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Ranking Exponential K-Trapezoidal-Triangular Fuzzy Numbers

By

Duha Bedo

Supervisor

Dr. Abdelhalim Ziqan

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This thesis was defended successfully on 1 / 11 /2015 and approved by:

Committee Members	**Signature**
1. Dr. Abdelhalim Ziqan/Supervisor Assistant Prof. of Mathematics	………………………..
2. Dr. Mahmoud Almanasra/ Internal Examiner Associated Prof. of Mathematics	………………………..
3. Dr. Mohammad Assad / External Examiner Assistant Prof. of Mathematics, An-Najah National University	………………………..

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Dedication

This thesis is dedicated to my family and many friends for endless love, support and encouragement.

Acknowledgment

All praise for almighty Allah who gave me the ability to handle all difficulties during my study and preparing this work. My sincere appreciation goes to my supervisor Dr. Abdelhalim Ziqan for his encouragement and helpful suggestions throughout this work and my study period. I would like to thank Dr. Saed Mallak for his great support and encouragement before and during the preparation of this thesis. I wish to express my deepest thanks and appreciation to the staff of the Department of Mathematics and Statistics

This all is the fruit of untiring efforts, lot of prayers, encouragement, sacrifices, guidance, moral and financial support of my great, respectful and loving parents. I have no words to thank my parents. I wish to thank my entire extended family for providing a loving environment for me.

Ranking Exponential K-Trapezoidal-Triangular Fuzzy Numbers

Abstract

In this thesis, the exponential K -trapezoidal-triangular and exponential $K+1$-trapezoidal Fuzzy Numbers have been defined. Then, the ranking methods- cardinality, TRD distance, median and integral- have been generalized on the defined fuzzy numbers. In addition to that, we illustrate examples on each method. Finally, the ranking result has been made choose**.**

ترتيب الارقام شبه المنحرفة المثلثية الاسية الضبابية

ملخص

في هذه الرسالة تم تعريف الاعداد ك - شبه المنحرفة - المثلثية الضبابية و ك+1- شبه المنحرفة الاسية الضبابية. بعد ذلك، طرق الترتيب – عدد العناصر، المسافة، المتوسط والتكامل – تم تعميمها على الارقام الضبابية المعرفة. بالاضافة الى ذلك، قدمنا أمثلة توضيحية على كل طريقة. أخيرا، تم اختيار نتيجة المقارنة.

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Introduction

In most situations of our life, only approximately data can be taken for decision making. To recognize those problems, Lotfi A. Zadeh in 1965 defined the fuzzy logic and the concept of fuzzy set by working on the problem of computer understanding of natural language [26]. Then, the researcher continued to develop the application of fuzzy logic.

Fuzzy logic has many applications in the real life, such as aerospace, automotive, business, chemical industry, defense, electronics, financial, industrial, manufacturing, marine, medical, mining and metal processing, robotics, securities, signal processing, telecommunication and transportation [11, 23] .

The problems of decision making need to optimize some procedure and so it is need to have ranking between imprecise quantities. Ranking fuzzy numbers began by Jain [9] for decision making in fuzzy situations by representating the ill-defined quantity as a fuzzy sets. Bortolan and Degani [4] and Dadgostar [7] reviewed some methods of ranking fuzzy sets. Yagar[25] is a first researcher, who contributed the centroid concept in ranking method. Ranking fuzzy numbers with interval values were presented by Liou and Wang [14]. Chu and Tsao [6] proposed an approach for ranking fuzzy numbers with in the area between the centroid point and original point. Abbasbandy and Asady [1] introduced an approach to rank fuzzy numbers by sign distance. Wang and Luo [24] proposed an area ranking of fuzzy numbers based on positive and negative ideal points. Mallak and Bedo [15-17] generalized some methods for ranking particular fuzzy numbers.

Ranking fuzzy numbers creates many applications in data analysis, artificial intelligence, management, engineering, basic sciences [2] and other different fields in operations research. Also, it plays key tool in a real life problems where uncertainty appears as in decision making, clustering, optimization, transportation problems, etc [22].

Recently, ranking of exponential trapezoidal fuzzy numbers has been studied by many authors. The researchers proposed many methods of ranking; for example, TRD distance, median interval, Median value, the cardinality of the fuzzy numbers and the integral value of the fuzzy numbers [3,12,19,20,21].

The aim of this thesis is to generalize the methods of ranking between exponential trapezoidal fuzzy numbers to rank exponential particular fuzzy numbers and to choose the ranking result that agrees with almost all methods, since fuzzy numbers is imprecision.

In chapter one, the main concepts of fuzzy set and fuzzy numbers, the definitions of particular fuzzy numbers, exponential k+1-trapoziodal fuzzy number are presented Also, we defined exponential k-trapezoidal-triangular fuzzy numbers.

In chapter two, three, four and five the methods of ranking exponential trapezoidal fuzzy numbers using cardinality, TRD distance, median value and integral value are extended to rank exponential particular fuzzy numbers. In addition, numerical examples have been solved using the generalized methods. Finally, in chapter six, the ranking result we choose that agrees with almost all methods.

Chapter one
Fuzzy Set and Fuzzy Numbers

This chapter consists of seven sections. In sections 1.1-1.6 we give the definitions of fuzzy sets, operations on fuzzy sets, fuzzy relations, alpha cuts, convex fuzzy sets, fuzzy numbers, triangular and trapezoidal fuzzy numbers and some particular fuzzy numbers. In section 1.7, the exponential $K+1$-Trapezoidal fuzzy numbers is presented, also we introduced the definition of a new fuzzy numbers, which we call it, Exponential K-trapezoidal-triangular fuzzy numbers.

1.1 Fuzzy set

The fuzzy set theory is built to deal with uncertainty phenomenons such as randomness, ambiguity and imprecision. It was first revealed by Lotifi A. Zadeh in his well-known paper entitled "Fuzzy Sets" in 1965 [31]. In crisp set theory, each element either belongs or doesn't to the set which can be represented by the membership function

$$\mu_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A \end{cases}.$$

A fuzzy set is a generalization of the concept of the crisp set, so each element in the overall set takes membership to the set in the unit interval $[0,1]$. Throughout this thesis, we use the bar above the letter to distinguish between the fuzzy set and the crisp set.

Definition 1.1.1 [8]: If X is a collection of objects, then a fuzzy subset $\overline{A}$ of X is defined by the membership function $\mu_{\overline{A}} : X \to [0,1]$, where $\mu_{\overline{A}}(x)$ is the degree of membership of x in $\overline{A}$.

Remark 1.1.2 [13]: If $\overline{A}$ is a fuzzy set of X, then $\overline{A}$ can be represented by different ways:

1. $\overline{A} = \{(x, \mu_{\overline{A}}(x)) : x \in X\}$

2. $\overline{A} = \dfrac{\mu_{\overline{A}}(x_1)}{x_1} + \dfrac{\mu_{\overline{A}}(x_2)}{x_2} + \ldots = \sum_{i \in N} \dfrac{\mu_{\overline{A}}(x_i)}{x_i}$

3. If X is continuous set then

$$\overline{A} = \int_R \frac{\mu_{\overline{A}}(x)}{x} dx$$

We notice here that the symbol '+' means the usual union between sets.

Example 1.1.3 [14]: $\overline{A} = \{\text{real numbers near to } 0\}$.

$$\overline{A} = \{(x, \mu_{\overline{A}}(x)) : \mu_{\overline{A}}(x) = \frac{1}{1+x^2}, x \in R\}$$

Example 1.1.4 [13]: $\overline{A} = \{\text{real numbers near to } 0\}$.

$$\overline{A} = \int_R \frac{\frac{1}{1+x^2}}{x} dx$$

Example 1.1.5 [13]: $\overline{A} = \{\text{real numbers near to } 0\}$.

$$\overline{A} = \{(x, \mu_{\overline{A}}(x)) : \mu_{\overline{A}}(x) = \frac{1}{(1+x^2)^2}, x \in R\}$$

1.2 Operation of Fuzzy Set and Fuzzy Relations

In this section, we list several definitions involving fuzzy sets which are obvious extensions of the corresponding definitions for crisp sets.

Definition 1.2.1 [13]: A fuzzy set $\overline{A}$ is empty if and only if $\mu_{\overline{A}}(x) = 0, \forall x \in X$.

Definition 1.2.2 [13]: Two fuzzy numbers $\overline{A}$ and $\overline{B}$ are equivalent if and only if $\mu_{\overline{A}}(x) = \mu_{\overline{B}}(x), \forall x \in X$ **,** and in this case we write $\overline{A} = \overline{B}$

Remark 1.2.3 [13]: If $\mu_{\overline{A}}(x) \neq \mu_{\overline{B}}(x), \forall x \in X$ then $\overline{A} \neq \overline{B}$.

Definition 1.2.4 [13]: $\overline{A}$ is a subset of $\overline{B}$, $\overline{A} \subseteq \overline{B}$ if and only if $\mu_{\overline{A}}(x) \leq \mu_{\overline{B}}(x), \forall x \in X$ **.**

Definition 1.2.5 [13]: $\overline{A}$ is a proper subset of $\overline{B}$ denoted by $\overline{A} \subset \overline{B}$ if and only if $\mu_{\overline{A}}(x) < \mu_{\overline{B}}(x), \forall x \in X$, i.e. $\overline{A} \subset \overline{B}$ if and only if $\overline{A} \subset \overline{B}$ and $\overline{A} \neq \overline{B}$**.**

Definition 1.2.7 [13]: The union of two fuzzy sets $\overline{A}$ and $\overline{B}$ is a fuzzy set $\overline{C} = \overline{A} \cup \overline{B}$ and is defined by $\mu_{\overline{C}}(x) = \max\{\mu_{\overline{A}}(x), \mu_{\overline{B}}(x)\}, \forall x \in X$.

Definition 1.2.8 [13]: The intersection of two fuzzy sets $\overline{A}$ and $\overline{B}$ is a fuzzy set $\overline{C} = \overline{A} \cap \overline{B}$ and is defined by $\mu_{\overline{C}}(x) = \min\{\mu_{\overline{A}}(x), \mu_{\overline{B}}(x)\}, \forall x \in X$.

Definition 1.2.9 [26]: The union of $\overline{A}$ and $\overline{B}$ is the smallest fuzzy set containing both $\overline{A}$ and $\overline{B}$. More precisely, if $\overline{D}$ is any fuzzy set which contains both $\overline{A}$ and $\overline{B}$, then it is also contains $\overline{A} \cup \overline{B}$.

Remark 1.2.10: Definition 1.2.9 is equivalent to the definition of the union of two fuzzy set 1.2.7 and has the following proof:

Proof: Let $\overline{C} = \overline{A} \cup \overline{B}$, then $\mu_{\overline{C}}(x) = \max\{\mu_{\overline{A}}(x), \mu_{\overline{B}}(x)\}$, so $\mu_{\overline{C}}(x) \geq \mu_{\overline{A}}(x)$ and $\mu_{\overline{C}}(x) \geq \mu_{\overline{B}}(x)$, i.e. $\overline{C}$ contains both $\overline{A}$ and $\overline{B}$. On the other hand, let $\overline{D}$ be any fuzzy set containing both $\overline{A}$ and $\overline{B}$, then $\mu_{\overline{D}}(x) \geq \mu_{\overline{A}}(x)$ and $\mu_{\overline{D}}(x) \geq \mu_{\overline{B}}(x)$. Hence $\mu_{\overline{D}}(x) \geq \mu_{\overline{C}}(x)$ and the proof of the remark is completed.

Definition 1.2.11[26]: The intersection of $\overline{A}$ and $\overline{B}$ is the largest fuzzy set which contained in both $\overline{A}$ and $\overline{B}$. More precisely, if $\overline{D}$ is any fuzzy set which is contained in both $\overline{A}$ and $\overline{B}$, then it is contained in $\overline{A} \cap \overline{B}$.

Remark 1.2.12: Definition 1.2.11 and definition 1.2.8 are equivalent.

Proof of remark 1.2.12: Let $\overline{C} = \overline{A} \cap \overline{B}$, then $\mu_{\overline{C}}(x) = \min\{\mu_{\overline{A}}(x), \mu_{\overline{B}}(x)\}$, so $\mu_{\overline{C}}(x) \leq \mu_{\overline{A}}(x)$ and $\mu_{\overline{C}}(x) \leq \mu_{\overline{B}}(x)$, i.e. $\overline{C}$ is contained in both $\overline{A}$ and $\overline{B}$. Let $\overline{D}$ be any fuzzy set contained in both $\overline{A}$ and $\overline{\boldsymbol{B}}$, we show that $\overline{C} \supset \overline{D}$. Since $\mu_{\overline{D}}(x) \leq \mu_{\overline{A}}(x)$ and $\mu_{\overline{D}}(x) \leq \mu_{\overline{B}}(x)$ we have $\mu_{\overline{D}}(x) \leq \mu_{\overline{C}}(x)$, then from definition $\overline{C} \supset \overline{D}$.

It is easy to extend many of the rules which are held in crisp set to fuzzy by using operations of union, complement and intersection :

1. De Morgan's law

   $(A \bigcap B)^c = A^c \bigcup B^c$

   $(A \bigcup B)^c = A^c \bigcap B^c$

2. Associatively

$(A \cap B) \cap C = A \cap (B \cap C)$

$(A \cup B) \cup C = A \cup (B \cup C)$

3. Commutatively

$A \cap B = B \cap A$

$A \cup B = B \cup A$

4. Distributively

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

These and similar rules can readily be established by showing the corresponding relations for the membership functions of $\overline{A}, \overline{B} \& \overline{C}$ [26].

Example 1.2.13: Let $X = R$ and $\overline{A} = \{(1,0.2),(2,0.5),(3,0.8),(4,1),(5,0.7),(6,0.3)\}$, $\overline{B} = \{(3,0.2),(4,0.5),(5,0.6),(6,0.8),(7,1),(8,1)\}$ are fuzzy subsets of X , then
The intersection $\overline{C} = \overline{A} \cap \overline{B}$ is

$$\overline{C} = \{(3,0.2),(4,0.5),(5,0.6),(6,0.3)\}$$

The union $\overline{M} = \overline{A} \cup \overline{B}$ is

$$\overline{A} = \{(1,0.2),(2,0.5),(3,0.8),(4,1),(5,0.7),(6,0.8).(7,1),(8,1)\}$$

$\overline{D} = \{(3,0.2),(4,0.5),(5,0.6)\}$ is continued in both $\overline{A}$, $\overline{B}$ and smaller than $\overline{C}$
$\overline{L} = \{(1,0.2),(2,0.5),(3,0.8),(4,1),(5,0.7),(6,0.8).(8,1)\}$ is containing both $\overline{A}$, $\overline{B}$ and larger than $\overline{M}$.

Definition 1.2.14[26]: A fuzzy relation is a fuzzy set whose membership mapped from $X \times X$ to [0,1].

Definition 1.2.15 [26]: The composition of two fuzzy relations $\overline{A}$ and $\overline{B}$ is denoted by $\overline{B} \circ \overline{A}$ is defined as a fuzzy relation in whose membership function is given by

$$\mu_{\overline{B} \circ \overline{A}}(x,y) = \sup_z \min\{\mu_{\overline{A}}(x,z), \mu_{\overline{B}}(z,y)\}$$, where $x, y, z \in X$.

1.3 Alpha Cuts and Convexity

The definition of $\alpha-$cut set A_α and some remarks about this definition are to be introduced in the current section, then we give the definition of the convexity of $\alpha-$cut set and convexity of fuzzy set.

Definitions 1.3.1 [13]: The set $A_\alpha=\{x\in X: \mu_A(x)\geq\alpha\}$ is called the $\alpha-$cut and is denoted by A_α.

Remark 1.3.2 [13, 5]:

1. A_α is crisp set.

2. If $\alpha=0$ then A_0 is called the support of $\overline{A}$. If $\alpha=1$ then A_1 is called the core of $\overline{A}$.

3. If $\alpha>\alpha'$ then $\overline{A}_\alpha\subseteq\overline{A}_{\alpha'}$, $0<\alpha'<\alpha<1$.

Example 1.3.3: Let $\overline{A}=\{(1,0.1),(2,0.7),(8,0.9),(10,1),(3,0.5)\}$ then

$A_0=\{1,2,3,8,10\}$, $A_{0.1}=\{1,2,3,8,10\}$, $A_{0.7}=\{2,8,10\}$, $A_{0.5}=\{2,3,8,10\}$, $A_{0.9}=\{8,10\}$,

$A_1=\{10\}$.

Definition 1.3.4 [26]: The α-cut, A_α of a fuzzy number $\overline{A}$ is convex if $t=\lambda r+(1-\lambda)s\in A_\alpha$ where $r \,\&\, s\in A_\alpha$, $\forall\lambda\in[0,1]$.

Remark 1.3.5 [26]: A fuzzy set is convex if and only if all α - cut; A_α is convex.

1.4 Fuzzy Number

Definition 1.4.1 [13]: A fuzzy number $\overline{A}$ is a fuzzy set satisfies the following conditions:

1. Convex fuzzy set.
2. Normalized fuzzy set (i.e. $\exists x\in\Re$, $\mu_{\overline{A}}(x)=1$.)
3. Its membership function is piecewise continuous.
4. Its membership function is defined on the real number.

Example 1.4.3[13]: Triangular fuzzy number (figure 1) is a fuzzy number represented by three real numbers $a_1 < a_2 < a_3$ where

$$\mu_{\bar{A}}(x) = \begin{cases} 0 & , x \le a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & , a_1 \le x \le a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & , a_2 \le x \le a_3 \\ 0 & , x \ge a_3 \end{cases},$$

and denoted by $\bar{A} = (a_1, a_2, a_3)$.

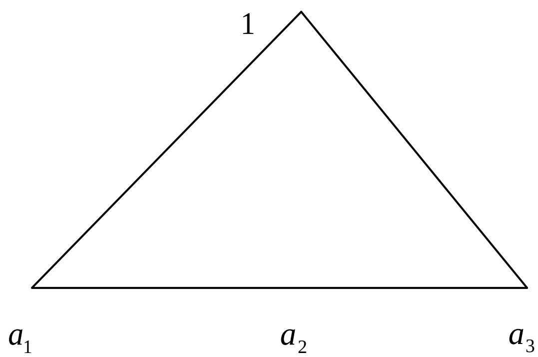

Figure1: Triangular fuzzy number

Definition 1.4.4[13]: Trapezoidal fuzzy number (Figure 2) is a fuzzy number represented by four real numbers $a_1 < a_2 < a_3 < a_4$, denoted by $\bar{A} = (a_1, a_2, a_3, a_4)$, and whose membership function is

$$\mu_{\bar{A}}(x) = \begin{cases} 0 & , x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & , a_1 \le x \le a_2 \\ 1 & , a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & , a_3 \le x \le a_4 \\ 0 & , x > a_5 \end{cases}.$$

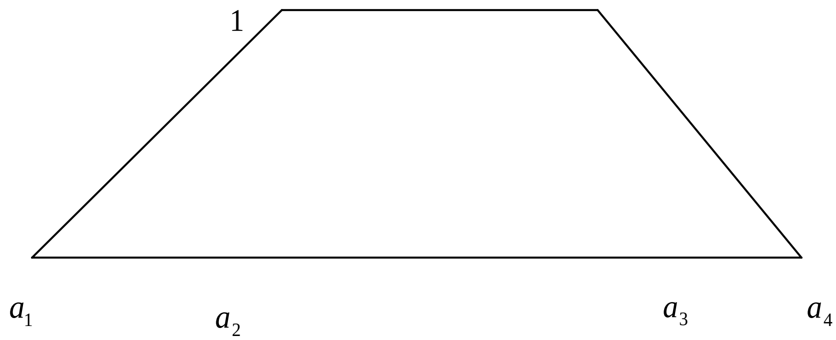

Figure2: Trapezoidal Fuzzy Number

Definition 1.4.5[19]: Exponential Trapezoidal fuzzy number is a fuzzy number represented by four real numbers $a_1 < a_2 < a_3 < a_4$ where its membership function is

$$\mu_{\overline{A}}(x) = \begin{cases} e^{\frac{x-a_1}{a_2-a_1}}, a_1 \le x \le a_2 \\ 1, a_2 \le x \le a_3 \\ e^{\frac{a_4-x}{a_4-a_3}}, a_3 \le x \le a_4 \end{cases}$$

Exponential trapezoidal fuzzy number denoted by $\overline{A} = (a_1, a_2, a_3, a_4)_E$

1.5 Operations on Fuzzy Numbers

In this section we give some of the operations on fuzzy numbers.

Definition 1.5.1 [13]: The maximum of two fuzzy numbers $\overline{A}$ and $\overline{B}$ is a fuzzy set and the membership function is $\mu_{\overline{A}\vee\overline{B}}(z) = \bigvee_{z=x\vee y} (\mu_{\overline{A}}(x) \wedge \mu_{\overline{B}}(y))$, $\forall x, y \in X$, $\vee$ is the maximum operation and $\wedge$ is the minimum operation.

Definition 1.5.2 [13]: The minimum of two fuzzy numbers $\overline{A}$ and $\overline{B}$ is a fuzzy set and the membership function is $\mu_{\overline{A}\vee\overline{B}}(z) = \bigvee_{z=x\wedge y} (\mu_{\overline{A}}(x) \wedge \mu_{\overline{B}}(y))$, $\forall x, y \in X$

Definition 1.5.3 [13]: The addition of two fuzzy numbers $\overline{A}$ and $\overline{B}$ is a fuzzy set and the membership function is $\mu_{\overline{A}\vee\overline{B}}(z) = \bigvee_{z=x+y} (\mu_{\overline{A}}(x) \wedge \mu_{\overline{B}}(y))$, $\forall x, y \in X$

Definition 1.5.4 [13]: The subtraction of two fuzzy numbers $\overline{A}$ and $\overline{B}$ is a fuzzy set and the membership function is $\mu_{\overline{A}\vee\overline{B}}(z) = \bigvee_{z=x-y} (\mu_{\overline{A}}(x) \wedge \mu_{\overline{B}}(y))$, $\forall x, y \in X$

Definition 1.5.5 [13]: The multiplication of two fuzzy numbers $\overline{A}$ and $\overline{B}$ is a fuzzy set and the membership function is $\mu_{\overline{A}\vee\overline{B}}(z) = \bigvee_{z=x*y} (\mu_{\overline{A}}(x) \wedge \mu_{\overline{B}}(y))$, $\forall x, y \in X$

Definition 1.5.6 [13]: The division of two fuzzy numbers $\overline{A}$ and $\overline{B}$ is a fuzzy set and the membership function is $\mu_{\overline{A}\vee\overline{B}}(z) = \bigvee_{z=x/y} (\mu_{\overline{A}}(x) \wedge \mu_{\overline{B}}(y))$, $\forall x, y \in X$

Definition 1.5.7 [13]: If $\overline{A} = (a_1, a_2, a_3)$ and $\overline{B} = (b_1, b_2, b_3)$ are triangular fuzzy numbers then

1. Addition: $\overline{A} + \overline{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2. Symmetric image: $-(\overline{A}) = (-a_3, -a_2, -a_1)$
3. Subtraction : $\overline{A} + \overline{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

Remark 1.5.8 [13]: The same manner defined in definition 1.5.2 is used to define operations on trapezoidal fuzzy numbers.

1.6 Particular Fuzzy Numbers

In this section we present some particular fuzzy numbers which are given in [16].

Definition 1.6.1: A fuzzy number that is determined by n real numbers $a_1, a_2, ..., a_{2k+3}$ and w_i, $0 \le w_i \le 1$,i=1,2,....,k, $n = 3(k+1)$ and $a_1 < a_2 < .... < a_{2k+3}$. Denote it by

$\overline{A} = (a_1, a_2, ..., a_{2k+3}; w_1, .., w_k)$ and whose membership function is given by

$$\mu_{\bar{A}}(x)=\begin{cases}0 & ,x\le a_1\\ \dfrac{c_1(x-a_1)}{a_2-a_1} & ,a_1\le x\le a_2\\ \vdots & \\ \dfrac{(1-w_k)(x-a_{k+1})}{a_{k+2}-a_{k+1}}+w_k & ,a_{k+1}\le x\le a_{k+2}\\ \dfrac{(w_k-1)(x-a_{k+3})}{a_{k+3}-a_{k+2}} & ,a_{k+2}\le x\le a_{k+3}\\ \vdots & \\ \dfrac{-w_1(x-a_{2k+3})}{a_{2k+3}-a_{2k+2}} & ,a_{2k+2}\le x\le a_{2k+3}\\ 0 & x\ge a_{2k+3}\end{cases}$$

is called K -Trapezoidal-Triangular fuzzy number

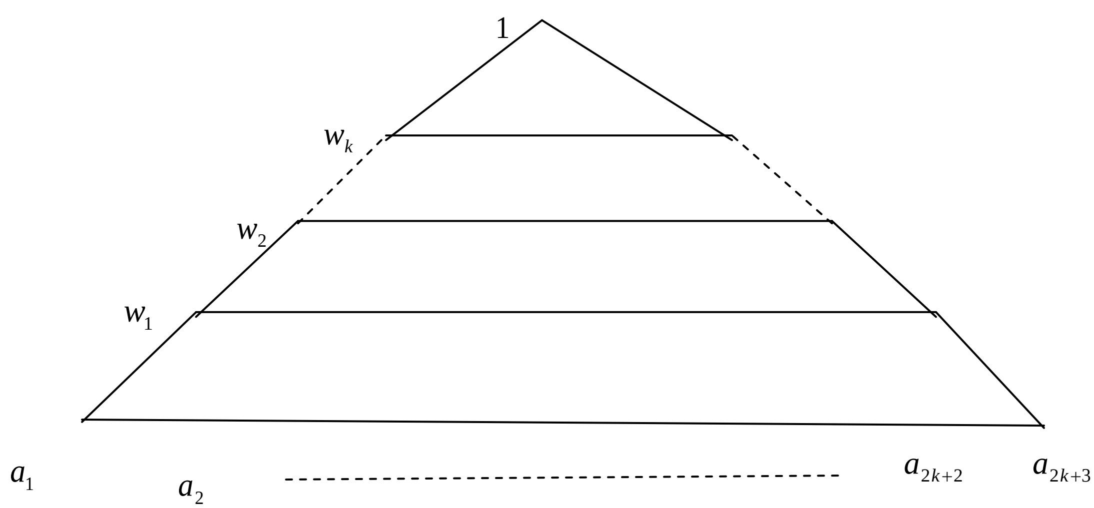

Figure3: K -Trapezoidal-Triangular fuzzy number

Definition 1.6.2: A fuzzy number that is determined by n real numbers $a_1, a_2, ..., a_{2k+4}$ and w_i, $0 \le w_i \le 1$, i=1,2,…,k where $n = 3(k+1)+1$, and $a_1 < a_2 < .... < a_{2k+4}$. Denote it by $\overline{A} = (a_1, ..., a_{2k+4}; w_1, .., w_k)$ and whose membership function is given by

$$\mu_{\overline{A}}(x) = \begin{cases} 0 & , x \le a_1 \\ \dfrac{c_1(x-a_1)}{a_2-a_1} & , a_1 \le x \le a_2 \\ \vdots & \\ \dfrac{(1-w_k)(x-a_{k+1})}{a_{k+2}-a_{k+1}} + w_k & , a_{k+1} \le x \le a_{k+2} \\ 1 & , a_{k+2} \le x \le a_{k+3} \\ \dfrac{(w_k-1)(x-a_{k+4})}{a_{k+4}-a_{k+3}} & , a_{k+3} \le x \le a_{k4} \\ \vdots & \\ \dfrac{-w_1(x-a_{2k+4})}{a_{2k+4}-a_{2k+3}} & , a_{2k+3} \le x \le a_{2k+4} \\ 0 & x \ge a_{2k+4} \end{cases}$$

is called $K+1$-Trapezoidal fuzzy number.

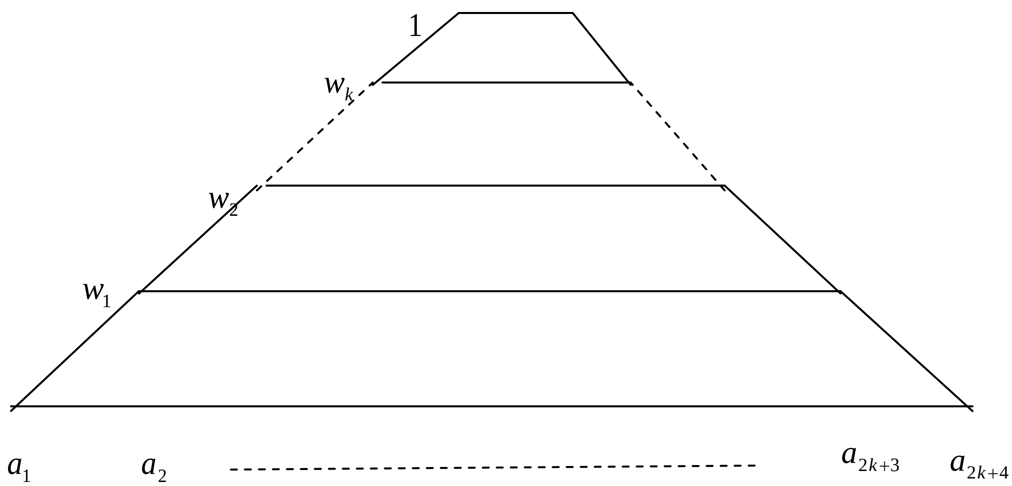

Figure4: $K+1$ -Trapziodal fuzzy number

1.7 Exponential Fuzzy Numbers

In this section, we introduce an Exponential $K+1$-Trapezoidal fuzzy number and we define exponential k-Trapezoidal-Triangular fuzzy number.

Definition 1.7.1: The Exponential K-Trapezoidal-Triangular Fuzzy Number is a fuzzy number whose membership function is given by the following formula:

$$\mu_{\bar{A}}(x)=\begin{cases} w_1 e^{\frac{x-a_2}{\lambda_1}} & ,a_1 \le x \le a_2 \\ e^{\frac{(a_3-x)\ln w_1+(x-a_2)\ln w_2}{\lambda_2}} & ,a_2 \le x \le a_3 \\ \vdots & \\ e^{\frac{(a_{k+1}-x)\ln w_{k-2}+(x--a_k)\ln w_{k-1}}{\lambda_k}} & ,a_k \le x \le a_{k+1} \\ e^{\frac{(a_{k+2}-x)\ln w_k}{\lambda_{k+1}}} & ,a_{k+1} \le x \le a_{k+2} \\ e^{\frac{(x-a_{k+2})n w_k}{\beta_{k+1}}} & ,a_{k+2} \le x \le a_{k+3} \\ e^{\frac{(x-a_{k+3})\ln w_{k-1}+(a_{k+4}-x)\ln w_k}{\beta_k}} & ,a_{k+3} \le x \le a_{k+4} \\ \vdots & \\ e^{\frac{(a_{2k+2}-x)\ln w_2+(x-a_{2k+1})\ln w_1}{\beta_1}} & ,a_{2k+1} \le x \le a_{2k+2} \\ w_1 e^{\frac{a_{2k+2}-x}{\beta_1}} & ,a_{2k+2} \le x \le a_{2k+3} \end{cases}$$

Where $\lambda_i = a_{i+1} - a_i$ and $\beta_i = a_{2k+4-i} - a_{2k+3-i}$, $i = 1,2,..,k+1.$

The fuzzy number given up is denoted by $\overline{A} = (a_1,...,a_{2k+3}; w_1,..,w_k)_E$.

Definition 1.7.2 [18]: The Exponential $K+1$-Trapezoidal Fuzzy Number is a fuzzy number which is defined by the following membership function:

$$\mu_{\overline{A}}(x) = \begin{cases} w_1 e^{\frac{x-a_2}{\lambda_1}} & , a_1 \le x \le a_2 \\ e^{\frac{(a_3-x)\ln w_1 + (x-a_2)\ln w_2}{\lambda_2}} & , a_2 \le x \le a_3 \\ \vdots & \\ e^{\frac{(a_{k+1}-x)\ln w_{k-2} + (x--a_k)\ln w_{k-1}}{\lambda_k}} & , a_k \le x \le a_{k+1} \\ e^{\frac{(a_{k+2}-x)\ln w_k}{\lambda_{k+1}}} & , a_{k+1} \le x \le a_{k+2} \\ 1 & , a_{k+2} \le x \le a_{k+3} \\ e^{\frac{(x-a_{k+3})\ln w_k}{\beta_{k+1}}} & , a_{k+3} \le x \le a_{k+4} \\ e^{\frac{(x-a_{k+4})\ln w_{k-1} + (a_{k+5}-x)\ln w_k}{\beta_k}} & , a_{k+4} \le x \le a_{k+5} \\ \vdots & \\ e^{\frac{(a_{2k+3}-x)\ln w_2 + (x-a_{2k+2})\ln w_1}{\beta_1}} & , a_{2k+2} \le x \le a_{2k+3} \\ w_1 e^{\frac{a_{2k+3}-x}{\beta_1}} & , a_{2k+3} \le x \le a_{2k+4} \end{cases}$$

Where $\lambda_i = a_{i+1} - a_i$ and $\beta_i = a_{2k+5-i} - a_{2k+4-i}$, $i = 1,2,..,k+1.$

We denoted it by $\overline{A} = (a_1,..., a_{2k+4}, w_1,.., w_k)_E$.

Example 1.7.3: Let $\overline{A} = (1,2,3,4,5;0.5)_E$ then

$$\mu_{\overline{A}}(x) = \begin{cases} 0.5e^{x-2} & ,1 \le x \le 2 \\ e^{(3-x)\ln 0.5} & ,2 \le x \le 3 \\ e^{(x-3)\ln 0.5} & ,3 \le x \le 4 \\ 0.5e^{4-x} & ,4 \le x \le 5 \end{cases}$$

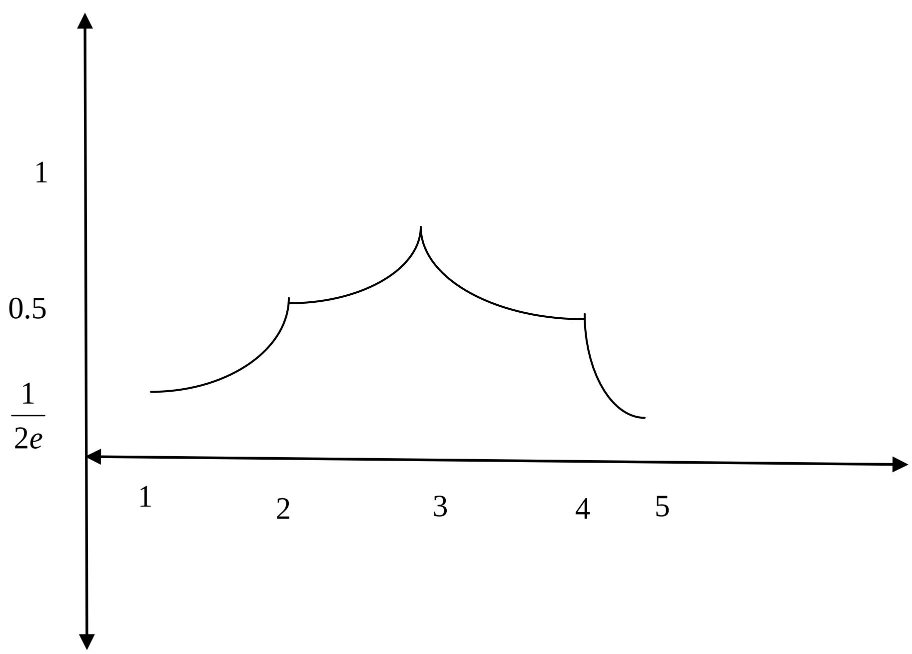

Figure5: 1-Trapezoidal-Triangular fuzzy number

Example 1.7.4: Let $\overline{A} = (1,2,3,4,5,6;0.5)_E$ then

$$\mu_{\bar{A}}(x)=\begin{cases} 0.5e^{x-2} & ,1\le x\le 2 \\ e^{(3-x)\ln 0.5} & ,2\le x\le 3 \\ 1 & ,3\le x\le 4 \\ e^{(x-4)\ln 0.5} & ,4\le x\le 5 \\ 0.5e^{5-x} & ,5\le x\le 6 \end{cases}$$

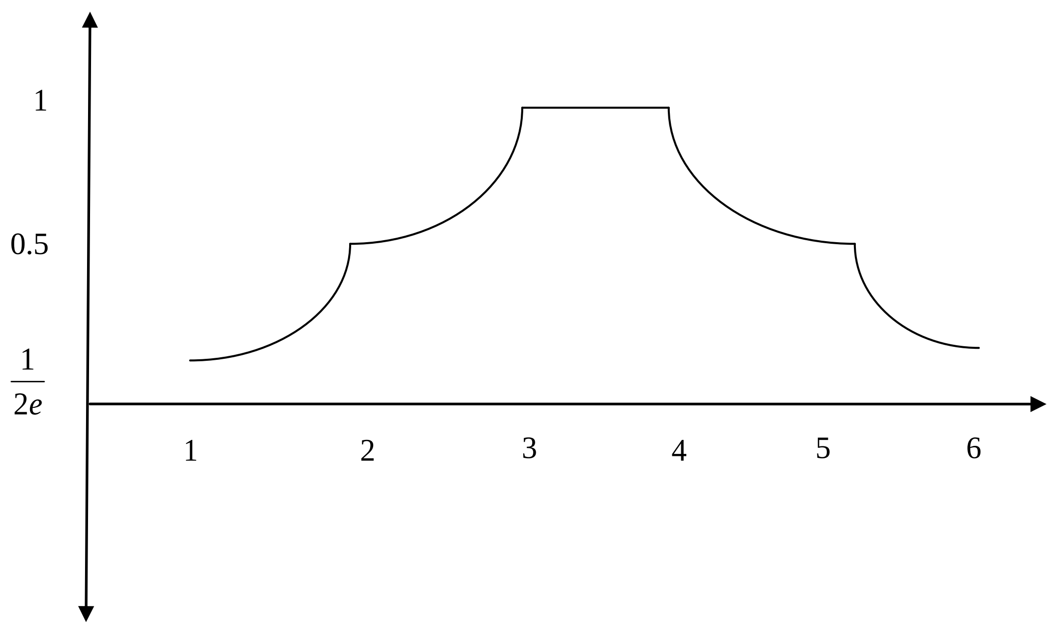

Figure6: 2-Trapezoidal fuzzy number

Chapter Two

Ranking Exponential Particular Fuzzy Numbers Using Cardinality

In this chapter, the definition of the cardinality of fuzzy numbers and ranking exponential trapezoidal fuzzy numbers using cardinality are introduced. Then we extend the method in section 2.1 to Exponential K-Trapezoidal-Triangular fuzzy numbers. Finally, we compare between Exponential $K+1$-Trapezoidal Fuzzy numbers using cardinality.

2.1 The Cardinality Method

In the present section, we describe the basis of the cardinality method for ranking of Exponential Trapezoidal fuzzy numbers.

Definition 2.1.1 [20]: Cardinality of a fuzzy number $\overline{A}$ is the value of the integral

$$card\,\overline{A} = \int_{a}^{b} A(x)dx = \int_{0}^{1} (b_\alpha - a_\alpha)d\alpha$$

Where a_α and b_α are the boundary value of the $\alpha-$cut set of $\overline{A}$.

The following theorem gives a simple formula of the cardinality of the Exponential Trapezoidal fuzzy number.

Theorem 2.1.2 [20]: Cardinality of an Exponential Trapezoidal fuzzy number $\overline{A} = (a,b,c,d;w)_E$ is the value of the integral

$$card\,\overline{A} = w(c-b) + \frac{w}{e}(b-a)(e-1) + (c-d)(1-e) . \qquad (2.1)$$

Where w is the height of the trapezoidal and $e = e^1$.

The following theorem shows how to compare between two Exponential Trapezoidal fuzzy numbers.

Theorem 2.1.3 [20]: if $\overline{A}=(a,b,c,d)_E$ and $\overline{B}=(a_1,b_1,c_1,d_1)_E$ are two Exponential Trapezoidal fuzzy numbers, then

1. If $card\overline{A}>card\overline{B}$ then $\overline{A}\succ\overline{B}$.

2. If $card\overline{A}<card\overline{B}$ then $\overline{A}\prec\overline{B}$.

3. If $card\overline{A}\approx card\overline{B}$ then $\overline{A}\approx\overline{B}$

2.2 Ranking of Exponential k-Trapezoidal-Triangular fuzzy numbers

The aim of this section is to extend an exact the cardinality method to an Exponential K - Trapezoidal- Triangular fuzzy number. So, we have the following theorem.

Theorem 2.2.1: the cardinality of exponential K -Trapezoidal-Triangular fuzzy number $\overline{A}=(a_1,a_2,..,a_{2k+3};w_1,..,w_k)_E$ is

$$card\overline{A}=\begin{cases} w_1(1-\frac{1}{e})(a_5-a_4+a_2-a_1)+\frac{(w_1-1)(a_4-a_2}{\ln w_1} & ,k=1 \\ w_1(1-\frac{1}{e})(a_{2k+3}-a_{2k+2}+a_2-a_1)+\frac{(w_1-1)(a_{k+3}-a_{k+1})}{\ln w_k} & \\ +\sum_{i=2}^{k}\frac{(w_i-w_{i-1})(a_{2k+4-i}-a_{2k+3-i}+a_{i+1}-a_i)}{\ln w_i-\ln w_{i-1}}+ & ,k>1 \end{cases} \tag{2.2}$$

Proof: Let $\boldsymbol{k=1}$, from definition 2.1.1 and the definition of K -Trapezoidal-Triangular fuzzy number, we have

$$card\overline{A}=\int_{a_1}^{a_5}f_A(x)dx$$

$$card\overline{A}=\int_{a_1}^{a_2}w_1e^{[(x-a_2)/(a_2-a_1)]}dx+\int_{a_2}^{a_3}e^{([(a_3-x)\ln w_1]/(a_3-a_2))}$$

$$+\int_{a_3}^{a_4}e^{([(x-a_3)\ln w_1]/(a_4-a_3))}+\int_{a_4}^{a_5}w_1e^{[(a_4-x)/(a_5-a_4)]}dx$$

Integrate the last equation by substitution to produce equation (2.2).

Now, assume that $k > 1$, from definition 2.1.1 and the definition of K-Trapezoidal-Triangular fuzzy number, we have

$$card\overline{A} = \int_{a_1}^{a_2} w_1 e^{[(x-a_2)/(a_2-a_1)]}dx + \int_{a_2}^{a_3} e^{[((a_3-x)\ln w_1+(x-a_2)\ln w_2)/(a_3-a_2)]}dx + ...$$

$$+ \int_{a_k}^{a_{k+1}} e^{[((a_{k+1}-x)\ln w_{k-1}+(x-a_k)\ln w_k)/(a_{k+1}-a_k)]}dx + \int_{a_{k+1}}^{a_{k+2}} e^{[((a_{k+2}-x)\ln w_k)/(a_{k+2}-a_{k+1})]}dx$$

$$+ \int_{a_{k+2}}^{a_{k+3}} e^{[((x-a_{k+2})\ln w_k)/(a_{k+3}-a_{k+2})]}dx + \int_{a_{k+3}}^{a_{k+4}} e^{[((x-a_{k+3})\ln w_{k-1}+(a_{k+4}-x)\ln w_k)/(a_{k+4}-c_{k+3})]}dx + ...$$

$$+ \int_{a_{2k+1}}^{a_{2k+2}} e^{[((a_{2k+2}-x)\ln w_2+(x-a_{2k+1})\ln w_1)/(a_{2k+2}-a_{2k+1})]} + \int_{a_{2k+2}}^{a_{2k+3}} w_1 e^{[(a_{2k+2}-x)/(a_{2k+3}-a_{2k+2})]}dx$$

Integrate the last equation by substitution to get equation (2.2). □

Remark 2.2.2: The algorithm of ranking represented in theorem 2.1.3 is still applicable when the fuzzy number is an exponential K-Trapezoidal-Triangular.

Below we give some illustrative examples.

Example 2.2.3:

Let $\overline{A} = (1,2,3.5,5,6.7;0.5)_E$ and $\overline{B} = (2,2.5,3,3.6,4;0.5)_E$.
Then

$$card\overline{A} = w_1(1-\frac{1}{e})(a_5 - a_4 + a_2 - a_1) + \frac{(w_1-1)(a_4-a_2)}{\ln w_1}$$

$$card\overline{A} = 0.5(1-\frac{1}{e})(6.7-5+2-1) + \frac{(0.5-1)(5-2)}{\ln 0.5}$$

$$card\overline{A} = 3.017$$

$$card\overline{B} = w_1(1-\frac{1}{e})(b_5 - b_4 + b_2 - b_1) + \frac{(w_1-1)(b_4-b_2)}{\ln w_1}$$

$$card\overline{B} = 0.5(1-\frac{1}{e})(4-3.6+2.5-2) + \frac{(0.5-1)(3.6-2.5)}{\ln 0.5}$$

$card\overline{B} = 1.077$

Since $card\overline{A} > card\overline{B}$ then $\overline{A} \succ \overline{B}$.

Example 2.2.4:

Let $\overline{A} = (1,2,2.3,4,5;0.5)_E$ and $\overline{B} = (4,4.5,6,6.5,6.7;0.5)_E$.
Then

$$card\overline{A} = w_1(1-\frac{1}{e})(a_5 - a_4 + a_2 - a_1) + \frac{(w_1 - 1)(a_4 - a_2)}{\ln w_1}$$

$$card\overline{A} = 0.5(1-\frac{1}{e})(5-4+2-1) + \frac{(0.5-1)(4-2)}{\ln 0.5}$$

$$card\overline{A} = 2.075$$

$$card\overline{B} = w_1(1-\frac{1}{e})(b_5 - b_4 + b_2 - b_1) + \frac{(w_1 - 1)(b_4 - b_2)}{\ln w_1}$$

$$card\overline{B} = 0.5(1-\frac{1}{e})(6.7-6.5+4.5-4) + \frac{(0.5-1)(6.5-4.5)}{\ln 0.5}$$

$$card\overline{B} = 1.664$$

Since $card\overline{A} > card\overline{B}$ then $\overline{A} \succ \overline{B}$.

Example 2.2.5:

Let $\overline{A} = (1,2,2.3,4,5,5.5,7;0.5,0.8)_E$ **and** $\overline{B} = (4,4.5,6,6.5,6.7,7,8;0.5,0.8)_E$

Then

$$card\overline{A} = w_1(1-\frac{1}{e})(a_7 - a_6 + a_2 - a_1) + \frac{(w_2 - 1)(a_5 - a_3)}{\ln w_2} + \frac{(w_2 - w_1)(a_6 - a_5 + a_3 - a_2)}{\ln w_2 - \ln w_1}$$

$$card\overline{A} = 0.5(1-\frac{1}{e})(7-5.5+2-1) + \frac{(0.8-1)(5.5-2.3)}{\ln 0.8} + \frac{(0.8-0.5)(5.5-5+2.3-2)}{\ln 0.8 - \ln 0.5}$$

$$card\overline{A} = 4.169$$

$$card\overline{B} = w_1(1-\frac{1}{e})(b_7 - b_6 + b_2 - b_1) + \frac{(w_2 - 1)(b_5 - b_3)}{\ln w_2} + \frac{(w_2 - w_1)(b_6 - b_5 + b_3 - b_2)}{\ln w_2 - \ln w_1}$$

$$card\overline{B} = 0.5(1-\frac{1}{e})(8-7+4.5-4)+\frac{(0.8-1)(6.7-6.5)}{\ln 0.8}+\frac{(0.8-0.5)(7-6.7+6-4.5)}{\ln 0.8-\ln 0.5}$$

$$card\overline{B} = 1.802$$

Since $card\overline{A} > card\overline{B}$ then $\overline{A} \succ \overline{B}$.

2.3 Ranking of Exponential $K+1$-Trapezoidal Fuzzy Numbers

The goal of this section is to introduce a formula for ranking of Exponential $K+1$-Trapezoidal fuzzy numbers using the cardinality method. So, we have the following theorem.

Theorem 2.3.1: The cardinality of the Exponential $K+1$-Trapezoidal fuzzy number $\overline{A} = (a_1, a_2, .., a_{2k+4}; w_1, .., w_k)_E$ is

$$card\overline{A} = \begin{cases} (a_4-a_3)+w_1(1-\frac{1}{e})(a_6-a_5+a_2-a_1)+\frac{(w_1-1)(a_5-a_4+a_3-a_2)}{\ln w_1} & ,k=1 \\ (a_{k+3}-a_{k+2})+w_1(1-\frac{1}{e})(a_{2k+4}-a_{2k+3}+a_2-a_1)+\frac{(w_1-1)(a_{k+4}-a_{k+3}+a_{k+2}-a_{k+1})}{\ln w_k} & \\ +\sum_{i=2}^{k}\frac{(w_i-w_{i-1})(a_{2k+5-i}-a_{2k+4-i}+a_{i+1}-a_i)}{\ln w_i - \ln w_{i-1}} & ,k>1 \end{cases}$$

(2.3)

Proof : when $k=1$, from definition 2.1.1 and the definition of $K+1$-Trapezoidal fuzzy number , we have

$$card\overline{A} = \int_{a_1}^{a_6} f_A(x)dx$$

$$card\overline{A} = \int_{a_1}^{a_2} w_1 e^{[(x-a_2)/(a_2-a_1)]}dx + \int_{a_2}^{a_3} e^{([(a_3-x)]\ln w_1/(a_3-a_2))}dx$$

$$+\int_{a_3}^{a_4} 1dx + \int_{a_4}^{a_5} e^{([(x-a_4)\ln w_1]/(a_5-a_4))} + \int_{a_5}^{a_6} w_1 e^{[(a_5-x)/(a_6-a_5)]}dx$$

After integrating the last equation by substitution we get equation (2.3).

When k= 2, from definition 2.1.1 and definition of K+1-Trapezoidal fuzzy number, we have

$$card\,\overline{A} = \int_{a_1}^{a_2} w_1 e^{[(x-a_2)/(a_2-a_1)]}dx + \int_{a_2}^{a_3} e^{[((a_3-x)\ln w_1+(x-a_2)\ln w_2)/(a_3-a_2)]}dx + ...$$

$$+ \int_{a_k}^{a_{k+1}} e^{[((a_{k+1}-x)\ln w_{k-1}+(x-a_k)\ln w_k)/(a_{k+1}-a_k)]}dx + \int_{a_{k+1}}^{a_{k+2}} e^{[((a_{k+2}-x)\ln w_k)/(a_{k+2}-a_{k+1})]}dx$$

$$+ \int_{a_{k+2}}^{a_{k+3}} 1dx + \int_{a_{k+3}}^{a_{k+4}} e^{[((x-a_{k+3})\ln w_k)/(a_{k+4}-a_{k+3})]}dx + \int_{a_{k+4}}^{a_{k+5}} e^{[((x-a_{k+4})\ln w_{k-1}+(a_{k+5}-x)\ln w_k)/(a_{k+5}-c_{k+4})]}dx$$

$$+ \ldots + \int_{a_{2k+2}}^{a_{2k+3}} e^{[((a_{2k+3}-x)\ln w_2+(x-a_{2k+2})\ln w_1)/(a_{2k+3}-a_{2k+2})]} + \int_{a_{2k+3}}^{a_{2k+4}} w_1 e^{[(a_{2k+3}-x)/(a_{2k+4}-a_{2k+3})]}dx$$

After integrating the last equation by substitution we get equation (2.3).□

Example 2.3.3:

Let $\overline{A} = (0.2,0.3,0.4,0.6,0.7,0.8;0.35)_E$ and $\overline{B} = (0.1,0.16,0.2,0.3,0.35,0.4;0.35)_E$.
Then

$$card\,\overline{A} = (a_4 - a_3) + \frac{(w_1-1)(a_5 - a_4 + a_3 - a_2)}{\ln w_1} + w_1(1-\frac{1}{e})(a_6 - a_5 + a_2 - a_1)$$

$$card\,\overline{A} = (0.6-0.4) + \frac{(0.35-1)(0.7-0.6+0.4-0.3)}{\ln 0.35} + 0.35(1-\frac{1}{e})(0.8-0.7+0.3-0.2)$$

$$card\,\overline{A} = 0.368$$

$$card\,\overline{B} = (b_4 - b_3) + \frac{(w_1-1)(b_5 - b_4 + b_3 - b_2)}{\ln w_1} + w_1(1-\frac{1}{e})(b_6 - b_5 + b_2 - b_1)$$

$$card\,\overline{B} = (0.3-0.2) + \frac{(0.35-1)(0.35-0.3+0.2-0.16)}{\ln 0.35} + 0.35(1-\frac{1}{e})(0.4-0.35+0.16-0.1)$$

$$card\,\overline{B} = 0.18$$

Since $card\overline{A} > card\overline{B}$ then $\overline{A} \succ \overline{B}$.

Example 2.3.4:

Let $\overline{A} = (2,3,5,5.3,5.5,6;0.2)_E$ and $\overline{B} = (1,1.5,1.7,1.9,2,2.5;0.2)_E$.
Then

$$card\overline{A} = (a_4 - a_3) + \frac{(w_1 - 1)(a_5 - a_4 + a_3 - a_2)}{\ln w_1} + w_1(1 - \frac{1}{e})(a_6 - a_5 + a_2 - a_1)$$

$$card\overline{A} = (5.3 - 5) + \frac{(0.2 - 1)(5.5 - 5.3 + 5 - 3)}{\ln 0.2} + 0.2(1 - \frac{1}{e})(6 - 5.5 + 3 - 2)$$

$$card\overline{A} = 1.584$$

$$card\overline{B} = (b_4 - b_3) + \frac{(w_1 - 1)(b_5 - b_4 + b_3 - b_2)}{\ln w_1} + w_1(1 - \frac{1}{e})(b_6 - b_5 + b_2 - b_1)$$

$$card\overline{B} = (1.9 - 1.7) + \frac{(0.2 - 1)(2 - 1.9 + 1.7 - 1.5)}{\ln 0.2} + 0.2(1 - \frac{1}{e})(2.5 - 2 + 1.5 - 1)$$

$$card\overline{B} = 0.475$$

Since $card\overline{A} > card\overline{B}$ then $\overline{A} \succ \overline{B}$.

Example 2.3.5:

Let $\overline{A} = (2,3,4,4.3,5,6,7,8;0.1,0.5)_E$ and $\overline{B} = (6,7,7.5,7.9,8,9,9.3,10;0.1,0.5)_E$

$$card\overline{A} = (a_5 - a_4) + \frac{(w_2 - 1)(a_6 - a_5 + a_4 - a_3)}{\ln w_2} + w_1(1 - \frac{1}{e})(a_8 - a_7 + a_2 - a_1)$$
$$+ \frac{(w_2 - w_1)(a_7 - a_6 + a_3 - a_2)}{\ln w_2 - \ln w_1}$$

$$card\,\overline{A} = (5-4.3) + \frac{(0.5-1)(6-5+4.3-4)}{\ln 0.5} + 0.1(1-\frac{1}{e})(8-7+3-2)$$
$$+\frac{(0.5-0.1)(7-6+4-3)}{\ln 0.5 - \ln 0.1}$$

$$card\,\overline{A} = 2.261$$

$$card\,\overline{A} = (b_5 - b_4) + \frac{(w_2-1)(b_6-b_5+b_4-b_3)}{\ln w_2} + w_1(1-\frac{1}{e})(b_8-b_7+b_2-b_1)$$
$$+\frac{(w_2-w_1)(b_7-b_6+b_3-b_2)}{\ln w_2 - \ln w_1}$$

$$card\,\overline{A} = (8-7.9) + \frac{(0.5-1)(9-8+7.9-7.5)}{\ln 0.5} + 0.1(1-\frac{1}{e})(10-9.3+7-6)$$
$$+\frac{(0.5-0.1)(9.3-9+7.5-7)}{\ln 0.5 - \ln 0.1}$$

$$card\,\overline{B} = 1.416$$

Since $card\,\overline{A} > card\,\overline{B}$ then $\overline{A} \succ \overline{B}$.

Chapter 3

Ranking of Exponential Fuzzy Numbers Using By TRD Distance

In this chapter, we recall the definition of TRD distance and then we apply this method to compare between exponential particular fuzzy numbers.

3.1 TRD Distance

In this section, we present the definition of weighted averaged representative and weighted width of fuzzy numbers and then we introduce the definition of TRD distance.

Definition 3.1.1[19]: The following values constitute the weighted averaged representative and weighted width, respectively, of the fuzzy number $\overline{A}$:

$$I(\overline{A}) = \int_0^1 (cL_{\overline{A}}(\alpha) + (1-c)R_{\overline{A}}(\alpha))d\alpha \qquad (3.1)$$

and

$$D(\overline{A}) = \int_0^1 (R_{\overline{A}}(\alpha) - L_{\overline{A}}(\alpha))\, f(\alpha)d\alpha \qquad (3.2)$$

Where the constant c denotes the "optimism/pessimism" coefficient in conducting operations on fuzzy numbers and ranges between 0 and 1, while $f(\alpha)$ is called the weighting function. On the other hand, $f(\alpha)$ is a nonnegative and increasing function on $[0,1]$ with $f(0)=0$, $f(1)=1$ and $\int_0^1 f(\alpha)d\alpha$. In actual applications, the function can be chosen according to the actual situation. In practical cases, it may be assumed that $f(\alpha)=\alpha$.

Definition 3.1.2 [19]: The TRD distance between two fuzzy numbers $\overline{A}$ and $\overline{B}$ is

$$TRD(\overline{A},\overline{B}) = \sqrt{(I(\overline{A}) - I(\overline{B}))^2 + (D(\overline{A}) - D(\overline{B}))^2} \qquad (3.3)$$

The following properties can be followed immediately from the definition of TRD distance:

1. $TRD(\overline{A}, \overline{B}) \geq 0$
2. $\overline{A} \approx \overline{B} \Leftrightarrow TRD(\overline{A}, \overline{B}) = 0$
3. $TRD(\overline{A}, \overline{B}) + TRD(\overline{B}, \overline{C}) \geq TRD(\overline{A}, \overline{C})$
4. $TRD(\overline{A}, \overline{B}) = TRD(\overline{B}, \overline{A})$

Let a be any real number and we consider the fuzzy number A_α whose membership function is

$$\mu_{\overline{A}_a}(x) = \begin{cases} 1, x = a \\ 0, x \neq a \end{cases}$$

Hence, if $a = 0$ then

$$\mu_{\overline{A}_0}(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases}$$

The fuzzy number $\overline{A_0}$ is known as a fuzzy origin number. So, for each fuzzy number $\overline{A}$ the TRD distance is

$$TRD(\overline{A}, \overline{A}_0) = \sqrt{(I(\overline{A}))^2 + (D(\overline{A}))^2} \tag{3.4}$$

Now, the weighted average representative and weighted width of exponential trapezoidal fuzzy numbers is presented in the following two theorems.

Theorem 3.1.3[19]: The weighted average representative of exponential trapezoidal fuzzy numbers $\overline{A} = (a, b, c, d; w)_E$ is

$$I(\overline{A}) = cw(b-a)(e^{\frac{1-b}{b-a}} - e^{\frac{-b}{b-a}}) - w(1-c)(d-c)(e^{\frac{c-1}{d-c}} - e^{\frac{c}{d-c}}) \tag{3.5}$$

Theorem 3.1.4[19]: The weighted width of exponential trapezoidal fuzzy numbers $\overline{A}=(a,b,c,d;w)_E$ is

$$D(\overline{A}) = w(d-c)(c-d+1)e^{\frac{c-1}{d-c}} + w(b-a)(b-a-1)e^{\frac{1-b}{b-a}} + w(d-c)^2 e^{\frac{c}{d-c}} + w(b-a)^2 e^{\frac{-b}{b-a}} \tag{3.6}$$

Finally, the following theorem presents how to compare between exponential trapezoidal fuzzy numbers using TRD distance.

Theorem 3.1.5[19]: Let $\overline{A}=(a_1,a_2,a_3,a_4)_E$ and $\overline{B}=(b_1,b_2,b_3,b_4)_E$ be two exponential trapezoidal fuzzy numbers, then

1. $TRD(\overline{A}) < TRD(\overline{B})$ then $\overline{A} < \overline{B}$.
2. $TRD(\overline{A}) > TRD(\overline{B})$ then $\overline{A} > \overline{B}$.
3. $TRD(\overline{A}) = TRD(\overline{B})$ then $\overline{A} \approx \overline{B}$.

3.2 Ranking of Exponential K-Trapezoidal-Triangular Fuzzy Numbers

The purpose of the current section is to find formulas for the weighted average representative and the weighted width of exponential particular fuzzy number and then use them to find TRD distance between exponential K-Trapezoidal-Triangular fuzzy numbers. To accomplish that aim we follow the same of algorithm introduced in theorem 3.1.5.

Let $\overline{A}$ be K-Trapezoidal – Triangular fuzzy number and let $A_1, A_2, .., A_{k+1}$ such that $A_1=(a_1,a_2,a_{2k+2},a_{2k+3};w_1)$, $A_i=(a_i,a_{i+1},a_{2k+3-i},a_{2k+4-i};w_i-w_{i-1})$,i=2,..,k , $A_{k+1}=(a_{k+1},a_{k+2},a_{k+3};1-w_k)$.

Definition 3.2.1: The weighted averaged representative of the particular fuzzy number $\overline{A}$ is :

$$I(A)=\int_0^{w_1}(c_1L_{A_1}(\alpha)+(1-c_1)R_{A_1}(\alpha))d\alpha$$
$$+\sum_{i=1}^{k-1}\int_{w_i}^{w_{i+1}}(c_{i+1}L_{A_{i+1}}(\alpha)+(1-c_{i+1})R_{A_{i+1}}(\alpha))d\alpha \qquad (3.7)$$
$$+\int_{w_n}^{1}(c_{k+1}L_{A_{k+1}}(\alpha)+(1-c_{k+1})R_{A_{k+1}}(\alpha))d\alpha$$

Where c_i denotes the "optimism/pessimism" coefficient in conducting operations on fuzzy number and satisfy the condition
$0<c_1<w_1, w_1<c_2<w_2,\ldots, w_{k-1}<c_k<w_k, w_k<c_k<1$.

Definition 3.2.2: The weighted width of the particular fuzzy number $\overline{A}$ is:

$$D(A)=\int_0^{w_1}(R_{A_1}(\alpha)-L_{A_1}(\alpha))f_1(\alpha)d\alpha+\sum_{i=1}^{k-1}\int_{w_i}^{w_{i+1}}(R_{A_{i+1}}(\alpha)-L_{A_{i+1}}(\alpha))f_{i+1}(\alpha)d\alpha$$
$$+\int_{w_n}^{1}(R_{A_{k+1}}(\alpha)-L_{A_{k+1}}(\alpha))f_{k+1}(\alpha)d\alpha \qquad (3.8)$$

Where the functions $f_i(\alpha)$ are nonnegative and increasing functions on [0,1] with $f_i(0)=0,\ f_i(1)=1$ and $\int_0^{w_1}f_1(\alpha)d\alpha+\sum_{i=1}^{k-1}\int_{w_i}^{w_{i+1}}f_{i+1}(\alpha)d\alpha+\int_{w_k}^{1}f_k(\alpha)d\alpha=\frac{1}{2}$.The functions $f_i(\alpha)$ are also called weighting function, $i=1,\ldots,k.$.

In actual applications, functions can be chosen according to the actual situation. In practical cases, it may be assumed that

$$f_i(\alpha)=\alpha, i=1,2,\ldots,k$$

The following theorems give the exact formulas of the weighted averaged representative and the weighted width of exponential k-trapezoidal-triangular fuzzy number.

Theorem 3.2.3: The weighted averaged representative of the Exponential K-Trapezoidal-Triangular fuzzy number $\overline{A}=(a_1,a_2,..,a_{2k+3};w_1,..,w_k)_E$ is

$$I(A)=c_1w_1\lambda_1(e^{\frac{(w_1-a_2)}{\lambda_1}}-e^{\frac{-a_2}{\lambda_1}})-w_1\beta_1(1-c_1)(e^{\frac{(a_{2k+2}-w_1)}{\beta_1}}-e^{\frac{a_{2k+2}}{\beta_1}})$$

$$+\sum_{i=2}^{k}\frac{c_i\lambda_i}{\ln w_i-\ln w_{i-1}}(e^{\frac{((a_{i+1}-\alpha)\ln w_{i-1})+((\alpha-a_i)\ln w_i)}{\lambda_i}})\Big|_{w_{i-1}}^{w_i}$$

$$+\sum_{i=2}^{k}\frac{(1-c_i)\beta_i}{\ln w_{i-1}-\ln w_i}e^{\frac{((a_{2k+i}-\alpha)\ln w_i)+((\alpha-a_{2k+i-1})\ln w_{i-1})}{\beta_i}}\Big|_{w_{i-1}}^{w_i} \tag{3.9}$$

$$+\frac{c_{k+1}\lambda_{k+1}}{\ln w_k}(e^{\frac{(a_{k+2}-1)\ln w_k}{\lambda_{k+1}}}-e^{\frac{(a_{k+2}-w_k)\ln w_k}{\lambda_{k+1}}})$$

$$+\frac{(1-c_{k+1})\beta_{k+1}}{\ln w_k}(e^{\frac{(1-a_{k+2})\ln w_k}{\beta_{k+1}}}-e^{\frac{(w_k-a_{k+2})\ln w_k}{\beta_{k+1}}})$$

where i=1,2,..,k.

Proof: Followed directly from definition 3.2.1 and integration by substitution. □

Theorem 3.2.4: The weighted width of the Exponential K-Trapezoidal-Triangular fuzzy number $\overline{A}=(a_1,a_2,..,a_{2k+3};w_1,..,w_k)_E$ is

$$D(A)=w_1\lambda_1(-w_1+\lambda_1)e^{\frac{(w_1-a_2)}{\lambda_1}}-\beta_1w_1(w_1+\beta_1)e^{\frac{a_{2k+2}-w_1}{\beta_1}}$$

$$-\sum_{i=2}^{k}\frac{\alpha\lambda_i}{\ln w_i-\ln w_{i-1}}(e^{\frac{(a_{i+1}-\alpha)\ln w_{i-1}+(\alpha-a_i)\ln w_i}{\lambda_i}})-\frac{\lambda^2{}_i}{(\ln w_i-\ln w_{i-1})^2}e^{\frac{(a_{i+1}-\alpha)\ln w_{i-1}+(\alpha-a_i)\ln w_i}{\lambda_i}}\Big|_{w_{i-1}}^{w_i}$$

$$+\sum_{i=2}^{k}\frac{\beta_i\alpha}{\ln w_{i-1}-\ln w_i}e^{\frac{(a_{2k+i}-\alpha)\ln w_i+(\alpha-a_{2k+i-1})\ln w_{i-1}}{\beta_i}}+\frac{\beta^2{}_i}{(\ln w_{i-1}-\ln w_i)^2}e^{\frac{(a_{2k+i}-\alpha)\ln w_i+(\alpha-a_{2k+i-1})\ln w_{i-1}}{\beta_i}}\Big|_{w_{i-1}}^{w_i}$$

$$-\frac{\lambda_{k+1}}{\ln w_k}(e^{\frac{(a_{k+2}-1)\ln w_k}{\lambda_{k+1}}}-w_ke^{\frac{(a_{k+2}-w_k)\ln w_k}{\lambda_{k+1}}})+\frac{\beta_{k+1}}{\ln w_k}(e^{\frac{(1-a_{k+2})\ln w_k}{\beta_{k+1}}}-w_ke^{\frac{(w_k-a_{k+2})\ln w_k}{\beta_{k+1}}})$$

$$+\frac{\beta_{k+1}{}^2}{(\ln w_k)^2}(e^{\frac{(1-a_{k+2})\ln w_k}{\beta_{k+1}}}-e^{\frac{(w_k-a_{k+2})\ln w_k}{\beta_{k+1}}})-\frac{\lambda_{k+1}{}^2}{(\ln w_k)^2}(e^{\frac{(a_{k+2}-1)\ln w_k}{\lambda_{k+1}}}-e^{\frac{(a_{k+2}-w_k)\ln w_k}{\lambda_{k+1}}})$$

(3.10) , where i=2,..,k.

Proof: From definition **3.2.2**, we have

$$D(A)=\int_0^{w_1}(-w_1 e^{\frac{\alpha-a_2}{\lambda_1}}+w_1 e^{\frac{a_{2k+2}-\alpha}{\beta_1}})\alpha d\alpha$$
$$+\sum_{i=2}^{k-1}\int_{w_{i-1}}^{w_i}(-e^{\frac{((a_{i+1}-\alpha)\ln w_{i-1})+((\alpha-a_i)\ln w_i)}{\lambda_i}}d\alpha+e^{\frac{((a_{2k+i}-\alpha)\ln w_i)+((\alpha-a_{2k+i-1})\ln w_{i-1})}{\beta_i}})\alpha d\alpha$$
$$+\int_{w_k}^{1}(-e^{\frac{(a_{k+2}-\alpha)\ln w_k}{\lambda_{k+1}}}+e^{\frac{(\alpha-a_{k+2})\ln w_k}{\beta_{k+1}}})\alpha d\alpha$$

Integrate the last formula by substitution and by part to get equation (3.10). □

Below we give some illustrative examples.

Example 3.2.5: Let $\overline{A}=(1,2,3.5,5,6.7;0.5)_E$ and $\overline{B}=(2,2.5,3,3.6,4;0.5)_E$. Then

1- $I(A)=-4.0531c_1+5.887$ **,** $I(B)=-1156.305c_1-6.85c_2+1163.129$

2- $D(A)=-9.0742$ **,** $D(B)=-418.004$

3- $TRD(A)=\sqrt{(-4.0531c_1+5.887)^2+(-9.0742)^2}$

For

$c_1=0, c_2=0.5\rightarrow TRD(A)=10.817$

$c_1=0.5, c_2=1\rightarrow TRD(A)=9.861$

$TRD(B)=\sqrt{(-1156.305c_1-6.85c_2+1163.129)^2+(-418.004)^2}$

For

$c_1=0, c_2=0.5\rightarrow TRD(\overline{B})=1081.75$

$c_1=0.5, c_2=1\rightarrow TRD(\overline{B})=713.4$

. $TRD(\overline{A}) < TRD(\overline{B})$ then $\overline{A} < \overline{B}$

Example 3.2.6: Let $\overline{A} = (1,2,2.3,4,5;0.5)_E$ and $\overline{B} = (4,4.5,6,6.5,6.7;0.5)_E$.

Then

1- $I(A) = -10.697c_1 - 2.358c_2 + 13.084$ **,** $I(B) = -1.195*10^3 c_1 - 739.26c_2 + 1.195*10^3$

2- $D(A) = -24.765$ **,** $D(B) = 7.48*10^{11}$ **,**

3- $TRD(A) = \sqrt{(-10.6971c_1 - 2.358c_2 + 13.084)^2 + (-24.765)^2}$

For

$c_1 = 0, c_2 = 0.5 \to TRD(A) = 27.478$

$c_1 = 0.5, c_2 = 1 \to TRD(A) = 25.342$

$TRD(B) = \sqrt{(-1.195*10^{13} c_1 - 739.26c_2 + 1.195*10^{13})^2 + (7..48*10^{11})^2}$

For

$c_1 = 0, c_2 = 0.5 \to TRD(B) = 1.197*10^{13}$

$c_1 = 0.5, c_2 = 1 \to TRD(A) = 6.0216*10^{12}$

. $TRD(\overline{A}) < TRD(\overline{B})$ then $\overline{A} < \overline{B}$

Example3.2.7: Let $\overline{A} = (1,2,2.3,4,5,5.5,7;0.5,0.8)_E$ **and** $\overline{B} = (4,4.5,6,6.5,6.7,7,8;0.5,0.8)_E$

Then

1- $I(A) = -8.290c_1 + 14.384c_2 - 0.535c_3 - 5.629$ **,**
$I(B) = -215.746c_1 - 3179.755c_2 - 103.993c_3 + 3499.523$

2- $D(A) = -33.828$ **,** $D(B) = 447.993$

3- $TRD(A) = \sqrt{(-8.290c_1 + 14.384c_2 - 0.535c_3 + -5.629)^2 + (-33.828)^2}$

For

$c_1 = 0, c_2 = 0.5, c_3 = 0.8 \rightarrow TRD(A) = 33.847$

$c_1 = 0.5, c_2 = 0.8, c_3 = 1 \rightarrow TRD(A) = 33.986$

$TRD(B) = \sqrt{(-215.746c_1 - 3179.755c_2 - 103.993c_3 + 3499.523)^2 + (447.993)^2}$

For

$c_1 = 0, c_2 = 0.5, c_3 = 0.8 \rightarrow TRD(B) = 1880.591$

$c_1 = 0.5, c_2 = 0.8, c_3 = 1 \rightarrow TRD(A) = 1776.00913$

. $TRD(\overline{A}) < TRD(\overline{B})$ then $\overline{A} < \overline{B}$

3.3 Ranking of Exponential $K+1$-Trapezoidal fuzzy numbers

In this section , we find $I(\overline{A})$, $D(\overline{A})$ and the TRD distance of Exponential $K+1$-Trapezoidal fuzzy numbers by using definition 3.2.1, then we use TRD distance to rank between Exponential $K+1$-Trapezoidal fuzzy numbers.

Let $\overline{A}$ be $K+1$-Trapoziodal fuzzy numbers and $A_1, A_2, .., A_{k+1}$ such that $A_1 = (a_1, a_2, a_{2k+2}, a_{2k+4}; w_1)$, $A_i = (a_i, a_{i+1}, a_{2k+4-i}, a_{2k+5-i}; w_i - w_{i-1})$,i=2,..,k , $A_{k+1} = (a_{k+1}, a_{k+2}, a_{k+3}, a_{k+4}; 1 - w_k)$.

In the next theorems 3.3.1 and 3.3.2, use definition 3.2.1 to find an exact formula of weighted average representative and weighted width of Exponential $K+1$-Trapezoidal fuzzy numbers.

Theorem 3.3.1: The weighted averaged representative of the Exponential K-Trapezoidal fuzzy numbers $\overline{A} = (a_1, a_2, .., a_{2k+4}; w_1, .., w_k)_E$ is

$$I(A) = c_1 w_1 \lambda_1 (e^{\frac{(w_1 - a_2)}{\lambda_1}} - e^{\frac{-a_2}{\lambda_1}}) - w_1 \beta_1 (1 - c_1)(e^{\frac{(a_{2k+3} - w_1)}{\beta_1}} - e^{\frac{a_{2k+3}}{\beta_1}})$$
$$+ \sum_{i=2}^{k} \frac{c_i \lambda_i}{\ln w_i - \ln w_{i-1}} (e^{\frac{((a_{i+1} - \alpha)\ln w_{i-1}) + ((\alpha_i - a_i)\ln w_i)}{\lambda_i}}) \Big|_{w_{i-1}}^{w_i} +$$

$$\sum_{i=2}^{k}\frac{(1-c_i)\beta_i}{\ln w_{i-1}-\ln w_i}e^{\frac{((a_{2k+i+1}-\alpha)\ln w_i)+((\alpha-a_{2k+i})\ln w_{i-1})}{\beta_i}}\Big|_{w_{i-1}}^{w_i}+\frac{c_{k+1}\lambda_{k+1}}{\ln w_k}(e^{\frac{(a_{k+2}-1)\ln w_k}{\lambda_{k+1}}}-e^{\frac{(a_{k+2}-w_k)\ln w_k}{\lambda_{k+1}}})$$

$$+\frac{(1-c_{k+1})\beta_{k+1}}{\ln w_k}(e^{\frac{(1-a_{k+3})\ln w_k}{\beta_{k+1}}}-e^{\frac{(w_k-a_{k+3})\ln w_k}{\beta_{k+1}}})$$

(3.11)

where i=1,2,..,k.

Proof:

$$I(A)=\int_0^{w_1}(c_1w_1e^{\frac{\alpha-a_2}{\lambda_1}}+(1-c_1)w_1e^{\frac{a_{2k+3}-\alpha}{\beta_1}})d\alpha$$

$$+\sum_{i=2}^{k-1}\int_{w_{i-1}}^{w_i}(c_{i+1}e^{\frac{((a_{i+1}-\alpha)\ln w_{i-1})+((\alpha-a_i)\ln w_i)}{\lambda_i}}d\alpha+(1-c_{i+1})e^{\frac{((a_{2k+i+1}-\alpha)\ln w_i)+((\alpha-a_{2k+i})\ln w_{i-1})}{\beta_i}})d\alpha$$

$$+\int_{w_k}^{1}(c_{k+1}e^{\frac{(a_{k+2}-\alpha)\ln w_k}{\lambda_{k+1}}}+(1-c_{k+1})e^{\frac{(\alpha-a_{k+3})\ln w_k}{\beta_{k+1}}})d\alpha$$

Integrate the last formula by substitution to produce equation (3.11). □

Theorem 3.3.2: The weighted width of the Exponential K+1-Trapezoidal fuzzy numbers $\overline{A}=(a_1,a_2,..,a_{2k+4};w_1,..,w_k)_E$ is

$$D(A)=w_1\lambda_1(-w_1+\lambda_1)e^{\frac{(w_1-a_2)}{\lambda_1}}-\beta_1w_1(w_1+\beta_1)e^{\frac{a_{2k+3}-w_1}{\beta_1}}$$

$$-\sum_{i=2}^{k}\frac{\alpha_i\lambda_i}{\ln w_i-\ln w_{i-1}}(e^{\frac{(a_{i+1}-\alpha)\ln w_{i-1}+(\alpha-a_i)\ln w_i}{\lambda_i}})-\frac{\lambda^2{}_i}{(\ln w_i-\ln w_{i-1})^2}e^{\frac{(a_{i+1}-\alpha)\ln w_{i-1}+(\alpha-a_i)\ln w_i}{\lambda_i}}\Big|_{w_{i-1}}^{w_i}$$

$$+\sum_{i=2}^{k}\frac{\beta_i\alpha}{\ln w_{i-1}-\ln w_i}e^{\frac{(a_{2k+i+1}-\alpha)\ln w_i+(\alpha-a_{2k+i})\ln w_{i-1}}{\beta_i}}+\frac{\beta^2{}_i}{(\ln w_{i-1}-\ln w_i)^2}e^{\frac{(a_{2k+i+1}-\alpha)\ln w_i+(\alpha-a_{2k+i})\ln w_{i-1}}{\beta_i}}\Big|_{w_{i-1}}^{w_i}$$

$$-\frac{\lambda_{k+1}}{\ln w_k}\left(e^{\frac{(a_{k+2}-1)\ln w_k}{\lambda_{k+1}}}-w_k e^{\frac{(a_{k+2}-w_k)\ln w_k}{\lambda_{k+1}}}\right)+\frac{\beta_{k+1}}{\ln w_k}\left(e^{\frac{(1-a_{k+3})\ln w_k}{\beta_{k+1}}}-w_k e^{\frac{(w_k-a_{k+3})\ln w_k}{\beta_{k+1}}}\right)$$

$$+\frac{\beta_{k+1}{}^2}{(\ln w_k)^2}\left(e^{\frac{(1-a_{k+3})\ln w_k}{\beta_{k+1}}}-e^{\frac{(w_k-a_{k+3})\ln w_k}{\beta_{k+1}}}\right)-\frac{\lambda_{k+1}{}^2}{(\ln w_k)^2}\left(e^{\frac{(a_{k+2}-1)\ln w_k}{\lambda_{k+1}}}-e^{\frac{(a_{k+2}-w_k)\ln w_k}{\lambda_{k+1}}}\right)$$

(3.12) , where i=1,2,..,k.

Proof:

$$D(A)=\int_0^{w_1}(-w_1 e^{\frac{\alpha-a_2}{\lambda_1}}+w_1 e^{\frac{a_{2k+3}-\alpha}{\beta_1}})\alpha d\alpha$$

$$+\sum_{i=2}^{k-1}\int_{w_{i-1}}^{w_i}(-e^{\frac{((a_{i+1}-\alpha)\ln w_{i-1})+((\alpha-a_i)\ln w_i)}{\lambda_i}}d\alpha+e^{\frac{((a_{2k+i+1}-\alpha)\ln w_i)+((\alpha-a_{2k+i})\ln w_{i-1})}{\beta_i}})\alpha d\alpha$$

$$+\int_{w_k}^{1}(-e^{\frac{(a_{k+2}-\alpha)\ln w_k}{\lambda_{k+1}}}+e^{\frac{(\alpha-a_{k+3})\ln w_k}{\beta_{k+1}}})\alpha d\alpha$$

Integrate the last formula by substation and by parts to produce equation (3.13). □

Example3.3.3:
Let $\overline{A}=(0.2,0.3,0.4,0.6,0.7,0.8;0.35)_E$ and $\overline{B}=(0.1,0.16,0.2,0.3,0.35,0.4;0.35)_E$.

Then

1- $I(A)=-37.167c_1-0.758+37.909$ **,** $I(B)=-18.677c_1-50046291.47c_2+19.191$

2- $D(A)=-0.790$ **,** $D(B)=-1315455656$

3- $TRD(A)=\sqrt{(-37.167c_1-0.758c_2+37.909)^2+(-0.790)^2}$

For

$c_1=0, c_2=0.35\rightarrow TRD(A)=37.652$

$c_1=0.35, c_2=1\rightarrow TRD(A)=24.155$

$TRD(B)=\sqrt{(-18.677c_1-50046291.47c_2+19.191)^2+(-1315455656)^2}$

For

$c_1 = 0, c_2 = 0.35 \rightarrow TRD(B) = 1315572271$

$c_1 = 0.35, c_2 = 1 \rightarrow TRD(B) = 1316407313$

. $TRD(\overline{A}) < TRD(\overline{B})$ then $\overline{A} < \overline{B}$

Example 3.3.4: Let $\overline{A} = (2,3,5,5.3,5.5,6;0.2)_E$ and $\overline{B} = (1,1.5,1.7,1.9,2,2.5;0.2)_E$.

Then

1- $I(A) = -1973.928c_1 - 8.25*10^{16}c_2 + 8.25*10^{16}$,
$I(B) = -1.798c_1 - 4.736*10^{10}c_2 + 4.736*10^{10}$

2- $D(A) = 6.153*10^{15}$ **,** $D(B) = 6527739585$

3- $TRD(A) = \sqrt{(-1973.928c_1 - 8.25*10^{16}c_2 + 8.25*10^{16})^2 + (6.153*10^{15})^2}$

For

$c_1 = 0, c_2 = 0.2 \rightarrow TRD(A) = 6.629*10^{16}$

$c_1 = 0.2, c_2 = 1 \rightarrow TRD(A) = 6.153*10^{15}$

$TRD(B) = \sqrt{(-1.798c_1 - 4.736*10^{10}c_2 + 4.736*10^{10})^2 + (6527739585)^2}$

For

$c_1 = 0, c_2 = 0.2 \rightarrow TRD(B) = 3.845$

$c_1 = 0.2, c_2 = 1 \rightarrow TRD(B) = 6527739585$

. $TRD(\overline{A}) > TRD(\overline{B})$ then $\overline{A} > \overline{B}$

Example3.3.5:

Let $\overline{A} = (2,3,4,4.3,5,6,7,8;0.1,0.5)_E$ and $\overline{B} = (6,7,7.5,7.9,8,9,9.3,10;0.1,0.5)_E$

Then

1- $I(A) = -10.435c_1 - 1962.0403c_2 - 9.549c_3 + 1982.0285$,
$I(B) = -5492.733c_1 - 4.472*10^{19}c_2 - 76.616c_3 + 4.472*10^{19}$

2- $D(A) = -2022.879$ **,** $D(B) = -1.638*10^{18}$

3- $TRD(A)=\sqrt{(-10.435c_1-1962.0403c_2-9.549c_3+1982.0285)^2+(-2022.879)^2}$

For

$c_1=0, c_2=0.1, c_3=0.5 \rightarrow TRD(A)=2695.214$

$c_1=0.1, c_2=0.5, c_3=1 \rightarrow TRD(A)=2252.324$

$TRD(B)=\sqrt{(-5492.733c_1-4.472*10^{19}c_2-76.616c_3+4.472*10^{19})^2+(-1.638*10^{18})^2}$

For

$c_1=0, c_2=0.1, c_3=0.5 \rightarrow TRD(B)=4.0281*10^{19}$

$c_1=0.1, c_2=0.5, c_3=1 \rightarrow TRD(A)=2.242*10^{19}$

$TRD(\overline{A})<TRD(\overline{B})$ then $\overline{A}<\overline{B}$.

Chapter 4

Ranking of Exponential particular Fuzzy Numbers by Median Value

In this chapter, we generalize the ranking method in [17] from exponential trapezoidal fuzzy numbers to exponential particular fuzzy numbers.

4.1 Median Value Method

In this section, we define the median value of a fuzzy number, the median of trapezoidal and exponential trapezoidal fuzzy numbers. Finally, we compare between exponential trapezoidal fuzzy numbers with respect to the median values.

Definition 4.1.1[3]: The median value of a fuzzy number $m_{\overline{A}}$ is the real number $\overline{A}$ from the support of $\overline{A}$ such that

$$\int_{a}^{m_{\overline{A}}} \mu_{\overline{A}}(x)dx = \int_{m_{\overline{A}}}^{d} \mu_{\overline{A}}(x)dx \tag{4.1}$$

The above expressions can be rewritten as

$$\int_{a}^{m_{\overline{A}}} \mu_{\overline{A}}(x)dx = 0.5card(\overline{A}) \tag{4.2}$$

The authors [8] classify fuzzy numbers with respect to the "distribution" of their cardinality as follows:
A fuzzy number A is called

1. A fuzzy number with equally heavy tails if

$$\int_{a}^{b} \mu_{\overline{A}}(x)dx = \int_{c}^{d} \mu_{\overline{A}}(x)dx \tag{4.3}$$

2. A fuzzy number with light tails if

$$\max\left\{\int_a^b \mu_{\bar{A}}(x)dx, \int_c^d \mu_{\bar{A}}(x)dx\right\} \leq 0.5\int_a^d \mu_{\bar{A}}(x)dx \qquad (4.4)$$

3. A fuzzy number with heavy left tail if

$$\int_a^b \mu_{\bar{A}}(x)d > 0.5\int_a^d \mu_{\bar{A}}(x) \qquad (4.5)$$

4. A fuzzy number with heavy right tail if

$$\int_c^d \mu_{\bar{A}}(x)dx > 0.5\int_a^d \mu_{\bar{A}}(x)dx \qquad (4.6)$$

The next proposition gives the median of trapezoidal fuzzy number.

Proposition 4.1.2 [3]: If $\bar{A} = (a,b,c,d)$ is a trapezoidal fuzzy number with light tails, then

$$m_{\bar{A}} = \frac{a+b}{2} + 0.5(\int_c^d \mu_{\bar{A}}(x)dx - \int_a^b \mu_{\bar{A}}(x)dx) \qquad (4.7)$$

Proof:

Fuzzy number is light tails then $\max\left\{\int_a^b \mu_{\bar{A}}(x)dx, \int_c^d \mu_{\bar{A}}(x)dx\right\} \leq 0.5\int_a^d \mu_{\bar{A}}(x)dx$

From definition 5.1.2 we have $\int_a^{m_{\bar{A}}} \mu_{\bar{A}}(x)dx = \int_{m_{\bar{A}}}^d \mu_{\bar{A}}(x)dx$

$$\int_a^b \mu_{\bar{A}}(x)dx + \int_b^{m_{\bar{A}}} 1dx = \int_{m_{\bar{A}}}^c 1dx + \int_c^d \mu_{\bar{A}}(x)dx$$

Integrate the last inequality we get

$$\int_a^b \mu_{\bar{A}}(x)dx + (m_{\bar{A}} - b) = \int_c^d \mu_{\bar{A}}(x)dx + (c - m_{\bar{A}})$$

We rearrange the above expressions to produce equation (4.7). □

All the above definitions and propositions can be applied to exponential fuzzy number.

In theorem 4.1.3 we will find the median of the exponential trapezoidal fuzzy number.

Theorem 4.1.3 [18]: If $\overline{A}=(a,b,c,d;w)_E$ is a Exponential Trapezoidal fuzzy number with light tails then

$$m_{\overline{A}}=\frac{w(b+c)}{2}+\frac{w}{2e}[(c-d)(1-e)-(b-a)(e-1)] \tag{4.8}$$

Proof:

From proposition 4.1.2,

$$m_{\overline{A}}=\frac{a+b}{2}+0.5(\int_c^d A(x)dx-\int_a^b A(x)dx)$$

$$m_{\overline{A}}=\frac{a+b}{2}+0.5(\int_c^d we^{\frac{c-x}{d-c}}dx-\int_a^b we^{\frac{x-b}{b-a}}dx)$$

Integrate the last expression and rearrange, then we produce equation (4.8). □

In theorem 4.1.4, we introduce the algorithm of ranking between exponential trapezoidal fuzzy numbers.

Theorem 4.1.4 [18]: If $\overline{A}=(a,b,c,d;w)_E$ and $\overline{B}=(e,f,g,h;w)_E$ are exponential trapezoidal fuzzy numbers, m_A and m_B are the median value of them, so

1. If $m_{\overline{A}}>m_{\overline{B}}$ then $\overline{A}>\overline{B}$

2. If $m_{\overline{A}}<m_{\overline{B}}$ then $\overline{A}<\overline{B}$

3. If $m_{\overline{A}}=m_{\overline{B}}$ then $\overline{A}\approx\overline{B}$

4.2 Ranking Between Exponential $K+1$-Trapezoidal Fuzzy Numbers

In this section, we define the median value of $K+1$ -Trapezoidal fuzzy numbers with light tails and the median value of Exponential $K+1$-Trapezoidal fuzzy number with light tails.

Remark 4.2.1: $K+1$-Trapezoidal fuzzy number is light tails if

$$\max\left\{\sum_{i=1}^{k}\int_{a_i}^{a_{i+1}}\mu_{\overline{A}}(x)dx,\sum_{i=1}^{k}\int_{a_{2k+4-i}}^{a_{2k+5-i}}\mu_{\overline{A}}(x)dx\right\}\leq 0.5\int_{a_1}^{a_{2k+4}}\mu_{\overline{A}}(x)dx$$

Proposition 4.2.2: If $\overline{A}=(a_1,a_2,..,a_{2k+4};c_1,c_2,..,c_k)$ is a $K+1$-Trapezoidal fuzzy numbers fuzzy number with light tails then

$$m_{\overline{A}}=\frac{a_{k+2}+a_{k+3}}{2}+0.5(\sum_{i=1}^{k+1}(\int_{a_{2k+4-i}}^{2k+5-i}\mu_{\overline{A}}(x)dx-\int_{a_i}^{a_{i+1}}\mu_{\overline{A}}(x)dx) \qquad (4.9)$$

Proof:

From definition 4.1.1 we have $\int_{a_1}^{m_{\overline{A}}}\mu_{\overline{A}}(x)dx=\int_{m_{\overline{A}}}^{a_{2k+3}}\mu_{\overline{A}}(x)dx$

Since the fuzzy numbers is light tails then

$$\max\left\{\sum_{i=1}^{k}\int_{a_i}^{a_{i+1}}\mu_{\overline{A}}(x)dx,\sum_{i=1}^{k}\int_{a_{2k+4-i}}^{a_{2k+5-i}}\mu_{\overline{A}}(x)dx\right\}\leq 0.5\int_{a_1}^{a_{2k+4}}\mu_{\overline{A}}(x)dx$$

This condition means that $m_{\overline{A}}$ lies between a_{k+2} and a_{k+3}

$$\int_{a_1}^{m_{\overline{A}}}\mu_{\overline{A}}(x)dx=\int_{m_{\overline{A}}}^{a_{2k+3}}\mu_{\overline{A}}(x)dx$$

$$\int_{a_1}^{a_2}\mu_{\overline{A}}(x)dx+..+\int_{a_{k+1}}^{a_{k+2}}\mu_{\overline{A}}(x)dx+\int_{a_{k+2}}^{m_{\overline{A}}}1dx=\int_{m_{\overline{A}}}^{a_{k+3}}1dx+...+\int_{a_{2k+3}}^{a_{2k+4}}\mu_{\overline{A}}(x)dx$$

Integrate the above formula by substitution, and we produce equation (4.9). □

Theorem 4.2.3: if $\overline{A} = (a_1, a_2, .., a_{2k+4}; c_1, c_2, .., c_k)_E$ is $K+1$-Trapezoidal fuzzy number with light tails then

$$m_{\overline{A}} = \frac{a_{k+2} + a_{k+3}}{2} + 0.5(\frac{(\beta_{k+1} - \lambda_{k+1})(w_k - 1)}{\ln w_k} + w_1(1 - \frac{1}{e})(\beta_1 - \lambda_1) + \sum_{i=2}^{k} \frac{(\beta_i - \lambda_i)(w_i - w_{i-1})}{\ln w_i - \ln w_{i-1}}) \quad (4.10)$$

Proof:

From the proposition 4.1.5 we have

$$m_{\overline{A}} = \frac{a_{k+2} + a_{k+3}}{2} + 0.5(\sum_{i=1}^{k+1} (\int_{a_{2k+4-i}}^{2k+5-i} \mu_{\overline{A}}(x)dx - \int_{a_i}^{a_{i+1}} \mu_{\overline{A}}(x)dx)$$

Substitute $\mu_{\overline{A}}(x)$ in the above expression

Integrate the above expression and rearrange to produce (4.10).

The algorithm in theorem 4.2.4 of ranking of Exponential $K+1$-Trapezoidal fuzzy number is similar to the algorithm introduced in theorem 4.1.4.

Example 4.2.5:

If $\overline{A} = (0.2, 0.3, 0.4, 0.6, 0.7, 0.8; 0.35)_E$ and $\overline{B} = (0.1, 0.16, 0.2, 0.3, 0.35, 0.4; 0.35)_E$

Then

$$m_{\overline{A}} = \frac{a_3 + a_4}{2} + 0.5(\frac{(\beta_2 - \lambda_2)(w_1 - 1)}{\ln w_1} + w_1(1 - \frac{1}{e})(\beta_1 - \lambda_1))$$

$$m_{\overline{A}} = \frac{0.6 + 0.4}{2} + 0.5(\frac{(0.2 - 0.1)(0.35 - 1)}{\ln 0.35} + 0.35(1 - \frac{1}{e})(0.1 - 0.1))$$

$$m_{\overline{A}} = 0.351$$

and

$$m_{\overline{B}} = \frac{b_3 + b_4}{2} + 0.5(\frac{(\beta_2 - \lambda_2)(w_1 - 1)}{\ln w_1} + w_1(1 - \frac{1}{e})(\beta_1 - \lambda_1))$$

$$m_{\overline{B}} = \frac{0.3 + 0.2}{2} + 0.5(\frac{(0.05 - 0.04)(0.35 - 1)}{\ln 0.35} + 0.35(1 - \frac{1}{e})(0.05 - 0.06))$$

$$m_{\overline{B}} = 0.252$$

$m_{\overline{A}} < m_{\overline{B}}$ then $\overline{A} < \overline{B}$.

Example 4.2.6:

If $\overline{A} = (2,3,5,5.3,5.5,6;0.2)_E$ and $\overline{B} = (1,1.5,1.7,1.9,2,2.5;0.2)_E$

Then

$$m_{\overline{A}} = \frac{a_3 + a_4}{2} + 0.5(\frac{(\beta_2 - \lambda_2)(w_1 - 1)}{\ln w_1} + w_1(1 - \frac{1}{e})(\beta_1 - \lambda_1))$$

$$m_{\overline{A}} = \frac{5 + 5.3}{2} + 0.5(\frac{(0.5 - 2)(0.2 - 1)}{\ln 0.2} + 0.2(1 - \frac{1}{e})(0.5 - 1))$$

$$m_{\overline{A}} = 4.745$$

and

$$m_{\overline{B}} = \frac{b_3 + b_4}{2} + 0.5(\frac{(\beta_2 - \lambda_2)(w_1 - 1)}{\ln w_1} + w_1(1 - \frac{1}{e})(\beta_1 - \lambda_1))$$

$$m_{\overline{B}} = \frac{1.7 + 1.9}{2} + 0.5(\frac{(0.3 - 0.2)(0.2 - 1)}{\ln 0.2} + 0.2(1 - \frac{1}{e})(0.5 - 0.5))$$

$$m_{\overline{B}} = 1.825$$

$m_{\overline{A}} > m_{\overline{B}}$ then $\overline{A} > \overline{B}$.

Example 4.2.7:

If $\overline{A} = (2,3,4,4.3,5,6,7,8;0.1,0.5)_E$ and $\overline{B} = (6,7,7.5,7.9,8,9,9.3,10;0.1,0.5)_E$

Then

$$m_{\overline{A}} = \frac{a_3 + a_4}{2} + 0.5(\frac{(\beta_3 - \lambda_3)(w_2 - 1)}{\ln w_2} + w_1(1 - \frac{1}{e})(\beta_1 - \lambda_1) + \frac{(\beta_2 - \lambda_2)(w_2 - w_1)}{\ln w_2 - \ln w_1})$$

$$m_{\bar{A}} = \frac{4.3+5}{2} + 0.5(\frac{(1-0.3)(0.8-1)}{\ln 0.8} + 0.1(1-\frac{1}{e})(0.1-0.1) + \frac{(1-1)(0.8-0.1)}{\ln 0.8 - \ln 0.1})$$

$$m_{\bar{A}} = 4.964$$

and

$$m_{\bar{B}} = \frac{b_3+b_4}{2} + 0.5(\frac{(\beta_3-\lambda_3)(w_2-1)}{\ln w_2} + w_1(1-\frac{1}{e})(\beta_1-\lambda_1) + \frac{(\beta_2-\lambda_2)(w_2-w_1)}{\ln w_2 - \ln w_1})$$

$$m_{\bar{B}} = \frac{7.9+8}{2} + 0.5(\frac{(1-0.5)(0.8-1)}{\ln 0.8} + 0.1(1-\frac{1}{e})(0.7-1) + \frac{(0.3-0.5)(0.8-0.1)}{\ln 0.8 - \ln 0.1})$$

$$m_{\bar{B}} = 8.131$$

$m_{\bar{A}} < m_{\bar{B}}$ then $\bar{A} < \bar{B}$.

4.3 Ranking of K-Trapezoidal-Triangular fuzzy numbers

In this section, we conclude the median value of triangular fuzzy number and then we generalize the classification of the light tails of triangular fuzzy number to light tail of K - Trapezoidal-Triangular fuzzy number.

Proposition 4.3.1: The median value triangular fuzzy number $\bar{A} = (a,b,d)$ with light tails is

$$m_{\bar{A}} = b \tag{4.11}$$

Proof:

Triangular fuzzy number is a special case of trapezoidal fuzzy number with $b = c$

A fuzzy number with light tails if $\max\left\{\int_a^b \mu_{\bar{A}}(x)dx, \int_b^d \mu_{\bar{A}}(x)dx\right\} \leq 0.5\int_a^d \mu_{\bar{A}}(x)dx$

In the above expression cannot less but equal since

$$\max\left\{\int_a^b \mu_{\bar{A}}(x)dx, \int_b^d \mu_{\bar{A}}(x)dx\right\} = 0.5\int_a^d \mu_{\bar{A}}(x)dx$$

$$\therefore \int_a^b \mu_{\bar{A}}(x)dx = \int_b^d \mu_{\bar{A}}(x)dx$$

$$\therefore m_{\overline{A}} = b \qquad \square$$

Remark 4.3.2: K - Trapezoidal-Triangular fuzzy number is light tail if

$$\max\left\{\sum_{i=1}^{k}\int_{a_i}^{a_{i+1}}\mu_{\overline{A}}(x)dx, \sum_{i=1}^{k}\int_{a_{2k+3-i}}^{a_{2k+4-i}}\mu_{\overline{A}}(x)dx\right\} \le 0.5\int_{a_1}^{a_{2k+3}}\mu_{\overline{A}}(x)dx$$

Now, we want to find the median of Exponential K-Trapezoidal-Triangular fuzzy number

Proposition 4.3.3: If $\overline{A} = (a_1,..,a_{2k+3}; w_1,..,w_k)$ is a fuzzy number with light tails then

$$m_{\overline{A}} = a_{k+2} \qquad (4.12)$$

Proof:

The fuzzy numbers is light tails then

$$\max\left\{\int_{a_i}^{a_{i+1}}\mu_{\overline{A}}(x)dx, \int_{a_{2k+3-i}}^{a_{2k+4-i}}\mu_{\overline{A}}(x)dx\right\} \le 0.5\int_{a_i}^{a_{2k+3-i}}\mu_{\overline{A}}(x)dx \qquad \text{i= 1,2,..,k+1 .}$$

The final condition

$$\max\left\{\int_{a_{k+1}}^{a_{k+2}}\mu_{\overline{A}}(x)dx, \int_{a_{k+2}}^{a_{k+3}}\mu_{\overline{A}}(x)dx\right\} \le 0.5\int_{a_{k+1}}^{a_{k+3}}\mu_{\overline{A}}(x)dx$$

So, the condition in remark 4.3.2 is satisfied and hence

$$m_{\overline{A}} = a_{k+2} \qquad \square$$

In **Theorem 4.2.4** we introduce the algorithm of ranking between Exponential K-Trapezoidal-Triangular fuzzy number .

Theorem 4.2.4: If $\overline{A} = (a_1,..,a_{2k+3}; w_1,..,w_k)_E$ is an Exponential K-Trapezoidal-Triangular fuzzy number, and the median value of them, So

1. $m_{\overline{A}} > m_{\overline{B}}$ then $\overline{A} > \overline{B}$

2. $m_{\overline{A}} < m_{\overline{B}}$ then $\overline{A} < \overline{B}$

3. $m_{\overline{A}} = m_{\overline{B}}$ then $\overline{A} \approx \overline{B}$

Example 4.3.5: Let $\overline{A} = (1,2,3.5,5,6.7;0.5)_E$ and $\overline{B} = (2,2.5,3,3.6,4;05)_E$
Then
$m_{\overline{A}} = 3.5$ and $m_{\overline{B}} = 3$
$m_{\overline{A}} > m_{\overline{B}}$ then $\overline{A} > \overline{B}$

Example 4.3.6: Let $\overline{A} = (1,2,2.3,4,5;0.5)$ and $\overline{B} = (4,4.5,6,6.5,6.7;0.5)$
Then
$m_{\overline{A}} = 2.3$ and $m_B = 6$
$m_{\overline{A}} < m_{\overline{B}}$ then $\overline{A} < \overline{B}$

Example 4.3.7:
Let $\overline{A} = (1,2,2.3,4,5,5.5,7;0.5,0.8)_E$ and $\overline{B} = (4,4.5,6,6.5,6.7,7,8;0.5,0.8)_E$
Then
$m_{\overline{A}} = 4$ and $m_{\overline{B}} = 6.5$
$m_{\overline{A}} < m_{\overline{B}}$ then $\overline{A} < \overline{B}$

Chapter 5
Ranking Between Exponential Particular Fuzzy Numbers Using Integral Value

The goal of this chapter is to find an exact formula of the integral value of exponential particular fuzzy number and then use it to rank between exponential particular fuzzy numbers.

5.1 Ranking of Exponential Trapezoidal Fuzzy Numbers

In this section, we introduce the ranking method using integral value [12] between exponential trapezoidal fuzzy numbers.

Definition5.1.1 [12]: Let $\overline{A} = (a_1, a_2, a_3, a_4; w)_E$ be a generalized Exponential Trapezoidal fuzzy number then

$$L(x) = we^{\frac{x-b}{b-a}} \quad , \quad R(x) = we^{\frac{c-x}{d-c}}$$

the left inverse function of L(x) is $L^{-1}(\alpha) = b + (b-a)\ln(\frac{\alpha}{w})$,

The right inverse function of L(x) is $R^{-1}(\alpha) = c - (d-c))\ln(\frac{\alpha}{w})$.

Definition 5.1.2 [12]: The Ranking function of fuzzy number $\overline{A}$ is

$$R(\overline{A}) = \frac{1}{2}\int_0^w (L^{-1}(\alpha) + R^{-1}(\alpha))d\alpha$$

Theorem 5.1.3 [11]: The ranking function of $\overline{A} = (a_1, a_2, a_3, a_4; w)_E$ is

$$R(\overline{A}) = \frac{w(a+d)}{2} \tag{5.1}$$

Proof:

From definition 5.1.2

$$R(\overline{A}) = \frac{1}{2}\int_0^w (L^{-1}(\alpha) + R^{-1}(\alpha))d\alpha$$

Since $\overline{A}$ is exponential trapezoidal fuzzy number we have

$$R(\overline{A}) = \frac{1}{2}\int_0^w (b + (b-a)\ln(\frac{\alpha}{w}) + c - (d-c)\ln(\frac{\alpha}{w}))d\alpha$$

Integrate the last formula by substitution and parts to produce equation (5.1). □

Definition 5.1.4 [12]: Let $\overline{A} = (a_1, a_2, a_3, a_4; w)_E$ and $\overline{B} = (b_1, b_2, b_3, d_4; r)_E$ be two Exponential Trapezoidal fuzzy number then the ranking function of $\overline{A}$ is

$R(\overline{A}) = \frac{c(a_1 + a_4)}{2}$ and $\overline{B}$ is $R(\overline{B}) = \frac{c(b_1 + b_4)}{2}$, where $c = \min\{w, r\}$.

Now, the algorithm to rank between Exponential Trapezoidal fuzzy number presents in theorem 5.1.5.

Theorem 5.1.5: If $\overline{A} = (a_1, a_2, a_3, a_4; w_1)_E$ and $\overline{B} = (b_1, b_2, b_3, b_4; w_2)_E$ be two Exponential Trapezoidal fuzzy number then

1.If $R(\overline{A}) > R(\overline{B})$ then $\overline{A} > \overline{B}$

2. If $R(\overline{A}) < R(\overline{B})$ then $\overline{A} < \overline{B}$

3. If $R(\overline{A}) = R(\overline{B})$ then $\overline{A} \approx \overline{B}$

5.2 Ranking of Exponential K-Trapezoidal-Triangular Fuzzy Numbers

In the current section, we find the ranking formula of K-Trapezoidal- Triangular.

Definition 5.2.1: Let $\overline{A} = (a_1, a_2, .., a_{2k+3}; w_1, .., w_k)_E$ be an Exponential K-Trapezoidal-Triangular fuzzy number then

$$L_1(x) = w_1 e^{\frac{x-a_2}{a_2-a_1}}, \quad R_1(x) = w_1 e^{\frac{a_{2k+2}-x}{a_{2k+3}-a_{2k+2}}},$$

$$L_i(x) = e^{\frac{(a_{i+1}-x)\ln w_{i-1} + (x-a_i)\ln w_i}{a_{i+1}-a_i}}, R_i(x) = e^{\frac{(x-a_{2k+4-i})\ln w_{i-1} + (a_{2k+3-i}-x)\ln w_i}{a_{2k+4-i}-a_{2k+3-i}}}$$

i=2,..,k

$$, L_{k+1}(x) = e^{\frac{(a_{k+2}-x)\ln w_k}{a_{k+2}-a_{k+1}}}, \quad R_{k+1}(x) = e^{\frac{(x-a_{k+2})\ln w_k}{a_{k+3}-a_{k+2}}}$$

and the left inverse function of $L_1(x)$ is $L_1^{-1}(\alpha) = a_2 + (a_2 - a_1)\ln(\frac{\alpha}{w_1})$, the right inverse function of $R_1(x)$ is $R_1^{-1}(\alpha) = a_{2k+2} - (a_{2k+3} - a_{2k+2}))\ln(\frac{\alpha}{w_1})$, the left inverse function of $L_i(x)$ is

$L_i^{-1}(\alpha)=\frac{(a_{i+1}-a_i)\ln\alpha-a_{i+1}\ln w_{i-1}+a_i\ln w_i}{\ln w_i-\ln w_{i-1}}$, the right inverse function of $R_i(x)$ is $R_i^{-1}(\alpha)=\frac{(a_{2k+3-i}-a_{2k+4-i})\ln\alpha-a_{2k+4-i}\ln w_{i-1}+a_{2k+3-i}\ln w_i}{\ln w_i-\ln w_{i-1}}$, the left inverse function of $L_{k+1}(x)$ is $L_{k+1}^{-1}(\alpha)=a_{k+2}-\frac{(a_{k+2}-a_{k+1})\ln(\alpha)}{\ln w_k}$, the right inverse function of $R_{k+1}(x)$ is $R_{k+1}^{-1}(\alpha)=a_{k+2}+\frac{(a_{k+3}-a_{k+2}))\ln(\alpha)}{\ln w_k}$

Definition 5.2.2: The ranking function of particular fuzzy number $\overline{A}$ is

$$R(\overline{A})=\frac{1}{2}(\int_0^{w_1}(L_1^{-1}(\alpha)+R_1^{-1}(\alpha))+\sum_{i=2}^{k}\int_{w_{i-1}}^{w_i}(L_i^{-1}(\alpha)+R_i^{-1}(\alpha))d\alpha$$
$$+\int_{w_k}^{1}(L_{k+1}^{-1}(\alpha)+R_{k+1}^{-1}(\alpha))d\alpha)$$

In theorem 5.2.3, we find the exact ranking function of exponential K-Trapezoidal-Triangular fuzzy number.

Theorem5.2.3: The ranking function exponential K-Trapezoidal- Triangular fuzzy number $\overline{A}=(a_1,a_2,\ldots,a_{2k+3};w_1,w_2,\ldots,w_k)_E$ is

$$R(\overline{A})=\frac{w_1(a_1+a_{2k+3})}{2}+a_{k+2}(1-w_k)+\frac{(a_{k+3}-2a_{k+2}+a_{k+1})(w_k(1-\ln w_k)-1)}{2\ln w_k}$$
$$+0.5\sum_{i=2}^{k}((a_{i+1}-a_i+a_{2k+3-i}-a_{2k+4-i})(w_i\ln w_i-w_i-w_{i-1}\ln w_{i-1}+w_{i-1})$$
$$-(a_{i+1}+a_{2k+4-i})(w_i-w_{i-1})\ln w_{i-1}+(a_i+a_{2k+3-i})(w_i-w_{i-1})\ln w_i)$$
(5.2)

Proof:

From definition 5.1.2, we have

$$R(\overline{A})=0.5\int_0^{w_1}(a_2+(a_2-a_1)\ln(\frac{\alpha}{w_1})+a_{2k+2}-(a_{2k+3}-a_{2k+2})\ln(\frac{\alpha}{w_1}))d\alpha$$
$$+0.5\sum_{i=2}^{k}(\int_{w_{i-1}}^{w_i}\frac{(a_{i+1}-a_i+a_{2k+3-i}-a_{2k+4-i})\ln\alpha-(a_{i+1}+a_{2k+4-i})\ln w_{i-1}+(a_i+a_{2k+3-i})\ln w_i}{\ln w_i-\ln w_{i-1}}$$
$$+0.5\int_{w_k}^{1}(2a_{k+2}+\frac{(a_{k+3}-2a_{k+2}+a_{k+1})\ln\alpha}{\ln w_k}d\alpha)$$

Integrate the last formula by substitution and parts we produce equation (5.2). □

Remark5.2.4: Let $\overline{A}=(a_1,a_2,..,a_{2k+3};w_1,w_2,..,w_k)_E$ and $\overline{B}=(b_1,b_2,..,b_{2k+3};r_1,r_2,..,r_k)_E$ be two Exponential K-Trapezoidal-Triangular fuzzy numbers then

$$R(\overline{A})=\frac{m_1(a_1+a_{2k+3})}{2}+a_{k+2}(1-m_k)+\frac{(a_{k+3}-2a_{k+2}+a_{k+1})(m_k(1-\ln m_k)-1)}{2\ln m_k}$$
$$+0.5\sum_{i=2}^{k}((a_{i+1}-a_i+a_{2k+3-i}-a_{2k+4-i})(m_i\ln m_i-m_i-m_{i-1}\ln m_{i-1}+m_{i-1})$$
$$-(a_{i+1}+a_{2k+4-i})(m_i-m_{i-1})\ln m_{i-1}+(a_i+a_{2k+3-i})(m_i-m_{i-1})\ln m_i)$$

and

$$R(\overline{B})=\frac{m_1(b_1+b_{2k+3})}{2}+b_{k+2}(1-m_k)+\frac{(b_{k+3}-2b_{k+2}+b_{k+1})(m_k(1-\ln m_k)-1)}{2\ln m_k}$$
$$+0.5\sum_{i=2}^{k}((b_{i+1}-b_i+b_{2k+3-i}-b_{2k+4-i})(m_i\ln m_i-m_i-m_{i-1}\ln m_{i-1}+m_{i-1})$$
$$-(b_{i+1}+b_{2k+4-i})(m_i-m_{i-1})\ln m_{i-1}+(b_i+b_{2k+3-i})(m_i-m_{i-1})\ln m_i)$$

where $m_1=\min\{w_1,r_1\}$, $m_i=\min\{w_i-w_{i-1},r_i-r_{i-1}\}$ for i=2,...,k .

The algorithm of ranking presented in theorem 5.1.4 is also applicable in ranking between K-Trapezoidal- Triangular fuzzy numbers.

Example 5.2.5: Let $\overline{A}=(1,2,3.5,5,6.7;0.5)$ and $\overline{B}=(2,2.5,3,3.6,4;0.5)$.

Then

$$R(\overline{A})=\frac{m_1(a_1+a_5)}{2}+a_3(1-m_1)+\frac{(a_4-2a_3+a_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{A})=\frac{0.5(1+6.7)}{2}+3.5(1-0.5)+\frac{(5-2(3.5)+2)(0.5(1-\ln 0.5)-1)}{2\ln 0.5}$$

$$R(\overline{A})=0.5$$

$$R(\overline{B})=\frac{m_1(b_1+b_5)}{2}+b_3(1-m_1)+\frac{(b_4-2b_3+b_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{B}) = \frac{0.5(2+4)}{2} + 3(1-0.5) + \frac{(3.6-2(3)+2.5)(0.5(1-\ln 0.5)-1)}{2\ln 0.5}$$

$R(\overline{B}) = 3.011$

$\boldsymbol{R(\overline{A}) > R(\overline{B})}$ then $\overline{A} > \overline{B}$

Example 5.2.6: Let $\overline{A} = (1,2,2.3,4,5;0.5)$ and $\overline{B} = (4,4.5,6,6.5,6.7;0.5)$.

Then

$$R(\overline{A}) = \frac{m_1(a_1+a_5)}{2} + a_3(1-m_1) + \frac{(a_4-2a_3+a_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{A}) = \frac{0.5(1+5)}{2} + 2.3(1-0.5) + \frac{(4-2(2.3)+2)(0.5(1-\ln 0.5)-1)}{2\ln 0.5}$$

$R(\overline{A}) = 2.085$

$$R(\overline{B}) = \frac{m_1(b_1+b_5)}{2} + b_3(1-m_1) + \frac{(b_4-2b_3+b_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{B}) = \frac{0.5(4+6.7)}{2} + 6(1-0.5) + \frac{(6.5-2(6)+4.5)(0.5(1-\ln 0.5)-1)}{2\ln 0.5}$$

$R(\overline{B}) = 5.564$

$\boldsymbol{R(\overline{A}) < R(\overline{B})}$ then $\overline{A} < \overline{B}$

Example 5.2.7: Let $\overline{A} = (1,2,2.3,4,5,5.5,7;0.5,0.8)$ and $\overline{B} = (4,4.5,6,6.5,6.7,7,8;0.5,0.8)$

Then

$$R(\overline{A}) = \frac{m_1(a_1 + a_7)}{2} + a_4(1 - m_2) + \frac{(a_5 - 2a_4 + a_3)(m_2(1 - \ln m_2) - 1)}{2\ln m_2}$$
$$+ 0.5(a_3 - a_2 + a_5 - a_6)(m_2 \ln m_2 - m_2 - m_1 \ln m_1 + m_1)$$
$$-(a_3 + a_6)(m_2 - m_1)\ln m_1 + (a_2 + a_5)(m_2 - m_1)\ln m_2$$

$$R(\overline{A}) = \frac{0.5(1+7)}{2} + 4(1-0.3) + \frac{(5-2(4)+2.3)(0.3(1-\ln 0.3)-1)}{2\ln 0.3}$$
$$+0.5(2.3-2+5-5.5)(0.3\ln 0.3 - 0.3 - 0.5\ln 0.5 + 0.5)$$
$$-(2.3+5.5)(0.3-0.5)\ln 0.5 + (2+5)(0.3-0.5)\ln 0.5$$

$$R(\overline{A}) = 3.0148$$

$$R(\overline{B}) = \frac{m_1(b_1 + b_7)}{2} + b_4(1 - m_2) + \frac{(b_5 - 2b_4 + b_3)(m_2(1 - \ln m_2) - 1)}{2\ln m_2}$$
$$+ 0.5(b_3 - b_2 + b_5 - b_6)(m_2 \ln m_2 - m_2 - m_1 \ln m_1 + m_1)$$
$$-(b_3 + b_6)(m_2 - m_1)\ln m_1 + (b_2 + b_5)(m_2 - m_1)\ln m_2$$

$$R(\overline{B}) = \frac{0.2(2+5.7)}{2} + 3.6(1-0.3) + \frac{(4-2(3.6)+3.1)(0.3(1-\ln 0.3)-1)}{2\ln 0.6}$$
$$+0.5(3.1-2.5+4-5)(0.3\ln 0.3 - 0.3 - 0.5\ln 0.5 + 0.5)$$
$$-(3.1+5)(0.3-0.5)\ln 0.2 + (2.5+4)(0.3-0.5)\ln 0.6$$

$$R(\overline{B}) = 6.186$$

$\boldsymbol{R(\overline{A}) < R(\overline{B})}$ then $\overline{A} < \overline{B}$

5.3 Ranking of Exponential $K+1$-Trapezoidal Fuzzy Numbers

In this section, we find the Ranking function using integral value of Exponential $K+1$-Trapezoidal fuzzy and then use it to rank between Exponential $K+1$-Trapezoidal fuzzy.

Definition 5.3.1: Let $\overline{A} = (a_1, a_2, .., a_{2k+4}; w_1, .., w_k)_E$ be a generalized Exponential $K+1$-Trapezoidal fuzzy number then

$L_1(x) = w_1 e^{\frac{x-a_2}{a_2-a_1}}$, $R_1(x) = w_1 e^{\frac{a_{2k+3}-x}{a_{2k+4}-a_{2k+3}}}$,

$L_i(x) = e^{\frac{(a_{i+1}-x)\ln w_{i-1} + (x-a_i)\ln w_i}{a_{i+1}-a_i}}$

, $R_i(x) = e^{\frac{(x-a_{2k+5-i})\ln w_{i-1} + (a_{2k+4-i}-x)\ln w_i}{a_{2k+5-i}-a_{2k+4-i}}}$ i=2,..,k

, $L_{k+1}(x) = e^{\frac{(a_{k+2}-x)\ln w_k}{a_{k+2}-a_{k+1}}}$, $R_{k+1}(x) = e^{\frac{(x-a_{k+3})\ln w_k}{a_{k+4}-a_{k+3}}}$, the left inverse function of $L_1(x)$ is $L_1^{-1}(\alpha) = a_2 + (a_2 - a_1)\ln(\frac{\alpha}{w_1})$, the right inverse function of $R_1(x)$ is $R_1^{-1}(\alpha) = a_{2k+3} - (a_{2k+4} - a_{2k+3}))\ln(\frac{\alpha}{w_1})$, the left inverse function of $L_i(x)$ is $L_i^{-1}(\alpha) = \frac{(a_{i+1} - a_i)\ln\alpha - a_{i+1}\ln w_{i-1} + a_i \ln w_i}{\ln w_i - \ln w_{i-1}}$, the right inverse function of $R_i(x)$ is $R_i^{-1}(\alpha) = \frac{(a_{2k+5-i} - a_{2k+4-i})\ln\alpha - a_{2k+5-i}\ln w_{i-1} + a_{2k+4-i}\ln w_i}{\ln w_i - \ln w_{i-1}}$, the left inverse function of $L_{k+1}(x)$ is $L_{k+1}^{-1}(\alpha) = a_{k+2} - \frac{(a_{k+2} - a_{k+1})\ln(\alpha)}{\ln w_k}$, the right inverse function of $R_{k+1}(x)$ is $R_{k+1}^{-1}(\alpha) = a_{k+3} + \frac{(a_{k+4} - a_{k+3}))\ln(\alpha)}{\ln w_k}$.

In theorem 5.3.2, we will find the exact ranking function of Exponential $K+1$-Trapezoidal fuzzy number.

Theorem 5.3.2: The ranking function of $\overline{A} = (a_1, a_2, .., a_{2k+4}; w_1, .., w_k)_E$ is

$$\begin{aligned}
R(\overline{A}) &= \frac{w_1(a_1 + a_{2k+4})}{2} + \frac{(a_{k+2} + a_{k+3})(1 - w_k)}{2} \\
&+ \frac{(a_{k+4} - a_{k+3} - a_{k+2} + a_{k+1})(w_k(1 - \ln w_k) - 1)}{2\ln w_k} \\
&+ 0.5\sum_{i=2}^{k}((a_{i+1} - a_i + a_{2k+4-i} - a_{2k+5-i})(w_i \ln w_i - w_i - w_{i-1}\ln w_{i-1} + w_{i-1}) \\
&- (a_{i+1} + a_{2k+5-i})(w_i - w_{i-1})\ln w_{i-1} + (a_i + a_{2k+4-i})(w_i - w_{i-1})\ln w_i)
\end{aligned} \tag{5.3}$$

Proof:
From definition 5.2.2, we have

$$R(\overline{A}) = 0.5\int_0^{w_1} (a_2 + (a_2 - a_1)\ln(\frac{\alpha}{w_1}) + a_{2k+3} - (a_{2k+4} - a_{2k+3})\ln(\frac{\alpha}{w_1}))d\alpha$$
$$+ 0.5\sum_{i=2}^{k}(\int_{w_{i-1}}^{w_i} \frac{(a_{i+1} - a_i + a_{2k+4-i} - a_{2k+5-i})\ln\alpha - (a_{i+1} + a_{2k+5-i})\ln w_{i-1} + (a_i + a_{2k+4-i})\ln w_i}{\ln w_i - \ln w_{i-1}}$$
$$+ 0.5\int_{w_k}^{1} (a_{k+2} + a_{k+3} + \frac{(a_{k+4} - a_{k+3} - a_{k+2} + a_{k+1})\ln\alpha}{\ln w_k} d\alpha))$$

Integrate the last formula by substation and parts to produce equation (5.3). □

Remark 5.3.2: Let $\overline{A} = (a_1, a_2, .., a_{2k+4}; w_1, w_2, .., w_k)_E$ and $\overline{B} = (b_1, b_2, .., b_{2k+4}; r_1, r_2, .., r_k)_E$ be two Exponential K+1-Trapezoidal fuzzy numbers then

$$R(\overline{A}) = \frac{m_1(a_1 + a_{2k+4})}{2} + \frac{(a_{k+2} + a_{k+3})(1 - m_k)}{2}$$
$$+ \frac{(a_{k+4} - a_{k+3} - a_{k+2} + a_{k+1})(m_k(1 - \ln m_k) - 1)}{2\ln m_k}$$
$$+ 0.5\sum_{i=2}^{k}((a_{i+1} - a_i + a_{2k+4-i} - a_{2k+5-i})(m_i \ln m_i - m_i - m_{i-1}\ln m_{i-1} + m_{i-1})$$
$$-(a_{i+1} + a_{2k+5-i})(m_i - m_{i-1})\ln m_{i-1} + (a_i + a_{2k+4-i})(m_i - m_{i-1})\ln m_i)$$

and

$$R(\overline{B}) = \frac{m_1(b_1 + b_{2k+4})}{2} + \frac{(b_{k+2} + b_{k+3})(1 - m_k)}{2}$$
$$+ \frac{(b_{k+4} - b_{k+3} - b_{k+2} + b_{k+1})(m_k(1 - \ln m_k) - 1)}{2\ln m_k}$$
$$+ 0.5\sum_{i=2}^{k}((b_{i+1} - b_i + b_{2k+4-i} - b_{2k+5-i})(m_i \ln m_i - m_i - m_{i-1}\ln m_{i-1} + m_{i-1})$$
$$-(b_{i+1} + b_{2k+5-i})(m_i - m_{i-1})\ln m_{i-1} + (b_i + b_{2k+4-i})(m_i - m_{i-1})\ln m_i)$$

where $m_1 = \min\{w_1, r_1\}$, $m_i = \min\{w_i - w_{i-1}, r_i - r_{i-1}\}$ for i=,2,...,k .

The algorithm of ranking between Exponential $K+1$-Trapezoidal fuzzy number is similar to algorithm of ranking between Exponential Trapezoidal fuzzy numbers.

Example5.3.4:

Let $\overline{A} = (0.2,0.3,0.4,0.6,0.7,0.8;0.35)_E$ and $\overline{B} = (0.1,0.16,0.2,0.3,0.35,0.4;0.35)_E$.

Then

$$R(\overline{A})=\frac{m_1(a_1+a_6)}{2}+\frac{(a_3+a_4)(1-m_1)}{2}+\frac{(a_5-a_4-a_3+a_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{A})=\frac{0.35(0.2+0.8)}{2}+\frac{(0.4+0.6)(1-0.35)}{2}+\frac{(0.7-0.6-0.4+0.3)(0.35(1-\ln 0.35)-1)}{2\ln 0.35}$$

$R(\overline{A})=0.5$

$$R(\overline{B})=\frac{m_1(b_1+b_6)}{2}+\frac{(b_3+b_4)(1-m_1)}{2}+\frac{(b_5-b_4-b_3+b_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$R(\overline{B})=0.248$

$R(\overline{A})>R(\overline{B})$ then $\overline{A}>\overline{B}$

Example 5.3.5: Let $\overline{A}=(2,3,5,5.3,5.5,6;0.2)_E$ and $\overline{B}=(1,1.5,1.7,1.9,2,2.5;0.2)_E$.

Then

$$R(\overline{A})=\frac{m_1(a_1+a_6)}{2}+\frac{(a_3+a_4)(1-m_1)}{2}+\frac{(a_5-a_4-a_3+a_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{A})=\frac{0.2(2+6)}{2}+\frac{(5+5.3)(1-0.2)}{2}+\frac{(5.5-5.3-5+3)(0.2(1-\ln 0.2)-1)}{2\ln 0.2}$$

$R(\overline{A})=4..653$

$$R(\overline{B})=\frac{m_1(b_1+b_6)}{2}+\frac{(b_3+b_4)(1-m_1)}{2}+\frac{(b_5-b_4-b_3+b_2)(m_1(1-\ln m_1)-1)}{2\ln m_1}$$

$$R(\overline{B})=\frac{0.2(1+2.5)}{2}+\frac{(1.7+1.9)(1-0.2)}{2}+\frac{(2-1.9-1.7+1.5)(0.2(1-\ln 0.2)-1)}{2\ln 0.2}$$

$R(\overline{B})=1.775$

$R(\overline{A}) > R(\overline{B})$ then $\overline{A} > \overline{B}$

Example5.3.6:

Let $\overline{A} = (2,3,4,4.3,5,6,7,8;0.1,0.5)_E$ and $\overline{B} = (6,7,7.5,7.9,8,9,9.3,10;0.1,0.5)_E$

Then

$$R(\overline{A}) = \frac{m_1(a_1 + a_8)}{2} + \frac{(a_4 + a_5)(1 - m_2)}{2} + \frac{(a_6 - a_5 - a_4 + a_3)(m_2(1 - \ln m_2) - 1)}{2\ln m_2}$$
$$+ 0.5(a_3 - a_2 + a_6 - a_7)(m_2 \ln m_2 - m_2 - m_1 \ln m_1 + m_1)$$
$$-(a_3 + a_7)(m_2 - m_1)\ln m_1 + (a_2 + a_6)(m_2 - m_1)\ln m_2$$

$$R(\overline{A}) = \frac{0.1(2+8)}{2} + \frac{(4.3+5)(1-0.4)}{2} + \frac{(6-5-4.3+4)(0.4(1-\ln 0.4)-1)}{2\ln 0.5}$$
$$+ 0.5(4-3+6-7)(0.4\ln 0.4 - 0.4 - 0.1\ln 0.1 + 0.1)$$
$$-(4+7)(0.4-0.1)\ln 0.1 + (3+6)(0.4-0.1)\ln 0.4$$

$R(\overline{A}) = 10.539$

$$R(\overline{B}) = \frac{m_1(b_1 + b_8)}{2} + \frac{(b_4 + b_5)(1 - m_2)}{2} + \frac{(b_6 - b_5 - b_4 + b_3)(m_2(1 - \ln m_2) - 1)}{2\ln m_2}$$
$$+ 0.5(b_3 - b_2 + b_6 - b_7)(m_2 \ln m_2 - m_2 - m_1 \ln m_1 + m_1)$$
$$-(b_3 + b_7)(m_2 - m_1)\ln m_1 + (b_2 + b_6)(m_2 - m_1)\ln m_2$$

$$R(\overline{B}) = \frac{0.1(6+10)}{2} + \frac{(7.9+8)(1-0.4)}{2} + \frac{(9-8-7.9+7.5)(0.4(1-\ln 0.4)-1)}{2\ln 0.5}$$
$$+ 0.5(7.5-7+9-9.3)(0.4\ln 0.4 - 0.4 - 0.1\ln 0.1 + 0.1)$$
$$-(7.5+9.3)(0.4-0.1)\ln 0.1 + (7+9)(0.4-0.1)\ln 0.4$$

$R(\overline{B}) = 15.827$

$R(\overline{A}) < R(\overline{B})$ then $\overline{A} < \overline{B}$

Chapter six

Conclusion

In this work, we generalized the four methods of ranking between Exponential Trapezoidal fuzzy number (cardinality, TRD distance, median value and integral value) to rank Exponential Particular fuzzy numbers.

In this chapter, we choose the approximate ranking results obtained from different approaches in the previous chapters and applied approaches on the same example. The Ranking are given in the following two tables.

Table1: shows the results of ranking Exponential K-Trapezoidal-Triangular fuzzy number for different methods.

Table1

Methods			
Cardinality	Ex 2.2.3: $\overline{A} > \overline{B}$	Ex 2.2.4: $\overline{A} > \overline{B}$	Ex 2.2.5: $\overline{A} > \overline{B}$
TRD	Ex 3.2.6: $\overline{A} < \overline{B}$	Ex 3.2.7: $\overline{A} < \overline{B}$	Ex 3.2.8: $\overline{A} < \overline{B}$
Median	Ex 4.2.5: $\overline{A} > \overline{B}$	Ex 4.2.6: $\overline{A} < \overline{B}$	Ex 4.2.7: $\overline{A} < \overline{B}$
Integral value	Ex 5.2.5: $\overline{A} > \overline{B}$	Ex 5.2.6: $\overline{A} < \overline{B}$	Ex 5.2.7: $\overline{A} < \overline{B}$

It's clear from the table above that, three methods out of the four methods that have been discussed agree while the cardinality opposed them. So, as in the literature, we can consider the results of ranking that agree with three methods.

Table 2: shows the results of ranking Exponential $K+1$-Trapezoidal fuzzy number for different methods. We can take the results of ranking that agree with the three methods**.**

Table 2

Methods			
Cardinality	Ex 2.3.3: $\overline{A} > \overline{B}$	Ex 2.3.4: $\overline{A} > \overline{B}$	Ex 2.3.5: $\overline{A} > \overline{B}$
TRD	Ex 3.3.3: $\overline{A} < \overline{B}$	Ex 3.3.4: $\overline{A} > \overline{B}$	Ex 3.3.5: $\overline{A} < \overline{B}$
Median	Ex 4.3.5: $\overline{A} < \overline{B}$	Ex 4.3.6: $\overline{A} > \overline{B}$	Ex 4.3.7: $\overline{A} < \overline{B}$
Integral value	Ex 5.3.4: $\overline{A} > \overline{B}$	Ex 5.3.5: $\overline{A} > \overline{B}$	Ex 5.3.6: $\overline{A} < \overline{B}$

The results of Ranking Exponential Particular fuzzy numbers between different methods is not the same, since as mentioned in the introduction fuzzy logic and fuzzy set came to recognize the approximation of data, Can be get for decision making, then are imprecision. Since a fuzzy number is defined as a fuzzy subset, then fuzzy number is imprecision quantity. When fuzzy numbers overlap with each other then it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other.

The methods of ranking between exponential particular fuzzy numbers which are defined in this work can be applied in many applications of real life such as artificial intelligence , decision making , clustering , optimization , transportation problems to devolve these applications keep up with the big growth in life .

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