(x,t)C \simeq \tilde{C}\psi(x,t) \tag{4.26}$$

Lemma 4.1 *Let* $B = [B^{(i,k)}], i, k = 0, 1, ..., M,$ *such that* $B^{(i,k)} = [b_{ijkq}]_{j,q=0}^{N},\ j, q = 0, 1, ..., N,$ *then*

$$\psi^T(x,t)B\psi(x,t) \simeq \hat{B}\psi(x,t),$$

where

$$\hat{B} = \Big[B_{00} \quad B_{01} \quad ... \quad B_{0N}, \quad B_{10} \quad B_{11} \quad ... \quad B_{1N}, \quad ...., \quad B_{M0} \quad B_{M1} \quad ...B_{MN} \Big]$$

[42]Such that $B_{mn} = \frac{(2m+1)(2n+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} w_{ikm} w'_{jqn} b_{ijkq}$

$m = 0, 1, ..., M, \quad n = 0, 1, ..., N$

Proof:

$$\psi^T(x,t)B\psi(x,t) = \psi^T(x,t)\begin{bmatrix} \sum_{k=0}^{M}\sum_{q=0}^{N} b_{00kq}\psi_{kq}(x,t) \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} b_{0Nkq}\psi_{kq}(x,t) \\ \sum_{k=0}^{M}\sum_{q=0}^{N} b_{10kq}\psi_{kq}(x,t) \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} b_{1Nkq}\psi_{kq}(x,t) \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} b_{M0kq}\psi_{kq}(x,t) \\ \vdots \\ \sum_{k=0}^{M}\sum_{q=0}^{N} b_{MNkq}\psi_{kq}(x,t) \end{bmatrix}$$

$$\begin{aligned} &= \sum_{k=0}^{M}\sum_{q=0}^{N} b_{00kq}\psi_{kq}(x,t)\psi_{00}(x,t) + ... + \sum_{k=0}^{M}\sum_{q=0}^{N} b_{0Nkq}\psi_{kq}(x,t)\psi_{0N}(x,t) \\ &+ \sum_{k=0}^{M}\sum_{q=0}^{N} b_{10kq}\psi_{kq}(x,t)\psi_{10}(x,t) + ... + \sum_{k=0}^{M}\sum_{q=0}^{N} b_{1Nkq}\psi_{kq}(x,t)\psi_{1N}(x,t) \\ &+ ... + \sum_{k=0}^{M}\sum_{q=0}^{N} b_{M0kq}\psi_{kq}(x,t)\psi_{M0}(x,t) + ... + \sum_{k=0}^{M}\sum_{q=0}^{N} b_{MNkq}\psi_{kq}(x,t)\psi_{MN}(x,t) \end{aligned}$$

Now, [38] according to the equation 4.25, we have:

$$\psi_{ij}(x,t)\psi_{kq}(x,t) \simeq \sum_{m=0}^{M}\sum_{n=0}^{N} \frac{(2m+1)(2n+1)}{lT} w_{ikm} w\prime_{jqn}\psi_{mn}(x,t)$$

which implies that:

$$
\begin{aligned}
\psi^T(x,t)B\psi(x,t) \;\; \simeq \;\; & \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{00kq}\frac{(2m+1)(2n+1)}{lT} w_{0km}w'_{0qn}\psi_{mn}(x,t) \\
+ \;\; & ... + \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{0Nkq}\frac{(2m+1)(2n+1)}{lT} w_{0km}w'_{Nqn}\psi_{mn}(x,t) \\
+ \;\; & \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{10kq}\frac{(2m+1)(2n+1)}{lT} w_{1km}w'_{0qn}\psi_{mn}(x,t) \\
+ \;\; & ... + \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{1Nkq}\frac{(2m+1)(2n+1)}{lT} w_{1km}w'_{Nqn}\psi_{mn}(x,t) \\
+ \;\; & ... + \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{M0kq}\frac{(2m+1)(2n+1)}{lT} w_{Mkm}w'_{0qn}\psi_{mn}(x,t) \\
+ \;\; & ... + \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{MNkq}\frac{(2m+1)(2n+1)}{lT} w_{Mkm}w'_{Nqn}\psi_{mn}(x,t)
\end{aligned}
$$

Expanding the previous summations, and gathering the similar terms. Also, depending on some properties, the following form is obtained:

$$\begin{aligned}
\psi^T(x,t)B\psi(x,t) &\simeq \frac{(2(0)+1)(2(0)+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ik0}w'_{jqo}\psi_{00}(x,t) \\
&+ \ldots + \frac{(2(0)+1)(2(N)+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ik0}w'_{jqN}\psi_{0N}(x,t) \\
&+ \frac{(2(1)+1)(2(0)+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ik1}w'_{jq0}\psi_{10}(x,t) \\
&+ \ldots + \frac{(2(1)+1)(2(N)+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ik1}w'_{jqN}\psi_{1N}(x,t) \\
&+ \ldots + \frac{(2(M)+1)(2(0)+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ikM}w'_{jq0}\psi_{Mo}(x,t) \\
&+ \frac{(2(M)+1)(2(1)+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ikM}w'_{jq1}\psi_{M1}(x,t) \\
&= B_{00}\psi_{00} + \ldots + B_{0N}\psi_{0N} + B_{10}\psi_{10} + \ldots + B_{1N}\psi_{1N} \\
&+ \ldots + B_{M0}\psi_{M0} + \ldots + B_{MN}\psi_{MN} \\
\psi^T(x,t)B\psi(x,t) &\simeq \hat{B}\psi(x,t)
\end{aligned}$$

where

$$\hat{B} = \Big[B_{00} \quad B_{01} \quad \ldots \quad B_{0N}, \quad B_{10} \quad B_{11} \quad \ldots \quad B_{1N}, \quad \ldots., \quad B_{M0} \quad B_{M1} \quad \ldots B_{MN}\Big]$$

$$\psi(x,t) = \Big[\psi_{00} \quad \psi_{01} \quad \ldots \quad \psi_{0N}, \quad \psi_{10} \quad \psi_{11} \quad \ldots \quad \psi_{1N}, \quad \ldots., \quad \psi_{M0} \quad \psi_{M1} \quad \ldots\psi_{MN}\Big]^T$$

and

$$B_{mn} = \frac{(2m+1)(2n+1)}{lT}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{M}\sum_{q=0}^{N} b_{ijkq}w_{ikm}w'_{jqn}$$

□ Similarly, any matrix $B_1,\ B_2$, and B_3 of size $(M+1)(N+1) \times (M+1)(N+1)$, we get:

$$\psi^T(x,t)B_1\psi(x,t) \simeq \hat{B}_1\psi(x,t) \tag{4.27}$$

$$\psi^T(x,t)B_2\psi(x,t) \simeq \hat{B}_2\psi(x,t) \tag{4.28}$$

$$\psi^T(x,t)B_3\psi(x,t) \simeq \hat{B}_3\psi(x,t) \tag{4.29}$$

4.5 Method Of Solution

In this section, we present a numerical method to find an approximate solution to the problem 4.1 - 4.4 which corresponds with equation 4.6. Also, we assume that the known functions in equation 4.1 satisfy the conditions 4.2 - 4.4 that this equation has a unique solution.

Using the way mentioned previously, [42–44] the functions $u^p(x,t),\ F(x,t),\ K_1(x,t,y,z)$, $K_2(x,t,y)$, and $K_3(x,t,z)$ can be approximated depending on the bivatiate shifted Legendre functions as:

$$u^p(x,t) \simeq C^T\psi(x,t) = \psi^T(x,t)C \tag{4.30}$$

$$F(x,t) \simeq F^T\psi(x,t) \tag{4.31}$$

$$K_1(x,t,y,z) \simeq \psi^T(x,t)K_1\psi(y,z) \tag{4.32}$$

$$K_2(x,t,y) \simeq \psi^T(x,t)K_2\psi(y,t) \tag{4.33}$$

$$K_3(x,t,z) \simeq \psi^T(x,t)K_3\psi(x,z) \tag{4.34}$$

where F^T is $1 \times (M+1)(N+1)$ vectors.

$$F^T = \Big[F_{00} \quad F_{01} \quad ... \quad F_{0N}, \quad F_{10} \quad F_{11} \quad ... \quad F_{1N}, \quad ...., \quad F_{M0} \quad F_{M1} \quad ...F_{MN} \Big]$$

and F_{mn} is given by the formula:

$$F_{mn} = \frac{(2m+1)(2n+1)}{lT} \int_0^T \int_0^l F(x,t)\psi_{mn}(x,t)dxdt$$

Substituting equations 4.30- 4.34 into equation 4.6 to yield:

$$\begin{aligned} C^T\psi(x,t) &= \int_0^t \int_0^x \psi^T(x,t)K_1\psi(y,z)\psi^T(y,z)Cdydz + \int_0^x \psi^T(x,t)K_2\psi(y,t)\psi^T(y,t)Cdy \\ &+ \int_0^t \psi^T(x,t)K_3\psi(x,z)\psi^T(x,z)Cdz + F^T\psi(x,t) \end{aligned}$$

From equation (4.26) the last equation becomes

$$\begin{aligned} C^T\psi(x,t) &= \psi^T(x,t)K_1\tilde{C} \int_0^t \int_0^x \psi(y,z)dydz + \psi^T(x,t)K_2\tilde{C} \int_0^x \psi_(y,t)dy \\ &+ \psi^T(x,t)K_3\tilde{C} \int_0^t \psi(x,z)dz + F^T\psi(x,t) \end{aligned}$$

Applying equations(4.11-4.13) to obtain:

$$\begin{aligned} C^T\psi(x,t) &= \psi^T(x,t)K_1\bar{C}Q_1\psi(x,t) + \psi^T(x,t)K_2\tilde{C}Q_2\psi(x,t) + \psi^T(x,t)K_3\tilde{C}Q_3\psi(x,t) \\ &+ F^T\psi(x,t) \\ &= \psi^T(x,t)B_1\psi(x,t) + \psi^T(x,t)B_2\psi(x,t) + \psi^T(x,t)B_3\psi(x,t) + F^T\psi(x,t) \end{aligned}$$

On the other hand, from equations 4.27, 4.28,and 4.29 we have:

$$C^T\psi(x,t) = \hat{B}_1\psi(x,t) + \hat{B}_2\psi(x,t) + \hat{B}_3\psi(x,t) + F^T(x,t)\psi(x,t). \tag{4.35}$$

Hence $C^T = \hat{B}_1 + \hat{B}_2 + \hat{B}_3 + F^T$ which corresponds with a system of linear algebraic equations in terms of the unknown element of the vector C, and can be solved easily using direct methods.

The unkonwn function $u(x,t)$ can be approximated in terms of the bivariate shifted Legendre functions as:

$$u(x,t) = A^T \psi(x,t) \tag{4.36}$$

such that, the entries of the vector A are unknowns. Depending on equations 4.26, and 4.36

$$u^p(x,t) = A^T \tilde{A}^{p-1} \psi(x,t) \tag{4.37}$$

Finally, using equation 4.37, and 4.30, we get:

$$A^T \tilde{A}^{p-1} = C^T \tag{4.38}$$

Equation 4.38 forms a system of $(M+1)(N+1)$ nonlinear system of algebraic equations, which can be used be solved easily for the elements of A using numerical methods such as Newton's iterative method.

The result in equation 4.37 can be proved by induction, ie: $\forall k \in \mathbb{Z}^+$.

By using result 4.26 and replace C by A, we get:

$$\psi(x,t)\psi^T(x,t)A \simeq \tilde{A}\psi(x,t) \tag{4.39}$$

Take the transpose of both sides of equation 4.39, to yield:

$$A^T \psi(x,t)\psi^T(x,t) \simeq \psi^T(x,t)\tilde{A}^T$$

If the result 4.36 is applied with $p = 1$, [44] then we get:

$$u^1(x,t) \simeq A^T \tilde{A}^0 \psi(x,t)$$

For $p = 2$,

$$\begin{aligned}
u^2(x,t) &= u(x,t)u(x,t) \\
&\simeq A^T\psi(x,t)\psi^T(x,t)A \\
u^2(x,t) &\simeq A^T\tilde{A}^1\psi(x,t)
\end{aligned}$$

If $p = 3$, then,

$$\begin{aligned}
u^3(x,t) &= u^2(x,t)u(x,t) \\
&\simeq A^T\tilde{A}^1\psi(x,t)\psi^T(x,t)A \\
&\simeq A^T\tilde{A}\tilde{A}\psi(x,t) \\
u^3(x,t) &\simeq A^T\tilde{A}^2\psi(x,t)
\end{aligned}$$

For $p = 4$,

$$\begin{aligned}
u^4(x,t) &= u^3(x,t)u(x,t) \\
&\simeq A^T\tilde{A}^2\psi(x,t)\psi^T(x,t)A \\
&\simeq A^T\tilde{A}^2\tilde{A}\psi(x,t) \\
u^4(x,t) &\simeq A^T\tilde{A}^3\psi(x,t)
\end{aligned}$$

Suppose that $k \in \mathbb{Z}^+$, and let equation 4.37 be true for k, then,

$$\begin{aligned}
u^{k+1}(x,t) &= u^k(x,t)u^1(x,t) \\
&\simeq A^T\tilde{A}^k\psi(x,t)\psi^T(x,t)A \\
&\simeq A^T\tilde{A}^{k-1}\tilde{A}\psi(x,t) \\
u^{k+1}(x,t) &\simeq A^T\tilde{A}^k\psi(x,t)
\end{aligned}$$

Thus equation 4.37 is true for $p = k+1$, and the proof of the induction step is completed.

4.6 Numerical Examples

In this section, some examples are presented to show the reliability of this method. In order to show the error of the method, the following notation is introduced:

$$e_{M,N}(x,t) = |u(x,t) - u_{M,N}(x,t)|, \quad (x,t) \in \Omega$$

where $u(x,t)$ is the exact solution, and $u_{M,N}(x,t)$ is the computed result with M, and N.

To solve the examples, we consider $M \neq N$ or $M = N$ and Newton's method is used for solving the nonlinear system. The initial guess in Newton's method is for these examples is considered to be $A^0 = C$, but the number of iterations can be reduced by choosing a more closed A^0 to the exact solution.

Example 4.1 *[42] Consider the following $(2D)$ nonlinear Volterra integal equation of the second kind:*

$$\int_0^t \int_0^x 2e^{x+t} u^3(y,z)dydz = \frac{1}{9}(e^{x+t} - e^{x+7t} - e^{4x+t} + e^{4x+7t}), \quad (x,t) \in [0,1] \times [0,1]$$

The exact solution is $u(x,t) = e^{x+2t}$.

Applying Bivariate Shifted Legende Functions when $M = N = 2, 4, 6$, we have the following tables which show the exact solution, approximate solution, and absolute error at some particular points.

Tab. 4.1: Exact and numerical solutions at $M = N = 2$

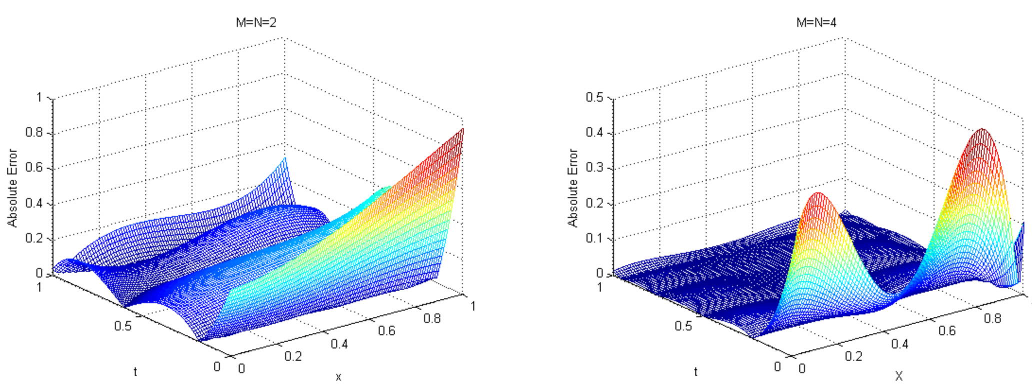
(x,t)	Exact	Approximation	Error
(0.5,0.5)	4.48168910	4.29143960	1.9024950×10^{-1}
(0.25,0.25)	2.11700002	1.97827371	1.3872631×10^{-1}
(0.125,0.125)	1.45499141	1.52152785	6.6536440×10^{-2}
(0.0625,0.0625)	1.20623025	1.41439968	2.0816943×10^{-1}
(0.03125,0.03125)	1.09828514	1.38620515	2.8792001×10^{-1}
(0.015625,0.015625)	1.04799100	1.37794336	3.2995236×10^{-1}

Tab. 4.2: Exact and numerical solutions at $M = N = 4$

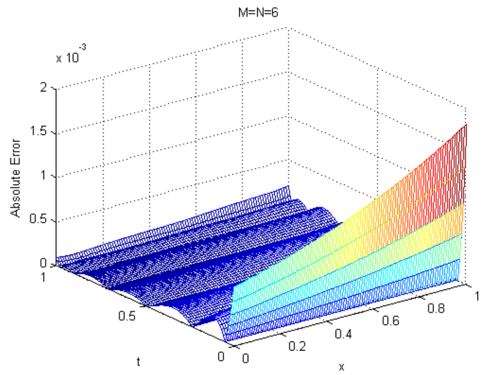
(x,t)	Exact	Approximation	Error
(0.5,0.5)	4.48168910	4.48464226	2.95316×10^{-3}
(0.25,0.25)	2.11700002	2.11638025	6.19770×10^{-4}
(0.125,0.125)	1.45499141	1.45143591	3.55550×10^{-3}
(0.0625,0.0625)	1.20623025	1.20415010	2.08015×10^{-3}
(0.03125,0.03125)	1.09828514	1.09825413	3.10100×10^{-5}
(0.015625,0.015625)	1.04799100	1.04939061	1.39961×10^{-3}

Tab. 4.3: Exact and numerical solutions at $M = N = 6$

(x,t)	Exact	Approximation	Error
(0.5,0.5)	4.48168910	4.48171956	3.0460×10^{-5}
(0.25,0.25)	2.11700002	2.11696376	3.6260×10^{-5}
(0.125,0.125)	1.45499141	1.45524307	2.5166×10^{-4}
(0.0625,0.0625)	1.20623025	1.20631829	8.8040×10^{-5}
(0.03125,0.03125)	1.09828514	1.09829225	7.1100×10^{-6}
(0.015625,0.015625)	1.04799100	1.04758734	4.0366×10^{-4}



(a) $M = N$ =2 (b) $M = N$=4



(c) $M = N = 6$

Fig. 4.1: The values of M=N versus absolute error at different x,t values

Example 4.2 *[41] Consider the following nonlinear (2D) nonlinear Volterra integral equation of the first kind:*

$$\int_0^t \int_0^x (y^2 + e^{-2z}) u^2(y,z) dy dz = \frac{1}{14} x^7 (e^{2t} - 1) + \frac{1}{5} x^5 t, \qquad (x,t) \in [0,2] \times [0,2]$$

The exact solution is $u(x,t) = x^2 e^t$. Applying Bivariate Shifted Legende Functions when $M = N = 2, 4, 6$, we have the following tables which show the exact solution, approximate solution, and absolute error at some particular points.

Tab. 4.4: Exact and numerical solutions at $M = N = 2$

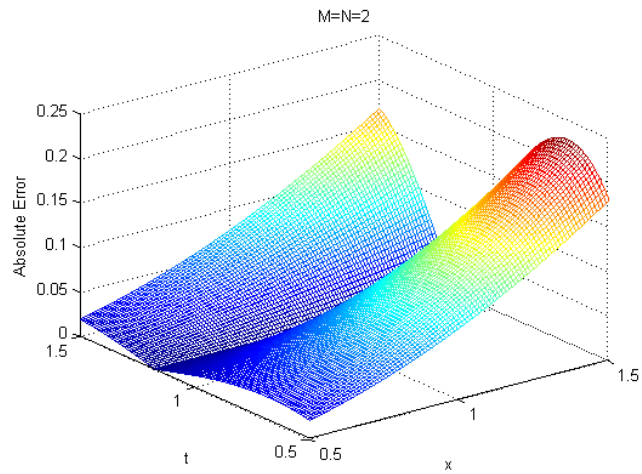
(x,t)	Exact	Approximation	Error
(0.4,0.4)	0.23869195	0.23113903	7.55292×10^{-3}
(0.55,0.55)	0.52430904	0.49716773	2.71413×10^{-2}
(0.75,0.75)	1.19081251	1.13593660	5.48759×10^{-2}
(0.85,0.85)	1.69039485	1.62912321	6.12716×10^{-2}
(1,1)	2.71828183	2.66724324	5.10386×10^{-2}
(1.2,1.2)	4.78096337	4.79204739	1.10840×10^{-2}
(1.45,1.45)	8.96319827	9.10682932	1.43631×10^{-1}
(1.6,1.6)	12.67976301	12.87292664	1.93164×10^{-1}
(1.8,1.8)	19.60085778	19.66223519	6.13774×10^{-2}
(1.95,1.95)	26.72658453	26.35777075	3.68814×10^{-1}

Tab. 4.5: Exact and numerical solutions at $M = N = 4$

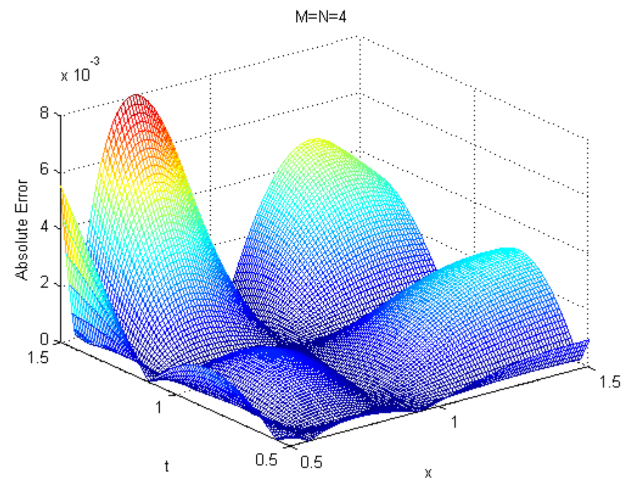
(x,t)	Exact	Approximation	Error
(0.4,0.4)	0.23869195	0.23474157	3.95038×10^{-3}
(0.55,0.55)	0.52430904	0.52432946	2.04200×10^{-5}
(0.75,0.75)	1.19081251	1.18996471	8.47800×10^{-4}
(0.85,0.85)	1.69039485	1.68974573	6.49120×10^{-4}
(1,1)	2.71828183	2.71848067	1.98840×10^{-4}
(1.2,1.2)	4.78096337	4.77980226	1.16111×10^{-3}
(1.45,1.45)	8.96319827	8.95923187	3.96640×10^{-3}
(1.6,1.6)	12.67976301	12.68000806	2.45050×10^{-4}
(1.8,1.8)	19.60085778	19.60873970	7.88192×10^{-3}
(1.95,1.95)	26.72658453	26.72136172	5.22281×10^{-3}

Tab. 4.6: Exact and numerical solutions at $M = N = 6$

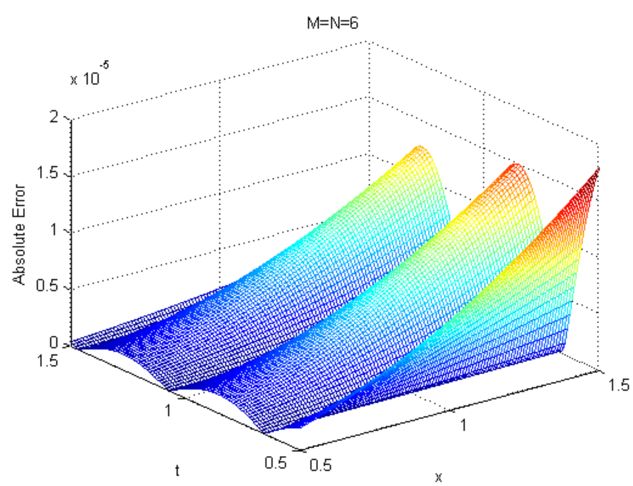
(x,t)	Exact	Approximation	Error
(0.4,0.4)	0.23869195	0.23869316	1.210×10^{-6}
(0.55,0.55)	0.52430904	0.52431089	1.850×10^{-6}
(0.75,0.75)	1.19081251	1.19081000	2.510×10^{-6}
(0.85,0.85)	1.69039485	1.69039006	4.790×10^{-6}
(1,1)	2.71828183	2.71827851	3.320×10^{-6}
(1.2,1.2)	4.78096337	4.78097515	1.178×10^{-5}
(1.45,1.45)	8.96319827	8.96319982	1.550×10^{-6}
(1.6,1.6)	12.67976301	12.67975010	1.291×10^{-5}
(1.8,1.8)	19.60085778	19.60086411	6.330×10^{-6}
(1.95,1.95)	26.72658453	26.72658890	4.370×10^{-6}



(a) M = N =2



(b) M = N=4



(c) M = N =6

Fig. 4.2: The values of N=M versus absolute error at different x,t values

Chapter 5

Three- dimensional Nonlinear Volterra Integral Equation

In this chapter, we present a numerical method for the solution of nonlinear three-dimensional Volterra integral equations **3D- VIE** of the first kind of the form:

$$f(x,t,w) = \int_0^w \int_0^t \int_0^x K(x,t,w,y,z,s)u^p(y,z,s)dydzds \tag{5.1}$$

where $x, y \in [0,l]$, $t, z \in [0,T]$, $w, s \in [0,W]$, and $(x,t,w) \in D := [0,l] \times [0,T] \times [0,W]$. $u(x,t,w)$ is an unknown function called the solution of the integral equation, p is a positive integer number, K is the kernel function and f is a smooth function.

Also, we assume that the following conditions are satisfied :

$$\begin{aligned}
f(x,t,0) &= 0,\ \forall x \in [0,l], \forall t \in [0,T] \\
f(x,0,w) &= 0 \quad,\ \forall x \in [0,l], \forall w \in [0,W] \\
f(0,t,w) &= 0 \quad,\ \forall t \in [0,T], \forall w \in [0,W] \\
f(x,0,0) &= 0 \quad,\ \forall x \in [0,l] \\
f(0,t,0) &= 0 \quad,\ \forall t \in [0,T] \\
f(0,0,w) &= 0 \quad,\ \forall w \in [0,W] \\
K(x,t,w,x,t,w) &\neq 0, \forall (x,t,w) \in D
\end{aligned} \tag{5.2}$$

Various problems in physics, mechanics, and biology arise to a nonlinear Volterra integral equations. Such equations also appear in modeling of the spatio- temporal development of an epidemic, theory of parabolic initial, boundary value problems. Population dynamics and Fourier problems.

The analytical solution of the thre-dimensional integral equations is usually difficult, and in many cases it is required to approximate the solution. [7, 35] Although several numerical methods for approximating the solutions of two dimensional Volterra integral equations were presented for the three dimensional ones, only a few methods have been discussed in the literature. The analysis of copmutational methods for several dimensional integral equations specially in the nonlinear case, has started more recently and is not so well developed.

The nonlinear **3D- VIE** of the first kind with conditions 5.2 to a nonlinear **3D-VIE** of the second kind. the nonlinear **3D- VIE** of the second kind can be obtained by making the derivative of 5.1 with respect to w, t and x.

Using Leibniz integral rule,

$$
\begin{aligned}
\frac{\partial^3 f(x,t,w)}{\partial x \partial t \partial w} &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\left[\frac{\partial}{\partial w}\int_0^w \left(\int_0^t \int_0^x K(x,t,w,y,z,s) u^p(y,z,s)dydz\right)ds\right]\right] \\
&= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\left[\int_0^w \int_0^t \int_0^x \frac{\partial K(x,t,w,y,z,s)}{\partial w} u^p(y,z,s)dydzds\right.\right. \\
&+ \left.\left.\int_0^t \int_0^x K(x,t,w,y,z,w) u^p(y,z,w)dydz\right]\right] \\
&= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}\int_0^t \left(\int_0^w \int_0^x \frac{\partial K(x,t,w,y,z,s)}{\partial w} u^p(y,z,s)dyds\right)dz\right. \\
&+ \left.\frac{\partial}{\partial t}\int_0^t \left(\int_0^x K(x,t,w,y,z,w) u^p((y,z,w)dy\right)dz\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\Big[\int_0^w \int_0^t \int_0^x \frac{\partial^2 K(x,t,w,y,z,s)}{\partial t \partial w} u^p(y,z,s)dydzds \\
&+ \int_0^w \int_0^x \frac{\partial K(x,t,w,y,t,s)}{\partial w} u^p(y,t,s)dyds \\
&+ \int_0^t \int_0^x \frac{\partial K(x,t,w,y,z,w)}{\partial t} u^p(y,z,w)dydz \\
&+ \int_0^x K(x,t,w,y,t,w)u^p(y,t,w)dy\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\partial}{\partial x}\int_0^x \left(\int_0^w \int_0^t \frac{\partial^2 K(x,t,w,y,z,s)}{\partial t \partial w} u^p(y,z,s)dzds\right)dy \\
&+ \frac{\partial}{\partial x}\int_0^x \left(\int_0^w \frac{\partial K(x,t,w,y,t,s)}{\partial w} u^p(y,t,s)ds\right)dy \\
&+ \frac{\partial}{\partial x}\int_0^x \left(\int_0^t \frac{\partial K(x,t,w,y,z,w)}{\partial t} u^p(y,z,w)dz\right)dy \\
&+ \frac{\partial}{\partial x}\int_0^x K(x,t,w,y,t,w)u^p(y,t,w)dy
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 f(x,t,w)}{\partial x \partial t \partial w} &= \int_0^w \int_0^t \int_0^x \frac{\partial^3 K(x,t,w,y,z,s)}{\partial x \partial t \partial w} u^p(y,z,s)dydzds \\
&+ \int_0^w \int_0^t \frac{\partial^2 K(x,t,w,x,z,s)}{\partial t \partial w} u^p(x,z,s)dzds \\
&+ \int_0^w \int_0^x \frac{\partial^2 K(x,t,w,y,t,s)}{\partial x \partial w} u^p(y,t,s)dyds + \int_0^w \frac{\partial K(x,t,w,x,t,s)}{\partial w} u^p(x,t,s)ds \\
&+ \int_0^t \int_0^x \frac{\partial^2 K(x,t,w,y,z,w)}{\partial x \partial t} u^p(y,z,w)dydz + \int_0^t \frac{\partial K(x,t,w,x,z,w)}{\partial t} u^p(x,z,w)dz \\
&+ \int_0^x \frac{\partial K(x,t,w,y,t,w)}{\partial x} u^p(y,t,w)dy + K(x,t,w,x,t,w)u^p(x,t,w) \qquad (5.3)
\end{aligned}
$$

Solving equation 5.3 for $u^p(x,t,w)$ to get the following form:

$$\begin{aligned}
u^p(x,t,w) &= \int_0^w \int_0^t \int_0^x K^1(x,t,w,y,z,s)u^p(y,z,s)dydzds \\
&+ \int_0^t \int_0^x K^2(x,t,w,y,z)u^p(y,z,w)dydz + \int_0^w \int_0^x K^3(x,t,w,y,s)u^p(y,t,s)dyds \\
&+ \int_0^w \int_0^t K^4(x,t,w,z,s)u^p(x,z,s)dzds + \int_0^x K^5(x,t,w,y)u^p(y,t,w)dy \\
&+ \int_0^t K^6(x,t,w,z)u^p(x,z,w)dz + \int_0^w K^7(x,t,w,s)u^p(x,t,s)ds + F(x,t,w)
\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}
K^1(x,t,w,y,z,s) &= -\frac{\partial^3 K(x,t,w,y,z,s)}{\partial x \partial t \partial w}/K(x,t,w,x,t,w) \\
K^2(x,t,w,y,z) &= -\frac{\partial^2 K(x,t,w,y,z,w)}{\partial x \partial t}/K(x,t,w,x,t,w) \\
K^3(x,t,w,y,s) &= -\frac{\partial^2 K(x,t,w,y,t,s)}{\partial x \partial w}/K(x,t,w,x,t,w) \\
K^4(x,t,w,z,s) &= -\frac{\partial^2 K(x,t,w,x,z,s)}{\partial t \partial w}/K(x,t,w,x,t,w) \\
K^5(x,t,w,y) &= -\frac{\partial K(x,t,w,y,t,w)}{\partial x}/K(x,t,w,x,t,w) \\
K^6(x,t,w,z) &= -\frac{\partial K(x,t,w,x,z,w)}{\partial t}/K(x,t,w,x,t,w) \\
K^7(x,t,w,s) &= -\frac{\partial K(x,t,w,x,t,s)}{\partial w}/K(x,t,w,x,t,w) \\
F(x,t,w) &= \frac{\partial^3 f(x,t,w)}{\partial x \partial t \partial w}/K(x,t,w,x,t,w)
\end{aligned}$$

such that $K^1 \in L^2(D \times D)$, $K^2 \in L^2(D \times [0,l] \times [0,T])$, $K^3 \in L^2(D \times [0,l] \times [0,W])$, $K^4 \in L^2(D \times [0,T] \times [0,W])$, $K^5 \in L^2(D \times [0,l])$, $K^6 \in L^2(D \times [0,T])$, and $K^7 \in L^2(D \times [0,W])$.

Now, the aim is to solve the nonlinear **3D- VIE** of the second kind using the bivariate shifted Legendre functions, and the obtained solution will be also the solution of the nonlinear **3D VIE** of the first kind. For that purpose, we introduce first the definition of three-dimensional shifted Legendre functions.

Definition 5.1 *(Three -dimensional Shifted Legendre Functions)*

The three -dimensional shifted Legendre functions are defined on the domain D as follows:

$$\begin{aligned}\psi_{mnr}(x,t,w) &= L_m\left(\frac{2}{l}x-1\right)L_n\left(\frac{2}{T}t-1\right)L_r\left(\frac{2}{W}w-1\right)\\ &= P_m(x)P_n(t)P_r(w)=\psi_m(x)\psi_n(t)\psi_r(w)\end{aligned}$$

where $m=0,1,...,M,\quad n=0,1,...,N,$ and $r=0,1,...,R.$

These functions are orthogonal with respect to the weight function $\omega(x,t,w)=1$ such that:

$$\int_0^W\int_0^T\int_0^l \omega(x,t,w)\psi_{mnr}(x,t,w)\psi_{ijk}(x,t,w)dxdtdw=\begin{cases}Ḩ, & \text{if i=m, j=n, k=r}\\ 0, & otherwise\end{cases}$$

where $Ḩ=\frac{lTW}{(2m+1)(2n+1)(2r+1)}$

Here $L_m,\ L_n$ and L_r are the well known Legendre polynomials respectively of order $m,\ n$ and r which are defined on the interval [-1,1] and satisfy the following recursive formla:

$$L_{n+1}(t)=\left(\frac{2n+1}{n+1}\right)tL_n(t)-\left(\frac{n}{n+1}\right)L_{n-1}(t),\quad n=1,2,....$$

where $L_0(t)=1,\quad L_1(t)=t.$

The shifted Legendre polynomials are defined on the interval $[0,s]$ as:

$$P_{m+1}(x)=\left(\frac{2m+1}{m+1}\right)\left(\frac{2}{s}x-1\right)P_m(x)-\left(\frac{m}{m+1}\right)P_{m-1}(x)$$

where $P_0(x) = 1,\ P_1(x) = \frac{2}{s}x - 1,\ P_m(x) = L_m\left(\frac{2}{s}x - 1\right)$, and $m = 1, 2, ...$

5.1 Function Approximation with Shifted Legendre Functions

A function $f(x, t, w) \in L^2(D)$ can be expanded in terms of Shifted Legendre series as follows [36]:

$$f(x, t, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} C_{mnr} \psi_{mnr}(x, t, w) \tag{5.5}$$

where C_{mnr} are constans given by:

$$C_{mnr} = \frac{< f(x, t, w), \psi_{mnr}(x, t, w) >}{< \psi_{mnr}, \psi_{mnr} >}$$

The inner product in space $L^2(D)$ is defined by:

$$< f(x, t, w), \psi_{mnr}(x, t, w) >= \int_0^W \int_0^T \int_0^l f(x, t, w) \psi_{mnr}(x, t, w) dx dt dw$$

and the norm, is defined as:

$$\begin{aligned} ||\psi_{mnr}(x, t, w)||_2 &= \left(< \psi_{mnr}(x, t, w), \psi_{mnr}(x, t, w) > \right)^{\frac{1}{2}} \\ &= \left(\int_0^W \int_0^T \int_0^l |\psi_{mnr}(x, t, w)|^2 dx dt dw \right)^{\frac{1}{2}}. \end{aligned}$$

If the infinite series in the equation 5.5 is truncated up to terms R, N, and M, then the function $f(x,t,w)$ may be written as:

$$\begin{aligned} f(x,t,w) &\simeq \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{mnr}\psi_{mnr}(x,t,w) = f_{MNR}(x,t,w) \\ f(x,t,w) &\simeq C^T\psi(x,t,w) \end{aligned}$$

where C, and $\psi(x,t,w)$ are $(M+1)(N+1)(R+1)\times 1$ vectors given by:

$$\begin{aligned} C = {} & [C_{000},...,C_{00R},C_{010},...,C_{01R},...,C_{0N0},...,C_{0NR},C_{100},...,C_{10R},C_{110},...,C_{11R}, \\ & ...,C_{1N0},...,C_{1NR},...,C_{M00},...,C_{M0R},C_{M10},...,C_{M1R},...,C_{MN0},...,C_{MNR}]^T \end{aligned}$$

$$\begin{aligned} \psi(x,t,w) = {} & [\psi_{000},...,\psi_{00R},\psi_{010},...,\psi_{01R},...,\psi_{0N0},...,\psi_{0NR},\psi_{100},\psi_{101},...,\psi_{10R},\psi_{110},..., \\ & \psi_{11R},...,\psi_{1N0},...,\psi_{1NR},...,\psi_{M00},...,\psi_{M0R},\psi_{M10},...,\psi_{M1R},...,\psi_{MN0},...,\psi_{MNR}]^T \end{aligned}$$

Now,

$$\begin{aligned} < f(x,t,w),\psi_{mnr(x,t,w)} > &= < \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} C_{ijk}\psi_{ijk}(x,t,w),\psi_{mnr}(x,t,w) > \\ &= < C_{mnr}\psi_{mnr}(x,t,w),\psi_{mnr}(x,t,w) > \\ &= C_{mnr} < \psi_{mnr}(x,t,w),\psi_{mnr}(x,t,w) > \\ < f(x,t,w),\psi_{mnr(x,t,w)} > &= C_{mnr}||\psi_{mnr}||_2^2 \end{aligned}$$

Solving the previous equation for the constants C_{mnr} to get:

$$C_{mnr} = \frac{< f(x,t,w),\psi_{mnr}(x,t,w) >}{||\psi_{mnr}||_2^2}$$

Depending on the definition of the inner product in space $L^2(D)$, the constants C_{mnr} can be expressed as:

$$C_{mnr} = \frac{(2m+1)(2n+1)(2r+1)}{lTW} \int_0^W \int_0^T \int_0^l f(x,t,w)\psi_{mnr}(x,t,w)dxdtdw$$

5.2 Approximating the Kernel Function with Shifted Legendre Functions

The aim of the current section is to express the kernel function $K(x,t,w,y,z,s)$ in terms of the shifted Legendre functions.

Lemma 5.1 *Let $K(x,t,w,y,z,s)$ is any function $\in L^2(D \times D)$, then*

$$K(x,t,w,y,z,s) \simeq \psi^T(x,t,w)K\psi(y,z,s) \tag{5.6}$$

where K is a block matrices of size $(M+1)(N+1)(R+1) \times (M+1)(N+1)(R+1)$ of the form $K = [K^{im}]_{i,m=0}^M$, $K^{im} = [K^{ijmn}]_{j,n=0}^N$, $K^{ijmn} = [K_{ijkmnr}]_{k,r=0}^R$,

$$K_{ijkmnr} = G\int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K(x,t,w,y,z,s)\psi_{mnr}(y,z,s)dydzds\right]\psi_{ijk}(x,t,w)dxdtdw$$

and $G = \frac{(2i+1)(2j+1)(2k+1)(2m+1)(2n+1)(2r+1)}{l^2T^2W^2}$.

Proof: Using the shifted Legendre functions, the kernel$K(x,t,w,y,z,s)$ can be expressed:

$$K(x,t,w,y,z,s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\psi_{ijk}(x,t,w)K_{ijkmnr}\psi_{mnr}(y,z,s) \tag{5.7}$$

Also, since the sequence $\psi_{ijk}(x,t,w)$ is orthonormal, the inner product of the function $K(x,t,w,y,z,s)$ together with the functions $\psi_{mnr}(y,z,s)$ and $\psi_{ijk}(x,t,w)$ can be reduced as follows:

$$
\begin{aligned}
&= \; << \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\sum_{e=0}^{\infty}\sum_{f=0}^{\infty} \psi_{abc}(x,t,w)K_{abcdef}\psi_{def}(y,z,s), \psi_{mnr}(y,z,s) >, \psi_{ijk}(x,t,w) > \\
&= \; < \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \psi_{abc}(x,t,w)K_{abcmnr} < \psi_{mnr}(y,z,s), \psi_{mnr}(y,z,s) >, \psi_{ijk}(x,t,w) > \\
&= \; < \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty} \psi_{abc}(x,t,w)K_{abcmnr}||\psi_{mnr}(y,z,s)||_2^2, \psi_{ijk}(x,t,w) > \\
&= \; K_{ijkmnr}||\psi_{mnr}(y,z,s)||_2^2 < \psi_{ikj}(x,t,w), \psi_{ikj}(x,t,w) > \\
&= \; K_{ijkmnr}||\psi_{mnr}(y,z,s)||_2^2||\psi_{ijk}(x,t,w)||_2^2.
\end{aligned}
$$

Therefore, the coefficients K_{ijkmnr} can be evaluated by

$$K_{ijkmnr} = \frac{<< K(x,t,w,y,z,s), \psi_{mnr}(y,z,s) >, \psi_{ijk}(x,t,w) >}{||\psi_{mnr}(y,z,s)||_2^2||\psi_{ijk}(x,t,w)||_2^2}$$

and using the integral notations these coefficients can be calculated by the following formula:

$$K_{ijkmnr} = G\int_0^W\int_0^T\int_0^l \left[\int_0^W\int_0^T\int_0^l K(x,t,w,y,z,s)\psi_{mnr}(y,z,s)dydzds\right]\psi_{ijk}(x,t,w)dxdtdw$$

where $G = \frac{(2i+1)(2j+1)(2k+1)(2m+1)(2n+1)(2r+1)}{l^2T^2W^2}$. If the infinite series in 5.7 is truncated, then it can be written as:

$$K(x,t,w,y,z,s) \simeq \sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} \psi_{ijk}(x,t,w)K_{ijkmnr}\psi_{mnr}(y,z,s). \tag{5.8}$$

The matrix form of the last equation is:

$$K(x,t,w,y,z,s) \simeq \psi^T(x,t,w)K\psi(y,z,s).$$

Similarly the functions K^i and $i = 1, 2, ..., 7$ can be expanded in terms of bivariate shifted Legendre functions respectively as:

$$\begin{aligned}
K^1(x,t,w,y,z,s) &\simeq \psi^T(x,t,w)K_1\psi(y,z,s) \\
K^2(x,t,w,y,z) &\simeq \psi^T(x,t,w)K_2\psi(y,z,w) \\
K^3(x,t,w,y,s) &\simeq \psi^T(x,t,w)K_3\psi(y,t,s) \\
K^4(x,t,w,z,s) &\simeq \psi^T(x,t,w)K_4\psi(x,z,s) \\
K^5(x,t,w,y) &\simeq \psi^T(x,t,w)K_5\psi(y,t,w) \\
K^6(x,t,w,z) &\simeq \psi^T(x,t,w)K_6\psi(x,z,w) \\
K^7(x,t,w,s) &\simeq \psi^T(x,t,w)K_7\psi(x,t,s)
\end{aligned}$$

where $K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_6$, and K_7 are matrices of size $(M+1)(N+1)(R+1) \times (M+1)(N+1)(R+1)$ of the form:

$K_q = [K_q^{(i,m)}]_{i,m=0}^{M}$, $q = 1, 2, ..., 7$

$K_q^{(i,m)} = [K_q^{ijmn}]_{j,n=0}^{N}$, $i, m = 0, 1, ..., M, \quad q = 1, 2, ..., 7$

$K_q^{ijmn} = [K_{ijkmnr}^{(q)}]_{k,r}^{R} = 0$, $j, n = 0, 1, ..., N, \quad q = 1, 2, ..., 7$

and Legendre coefficients $K_{ijkmnr}^{(q)}$, $q = 1, 2, ..., 7$ are given by:

$$\begin{aligned}
K_{ijkmnr}^{(1)} &= \frac{<< K^1(x,t,w,y,z,s), \psi_{mnr}(y,z,s) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(y,z,s)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G \int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^1(x,t,w,y,z,s)\psi_{mnr}(y,z,s)dydzds \right] \psi_{ijk}(x,t,w)dxdtdw
\end{aligned}$$

$$\begin{aligned}
K_{ijkmnr}^{(2)} &= \frac{<< K^2(x,t,w,y,z), \psi_{mnr}(y,z,w) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(y,z,w)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G \int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^2(x,t,w,y,z)\psi_{mnr}(y,z,w)dydzdw \right] \psi_{ijk}(x,t,w)dxdtdw
\end{aligned}$$

$$
\begin{aligned}
K^{(3)}_{ijkmnr} &= \frac{<< K^3(x,t,w,y,s), \psi_{mnr}(y,t,s) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(y,t,s)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G\int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^3(x,t,w,y,s)\psi_{mnr}(y,t,s)dydtds \right] \psi_{ijk}(x,t,w)dxdtdw
\end{aligned}
$$

$$
\begin{aligned}
K^{(4)}_{ijkmnr} &= \frac{<< K^4(x,t,w,z,s), \psi_{mnr}(x,z,s) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(x,z,s)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G\int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^4(x,t,w,z,s)\psi_{mnr}(x,z,s)dxdzds \right] \psi_{ijk}(x,t,w)dxdtdw
\end{aligned}
$$

$$
\begin{aligned}
K^{(5)}_{ijkmnr} &= \frac{<< K^5(x,t,w,y), \psi_{mnr}(y,t,w) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(y,t,w)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G\int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^5(x,t,w,y)\psi_{mnr}(y,t,w)dydtdw \right] \psi_{ijk}(x,t,w)dxdtdw
\end{aligned}
$$

$$
\begin{aligned}
K^{(6)}_{ijkmnr} &= \frac{<< K^6(x,t,w,z), \psi_{mnr}(x,z,w) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(x,z,w)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G\int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^6(x,t,w,z)\psi_{mnr}(x,z,w)dxdzdw \right] \psi_{ijk}(x,t,w)dxdtdw
\end{aligned}
$$

$$
\begin{aligned}
K^{(7)}_{ijkmnr} &= \frac{<< K^7(x,t,w,s), \psi_{mnr}(x,t,s) >, \psi_{ijk}(x,t,w)}{||\psi_{mnr}(x,t,s)||_2^2 ||\psi_{ijk}(x,t,w)||_2^2} \\
&= G\int_0^W \int_0^T \int_0^l \left[\int_0^W \int_0^T \int_0^l K^7(x,t,w,s)\psi_{mnr}(x,t,s)dxdtds \right] \psi_{ijk}(x,t,w)dxdtdw,
\end{aligned}
$$

where $G = \frac{(2i+1)(2j+1)(2k+1)(2m+1)(2n+1)(2r+1)}{l^2T^2W^2}$. $\square$

The matrix K in equation (5.6) will be in the form

$$
K = \begin{bmatrix} K^{(0,0)} & K^{(0,1)} & \dots & K^{(0,M)} \\ K^{(1,0)} & K^{(1,1)} & \dots & K^{(1,M)} \\ \vdots & \ddots & \vdots & \vdots \\ K^{(M,0)} & K^{(M,1)} & \dots & K^{(M,M)} \end{bmatrix},
$$

$$= \begin{bmatrix} K^{0000} & K^{0001} & ... & K^{000N} & & K^{00M0} & K^{00M1} & ... & K^{00MN} \\ K^{0100} & K^{0101} & ... & K^{010N} & ,..., & K^{01M0} & K^{01M1} & ... & K^{01MN} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ K^{0N00} & K^{0N01} & ... & K^{0N0N} & & K^{0NM0} & K^{0NM1} & ... & K^{0NMN} \\ & & \vdots & & \ddots & & & \vdots & \\ K^{M000} & K^{M001} & ... & K^{M00N} & & K^{M0M0} & K^{M0M1} & ... & K^{M0MN} \\ K^{M100} & K^{M101} & ... & K^{M10N} & ,..., & K^{M1M0} & K^{M1M1} & ... & K^{M1MN} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ K^{MN00} & K^{MN01} & ... & K^{MN0N} & & K^{MNM0} & K^{MNM1} & ... & K^{MNMN} \end{bmatrix}$$

Note that $K^{ijmn} = [K_{ijkmnr}]_{(R+1)\times(R+1)}$, $i, m = 0, 1, ..., M, \quad j, n = 0, 1, ..., N$ and $k, r = 0, 1, ..., R$.

the matrices $K_q, \ q = 1, 2, ..., 7$ can be obtained using the same manner as the matrix K.

5.3 Operational Matrices of Integration

The integration of the vector $\psi(x, t, w)$ can be approximately obtained by the following formulas [18]:

$$\int_0^w \int_0^t \int_0^x \psi(y, z, s) dydzds \simeq Q_1 \psi(x, t, w) \tag{5.9}$$

$$\int_0^t \int_0^x \psi(y, z, w) dydz \simeq Q_2 \psi(x, t, w) \tag{5.10}$$

$$\int_0^w \int_0^x \psi(y, t, s) dyds \simeq Q_3 \psi(x, t, w) \tag{5.11}$$

$$\int_0^w \int_0^t \psi(x, z, s) dzds \simeq Q_4 \psi(x, t, w) \tag{5.12}$$

$$\int_0^x \psi(y, t, w) dy \simeq Q_5 \psi(x, t, w) \tag{5.13}$$

$$\int_0^t \psi(x, z, w) dz \simeq Q_6 \psi(x, t, w) \tag{5.14}$$

$$\int_0^w \psi(x, t, s) ds \simeq Q_7 \psi(x, t, w) \tag{5.15}$$

where $x \in [0, l]$, $t \in [0, T]$, and $w \in [0, W]$. Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 and Q_7 are the $(M+1)(N+1)(R+1) \times (M+1)(N+1)(R+1)$ operational matrices of integration and have shapes:

$$
\begin{aligned}
Q_1 &= (P_1 \otimes P_2) \otimes P_3 \\
Q_2 &= (P_1 \otimes P_2) \otimes I_{R+1} \\
Q_3 &= (P_1 \otimes I_{N+1}) \otimes P_3 \\
Q_4 &= (I_{M+1} \otimes P_2) \otimes P_3 \\
Q_5 &= (P_1 \otimes I_{N+1}) \otimes I_{R+1} \\
Q_6 &= (I_{M+1} \otimes P_2) \otimes I_{R+1} \\
Q_7 &= (I_{M+1} \otimes I_{N+1}) \otimes P_3
\end{aligned}
$$

where I_{N+1}, I_{M+1} and I_{R+1} are the identity matrices of size $M+1$, $N+1$ and $R+1$. P_1, P_2 , and P_3 are the operational matrices of one dimensional Shifted Legendre polynomials defined respectively in $[0,l]$, $[0,T]$ and $[0,W]$ as follows:

$$
P_q = \frac{z}{2}\begin{bmatrix}
1 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\
\frac{-1}{3} & 0 & \frac{1}{3} & 0 & ... & 0 & 0 & 0 \\
0 & \frac{-1}{5} & 0 & \frac{1}{5} & ... & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 0 & ... & \frac{-1}{2h-1} & 0 & \frac{1}{2h-1} \\
0 & 0 & 0 & 0 & ... & 0 & \frac{-1}{2h+1} & 0
\end{bmatrix}
$$

for $z = l, T$and W, $h = M, N$ and R, $q = 1, 2, 3$. For more details see reference....

5.4 The Product of Operational Matrices

The following property of the product of two vectors $\psi(x,t,w)$ and $\psi^T(x,t,w)$ is very helpful

$$\psi(x,t,w)\psi^T(x,t,w)C \simeq \tilde{C}\psi(x,t,w) \tag{5.16}$$

where

$$\begin{aligned} C &= [C_{000},...,C_{00R},C_{010},...,C_{01R},...,C_{0N0},...,C_{0NR},C_{100},...,C_{10R},C_{110},...C_{11R},...,C_{1N0}, \\ &\quad ...,C_{M10},...,C_{M1R},...,C_{MN0} \quad C_{MN1} \quad ... \quad C_{MNR}]^T \end{aligned}$$

To explain the the above result, first notice that $\psi(x,t,w)\psi^T(x,t,w)C = [\psi_{ijk}][\psi_{hqv}]^T C$ is a column matrix and by elementary calculations the entries can be written as $\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{hqv}\psi_{hqv}\psi_{ijk}$ where $i=0,...,M,\ j=0,...,N$, and $k=0,...,R$.
On the other hand, the product $\psi_{ijk}(x,t,w)\psi_{hqv}(x,t,w)$ can be written as a linear combinations of three-dimensional shifted Legendre function. In fact, it can be expressed as follows

$$\psi_{ijk}(x,t,w)\psi_{hqv}(x,t,w) = \sum_{u=0}^{i+h}\sum_{s=0}^{j+q}\sum_{p=0}^{k+v} a_{usp}\psi_{usp}(x,t,w) \tag{5.17}$$

.

To compute the coefficients a_{usp}, multiply both sides of equation 5.17 by $\psi_{mnr}(x,t,w),\ \ m = 0,1,...,M,\ \ \ n=0,1,...,N,\ \ \ r=0,1,...,R$ and integrate from 0 to l, 0 to T, and 0 to W and taking in account the orthogonality property

$$\begin{aligned} a_{mnr} &= \frac{(2m+1)(2n+1)(2r+1)}{lTW}\int_0^W\int_0^T\int_0^l \psi_{ijk}\psi_{hqv}\psi_{mnr}\,dxdtdw \\ &= \frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{ihm}w'_{jqn}w''_{kvr} \end{aligned}$$

where

$$\begin{aligned}
w_{ihm} &= \int_0^l \psi_i(x)\psi_h(x)\psi_m(x)dx, \quad i,h,m = 0,1,...,M \\
w'_{jqn} &= \int_0^l \psi_j(t)\psi_q(t)\psi_n(t)dt, \quad j,q,n = 0,1,...,N \\
w''_{kvr} &= \int_0^l \psi_k(w)\psi_v(w)\psi_r(w)dw, \quad k,v,r = 0,1,...,R
\end{aligned}$$

So, equation 5.17 becomes

$$\psi_{ijk}\psi_{hqv} = \sum_{m=0}^{i+h}\sum_{n=0}^{j+q}\sum_{r=0}^{k+v} \frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{ihm}w'_{jqn}w''_{kvr}\psi_{mnr}(x,t,w)$$

Which implying to :

$$\psi_{ijk}\psi_{hqv} \simeq \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} \frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{ihm}w'_{jqn}w''_{kvr}\psi_{mnr}(x,t,w) \quad (5.18)$$

If we substuted equation 5.18 in $\psi(x,t,w)\psi^T(x,t,w)C$, we get the approximated

$$\begin{bmatrix}
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{0hm}w'_{0qn}w''_{0vr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{0hm}w'_{0qn}w''_{Rvr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{0hm}w'_{Nqn}w''_{0vr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{0hm}w'_{Nqn}w''_{Rvr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{Mhm}w'_{0qn}w''_{0vr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{Mhm}w'_{0qn}w''_{Rvr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{Mhm}w'_{Nqn}w''_{0vr}\psi_{mnr}(x,t,w) \\
\vdots \\
\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} C_{hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW} w_{Mhm}w'_{Nqn}w''_{Rvr}\psi_{mnr}(x,t,w)
\end{bmatrix}$$

It is clear that $w_{ihm} = w_{imh}$, $w'_{jqn} = w'_{jnq}$, and $w''_{kvr} = w''_{krv}$ and after a suitable reordering of lower indices, the last column matrix could be written as

$$\begin{bmatrix}
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{0hm}w'_{0qn}w''_{0vr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{0hm}w'_{0qn}w''_{Rvr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{0hm}w'_{Nqn}w''_{0vr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{0hm}w'_{Nqn}w''_{Rvr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{Mhm}w'_{0qn}w''_{0vr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{Mhm}w'_{0qn}w''_{Rvr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{Mhm}w'_{Nqn}w''_{0vr}\psi_{hqv}(x,t,w) \\
\vdots \\
\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} C_{mnr}\frac{(2h+1)(2q+1)(2v+1)}{lTW} w_{Mhm}w'_{Nqn}w''_{Rvr}\psi_{hqv}(x,t,w)
\end{bmatrix}$$

Hence we have:

$$\begin{bmatrix} \tilde{C}_{00} & \tilde{C}_{01} & ... & \tilde{C}_{0M} \\ \tilde{C}_{10} & \tilde{C}_{11} & ... & \tilde{C}_{1M} \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{C}_{M0} & \tilde{C}_{M1} & ... & \tilde{C}_{MM} \end{bmatrix} \begin{bmatrix} \psi_{000} \\ \cdots \\ \psi_{00R} \\ \vdots \\ \psi_{0N0} \\ \vdots \\ \psi_{0NR} \\ \vdots \\ \psi_{M00} \\ \vdots \\ \psi_{M0R} \\ \vdots \\ \psi_{MN0} \\ \vdots \\ \psi_{MNR} \end{bmatrix} = \tilde{C}\psi(x,t,w)$$

Finally

$$\psi(x,t,w)\psi^T(x,t,w)C \simeq \tilde{C}\psi(x,t,w) \tag{5.19}$$

where $\tilde{C}$ is a $(M+1)(N+1)(R+1) \times (M+1)(N+1)(R+1)$ product operational matrix.

$$\tilde{C} = [\tilde{C}_{ih}], \quad i,h = 0,1,...,M \tag{5.20}$$

such that $\tilde{C}_{ih}$ is given by:

$$\tilde{C}_{ih} = \frac{2h+1}{l} \sum_{m=0}^{M} w_{ihm} A_m$$

where A_m is a square matrix of size $(N+1)$ whose entries are of the form:

$$[A_m]_{j,q=0}^{N} = \frac{2q+1}{T}\sum_{n=0}^{N} w'_{jqn}B_{mn}$$

and B_{mn} is a square matrix of size $(R+1)$ with entries of the form:

$$[B_{mn}]_{k,v=0}^{R} = \frac{2v+1}{W}\sum_{r=0}^{R} w''_{kvr}C_{mnr}$$

Lemma 5.2 *For an* $(M+1)(N+1)(R+1)\times(M+1)(N+1)(R+1)$ *matrix* B *, we have:*

$$\psi^T(x,t,w)B\psi(x,t,w) \simeq \hat{B}\psi(x,t,w)$$

where $B = [B^{ih}]_{i,h=0}^{M}$, $\quad i,h = 0,1,...,M, \quad B^{ih} = [B^{ijhq}]_{j,q=0}^{N}, \quad j,q = 0,1,...,N,$
$B^{ijhq} = [b_{ijkhqv}]_{k,v=0}^{R}, \quad k,v = 0,1,...,R$

$$B = \begin{bmatrix} B^{00} & B^{01} & ... & B^{0M} \\ B^{10} & B^{11} & ... & B^{1M} \\ \vdots & \ddots & \vdots & \vdots \\ B^{M0} & B^{M1} & ... & B^{MM} \end{bmatrix},$$

$$B = \begin{bmatrix} B^{0000} & B^{0001} & ... & B^{000N} & & B^{00M0} & B^{00M1} & ... & B^{00MN} \\ B^{0100} & B^{0101} & ... & B^{010N} & ,..., & B^{01M0} & B^{01M1} & ... & B^{01MN} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ B^{0N00} & B^{0N01} & ... & B^{0N0N} & & B^{0NM0} & B^{0NM1} & ... & B^{0NMN} \\ & & \vdots & & \ddots & & & \vdots & \\ B^{M000} & B^{M001} & ... & B^{M00N} & & B^{M0M0} & B^{M0M1} & ... & B^{M0MN} \\ B^{M100} & B^{M101} & ... & B^{M10N} & ,..., & B^{M1M0} & B^{M1M1} & ... & B^{M1MN} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ B^{MN00} & B^{MN01} & ... & B^{MN0N} & \dots & B^{MNM0} & B^{MNM1} & ... & B^{MNMN} \end{bmatrix}$$

Note that $B^{ijhq} = [b_{ijkhqv}]_{(R+1)\times(R+1)}$, $i,h = 0,1,...,M$, $\quad j,q = 0,1,...,N$ *and* $k,v = 0,1,...,R$.

$\hat{B}$ *is a* $1\times(M+1)(N+1)(R+1)$ *vector defined as:*

$$\hat{B} = [B_{000} \;\; B_{001} \;\; ... \;\; B_{00R}, \; ..., \;\; B_{0N0} \;\; B_{0N1} \;\; ... \;\; B_{0NR}, \;\; B_{100} \;\; B_{101} ... \;\; B_{10R}, \;\; ..., \\ B_{1N0} \;\; B_{1N1} \;\; ... \;\; B_{1NR}, ..., \;\; B_{M00} \;\; B_{M01} \;\; ... \;\; B_{M0R} \;\; , \;\; ..., \; B_{MN0} \;\; B_{MN1} \;\; ... \;\; B_{MNR}]$$

with

$$B_{mnr} = \frac{(2m+1)(2n+1)(2r+1)}{lTW} \sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} w_{ihm} w'_{jqn} w''_{kvr} b_{ijkhqv}$$

$m = 0,1,...,M, \quad n = 0,1,...,N, \quad r = 0,1,...,R$

Proof:

$$\psi^T(x,t,w)B\psi(x,t,w) = \begin{bmatrix} \psi_{000} \\ \vdots \\ \psi_{00R} \\ \vdots \\ \psi_{0N0} \\ \vdots \\ \psi_{0NR} \\ \vdots \\ \psi_{M00} \\ \vdots \\ \psi_{M0R} \\ \vdots \\ \psi_{MN0} \\ \vdots \\ \psi_{MNR} \end{bmatrix}^T B \begin{bmatrix} \psi_{000} \\ \vdots \\ \psi_{00R} \\ \vdots \\ \psi_{0N0} \\ \vdots \\ \psi_{0NR} \\ \vdots \\ \psi_{M00} \\ \vdots \\ \psi_{M0R} \\ \vdots \\ \psi_{MN0} \\ \vdots \\ \psi_{MNR} \end{bmatrix}$$

$$\psi^T(x,t,w)B\psi(x,t,w) = \psi^T(x,t,w)\begin{bmatrix} \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{000hqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{00Rhqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{0N0hqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{0NRhqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{M00hqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{M0Rhqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{MN0hqv}\psi_{hqv} \\ \vdots \\ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{MNRhqv}\psi_{hqv} \end{bmatrix}$$

and the above product has the following form

$$\begin{aligned}\psi^T(x,t,w)B\psi(x,t,w) \quad &= \quad \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{000hqv}\psi_{000}\psi_{hqv} \\ &+\ldots+ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{00Rhqv}\psi_{00R}\psi_{hqv} \\ &+\ldots+ \sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{0N0hqv}\psi_{0N0}\psi_{hqv}\end{aligned}$$

$$+\ldots+\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{0NRhqv}\psi_{0NR}\psi_{hqv}$$
$$+\ldots+\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{M00hqv}\psi_{M00}\psi_{hqv}$$
$$+\ldots+\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{M0Rhqv}\psi_{M0R}\psi_{hqv}$$
$$+\ldots+\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{MN0hqv}\psi_{MN0}\psi_{hqv}$$
$$+\ldots+\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{MNRhqv}\psi_{MNR}\psi_{hqv}$$

According to equation 5.18, we have:

$$\begin{aligned}
&\psi^T(x,t,w)B\psi(x,t,w)\simeq\\
&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{000hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{0hm}w'_{0qn}w''_{0vr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{00Rhqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{0hm}w'_{0qn}w''_{Rvr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{0N0hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{0hm}w'_{Nqn}w''_{0vr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{0NRhqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{0hm}w'_{Nqn}w''_{Rvr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{M00hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{Mhm}w'_{0qn}w''_{0vr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{M0Rhqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{Mhm}w'_{0qn}w''_{Rvr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{MN0hqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{Mhm}w'_{Nqn}w''_{0vr}\psi_{mnr}(x,t,w)\\
+\ldots+\;&\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R}b_{MNRhqv}\frac{(2m+1)(2n+1)(2r+1)}{lTW}w_{Mhm}w'_{Nqn}w''_{Rvr}\psi_{mnr}(x,t,w)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(2(0)+1)(2(0)+1)(2(0)+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ik0}w'_{jq0}w''_{kv0}\psi_{000}(x,t,w) \\
&+\ldots+ \frac{(2(0)+1)(2(0)+1)(2R+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ik0}w'_{jq0}w''_{kvR}\psi_{00R}(x,t,w) \\
&+\ldots+ \frac{(2(0)+1)(2N+1)(2(0)+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ik0}w'_{jqN}w''_{kv0}\psi_{0N0}(x,t,w) \\
&+\ldots+ \frac{(2(0)+1)(2N+1)(2R+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ik0}w'_{jqN}w''_{kvR}\psi_{0NR}(x,t,w) \\
&+\ldots+ \frac{(2M+1)(2(0)+1)(2(0)+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ikM}w'_{jq0}w''_{kv0}\psi_{M00}(x,t,w) \\
&+\ldots+ \frac{(2M+1)(2(0)+1)(2(N)+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ikM}w'_{jq0}w''_{kvR}\psi_{M0R}(x,t,w) \\
&+\ldots+ \frac{(2M+1)(2N+1)(2(0)+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ikM}w'_{jqN}w''_{kv0}\psi_{MN0}(x,t,w) \\
&+\ldots+ \frac{(2M+1)(2N+1)(2R+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ikM}w'_{jqN}w''_{kvR}\psi_{MNR}(x,t,w) \\
&= \sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{r=0}^{R} B_{mnr}\psi_{mnr}(x,t,w) = \hat{B}\psi(x,t,w)
\end{aligned}
$$

where

$$
B_{mnr} = \frac{(2m+1)(2n+1)(2r+1)}{lTW}\sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{R}\sum_{h=0}^{M}\sum_{q=0}^{N}\sum_{v=0}^{R} b_{ijkhqv}w_{ihm}w'_{jqn}w''_{kvr}
$$

Finally,

$$
\psi^T(x,t,w)B\psi(x,t,w) \simeq \hat{B}\psi(x,t,w)
$$

□

Similarly, any matrix of the size $(M+1)(N+1)(R+1) \times (M+1)(N+1)(R+1)$

we have:

$$\begin{aligned}
\psi^T(x,t,w)B_1\psi(x,t,w) &\simeq \hat{B}_1\psi(x,t,w)\\
\psi^T(x,t,w)B_2\psi(x,t,w) &\simeq \hat{B}_2\psi(x,t,w)\\
\psi^T(x,t,w)B_3\psi(x,t,w) &\simeq \hat{B}_3\psi(x,t,w)\\
\psi^T(x,t,w)B_4\psi(x,t,w) &\simeq \hat{B}_4\psi(x,t,w)\\
\psi^T(x,t,w)B_5\psi(x,t,w) &\simeq \hat{B}_5\psi(x,t,w)\\
\psi^T(x,t,w)B_6\psi(x,t,w) &\simeq \hat{B}_6\psi(x,t,w)\\
\psi^T(x,t,w)B_7\psi(x,t,w) &\simeq \hat{B}_7\psi(x,t,w) \qquad (5.21)
\end{aligned}$$

5.5 Method of Solution

In this section, we present a numerical method to find an approximate solution to 5.1 which corresponds with equation 5.4. Also, it is assumed that the included functions this equation satisfy the conditions (5.2) to garntee that the solution is unique.
Using the Bivariate shifted Legendre functions, $u^P(x,t,w)$ and $F(x,t,w)$ can be approximated by as:

$$u^p(x,t,w) \simeq C^T\psi(x,t,w) = \psi^T(x,t,w)C \qquad (5.22)$$

$$F(x,t,w) \simeq F^T\psi(x,t,w) \qquad (5.23)$$

where F^T is a $1 \times (M+1)(N+1)(R+1)$ vector of constants of the form

$$F^T = [F_{000}\ \ldots\ F_{00R},\ \ldots,\ F_{0N0}\ \ldots\ F_{0NR},\ \ldots,\ F_{M00}\ \ldots\ F_{M0R},\ \ldots,\ F_{MN0}\ \ldots\ F_{MNR}]$$

and such that $F_{mnr} = \frac{(2m+1)(2n+1)(2r+1)}{lTW}\int_0^W\int_0^T\int_0^l F(x,t,w)\psi_{mnr}(x,t,w)dxdtdw$.

Also, the approximation of the other functions in equation 5.4 are needed, so we list them below:

$$K^1(x,t,w,y,z,s) \simeq \psi^T(x,t,w)K_1\psi(y,z,s) \tag{5.24}$$

$$K^2(x,t,w,y,z) \simeq \psi^T(x,t,w)K_2\psi(y,z,w) \tag{5.25}$$

$$K^3(x,t,w,y,s) \simeq \psi^T(x,t,w)K_3\psi(y,t,s) \tag{5.26}$$

$$K^4(x,t,w,z,s) \simeq \psi^T(x,t,w)K_4\psi(x,z,s) \tag{5.27}$$

$$K^5(x,t,w,y) \simeq \psi^T(x,t,w)K_5\psi(y,t,w) \tag{5.28}$$

$$K^6(x,t,w,z) \simeq \psi^T(x,t,w)K_6\psi(x,z,w) \tag{5.29}$$

$$K^7(x,t,w,s) \simeq \psi^T(x,t,w)K_7\psi(x,t,s) \tag{5.30}$$

Substituting equations (5.22-5.30) into equation 5.4 to get:

$$\begin{aligned}
C^T\psi(x,t,w) &= \int_0^w\int_0^t\int_0^x \psi^T(x,t,w)K_1\psi(y,z,s)\psi^T(y,z,s)Cdydzds \\
&+ \int_0^t\int_0^x \psi^T(x,t,w)K_2\psi(y,z,w)\psi^T(y,z,w)Cdydz \\
&+ \int_0^w\int_0^x \psi^T(x,t,w)K_3\psi(y,t,s)\psi^T(y,t,s)Cdyds \\
&+ \int_0^w\int_0^t \psi^T(x,t,w)K_4\psi(x,z,s)\psi^T(x,z,s)Cdzds \\
&+ \int_0^x \psi^T(x,t,w)K_5\psi(y,t,w)\psi^T(y,t,w)Cdy \\
&+ \int_0^t \psi^T(x,t,w)K_6\psi(x,z,w)\psi^T(x,z,w)Cdz \\
&+ \int_0^w \psi^T(x,t,w)K_7\psi(x,t,s)\psi^T(x,t,s)Cds + F^T\psi(x,t,w)
\end{aligned}$$

$$
\begin{aligned}
C^T\psi(x,t,w) &= \psi^T(x,t,w)K_1\int_0^w\int_0^t\int_0^x \psi(y,z,s)\psi^T(y,z,s)C dydzds \\
&+ \psi^T(x,t,w)K_2\int_0^t\int_0^x \psi(y,z,w)\psi^T(y,z,w)C dydz \\
&+ \psi^T(x,t,w)K_3\int_0^w\int_0^x \psi(y,t,s)\psi^T(y,t,s)C dyds \\
&+ \psi^T(x,t,w)K_4\int_0^w\int_0^t \psi(x,z,s)\psi^T(x,z,s)C dzds \\
&+ \psi^T(x,t,w)K_5\int_0^x \psi(y,t,w)\psi^T(y,t,w)C dy \\
&+ \psi^T(x,t,w)K_6\int_0^t \psi(x,z,w)\psi^T(x,z,w)C dz \\
&+ \psi^T(x,t,w)K_7\int_0^w \psi(x,t,s)\psi^T(x,t,s)C ds + F^T\psi(x,t,w)
\end{aligned}
$$

According to equation 5.19 the above expression took the following form

$$
\begin{aligned}
C^T\psi(x,t,w) &= \psi^T(x,t,w)K_1\int_0^w\int_0^t\int_0^x \tilde{C}\psi(y,z,s)dydzds \\
&+ \psi^T(x,t,w)K_2\int_0^t\int_0^x \tilde{C}\psi(y,z,w)dydz + \psi^T(x,t,w)K_3\int_0^w\int_0^x \tilde{C}\psi(y,t,s)dyds \\
&+ \psi^T(x,t,w)K_4\int_0^w\int_0^t \tilde{C}\psi(x,z,s)dzds + \psi^T(x,t,w)K_5\int_0^x \tilde{C}\psi(y,t,w)dy \\
&+ \psi^T(x,t,w)K_6\int_0^t \tilde{C}\psi(x,z,w)dz + \psi^T(x,t,w)K_7\int_0^w \tilde{C}\psi(x,t,s)ds \\
&+ F^T\psi(x,t,w)
\end{aligned}
$$

Using equations 5.9-5.15, we get:

$$
\begin{aligned}
C^T\psi(x,t,w) &= \psi^T(x,t,w)K_1\tilde{C}Q_1\psi(x,t,w) + \psi^T(x,t,w)K_2\tilde{C}Q_2\psi(x,t,w) \\
&+ \psi^T(x,t,w)K_3\tilde{C}Q_3\psi(x,t,w) + \psi^T(x,t,w)K_4\tilde{C}Q_4\psi(x,t,w) \\
&+ \psi^T(x,t,w)K_5\tilde{C}Q_5\psi(x,t,w) + \psi^T(x,t,w)K_6\tilde{C}Q_6\psi(x,t,w) \\
&+ \psi^T(x,t,w)K_7\tilde{C}Q_7\psi(x,t,w) + F^T\psi(x,t,w).
\end{aligned}
$$

Finally, from equations 5.21 the product $C^T\psi(x,t,w)$

$$\begin{aligned}
C^T\psi(x,t,w) &= \hat{B}_1\psi(x,t,w)+\hat{B}_2\psi(x,t,w)+\hat{B}_3\psi(x,t,w)+\hat{B}_4\psi(x,t,w)\\
&+ \hat{B}_5\psi(x,t,w)+\hat{B}_6\psi(x,t,w)+\hat{B}_7\psi(x,t,w)+F^T\psi(x,t,w)\\
C^T\psi(x,t,w) &= \left(\sum_{i=1}^{7}\hat{B}_i+F^T\right)\psi(x,t,w). \qquad (5.31)
\end{aligned}$$

Hence we have:

$$C^T=\sum_{i=1}^{7}\hat{B}_i+F^T$$

which corresponds with a system of linear algebraic equations in terms of the unknown elements of the vector C and can be solved easily using direct methods.

The unknown function $u(x,t,w)$ can be approximated in terms of the Bivariate shifted Legendre functions as:

$$u(x,t,w)\simeq A^T\psi(x,t,w) \qquad (5.32)$$

Such that the entries of the vector A are unknowns.

Using equations 5.19, and 5.32 , it is can be easily obtained that:

$$u^p(x,t,w)\simeq A^T\tilde{A}^{p-1}\psi(x,t,w) \qquad (5.33)$$

Finally, using equations 5.33, 5.22, we get:

$$A\tilde{A}^{p-1}=C^T \qquad (5.34)$$

Equation 5.34 forms a system of $(M+1)(N+1)(R+1)$ system of nonlinear equations.

The result in equation 5.33 can be proved by induction, that $\forall$ K $\in \mathbb{Z}^+$,

Using the result 5.19, and replace C by A to have:

$$\psi(x,t,w)\psi^T(x,t,w)A\simeq\tilde{A}\psi(x,t,w) \qquad (5.35)$$

Apply the transpose for both sides of equation 5.35 to yield:

$$A^T\psi(x,t,w)\psi^T(x,t,w) \simeq \psi^T(x,t,w)\tilde{A}^T \tag{5.36}$$

Apply the result of equation 5.32 with $p = 1$ to get:

$$u(x,t,w) \simeq A^T\tilde{A}^0\psi(x,t,w) \tag{5.37}$$

For $p = 2$,

$$\begin{aligned} u^2(x,t,w) &= u(x,t,w)u(x,t,w) \\ &\simeq A^T\psi(x,t,w)\psi^T(x,t,w)A \\ u^2(x,t) &\simeq A^T\tilde{A}^1\psi(x,t,w) \end{aligned}$$

If $p = 3$, then,

$$\begin{aligned} u^3(x,t) &= u^2(x,t,w)u(x,t,w) \\ &\simeq A^T\tilde{A}^1\psi(x,t,w)\psi^T(x,t,w)A \\ &\simeq A\tilde{A}\tilde{A}\psi(x,t,w) \\ u^3(x,t,w) &\simeq A^T\tilde{A}^2\psi(x,t,w) \end{aligned}$$

For $p = 4$,

$$\begin{aligned} u^4(x,t,w) &= u^3(x,t,w)u(x,t,w) \\ &\simeq A^T\tilde{A}^2\psi(x,t,w)\psi^T(x,t,w)A \\ u^4(x,t,w) &\simeq A^T\tilde{A}^3\psi(x,t,w) \end{aligned}$$

Suppose that $k \in \mathbb{Z}^+$, and let equation 5.33 be true for k, then,

$$\begin{aligned}
u^{k+1}(x,t,w) &= u^k(x,t,w)u^1(x,t,w) \\
&\simeq A^T \tilde{A}^{k-1}\psi(x,t,w)\psi^T(x,t,w)A \\
&\simeq A^T \tilde{A}^{k-1}\tilde{A}\psi(x,t,w) \\
u^{k+1}(x,t,w) &\simeq A^T \tilde{A}^{k}\psi(x,t,w)
\end{aligned}$$

Thus equation 5.33 is true for $p = k+1$, and the proof is completed.

5.6 Numerical Examples

In this section, some examples are presented to show the reliability of this method. In order to show the error of the method, the following notation is introduced:

$$e_{M,N,R}(x,t,w) = |u(x,t,w) - u_{M,N,R}(x,t,w)|, \quad (x,t,w) \in D$$

where $u(x,t,w)$ is the exact solution, and $u_{M,N,R}(x,t,w)$ is the computed result with M, and N.
Throughout this section, we assume that $M \neq N \neq R$ or $M = N = R$. Also, in order to apply Newton's method to solve the resultant nonlinear system, we assume that the initial guess is $A^0 = C$.

Example 5.1 *Consider the following **3D** nonlinear volterra integral equation of the first kind:*

$$\int_0^w \int_0^t \int_0^x \frac{u^3(y,z,s)}{y+z+s+2} dydzds = \frac{xtw}{6}(2t^2+3tw+3tx+12t+2w^2+3wx+12w+2x^2+12x+24)$$

$(x,t,w) \in [0,3] \times [0,3] \times [0,3]$.

The exact solution is $u(x,t,w) = x+t+w+2$.

Applying the presented method with $M = N = R = 1$, and obtaining the linear system in terms of the unknown coefficiets of the function $u^3(x, t, w)$ as:

$c_{000} = \frac{637}{2}, \quad c_{001} = c_{010} = c_{100} = \frac{1989}{10}, \quad c_{011} = c_{101} = c_{110} = \frac{351}{4}, \quad c_{111} = \frac{81}{4}.$

Substituting obtained values for $c_{ijk}, \quad i, j, k = 0, 1$ in to equation $A^T\tilde{A}^2 = C^T$, then we get a system of nonlinear equations in terms of the unknown coefficients of $u(x, t, w)$. Finally, Newton's method is used with the initial guess $A^{(0)}$, eight ierations and presicion 10^{-9}, are considered.

$a_{000} = 6.495628455, \quad a_{001} = a_{010} = a_{110} = 1.510686315,$

$a_{011} = a_{1o1} = a_{110} = -0.009707900, \quad a_{111} = 0.009578177.$

Therefor, we have $u_{1,1,1}(x, t, w) = 1.026543529(x+t+w) - 0.00857159(tw+tx+xw) + 0.002837978(twx) + 1.924867633.$

Also,

$e_{1,1,1} = |0.026453529(x + t + w) - 0.008590(tx + tw + xw) + 0.002837978(txw) - 0.075132367| \leq 0.075132367.$

For $M = N = R = 2$, we have

$a_{000} = 6.5, \quad a_{001} = a_{010} = a_{100} = 1.5, \quad a_{002} = a_{011} = a_{012} = a_{020} = a_{021} = a_{022} = a_{101} = a_{102} = a_{110} = a_{111} = a_{112} = a_{120} = a_{121} = a_{122} = a_{200} = a_{201} = a_{202} = a_{210} = a_{211} = a_{212} = a_{220} = a_{221} = a_{222} = 0$

So, $u_{222}(x, t, w) = x + t + w + 2$ which is the exact solution.

Example 5.2 *[57]*

$$\int_0^w \int_0^t \int_0^x yz^2u(y, z, s)dydzds = \frac{x^3t^3}{9}sinw, \quad (x, t, w) \in [0, 1] \times [0, 1] \times [0, 1]$$

The exact solution is $u(x, t, w) = xcosw$

Applyinf Bivariate shifted Legendre function for $M = N = R = 1, 2, 3$, The following tables show the exact solution, the approximate solution, and the absolute error at some particular points.

Tab. 5.1: Exact and numerical solutions at $M = N = R = 1$

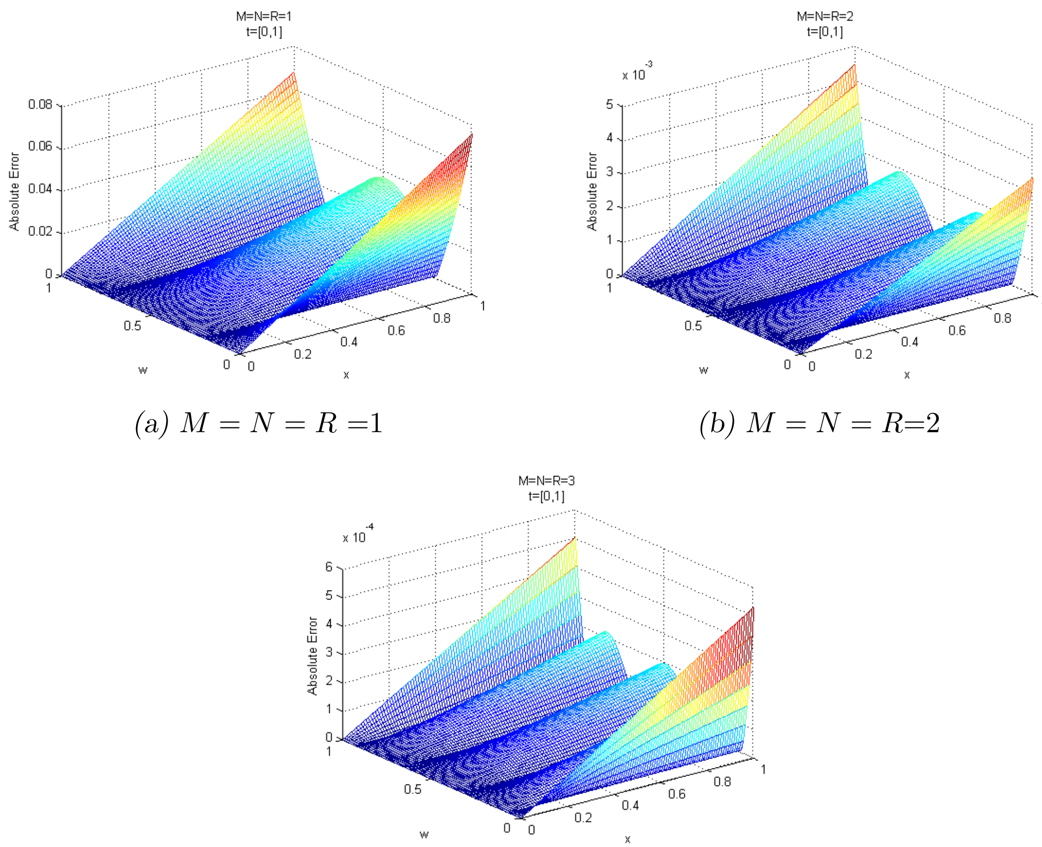
$x = t = w$	Exact	Approximation	Error
10^{-7}	10^{-7}	1.0752000×10^{-7}	7.520000×10^{-9}
0.1	9.9500420×10^{-2}	1.0284900×10^{-1}	3.348580×10^{-3}
0.2	1.9601332×10^{-1}	1.9634698×10^{-1}	3.336600×10^{-4}
0.3	2.8660095×10^{-1}	2.8049408×10^{-1}	6.106870×10^{-3}
0.4	3.6842440×10^{-1}	3.5529025×10^{-1}	1.313415×10^{-2}
0.5	4.3879128×10^{-1}	4.2073549×10^{-1}	1.805579×10^{-2}
0.6	4.9520137×10^{-1}	4.7682881×10^{-1}	1.837256×10^{-2}
0.7	5.3538953×10^{-1}	5.2357319×10^{-1}	1.181634×10^{-2}
0.8	5.5736537×10^{-1}	5.6096565×10^{-1}	3.600280×10^{-3}
0.9	5.5944897×10^{-1}	5.8900717×10^{-1}	2.955820×10^{-2}
1	5.4030231×10^{-1}	6.0769777×10^{-1}	6.739546×10^{-2}

Tab. 5.2: Exact and numerical solutions at $M = N = R = 2$

$x = t = w$	Exact	Approximation	Error
10^{-7}	10^{-7}	1.0034000×10^{-7}	3.40000×10^{-10}
0.1	9.9500420×10^{-2}	9.9544550×10^{-2}	4.41300×10^{-5}
0.2	1.9601332×10^{-1}	1.9577230×10^{-1}	2.41020×10^{-4}
0.3	2.8660095×10^{-1}	2.8609721×10^{-1}	5.03740×10^{-4}
0.4	3.6842440×10^{-1}	3.6793321×10^{-1}	4.91190×10^{-4}
0.5	4.3879128×10^{-1}	4.3869424×10^{-1}	9.70400×10^{-5}
0.6	4.9520137×10^{-1}	4.9579424×10^{-1}	5.92870×10^{-4}
0.7	5.3538953×10^{-1}	5.3664716×10^{-1}	1.25763×10^{-3}
0.8	5.5736537×10^{-1}	5.5866693×10^{-1}	1.30156×10^{-3}
0.9	5.5944897×10^{-1}	5.5926749×10^{-1}	1.81480×10^{-4}
1	5.4030231×10^{-1}	5.3586279×10^{-1}	4.43952×10^{-3}

Tab. 5.3: Exact and numerical solutions at $M = N = R = 3$

$x = t = w$	Exact	Approximation	Error
10^{-7}	10^{-7}	9.9947000×10^{-8}	5.3000×10^{-11}
0.1	9.9500420×10^{-2}	9.9513030×10^{-2}	1.2610×10^{-5}
0.2	1.9601332×10^{-1}	1.9605599×10^{-1}	4.2670×10^{-5}
0.3	2.8660095×10^{-1}	2.8661729×10^{-1}	1.6340×10^{-5}
0.4	3.6842440×10^{-1}	3.6837449×10^{-1}	4.9910×10^{-5}
0.5	4.3879128×10^{-1}	4.3869424×10^{-1}	9.7040×10^{-5}
0.6	4.9520137×10^{-1}	4.9513232×10^{-1}	6.9050×10^{-5}
0.7	5.3538953×10^{-1}	5.3543363×10^{-1}	4.4100×10^{-5}
0.8	5.5736537×10^{-1}	5.5753220×10^{-1}	1.6683×10^{-4}
0.9	5.5944897×10^{-1}	5.5955117×10^{-1}	1.0220×10^{-4}
1	5.4030231×10^{-1}	5.3980282×10^{-1}	4.9949×10^{-4}



(a) $M = N = R$ =1

(b) $M = N = R$=2

(c) $M = N = R$=3

Fig. 5.1: The values of N=M=R versus absolute error at different x,w values and $t = [0, 1]$

Example 5.3

$$\int_0^w \int_0^t \int_0^x (xtw - yzs - 1)u^2(y,z,s)dydzds = \frac{-twx}{144}(-39wt^3x - 56wt^2x^2 + 48t^2 - 39wtx^3 + 72tx + 48x^2),$$

$(x,t,w) \in [0,1] \times [0,1] \times [0,1]$. The exact solution is $u(x,t,w) = x + t$.

Applyinf Bivariate shifted Legendre function for $M = N = R = 1, 2, 3$, The following tables show the exact solution, the approximate solution, and the absolute error at some particular points.

Tab. 5.4: Exact and numerical solutions at $M = N = R = 1$

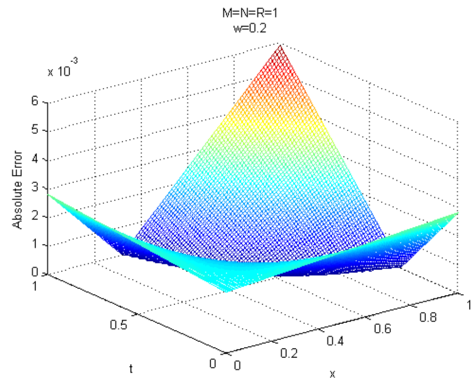
(x,t,w)	Exact	Approximation	Error
$(0,0,0)$	0	$7.42393294 \times 10^{-4}$	$7.42393294 \times 10^{-4}$
(0.2,0.2,0.2)	0.4	0.40202067	$2.02067000 \times 10^{-3}$
(0.4,0.4,0.4)	0.8	0.803343470	$3.34347000 \times 10^{-3}$
(0.6,0.6,0.6)	1.2	0.120237667	$2.37667000 \times 10^{-3}$
(0.8,0.8,0.8)	1.6	1.59678614	$3.21386000 \times 10^{-3}$
(1,1,1)	2	1.98423777	$1.57622300 \times 10^{-2}$
(0.1,0.2,0.3)	0.3	0.303174610	$3.17461000 \times 10^{-3}$
(0.3,0.5,0.4)	0.8	0.803534260	$3.53426000 \times 10^{-3}$
(0.6,0.4,0.2)	1	1.00055495	$5.54950000 \times 10^{-4}$
(0.5,0.7,0.6)	1.2	1.20266471	$2.66471000 \times 10^{-3}$
(0.7,0.8,0.3)	1.5	1.49806945	$1.93055000 \times 10^{-3}$
(0.8,0.9,0.7)	1.7	1.69496634	$5.03366000 \times 10^{-3}$
(0.9,1,0.5)	1.9	1.89230119	$7.69881000 \times 10^{-3}$

Tab. 5.5: Exact and numerical solutions at $M = N = R = 2$

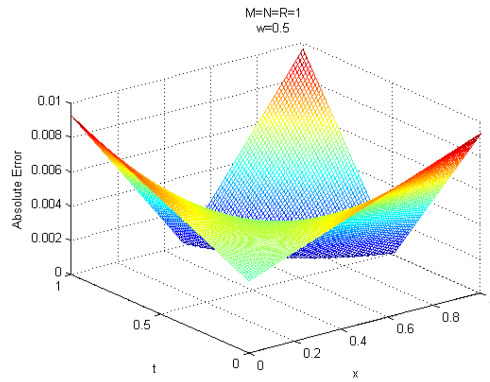
(x, t, w)	Exact	Approximation	Error
$(0, 0, 0)$	0	$7.93688432 \times 10^{-4}$	$7.93688432 \times 10^{-4}$
(0.2,0.2,0.2)	0.4	0.400162800	$1.62800000 \times 10^{-4}$
(0.4,0.4,0.4)	0.8	0.800007130	$7.13000000 \times 10^{-6}$
(0.6,0.6,0.6)	1.2	1.20000476	$4.76000000 \times 10^{-6}$
(0.8,0.8,0.8)	1.6	1.60000280	$2.80000000 \times 10^{-6}$
(1,1,1)	2	2.00001740	$1.74000000 \times 10^{-5}$
(0.1,0.2,0.3)	0.3	0.300252630	$2.52630000 \times 10^{-4}$
(0.3,0.5,0.4)	0.8	0.799997140	$2.86000000 \times 10^{-6}$
(0.6,0.4,0.2)	1	0.999994380	$5.62000000 \times 10^{-6}$
(0.5,0.7,0.6)	1.2	1.20000114	$1.14000000 \times 10^{-6}$
(0.7,0.8,0.3)	1.5	1.50000430	$4.30000000 \times 10^{-6}$
(0.8,0.9,0.7)	1.7	1.69999907	$9.30000000 \times 10^{-7}$
(0.9,1,0.5)	1.9	1.90000252	$2.52000000 \times 10^{-6}$

Tab. 5.6: Exact and numerical solutions at $M = N = R = 3$

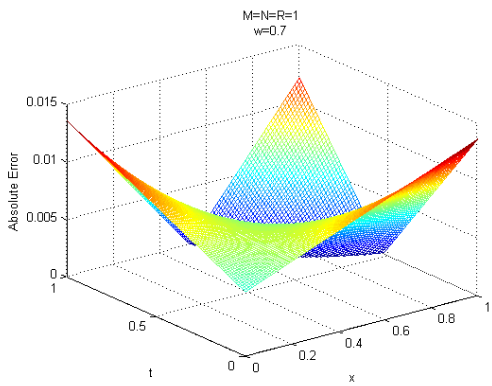
(x, t, w)	Exact	Approximation	Error
$(0, 0, 0)$	0	8.3855×10^{-16}	8.3855×10^{-16}
(0.2,0.2,0.2)	0.4	0.39999999999999600	$3.99680289 \times 10^{-15}$
(0.4,0.4,0.4)	0.8	0.799999999999984	$1.60982339 \times 10^{-14}$
(0.6,0.6,0.6)	1.2	1.1999999999999700	$2.99760217 \times 10^{-14}$
(0.8,0.8,0.8)	1.6	1.5999999999999400	$6.01740879 \times 10^{-14}$
(1,1,1)	2	1.9999999999999000	$9.99200722 \times 10^{-14}$
(0.1,0.2,0.3)	0.3	0.2999999999999980	$1.99840144 \times 10^{-15}$
(0.3,0.5,0.4)	0.8	0.7999999999999850	$1.50990331 \times 10^{-14}$
(0.6,0.4,0.2)	1	0.9999999999999780	$2.19824159 \times 10^{-14}$
(0.5,0.7,0.6)	1.2	1.1999999999999700	$2.99760217 \times 10^{-14}$
(0.7,0.8,0.3)	1.5	1.4999999999999500	$4.99600361 \times 10^{-14}$
(0.8,0.9,0.7)	1.7	1.6999999999999300	$6.99440506 \times 10^{-14}$
(0.9,1,0.5)	1.9	1.8999999999999100	$8.99280650 \times 10^{-14}$



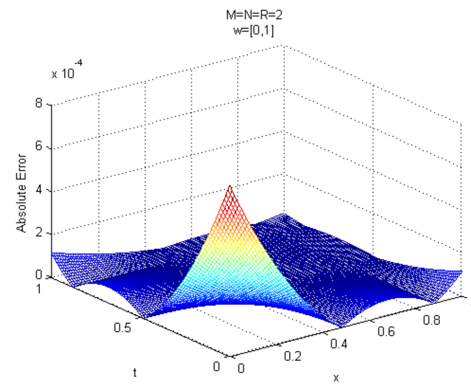
(a) $M = N = R$ =1



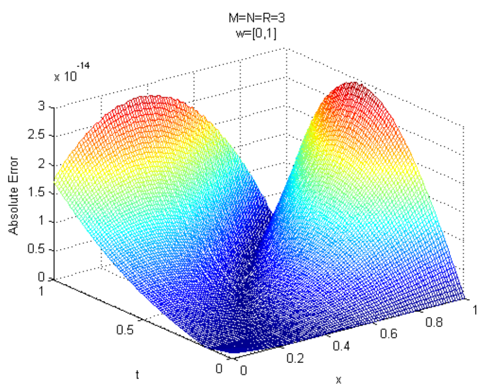
(b) $M = N = R$=1



(c) $M = N = R$=1



(d) $M = N = R$=2



(e) $M = N = R$=3

Fig. 5.2: The values of M=N=R versus absolute error at different x,t values and $w = [0, 1]$

5.7 Conclusions

In the presented method of one, two, and three-dimensional nonlinear VIE of the first kind are transformed to a nonlinear VIE of the second-kind. The bivariate shifted Legendre functions operational matrices have been used to approximate the solution of problem. The approximated non-linear VIE of the second-kind is transformed to a linear system of algebraic equations with unknown coefficients which solved using Gauss-Jordan elimination method. Finally, a system of nonlinear algebraic equations with unknown coefficients of the solution of the main problem has been obtained which can be solved using the Newton's iterative method.

The applicability and accuracy of the method have been checked for some examples in one, two, and three dimensional VEI. It was noticed that that the present method gives more accurate results than the methods presented even when we use a small number of basis functions.

The method can be applied to the first-kind VIE integral equations in one, two, and three dimensions. Morever, it will not difficult to extend this approach to nonlinear integral equations of diffierent forms.

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ملخص

طريقة لجندر المزاحة لحل المعادلات غير الخطية

محمود طاهر ابراهيم

في هذه الأطروحة، تم تقديم وتحليل طريقة لاجندر ثنائية المتغير المتحولة لحل معادلة فولتيرا التكاملية غير الخطية احادية ، ثنائية ، ثلاثية البعد . اكثر تحديدا تم دراسة طريقة لاجندر ثنائية المتغير المتحولة بالتفصيل وايضا تم استخلاص بعض الصيغ المتعلقة بمعادلة فولتيرا التكاملية ثلاثية الابعاد . اضافة الى ذلك ولايجاد حل تقريبي تم تحويل معادلة فولتيرا من النوع الاول الى النوع الثاني ومن ثم الوصول الى نظام خطي من المعادلات الجبرية . واخيرا ، قدمت امثلة عددية لتوضيح الطريقة ودقتها .