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A Simulation Study of Power Comparison of Goodness-of-Fit Tests

by

Yousef Jawad Atatrah

Supervisor : Dr. Elias Dabeet

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Committee members	Signature
1. Dr. Elias Dabeet (Supervisor)	....
2. Dr. Mahmoud Almanassra (Internal Examiner)	....
3. Dr. Khalid Salah (External Examiner)	....

Al-Quds University

Dedication

I dedicate this thesis to my family and my friend Hayel. A special feeling of gratitude to my lovely parents, Hind and Jawad, my aunt, Suhaila, my uncles, my brothers and my sisters.

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All Praise to Allah for His blessing until I managed to complete this thesis successfully.

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Contents

Abstract

The main purpose of this thesis is to study and compare the power of five goodness-of-fit(GOF) tests: Chi-square(χ^2) test, Kolmogorov-Smirnov(KS) test, Cramér-von Mises (CVM) test, Anderson-Darling(AD) test and Bickel-Rosenblatt(BR) test under several parametric and non-parametric alternatives. Power comparisons of these five tests were obtained by using Monte Carlo simulation method of sample data generated from parametric and non-parametric alternatives and the parametric alternatives follow symmetric and non-symmetric distributions, R software was used to generate data for simulations purpose. Two significance levels 5% and 10% were used and the critical values for power comparisons were obtained based on 10000 simulated samples from different null distributions. 10000 samples each of size n = 10, 20, 30, 40, 50, 100, 200, 300, 400, 500, 1000, and 2000 were generated from each of the given alternatives. The power of each test was then obtained by comparing the GOF test statistics with the respective critical values. Simulation results show that the AD test has a higher power in the case of testing symmetric distributions and the data were generated from parametric alternative distributions followed by the CVM and the KS tests while the χ^2 test has the lowest power. The BR test has a higher power in the case of testing symmetric distributions and the data were generated from some non-parametric alternative distributions and the AD test has a higher power under other non-parametric alternative distributions. This study also shows that the BR test has a higher power when using the Epanechnikov kernel compared to the uniform kernel.

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List of Abbreviations

PDF	Probability density function
CDF	Cumulative distribution function
EDF	Empirical distribution function
KDE	Kernel density estimation
MSE	Mean squared error
MISE	Mean integrated squared error
AMISE	Asymptotic mean integrated squared error
GOF	Goodness-of-fit
χ^2	Chi-square test
KS	Kolmogorov-Smirnov test
AD	Anderson-Darling test
CVM	Cramér–von Mises test
BR	Bickel-Rosenblatt test

Chapter 1

Introduction and Kernel Density Estimation

1.1 Introduction

There are significant amount of Goodness-of-fit(GOF) tests available in the literature. Some of these tests can only be applied under certain conditions or assumptions. The main purpose of this thesis is to study and compare the power of GOF tests: Chi-square test, Kolmogorov-Smirnov (KS) test, Cramér-von Mises (CVM) test, Anderson-Darling (AD) test and the Bickel-Rosenblatt (BR) test.

Chapter 1 provides the background of the study and briefly discusses the histogram and kernel density estimation as a method for estimating the probability density function. The basic properties of the kernel density estimator will be presented in this chapter.

In Chapter 2, the following five Goodness-of-fit tests will be defined: Chi-square test, Kolmogorov-Smirnov (KS) test, Cramér-von Mises (CVM) test, Anderson-Darling (AD) test and the Bickel-Rosenblatt (BR) test. In addition, the asymptotic null distribution and also the asymptotic power for each of these tests are presented in this chapter.

In Chapter 3, we discuss in details the Monte Carlo simulation methodology for power comparisons of the above mentioned Goodness-of-fit tests. The algorithms involved in the simulation study will be described. This chapter includes all the simulation results. Some analyses to rank the power of the tests and graphs of the power of tests are also conducted and presented in this chapter. In addition, discussion of the results and a conclusion based on the findings obtained are also discussed in this chapter.

1.2 Literature Review

There are significant amount of Goodness-of-fit(GOF) tests available in the literature. The effort of developing techniques to detect departures from normality has begun as early as the late 19th century. This effort was initiated by Pearson [1] who worked on the skewness and kurtosis coefficients, Althouse et al, [2]. In 1900, Pearson extended his work and introduced the Chi-square test of normality. Kolmogorov and Smirnov then introduced the Kolmogorov-Smirnov test in 1933. Conover [3] stated that the Cramer-von Mises test was developed based on the contributions by Cramér in [4] and von Mises [5]. In 1954, Anderson and Darling proposed their test which was a modification of the Cramér-von Mises test, Farrel and Stewart [6], Anderson-Darling test was an improvement of Kolmogorov-Smirnov tests. Fix and Hodges [7] introduced the basic algorithm of nonparametric density estimation. During the following decade, several general algorithms and alternative theoretical modes of analysis were introduced by Rosenblatt [8], Parzen [9] and Cencov [10]. These were followed by the most important theoretical papers by Watson and Leadbetter [11], Loftsgaarden and Quesenberry [12], Schwartz [13], Epanechnikov [14], Tarter and Kronmal [15] and Wahba [16]. In 1973 and 1975 Bickel and Rosenblatt introduced their test which was called Bickel-Rosenblatt test and this test depends on the kernel density estimator that was introduced by Rosenbaltt [8] and Parzen [9].

1.3 Histogram

A histogram is a graphical representation of the distribution of the data. It is an estimate of the probability distribution of a continuous variable and was first introduced by Karl Pearson in 1895. A histogram is considered as the simplest form for estimating the probability density function.

A histogram can be thought of as a special case of a kernel density estimation which will be defined in section 1.4, which uses a kernel to smooth frequencies over the bins. This will yield a smoother probability density function, which will in general more accurately reflect the distribution of the underlying variable. The density estimate could be plotted as an alternative to the histogram, and is usually drawn as a curve rather than a set of boxes [17, 18]. In the literature, there are several formulas for the number of bins. A few of them are listed below.

Number of bins and width

The number of bins k can be assigned directly or can be calculated from a suggested bin width h as:

$$k = \left\lceil \frac{\max x - \min x}{h} \right\rceil \tag{1.1}$$

Where x is the data set and the braces indicate the ceiling function.

Square-root choice

$$k = \sqrt{n} \tag{1.2}$$

Which takes the square root of the number of data points in the sample.

Sturges' formula

Sturges' formulas derived from a binomial distribution and implicitly assumes an approximately normal distribution.

$$k = \lceil \log_2(n+1) \rceil \tag{1.3}$$

It implicitly bases the bin sizes on the range of the data and can perform poorly if n <30, because the number of bins will be small-less than seven-and unlikely to show trends in the data well. It may also perform poorly if the data are not normally distributed [19].

Rice Rule

$$k = \lceil 2n^{1/3} \rceil \tag{1.4}$$

The Rice Rule is presented as a simple alternative to Sturges's rule.

Scott's normal reference rule

$$h = \frac{3.5\hat{\sigma}}{n^{1/3}} \tag{1.5}$$

Where $\hat{\sigma}$ is the sample standard deviation. Scott's normal reference rule is optimal for random samples of normally distributed data, in the sense that it minimizes the integrate mean squared error of the density estimate.

1.4 Kernel Density Estimation(KDE)

Kernel density estimation (KDE) is a non-parametric method to estimate the probability density function of a random variable. Kernel density estimation is a fundamental data smoothing problem where inferences about the population are made, based on a finite data sample. It has some applications in fields such as signal processing and econometrics. It is also termed the Parzen-Rosenblatt window method, after Emanuel Parzen and Murray Rosenblatt [8,9] who are usually credited with independently creating it in its current form.

Definition of Kernel Density Estimator(KDE)

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a specified continuous probability density function $f(x)$. Rosenblatt's(1956) and Parzen's (1962) introduced the kernel density estimate $f_n(x)$ for estimating $f(x)$ at a fixed point $x \in \mathbb{R}$ using the data $(X_1, X_2, \ldots, X_n)$, as follows:

$$f_n(x) = \frac{1}{n}\sum_{i=1}^{n} K_{b_n}(x - X_i) = \frac{1}{nb_n}\sum_{i=1}^{n} K\left[\frac{x-X_i}{b_n}\right]$$

$$= \frac{1}{b_n}\int K\left[\frac{x-X_i}{b_n}\right] dF_n(t) \qquad (1.6)$$

Where $K_{b_n}(x) = 1/b_n\, K(x/b_n)$ and $F_n(t)$ is a sample distribution function and K is a suitable kernel function on ℝ such that:

1)$\int_{-\infty}^{\infty} K(u)du = 1$

2) $K(u) = K(-u)$ for all values of u

And $b_n > 0$ is a smoothing parameter called the bandwidth such that $b_n \to 0$ and

$nb_n \to \infty \;\; as\, n \to \infty$

The most important choice is the bandwidth $b_n > 0$ which controls the amount of smoothing.

Practical estimation of the bandwidth

If Gaussian basis functions are used to approximate univariate data, and the underlying density being estimated is Gaussian, then it can be shown that the optimal choice for b_n is

$$b_n = \left(\frac{4\hat{\sigma}^5}{3n}\right)^{1/5} \approx 1.06\, \hat{\sigma}\, n^{-1/5} \qquad (1.7)$$

Where $\hat{\sigma}$ is the standard deviation of the samples. This approximation is termed the normal distribution approximation, Gaussian approximation [20].

1.4.1 Examples of Kernel Functions

Several kernel functions were commonly used in the literature, among them are:

Uniform	$K(u) = \frac{1}{2} \;\; , -1 \le u \le 1$
Triangular	$K(u) = 1 - \lvert u \rvert \;\; , -1 \le u \le 1$
Epanechnikov	$K(u) = \frac{3}{4}\,(1 - u^2) \;\; , -1 \le u \le 1$

Biweight $K(u) = \frac{15}{16}\,(1-u^2)^2 \;\; , -1 \le u \le 1$

Triweight $K(u) = \frac{35}{32}\,(1-u^2)^3 \;\; , -1 \le u \le 1$

Tricube $K(u) = \frac{70}{81}\,(1-|u|^3)^3 \;\; , -1 \le u \le 1$

Gaussian $K(u) = \frac{1}{\sqrt{2\pi}}\,exp(-\frac{1}{2}u^2) \;\; , -1 \le u \le 1$

Cosine $K(u) = \frac{\pi}{4}\,cos(\frac{\pi}{2}u) \;\; , -1 \le u \le 1$

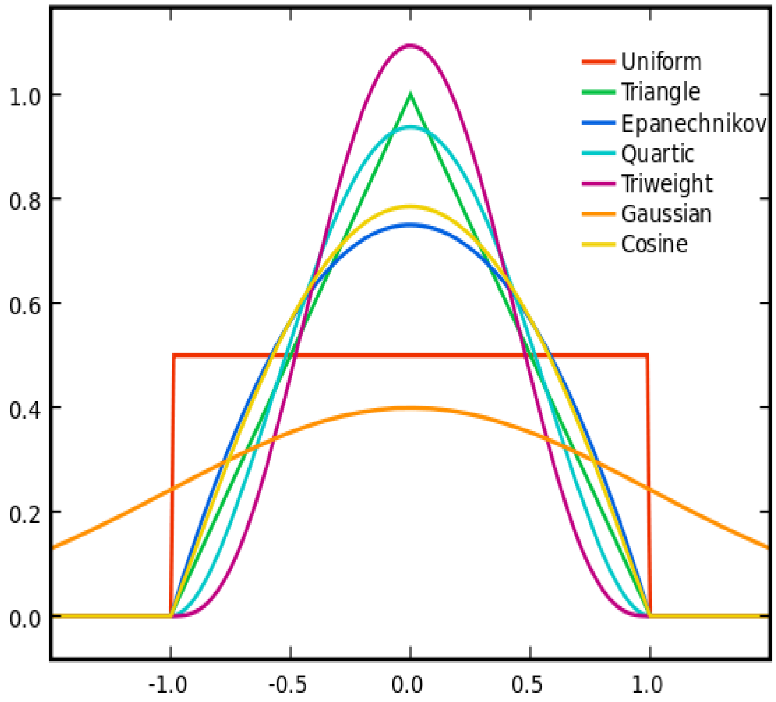


Figure 1.1: Some of the kernels mentioned above in a common coordinate system.

1.4.2 properties of the kernel density estimator

There is a vast amount of literature on the general properties of the (KDE), which can be summarized as follows:

1.4.2.1 Bias

For the bias we have

$$Bias\,\{f_n(x)\} = E\{f_n(x)\} - f(x)$$

But,

$$E\{f_n(x)\} = \frac{1}{nb_n}\sum_{i=1}^{n} E\{K(\frac{x-X_i}{nb_n})\} = \frac{1}{b_n}E\{K(\frac{x-X}{b_n})\} \quad (1.8)$$

$$E\{f_n(x)\} = \frac{1}{b_n}\int K(\frac{x-u}{b_n})\, f(u)du$$

Using the variable $y = \frac{x-u}{b_n}$ and the symmetry of the kernel, i.e. $K(-s) = K(s)$, we get

$$E\{f_n(x)\} = \int K(y)\, f(x - yb_n)dy \quad (1.9)$$

Now, assume that the second derivative $f''$of the underlying density f is absolutely continuous and square integrable. Then, expanding $f(x - yb_n)$ in a Taylor series about x we have

$$f(x - yb_n) = f(x) - b_n y f'(x) + \frac{1}{2}{b_n}^2 y^2 f''(x) + o({b_n}^2)$$

Where, $o({b_n}^2)$ is a little-o notation of order 2.

Now, $$E\{f_n(x)\} = \int K(y)\{f(x) - b_n\, y f'(x) + \frac{1}{2}{b_n}^2 y^2 f''(x) + o({b_n}^2)\}dy$$

Then, using the properties of the Kernel, the bias of the density estimator is

$$Bias\,\{f_n(x)\} = \frac{{b_n}^2}{2} f''(x)\mu_2(K) + o({b_n}^2),\ as\ \ b_n \to 0 \quad (1.10)$$

Here we denote $\mu_2(K) = \int_{-\infty}^{\infty} x^2 K(x)dx$.

Observe from equation (1.10) that the bias is proportional to ${b_n}^2$. Thus, we have to choose a small b_n to reduce the bias. Moreover, $Bias\,\{f_n(x)\}$ depends on $f''(x)$. The effects of this dependence are illustrated in Figure 1.2 where the dashed lines mark $E\{f_n(*)\}$ and the solid line the true density $\{f(*)\}$. The bias is thus given by the vertical difference between the dashed and the solid line.

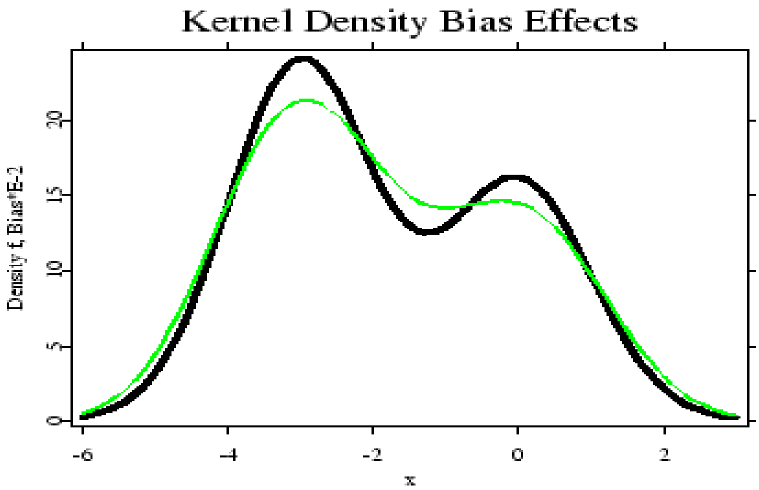


Figure 1.2: Bias effects

Note that in "valleys" of f, the bias is positive since $f'' > 0$ around a local minimum of f. Consequently, the dashed line is always above the solid line. Near peaks of f the opposite is true. The magnitude of the bias depends on the curvature of f, reflected in the absolute value of f''. Obviously, large values of $|f''|$ imply large values of bias of $\{f_n(x)\}$.

1.4.2.2 Variance

For the variance we calculate

$$Var\{f_n(x)\} = Var\left\{\frac{1}{n}\sum_{i=1}^{n} K\left[\frac{x-X_i}{b_n}\right]\right\}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var\left\{K\left[\frac{x-X_i}{b_n}\right]\right\}$$

$$Var\{f_n(x)\} = \frac{1}{n}\, Var\left\{K\left[\frac{x-X}{b_n}\right]\right\}$$

$$= \frac{1}{n}\left(E\left\{K^2\left[\frac{x-X}{b_n}\right]\right\} - \left\{E\left(K\left[\frac{x-X}{b_n}\right]\right)\right\}^2\right) \tag{1.11}$$

Using

$$\frac{1}{n}\, E\left\{K^2\left[\frac{x-X}{b_n}\right]\right\} = \frac{1}{n}\frac{1}{{b_n}^2}\int K^2\left(\frac{x-u}{b_n}\right) f(u)du$$

and

$$E\left\{K\left[\frac{x-X}{b_n}\right]\right\} = f(x) + o(b_n)$$

and similar variable substitution and Taylor expansion arguments as in the derivation of the bias, it can be shown that

$$Var\{f_n(x)\} = \frac{1}{nb_n}\|K\|_2{}^2 f(x) + \left(\frac{1}{nb_n}\right), as\ nb_n \to \infty \quad (1.12)$$

Here, $\|K\|_2{}^2$ is shorthand for $\int_{-\infty}^{\infty} K^2(x)dx$, the squared L_2 norm of K.

Notice that the variance of the kernel density estimator is nearly proportional to $nb_n{}^{-1}$. Hence, in order to make the variance small we have to choose a fairly large b_n . Large values of b_n mean bigger intervals $[x - b_n , x + b_n]$, more observations in each interval and hence more observations that get non-zero weight in the sum $K\left[\frac{x-X_i}{b_n}\right]$. But, as you may recall from the analysis of the properties of the sample mean in basic statistics, using more observations in a sum will produce sums with less variability.

Similarly, for a given value of b_n (be it large or small), increasing the sample size n will decrease $\frac{1}{nb_n}$ and therefore reduce the variance. But this makes sense because having a greater total number of observations means that, on average, there will be more observations in each interval $[x - b_n ,\ x + b_n]$.

Also observe that the variance is increasing in $\|K\|_2{}^2$. This term will be rather small for flat kernels such as the Uniform kernel. Intuitively speaking, we might say that smooth and flat kernels will produce less volatile estimates in repeated sampling since in each sample all realizations are given roughly equal weight.

1.4.2.3 Mean Squared Error(MSE)

As mentioned before for the histograms that the choice of the bandwidth b_n is an important issue in nonparametric density estimation. The kernel density estimator is no exception. If we look at formula (1.10) and (1.12) we can see that we face the familiar trade-off between variance and bias. We would surely like to keep both variance and bias small but increasing b_n will lower the variance while it will raise the bias (decreasing b_n will do the opposite). Minimizing the MSE which is the sum between variance and squared bias, represents a compromise between over and undersmoothing. Figure 1.3 puts variance, bias and MSE onto one graph.

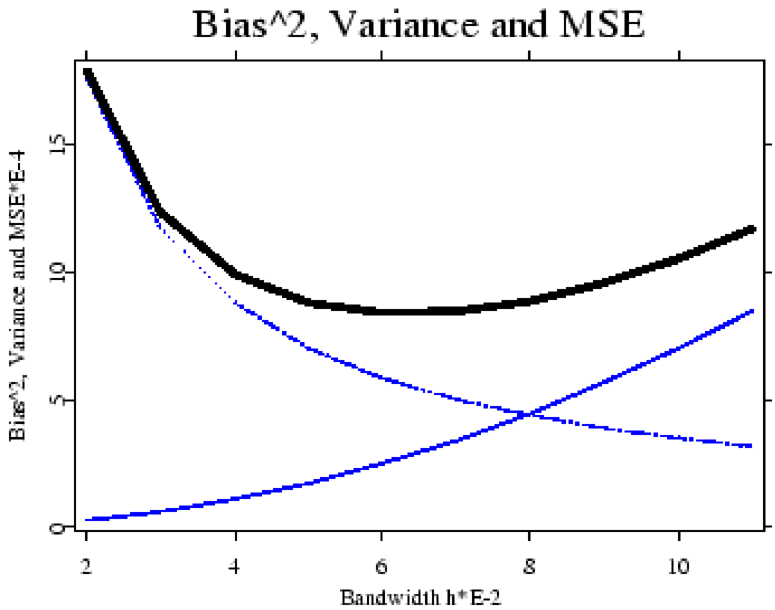


Figure 1.3: Squared bias part (thin solid), variance part (thin dashed) and MSE (thick solid) for kernel density estimate.

Moreover, looking at the MSE provides a way of assessing whether the kernel density estimator is consistent. Convergence in mean square implies convergence in probability which is consistency. Equations (1.10) and (1.12) yield to:

$$MSE\{f_n(x)\} = Bias^2\{f_n(x)\} + Var\{f_n(x)\}$$

$$= \frac{{b_n}^4}{4}(f''(x))^2(\mu_2(K))^2 + \frac{1}{nb_n}\|K\|_2{}^2 f(x) + o({b_n}^4) + o\left(\frac{1}{nb_n}\right) \quad (1.13)$$

If we look at equation (1.13) we can see that the MSE of the kernel density estimator goes to zero as $b_n \to 0$ and $nb_n \to \infty$. Hence, the kernel density estimator is indeed consistent. Unfortunately, by looking at equation (1.13) we can also observe that the MSE depends on f and f'', both functions being unknown in practice. If we derive the value of b_n that is minimizing the MSE,call it(${b_n}_{opt}$), you will discover that both f and f'' do not drop out in the process of deriving ${b_n}_{opt}(x)$. Consequently, ${b_n}_{opt}(x)$ is not applicable in practice unless we find a way of obtaining suitable substitutes for $f(x)$ and $f''(x)$. Note further that ${b_n}_{opt}(x)$ depends on x and is thus a local bandwidth [21].

The Epanechnikov kernel is optimal in a mean square error sense, though the loss of efficiency is small for the kernels listed previously, and due to its convenient mathematical properties, the normal kernel is often used $K(x) = \phi(x)$, where ϕ is the standard normal density function.

1.4.2.4 Mean Integrated Squared Error($MISE$)

For the kernel density estimator the $MISE$ is given by

$$MISE\{f_n(x)\} = \int MSE\{f_n(x)\}\ dx$$

$$= \int \left[\frac{{b_n}^4}{4}\{\mu_2(K)\}^2\{f''(x)\}^2 + \frac{1}{nb_n}{\|K\|_2}^2 f(x) + o\left({b_n}^4\right) + o\left(\frac{1}{nb_n}\right)\right] dx$$

$$= \frac{{b_n}^4}{4}\{\mu_2(K)\}^2 \int \{f''(x)\}^2 dx + \frac{1}{nb_n}{\|K\|_2}^2 \int f(x)dx + o\left({b_n}^4\right) + o\left(\frac{1}{nb_n}\right)$$

$$= \frac{{b_n}^4}{4}\{\mu_2(K)\}^2{\|f''\|_2}^2 + \frac{1}{nb_n}{\|K\|_2}^2 + o\left({b_n}^4\right) + o\left(\frac{1}{nb_n}\right) \tag{1.14}$$

Ignoring higher order terms an approximate formula for the $MISE$, called asymptotic mean integrated squared error($AMISE$), can be given as

$$AMISE\{f_n(x)\} = \frac{{b_n}^4}{4}\{\mu_2(K)\}^2{\|f''\|_2}^2 + \frac{1}{nb_n}{\|K\|_2}^2 \tag{1.15}$$

In order to get an optimal bandwidth differentiate $AMISE$ with respect to b_n and solving the first order condition for b_n yields the $AMISE$ optimal bandwidth

$${b_n}^3\{\mu_2(K)\}^2\ {\|f''\|_2}^2 - \frac{1}{n{b_n}^2}{\|K\|_2}^2 = 0$$

$${b_n}^3\{\mu_2(K)\}^2\ {\|f''\|_2}^2 = \frac{1}{n{b_n}^2}\ {\|K\|_2}^2$$

$${b_n}^5 n\{\mu_2(K)\}^2\ {\|f''\|_2}^2 = {\|K\|_2}^2$$

$${b_n}^5 = \frac{{\|K\|_2}^2}{\{\mu_2(K)\}^2\ {\|f''\|_2}^2\ n}$$

$$b_{n_{opt}} = \left(\frac{{\|K\|_2}^2}{\{\mu_2(K)\}^2\ {\|f''\|_2}^2\ n}\right)^{1/5} \sim n^{-1/5} \tag{1.16}$$

Apparently, the problem of having to deal with unknown quantities has not been solved completely as $b_{n_{opt}}$ still depends on ${\|f''\|_2}^2$. At least we can use $b_{n_{opt}}$ to get a further theoretical result regarding the statistical properties of the kernel density estimator. Inserting $b_{n_{opt}}$ into equation (1.15) gives

$$AMISE\left\{f_{b_{n_{opt}}}\right\} = \frac{5}{4}\left({\|K\|_2}^2\right)^{4/5}\left(\mu_2(K)\right){\|f''\|_2}^{2/5} \tag{1.17}$$

Where we indicated that the terms proceeding $n^{-4/5}$ are constant with respect to n. Obviously, if we let the sample size get larger and larger $AMISE$ is converging at the rate $n^{-4/5}$. If we take the $AMISE$ optimal bandwidth of the histogram and plug it into equation (1.15) we will find out that for the histogram, $AMISE$ is converging at a slower rate of $n^{-4/5}$.

In chapter 2, we will define the following five goodness-of-fit tests, and study their asymptotic null distributions and their powers: Chi-square test, Kolmogorov-Smirnov (KS) test, Cramér-von Mises (CVM) test, Anderson-Darling (AD) test and Bickel-Rosenblatt (BR) test.

Chapter 2

Goodness-of-Fit Tests

2.1 Goodness-of-Fit (GOF) Problem

The goodness-of-fit of a statistical model describes how well it fits a set of observations. Measures of goodness of fit typically summarize the discrepancy between observed values and the values expected under the model in question. Such measures can be used in statistical hypothesis testing, testing for normality of a set of data is important in some statistical inference procedures.

Distribution assumptions are commonly made when conducting data analysis like the independent samples t-test, and the analysis of variance. Those distribution assumptions which include both statistical distributions and models specifications, are very often critical in statistical inferences. Failure of a distribution assumption or model specification may lead to invalid conclusions. Goodness-of-fit tests are used to examine how well a sample of data agrees with a given distribution of a population. The importance of goodness-of-fit test have also been emphasized by many authors, including Anderson and Darling [22], Stephens [23], D'Agostino and Stephens [24], Read and Cressie [25], Thode [26], Lehmann and Romano [27].

In hypothesis testing, two hypotheses are usually studied: the null hypothesis H_0 and the alternative hypothesis H_1. H_0 is that a given random variable follows from a stated distribution with cumulative distribution function F_0, the goodness-of-fit tests for testing H_0 are then made based on measuring the correspondence of the sample data to the hypothesized distribution, D'Agostino and Stephens [24]. A primary goal of constructing goodness-of-fit test is to reduce the Type II error while controlling Type I error.

In this thesis, the purpose is to test the hypothesis:

$$H_0 : f = f_0$$

Against several types of parametric and nonparametric alternatives at a specified significance level α, where f_0 is completely specified and satisfied certain regularity conditions.

A number of goodness of fit tests exists for testing $H_0 : f = f_0$.

In this thesis, the following five tests will be studied and compared in terms of power using Monte Carlo simulation method: Chi-square test, Kolmogorov-Smirnov (KS) test, Cramér-von Mises (CVM) test, Anderson-Darling (AD) test and Bickel-Rosenblatt (BR) test.

2.2 Chi-Square Test

The statistical procedure that is used to test whether an assumed distribution is correct is called goodness-of-fit test. A well-known goodness of fit test is called Pearson chi-square test. It was proposed by Pearson in 1900. The chi-square goodness-of-fit test, also referred to as the chi-square test for a single sample, is employed in a hypothesis testing situation consisting of a single sample. Based on some preexisting characteristic or measure of performance, each of n observations that is randomly selected from a population consisting of N observations is assigned to one of K mutually exclusive categories [28, 29].

The chi-square goodness-of-fit test is based on the following assumptions:

1. Categorical data are employed in the analysis. This assumption reflects the fact that the test data should represent frequencies for k mutually exclusive categories;

2. The data evaluated consist of a random sample of n independent observations. This assumption reflects the facts that each observation can only be represented once in the data; and

3. The expected frequency of each cell is 5 or greater. This is related to the normal approximation to the binomial distribution.

Definition 1

Chi-square goodness-of-fit is used to check whether or not an observed frequency distribution differs from a theoretical distribution [30, 31, 32].

Chi-square test-statistic

The value of the test-statistic is

$$\sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \tag{2.1}$$

Where

χ^2 : Pearson's cumulative test statistics, which is asymptotically approaches a χ^2 distribution.

O_i: is an observed frequency.

$E_i = np_i$: is an expected frequency, asserted by the null hypothesis.

k = the number of cells.

In the case of binomial distribution, the Chi-square test-statistic reduces to

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - np_i)^2}{np_i}$$

Where p_i is the success probability in each trial i.

Chi-Square-Goodness-of-Fit Test

Theorem 1. Let $X_1, X_2, \dots, X_k$ are iid observations, and let O_i be an observed frequency and E_i be an expected frequency, $i = 1, 2, \dots, k$ asserted by the null hypothesis. Consider the problem of testing [33]

$H_0 : O_i = E_i$ for all cells

Versus

$H_1 : O_i \neq E_i$ at least one cell

at the level of significance α, reject H_0 if

$$\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \geq \chi^2{}_{1-\alpha, k-1}$$

Remark: The critical values of the Chi-square test are obtained from table A.1 in the appendix**.**

Asymptotic Distribution of Pearson's Chi-square Statistic

The following theorem tell us the asymptotic null distribution of the Pearson's Chi-square statistic.

$$\sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Theorem 2. Suppose $X_1, X_2, \ldots, X_k$ are iid observations from some distribution F in a finite-dimensional Euclidian space. consider testing $H_0 : F = F_0$ (specified). Let χ^2 be the Pearson χ^2 statistic defined previously. Then $\chi^2 \xrightarrow{d} \chi^2{}_{k-1}$ under H_0 .

This theorem shows that χ^2 converges in distribution to the chi-square distribution with $k-1$ degrees of freedom. The idea of the proof of the theorem is to use the definition of the covariance and the projection matrices besides using the central limit theorem and the fact that $\chi^2{}_{k-1}$ distribution is the distribution of the sum of the square of $k-1$ independent standard normal random variables.

2.3 Kolmogorov-Smirnov (KS) Test

Kolmogorov-Smirnov test is another non-parametric test which can be used to check whether a given sample come from a certain specified population. The name of Kolmogorov-Smirnov test is referred to Andrey Kolmogorov who proposed in 1933 the KS test-statistic defined below and it's asymptotic distribution under the null hypothesis, and to Nikolai Smirnov who published the table of the distribution. The test is designed to test the goodness-of-fit of an empirical to a theoretical distribution function [34, 35]. The statistical model underlying the test assumes a continuous distribution so that the sample observations have zero probability of being equal.

KS Test-Statistic

Let $X_1, X_2, \ldots, X_n$ be a sample of independent observations in ascending order all come from the same continuous population with cumulative distribution function $F(x)$. Then the KS test-statistic is defined by:

$$D_n = \sup_x |F_n(x) - F(x)| \tag{2.2}$$

Where, F_n is the empirical distribution function of the sample, and $F(x)$ is the cumulative distribution function [36, 37, 38].

By using the definition of $F_n(x) = \frac{i}{n}$, $i = 0, 1, \ldots, n$, We can get

$$D_n = \sup_x |F_n(x) - F(x)|$$

$$D_n = \max_i \left|\frac{i}{n} - F(x_i)\right|$$

$$D_n = \frac{1}{n} \max_i |i - nF(x_i)|$$

$$D_n = \frac{1}{n} \max_i |D_i| \tag{2.3}$$

The largest absolute cumulative differences divided by n.

Kolmogorov-Smirnov Goodness-of-Fit Test

Theorem 3. Let $X_1, X_2, \dots, X_n$ be a sample of independent observations in ascending order all come from the same continuous population with cumulative distribution function $F(x)$. Consider the problem of testing

$H_0 : F = F_0$

Versus

$H_1 : F \neq F_0$

Where F_0 is some specified distribution function,

Then, reject H_0 if the test statistic D_n exceeds the critical value.

Remark: The critical values of the KS test are obtained from table A.2 in the appendix**.**

Asymptotic Null Distribution of Kolmogorov-Smirnov Statistic

Theorem 4. Let $X_1, X_2, \dots, X_n$ be iid random variables with c.d.f. F, then under the null hypothesis

$$\sqrt{n}\, D_n \xrightarrow{D} \sup_t |B(F(t)| \quad \text{as } n \to \infty \tag{2.4}$$

Where $B(t)$ is the Brownian bridge which is defined on [0,1] and is given by:

$$B_t = \sum_{k=1}^{\infty} Z_k \frac{\sqrt{2}\, sin(k\pi t)}{k\pi} \tag{2.5}$$

Where $Z_1, Z_2, \dots$ are iid standard normal random variables.

In other words, under the null hypothesis $\sqrt{n}\, D_n$ converges in distribution to the Kolmogorov distribution [39, 40].

Remark: The asymptotic null distribution of the Kolmogorov-Smirnov test statistic does not depend on the cumulative distribution function $F(x)$.

Asymptotic Power of the Kolmogorov-Smirnov Test

Theorem 5. Let $X_1, X_2, \dots, X_n$ be iid random variables with c.d.f. F and consider the problem of testing

$H_0 : F = F_0$

Versus

$H_1 : F \neq F_0$

Then the power of the Kolmogorov-Smirnov test tends to one uniformly over all alternatives F satisfying $\sqrt{n}\, d_k(F, F_0) \geq \Delta_n$ if $\Delta_n \to \infty$ as $n \to \infty$

Where d_k is the Kolmogorov-Smirnov distance

$$d_k(F, F_0) = \sup_x |F(x) - F_0(x)| \qquad (2.6)$$

2.4 Anderson-Darling (AD) Test

Anderson-Darling (AD) test was proposed by Theodore Anderson and Donald Darling in 1954. This test is considered as an improvement of the Kolmogorov-Smirnov test. The Anderson-Darling test is used to check whether or not a sample of data come from a certain population with specific distribution function [41, 42]. Like the KS test, the Anderson-Darling test is based on the empirical distribution function (EDF). The Anderson-Darling test is known as a "quadratic" test because it is based on the weighted square of the distance between the empirical function and hypothesized cumulative distribution function.

Definition 2

Anderson-Darling (AD) test is used to test if a sample of data follows a population with a specified distribution.

Anderson-Darling(AD) Test-Statistic

Let $X_1, X_2, \dots, X_n$ be a sample of independent observations in ascending order selected from the same continuous population with cumulative distribution function $F(x)$. Then the AD test-statistic is defined as follows:

$$A_n{}^2 = n \int_{-\infty}^{\infty} (F_n(x) - F_0(x))^2\, w\big(F_0(x)\big) dF_0(x) \qquad (2.7)$$

Where,

w is a non-negative weight function , $w(x) = [F_0(x)(1 - F_0(x)]^{-1}$

And

F_n is the empirical distribution function of the sample, and $F(x)$ is the cumulative distribution function [43, 44, 45, 46].

Note that $F_n(x) = \frac{i}{n}$, $i = 1, 2, \ldots, n$ is a piecewise constant(with the order statistics corresponding to the discontinuity points) and $F_0(x)$ is a non-decreasing function. For convenience define $Z_i = F_0(x_i)$, and let $Z_{(i)}$ denote the corresponding order statistics and since F_0 is monotone $Z_{(i)} = F_0(x_{(i)})$ and the order is not disorganized, we can therefore write equation (2.7) as

$$A_n{}^2 = -n - \frac{1}{n}\sum_{i=1}^{n}\{(2i-1)ln(Z_i) + (2n+1-2i)ln(1-Z_i)\} \qquad (2.8)$$

Theorem 6. Let $X_1, X_2, \ldots, X_n$ be a sample of independent observations in ascending order selected from the same continuous population with cumulative distribution function $F(x)$. Suppose that we need to test

$$H_0 : F = F_0$$

Versus

$$H_1 : F \neq F_0$$

Where, F_0 is some specified distribution function,

then, reject H_0 if the modified test statistic A^* greater than the critical value.

Where, the modified test statistic can be evaluated as follows

$$\mathrm{A}^* = \mathrm{A}_n{}^2(1+ \frac{0.75}{n} + \frac{2.25}{n^2}) \qquad (2.9)$$

According to D'Agostino [24], formula (2.9) is used if the parameters of the null distribution are unknown and need to be estimated from the sample data. Moreover, when the sample size n is large enough then the modified test statistic is close to the original test statistic.

Remark: The critical values of the AD test are obtained from tables A.3.1 and A.3.2 in the appendix**.**

Asymptotic Null Distribution of Anderson-Darling Test-Statistic

Theorem 7. Let $X_1, X_2, \ldots, X_n$ be an iid random variables in ascending order with c.d.f. F, then

Under the null hypothesis [47, 48]

$$\text{as } n \to \infty, \qquad n\, {A_n}^2 \xrightarrow{D} \int_0^1 \frac{B_0^2(t)}{t(1-t)}\, dt \tag{2.10}$$

Where $B_0(t)$ is the Brownian bridge which is defined on [0,1] and is given by equation (2.5).

We can summarize the proof of this theorem as follows

Let $u = F^0(x)$, $u_i = F^0(x_i)$, and $u_{(i)} = F^0(x_{(i)}), i = 1, 2, \ldots, n$. Let $G_n(u)$ be the empirical distribution function of $u_1, \ldots, u_n$; that is,

$$G_n(u) = \frac{k}{n},\; 0 \le u \le 1$$

If k of $u_1, \ldots, u_n$ are $\le u$. Thus

$$G_n[F^0(x] = {F_n}^0(x)$$

And
$$ {A_n}^2 = n \int_0^1 (G_n(u) - u)^2\, \psi(u) du, \tag{2.11}$$

When the null hypothesis $F(x) = F^0(x)$ is true. For every u $(0 \le u \le 1)$

$$Y_n(u) = \sqrt{n}[G_n(u) - u] \tag{2.12}$$

is a random variable, and the set of these may be considered as a stochastic process with parameter u. Thus

$$P\{{A_n}^2 \le z\} = P\left\{\int_0^1 {Y_n}^2(u)\psi(u)du \le z\right\} = A_n(z), \tag{2.13}$$

Say. for a fixed set $u_1, \ldots, u_k$ the k-variate distribution of $Y_n(u_1), \ldots, Y_n(u_k)$ approaches a multivariate normal distribution as $n \to \infty$ with mean and covariance function

$$E[Y_n(u)] = 0,\; E[Y_n(u)Y_n(v)] = min(u, v) - uv.$$

The limiting process of $\{Y_n(u)\}$ is a Gaussian process $y(u)$, $0 \le u \le 1$, and $E[y(u)] = 0$ and $E[y(u)y(v)] = min(u, v) - uv$. Let

$$a(z) = P\left\{\int_0^1 y^2(u)\psi(u)du \le z\right\} \tag{2.14}$$

Then, $A_n(z) \to a(z)$, $0 \le z < \infty$. The mathematical problem for the Anderson-Darling statistic is to find the distribution function $a(z)$ when $\psi(u) = \frac{1}{u(1-u)}$.

Then, we follow the procedure to find the distribution of $\int_0^1 z^2(u)du$, where $z(u)$ is a Gaussian stochastic process, and we obtain the characteristic function of the limiting distribution of ${A_n}^2$ which is

$$\sqrt{\frac{\sqrt{2it}}{sin(\sqrt{2it})}}$$

Asymptotic Power of the Anderson-Darling Test

Theorem 8. Let $X_1, X_2, \dots, X_n$ be an iid random variables in ascending order with c.d.f. F. consider the problem of testing

$$H_0 : F = F_0$$

Versus

$$H_1 : F \neq F_0$$

Then, the statistic ${A_n}^2$ will detect alternatives which produce observations towards 0 or 1.

We can summarize the proof as follows, since ${A_n}^2$ is a quadratic statistic and gives more weight to the tails, then it's good at detecting irregularity in the tails of the distribution. When the basic problem is to test an X-values for a distribution $F(x)$ so that the observation u_i where $0 \leq u_i \leq 1$ have been obtained by probability integral transformation. ${A_n}^2$ will detect shifts in the mean of hypothesized distribution from the true mean. In addition, ${A_n}^2$ is powerful test for tests F_0 departs from the true distribution in the tails, especially when there appears to be too many outlying X-values for the F_0 as specified. In goodness-of-fit work, departure in the tails is often important to detect, and ${A_n}^2$ is the recommended statistic. The statistic will also detect a shift of values towards 0 or 1.

Remark:

1) ${A_n}^2$ can be expected to be powerful in detecting alternatives which have high probability of giving observations in the tails.

2) Other statistics of the EDF class are more suitable for alternatives which produce a cluster near 0.5, Stephens [23].

Cramér von Mises (CVM) Test

Definition 3

Cramér von Mises (CVM) test is a special case of Anderson-Darling test when the weighting function $w(x) = 1$.

The Cramér–von Mises test statistic is defined as follows [49, 50]:

$$w^2 = n\int_{-\infty}^{\infty}(F_n(x) - F_0(x))^2 \ dF_0(x) \tag{2.15}$$

2.5 Bickel-Rosenblatt (BR) Test

Bickel-Rosenblatt test is a member of the group of goodness-of-fit tests and was proposed by Peter Bickel and Murray Rosenblatt in(1973) and (1975). This test depends on the kernel density estimate $f_n(x)$ of a probability density function $f(x)$ which was defined in chapter one in this thesis. The Bickel-Rosenblatt test depends on the distance between the kernel density estimate $f_n(x)$ and the expectation of the kernel density estimate $E_0 f_n(x)$ under f_0.

In the Bickel-Rosenblatt test, the value of test statistics depends on the specified distribution with probability density function that is being tested, kernel function, the smoothing parameter b_n, the weighted function $a(x)$ in addition to the sample size n. The computed value of Bickel-Rosenblatt test-statistics is compared with $\mu(K,a) + z_\alpha {b_n}^{1/2}\, \sigma(K,a)$ and null hypothesis is rejected if test statistic is greater than this value [51].

Definition 4

Bickel-Rosenblatt test is used to verify if a sample of data follows a population with a specified continuous probability density function f.

Bickel-Rosenblatt(BR)Test-Statistic

Let $X_1, X_2, \dots, X_n$ be iid random variables with a specified continuous probability density function $f(x)$. Bickel-Rosenblatt (BR) test-statistic is defined as follows:

$$T_n = nb_n\int_{-\infty}^{\infty}\left(f_n(x) - E_0 f_n(x)\right)^2 a(x)dx \tag{2.16}$$

Where a is a weight function on $\mathbb{R}$ and $f_n(x)$ is the kernel density estimator as defined in chapter one [52].

Bickel-Rosenblatt Goodness-of-Fit Test

Theorem 9. [53] Let $X_1, X_2, \dots, X_n$ be iid random variables with a specified continuous probability density function $f(x)$. Consider the problem of testing

$$H_0 : f = f_0$$

Versus

$$H_1 : f \neq f_0$$

where f_0 is completely specified at a specified significance level α. Then,

Reject H_0 if

$$T_n \geq \mu(K,a) + z_\alpha b_n^{1/2}\, \sigma(K,a) = d_\alpha \tag{2.17}$$

Where

$$\mu(K,a) = I(K) \int_{-\infty}^{\infty} f_0(x)\, a(x) dx\,, \tag{2.18}$$

$$\sigma^2(K,a) = 2\, J(K) \int_{-\infty}^{\infty} {f_0}^2(x) a^2(x) dx \tag{2.19}$$

$$I(K) = \int_{-\infty}^{\infty} K^2(x)\, dx \tag{2.20}$$

$$J(K) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} K(x+y) K(x) dx\right]^2 dy \tag{2.21}$$

also z_α is defined by :

$$\Phi(z_\alpha) = 1 - \alpha\,,\ \Phi(\mathrm{z}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-x^2/2\right)\, dx \tag{2.22}$$

To test $H_0 : f = f_0$ it is natural to compute $\widetilde{M}_n$ with $f = f_0$ and reject for large values of the statistic.

Where $$\widetilde{M}_n = max\{|Y_n(t)|: 0 \leqq t \leqq 1\} \tag{2.23}$$

And $$Y_n(t) = b_n^{-\frac{1}{2}} f^{-\frac{1}{2}}(t) \int_{-\infty}^{\infty} K\left(\frac{t-s}{b_n}\right) dZ_n^0(F(s)). \tag{2.24}$$

Where, $Z_n^0(t) = n^{\frac{1}{2}}({F_n}^*(t) - t),\ 0 \leqq t \leqq 1$ and ${F_n}^* = F_n(F^{-1})$ is the empirical distribution of $F(X_1), \ldots, F(X_n)$.

Now, in order to understand the asymptotic distribution of the Bickel-Rosenblatt statistic, consider the following regularity conditions (*a*-*d́*) that are defined as follows [53, 54]:

a) f_0 is bounded, either positive on $\mathbb{R}$ or positive only on some $[c_0, d_0]$, and continuous with a bounded continuous derivative in the interior of it's domain of positivity;

b) $a(x)$ is piecewise continuous, bounded and integrable on $\mathbb{R}$;

c) $\int_{-\infty}^{\infty} K(x) dx = 1\,, \int_{-\infty}^{\infty} x^2 K(x) dx < \infty\,, \int_{-\infty}^{\infty} K^2(x) < \infty$;

and either

d) $b_n = n^{-\delta}$ for some δ ∈ $(0, 1/4)$ and K is continuous on ℝ satisfying

$$\int_{-\infty}^{\infty} \left|\acute{K}(x)\right|^2 dx < \infty\,; \ \int_{3}^{\infty} |x|^{3/2} \left|\acute{K}(x)\right|^2 (log(|x|)^{1/2})\, dx < \infty\,,$$

or

$\acute{d}$) $b_n = n^{-\delta}$ for some $\delta \in (0,1)$ and K is bounded on some $[C_k, D_k]$ and 0 outside.

Asymptotic Null Distribution of Bickel-Rosenblatt Test-Statistic

Theorem 10. [54] Let $X_1, X_2, \dots, X_n$ be iid random variables with a specified continuous probability density function $f(x)$.

When α-level test is asymptotic then under the null hypothesis

T_n is asymptotically normal with mean $\mu(K, a)$ and variance $b_n\sigma^2(K, a)$, where

$\mu(K, a)$ and $\sigma^2(K, a)$are defined previously in equations (2.18) and (2.19).

The proof of this theorem is not direct and several theorems and results are needed, but we can summarize the proof as follows:

First, the technique that used is to consider the BR statistic as functionals of certain stochastic process on the interval $[0, 1]$.

According to theorem 3.2. in the paper of Bickel and Rosenblatt [54], the BR statistic is asymptotically unbiased for such alternatives. The reason is that $s(\eta) \geqq 2$ with $s(\eta) > 2$ unless $\eta = 0$ and family of distributions $e^{\theta e^{-x}}$ is an exponential family in θ.

Where $s(\eta) = \int_0^1 \left\{exp[\eta(x)/(f_0(x)\lambda(K))^{\frac{1}{2}}] + exp[-\eta(x)(f_0(x)\lambda(K))^{\frac{1}{2}}]\right\} dx$ (2.25)

And $\lambda(K) = \int_{-\infty}^{\infty} K^2(x)\, dx$, and η must be continuous on [0,1].

Unfortunately these tests are asymptotically inadmissible(have pitman efficiency 0) when compared to the test based on the quadratic functional that presented below. The reason is that alternatives there may be permitted to come in to f_0 at rate $n^{-\frac{1}{2}+\delta/4}$ rather than $n^{-\frac{1}{2}+\delta/2}$. However, this test for moderate sample and some alternatives may well preferable.

In addition, we are interested in the behavior of the quadratic functionals in the same paper [54]:

$$T_n = nb_n \int_{-\infty}^{\infty} \left(f_n(x) - E_0 f_n(x)\right)^2 a(x) dx = \int_{-\infty}^{\infty} {L_n}^2(x) a(x) dx, \qquad (2.26)$$

Where $L_n = f^{\frac{1}{2}} Y_n$ and Y_n is some Gaussian process and a is integrable. If the regularity conditions (a-$\acute{d}$) hold and (say) $b_n = n^{-\delta},\ \delta < \frac{1}{4}$, then.

$$a_n = \left|T_n - \int_{-\infty}^{\infty} (L_0^n(x))^2 a(x) dx\right| = o_p(\, b_n^{\frac{1}{2}}) \qquad (2.27)$$

Means that $\frac{a_n}{\left\|b_n^{\frac{1}{2}}\right\|}$ converges in probability to zero.

Moreover, if a is bounded as well as integrable and K and f are bounded, we can

replace L_n^0 by $L_n^1 = f_1^{\frac{1}{2}} Y_n$ and hence by $L_n^2 = f_2^{\frac{1}{2}} Y_n$, we get

$$\left|T_n - \int_{-\infty}^{\infty} (L_n^2(x))^2 a(x) dx\right| = o_p(b^{\frac{1}{2}}{}_n). \qquad (2.28)$$

Theorem 11. [54] Suppose that $Z_n(f_0)$ is defined as follows:

$$Z_n(f_0) = n^{\delta/2}\, \sigma_0{}^{-1} \left\{ n^{1-\delta} \int \left(f_n(x) - E_0 f_n(x)\right)^2 a(x) dx - \mu_0 \right\} \qquad (2.29)$$

Where μ_0 and σ_0 denote $\mu(K, a)$ and $\sigma(K, a)$ respectively defined in equations (2.18) and (2.19).

Then, the distribution of $Z_n(f_0)$ is asymptotically $N(0,1)$ under assumptions $(a) - (d)$.

Proof.

The proof of this theorem follows directly from theorem 10. By theorem 10 since T_n is asymptotically normal with mean μ_0 and variance $b_n {\sigma_0}^2$ and by using transformation we get

$$Z_n(f_0) = \frac{T_n - \mu_0}{\sqrt{b_n}\, \sigma_0}$$

$$Z_n(f_0) \sqrt{b_n}\, \sigma_0 = T_n - \mu_0$$

$$Z_n(f_0) = b_n{}^{-\frac{1}{2}} \sigma_0{}^{-1} \left\{ n b_n \int_{-\infty}^{\infty} \left(f_n(x) - E_0 f_n(x)\right)^2 a(x) dx - \mu_0 \right\} \qquad (2.30)$$

Now, under assumptions $(a) - (d)$, substitute $b_n = n^{-\delta}$ for some $\delta \in (0, 1/4)$ in equation(2.30) to get

$$Z_n(f_0) = (n^{-\delta})^{-\frac{1}{2}} \sigma_0{}^{-1} \left\{ n\, n^{-\delta} \int_{-\infty}^{\infty} \left(f_n(x) - E_0 f_n(x)\right)^2 a(x) dx - \mu_0 \right\}$$

$$Z_n(f_0) = n^{\delta/2}\, \sigma_0{}^{-1} \left\{ n^{1-\delta} \int_{-\infty}^{\infty} \left(f_n(x) - E_0 f_n(x)\right)^2 a(x) dx - \mu_0 \right\} \qquad (2.31)$$

Therefore, $Z_n(f_0)$ is asymptotically $N(0,1)$.

The Asymptotic Power of the Bickel-Rosenblatt Test

The asymptotic power of the Bickel-Rosenblatt test is a decreasing function of $J(K)$ in equation (2.21) for any fixed choice of $(f_0\,,\ g,\ a > 0,\ \delta\ )$,where g is an arbitrary alternative.

Since $Z_n(f_0)$ is asymptotically $N(0,1)$ under g, it follows that the power of the BR test against an arbitrary g tends to 1.

The following theorem illustrates the power of the BR-test against an arbitrary alternatives to $H\!: f = f_0$.

Theorem 12. [54] Let $g_n(x)$ be an alternative to H defined as follows:

$$g_n(x) = f_0(x) + n^{-\beta} w(x) + o\left(n^{-\beta}\right),\ \beta > 0 \tag{2.32}$$

Then, under assumptions $(a) - (d)$ and for each $\delta \in (0, 1/4)$ the power of the BR test against equation (2.32) is :

$$lim_{n\to\infty}\, \pi_n(g_n) = \begin{cases} \alpha,\ if\ \ \beta > (2-\delta)/4, \\ \Phi(l),\ if\ \ \beta = (2-\delta)/4, \\ 1,\ if\ \ 0 < \beta < (2-\delta)/4, \end{cases} \tag{2.33}$$

Where

$$l = {\sigma_0}^{-1} \int w^2(x) a(x)\, dx - z_\alpha \tag{2.34}$$

Remark: under assumptions $(a) - (c)$and $(\acute{d})$, equation (2.33) holds for each $\delta \in (0, 2/3)$.

The proof of this theorem is straightforward according to Bickel and Rosenblatt [54]. Since the test rejects H_0 when $T_n \geq \mu(K,a) + z_\alpha {b_n}^{1/2}\, \sigma(K,a)$ is locally strictly unbiased if $a(x) > 0$ for all x. Also as before asymptotic lead to choosing δ as large as possible and again this conclusion is shaken if one uses better approximation to the asymptotic mean. It is also clear for a fixed δ we can get power.

In addition, According to Ghosh and Huang [53], when $\alpha > 0$ we can conclude: first, the asymptotic power of the BR test against equation (2.32) is a decreasing function of $J(K)$, second as δ gets smaller (from $1/4$ or $2/3$ to 0), the power improves. This feature, incidentally, conflicts with the fact that the asymptotic normality of T_n under H improves as δ gets larger (from 0 to $1/4$ or 1).

In chapter 3, we will conduct Monte Carlo simulation to compare the power of Goodness-of-fit-tests: Chi-square test, Kolmogorov-Smirnov (KS) test, Cramér-von Mises (CVM) test, Anderson-Darling (AD) test and Bickel-Rosenblatt (BR) test.

Chapter 3

Results and Analysis

3.1 Simulation Methodology

In this study, Monte Carlo procedures were used to evaluate the power of χ^2, KS, AD, CVM and BR tests in testing if a random sample of n independent observation come from a population with specified distribution.

The null and alternative hypotheses are:

H_0: The data comes from a population with a specified distribution

H_1: The data does not come from a population with a specified distribution

Two levels of significance, α = 0.05 and 0.1 and different sample sizes were considered to investigate the effect of the significance level and the sample sizes on the power of the tests. The critical values for each test vary with the sample size, Yazici and Yolacan [55]. Therefore, first, choose critical values for each test statistic under the null distribution and sample sizes n =10, 20, 30, 40, 50, 100, 200, 300, 400, 500, 1000 and 2000. The critical values were obtained based on 50,000 simulated samples from a standard normal distribution, uniform distribution and T distribution [56, 57, 58].

The KS, CVM and AD tests are right-tailed test, so their critical values are the 100(1-α)th percentiles of the empirical distribution of the test statistics and it is available in the literature. The critical values of KS and CVM tests depend on the sample sizes n and the significance levels α, but the critical values of AD test depends on the distribution being tested, in addition to the sample sizes n and the significance levels α and the critical values of χ^2 test depend on the number of intervals when we grouped the data. For the BR test, there is a formula for calculating the critical values. These critical values depend on the sample size n, kernel function, bandwidth b_n and the probability density function of the distribution being tested.

In order to obtain the simulated power of the five Goodness-of-fit tests at α = 0.05 and α = 0.1 and for each sample size n, we can follow the following procedure:

a) Generate 10000 samples from each of the following:

1) Symmetric parametric alternative distributions: $N(0,1)$, $U(0,1)$, $U(0,2)$, $t(7)$, $t(15)$, $t(4)$.
2) Skewed parametric alternative distributions: $Exp(0.9)$, $Exp(5)$, $G\text{amma}(4,5)$, $\chi^2(4)$, $Weibull(10,2)$.
3) Family of non-parametric alternatives distributions $\mathrm{A}(\theta), \mathrm{B}(\theta), \mathrm{C}(\theta)$, $\theta = 1.5, 2$ which are presented in section 3.2.

b) Substitute these generated samples in each formula of goodness-of-fit test-statistics under the distribution being tested and then obtain the values of test-statistics for each sample.

c) Compare these values of test-statistics with corresponding critical values by counting the number of values of test-statistics that exceed the corresponding critical values, then the simulated power will be this number divided by the number of generated samples.

.

3.2 Results and Discussion

The main purpose of this thesis is to use Monte Carlo simulation method to compare the power of several goodness-of-fit tests based on the empirical distribution function and the kernel density function under parametric and nonparametric alternatives. The results were organized in three parts:

Part 1: Symmetric parametric null hypotheses vs. symmetric parametric alternative hypotheses.

Part 2: Symmetric parametric null hypotheses vs. skewed parametric alternative hypotheses.

Part 3: Symmetric parametric null hypothesis vs. nonparametric alternative hypotheses.

The Non-parametric alternative distributions that were used by Stephen are as follows:

$$A(\theta) = \theta(1-x)^{\theta-1}, \; 0 \le x \le 1, \; \theta \in \mathcal{R}$$

$$B(\theta) = \begin{cases} B_1(\theta) = \theta(2x)^{\theta-1} & , \quad 0 \le x < 1/2 \\ B_2(\theta) = \theta(2-2x)^{\theta-1} & , \quad 1/2 \le x \le 1 \end{cases} \quad ,\theta \in \mathcal{R}$$

$$C(\theta) = \begin{cases} C_1(\theta) = \theta(1-2x)^{\theta-1} & , \quad 0 \le x < 1/2 \\ C_2(\theta) = \theta(2x-1)^{\theta-1} & , \quad 1/2 \le x \le 1 \end{cases} \quad ,\theta \in \mathcal{R}$$

Stephen's used the following:

$$\theta = 3/2, 2$$

3.2.1 Comparison of power of GOF tests for testing symmetric distributions against symmetric parametric alternative distributions.

Case 1: Testing $N(0.5, 1)$ against $N(0, 1)$

Table 1(a): Critical values, $CV(\alpha, n)$ of the tests(χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and $\alpha = 0.05$.

n	χ^2 $K=5$	χ^2 $K=10$	KS	CVM	AD	BR $Uniform$	BR $Epan.$
20	9.488	16.919	0.294	0.217	2.492	1.58428	0.9431867
30	9.488	16.919	0.2417	0.218	2.492	1.527956	0.9101042
50	9.488	16.919	0.189	0.220	2.492	1.464661	0.8729264
100	9.488	16.919	0.1360	0.220	2.492	1.390732	0.8295028
200	9.488	16.919	0.0961	0.220	2.492	1.328565	0.7929881
300	9.488	16.919	0.0785	0.220	2.492	1.296892	0.7743844
400	9.488	16.919	0.0680	0.220	2.492	1.276289	0.762283
500	9.488	16.919	0.0608	0.220	2.492	1.261298	0.7534778

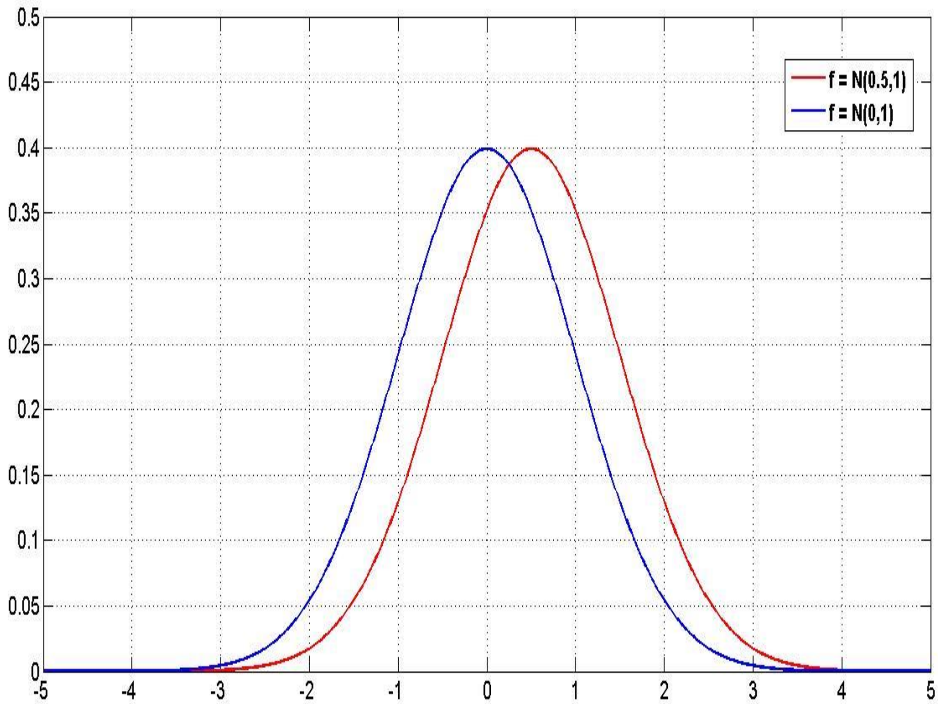


Figure 1(a): The graph of the null distribution $H_0: f = N(0.5,1)$ and the alternative distribution $H_1: f = N(0,1)$

Table 1(b): The power for tests ($\chi^2, KS,\ CVM, AD,\ BR$) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

$$H_0 : Normal(0.5,1)$$

$$H_1 : Normal(0,1)$$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	-	-	0.8825	0.8195	0.5868	0.0697	0.1509
30	0.8887	-	0.9860	0.9161	0.7599	0.1008	0.2185
50	0.9978	0.8970	1.0000	0.9828	0.9323	0.1670	0.3390
100	1.0000	1.0000	1.0000	0.9997	0.9983	0.3542	0.6113
200	1.0000	1.0000	1.0000	1.0000	1.0000	0.7056	0.9104
300	1.0000	1.0000	1.0000	1.0000	1.0000	0.8977	0.9878
400	1.0000	1.0000	1.0000	1.0000	1.0000	0.9653	0.9991
500	1.0000	1.0000	1.0000	1.0000	1.0000	0.9922	1.0000

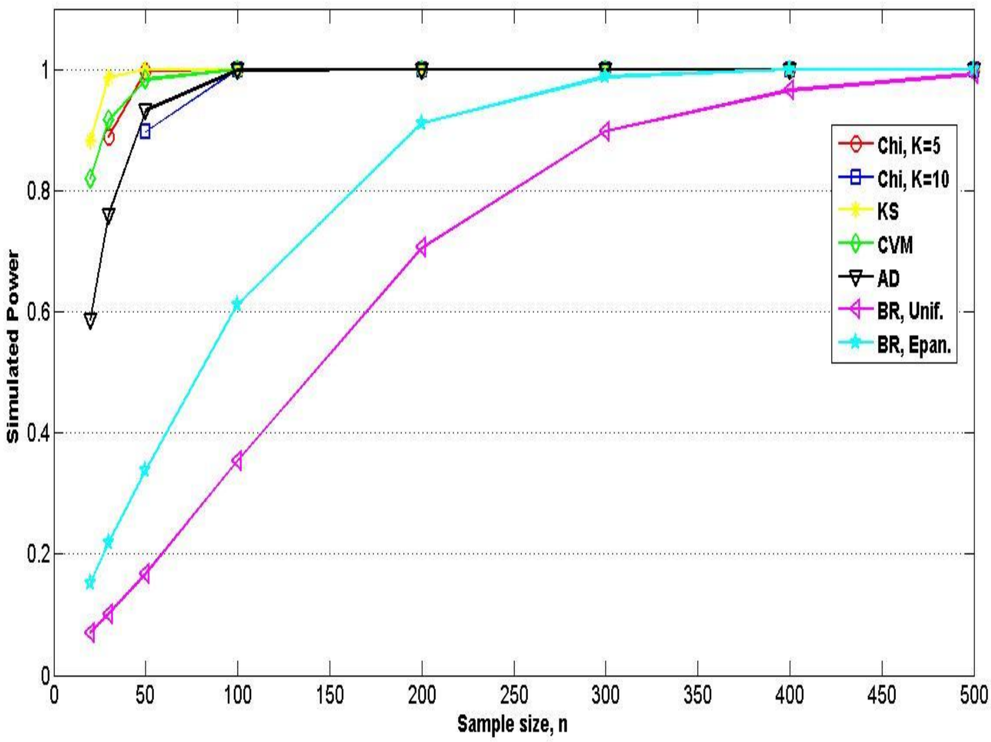


Figure 1(b): Comparison of power for different normality tests against $Normal(0,1)$

Summary

Table 1(b) and Figure 1(b) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD,\ BR$) for testing $H_0: N(0.5,1)$ against $H_1: N(0,1)$ at significance level $\alpha = 0.05$ and at different sample sizes n. We observe the following main findings:

1. Chi-square test is able to detect the null hypothesis at sample sizes $n \geq 100$.

2. CVM test has a higher power compared with (AD, BR) tests under different sample sizes n.

3. The power of the BR test is sensitive to the choices of the kernel functions, and BR test has a higher power at Epanechnikov kernel than that of uniform kernel.

4. In addition, all tests have good power when the sample size is greater than or equal 300.

Case 2(a): Testing Normality against $U(0,1)$

Table 2(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05.

	χ^2					BR	
n	$K=5$	$K=10$	KS	CVM	AD	$Uniform$	$Epan.$
10	9.488	16.919	0.409	0.212	0.752	2.293353	1.359673
20	9.488	16.919	0.232	0.217	0.752	2.087576	1.238806
30	9.488	16.919	0.19	0.218	0.752	1.982736	1.177227
50	9.488	16.919	0.149	0.220	0.752	1.864917	1.108024
100	9.488	16.919	0.107	0.220	0.752	1.727306	1.027196
200	9.488	16.919	0.075	0.220	0.752	1.611589	0.9592272
300	9.488	16.919	0.061	0.220	0.752	1.552633	0.9245984
400	9.488	16.919	0.053	0.220	0.752	1.514283	0.9020729
500	9.488	16.919	0.047	0.220	0.752	1.486379	0.8856829
1000	9.488	16.919	0.034	0.220	0.752	1.408994	0.8402297
2000	9.488	16.919	0.024	0.220	0.752	1.343922	0.8020083

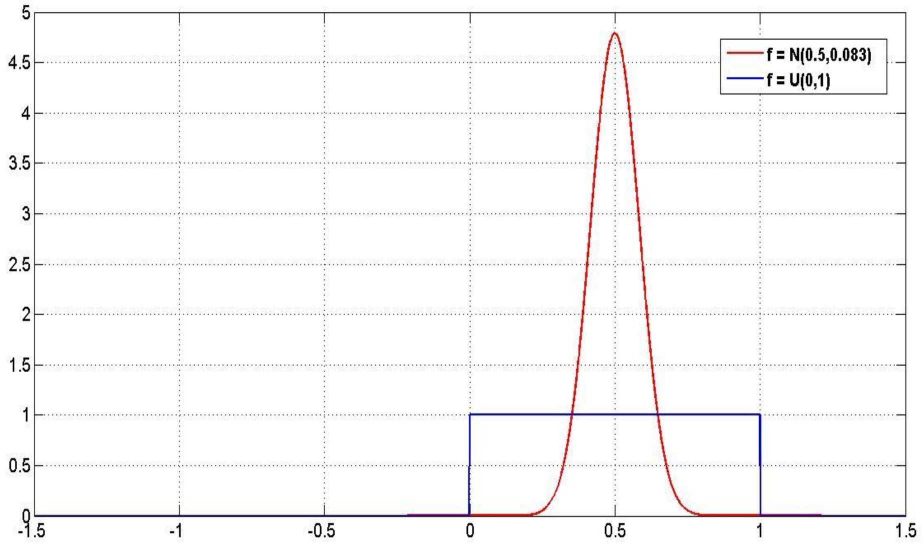


Figure 2(a): The graph of the null distribution H_0: $f = N(0.5,0.0833)$ and the alternative distribution H_1: $f = U(0,1)$

Table 2(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes(n) and number of intervals(K)and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $U(0,1)$

	χ^2					**BR**	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	-	-	0.0858	0.0024	0.0847	0.0009	0.0005
20	-	-	0.1074	0.0077	0.1708	0.0232	0.0180
30	0.2948	-	0.1239	0.0188	0.3022	0.0614	0.0812
50	0.6472	0.5223	0.1618	0.0633	0.5817	0.1890	0.2712
100	0.9614	0.9479	0.2562	0.3568	0.9523	0.5993	0.7634
200	0.9997	1.0000	0.4851	0.9077	1.0000	0.9690	0.9959
300	1.0000	1.0000	0.7045	0.9949	1.0000	0.9993	1.0000
400	1.0000	1.0000	0.8446	0.9998	1.0000	1.0000	1.0000
500	1.0000	1.0000	0.9331	1.0000	1.0000	1.0000	1.0000
1000	1.0000	1.0000	0.9996	1.0000	1.0000	1.0000	1.0000
2000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

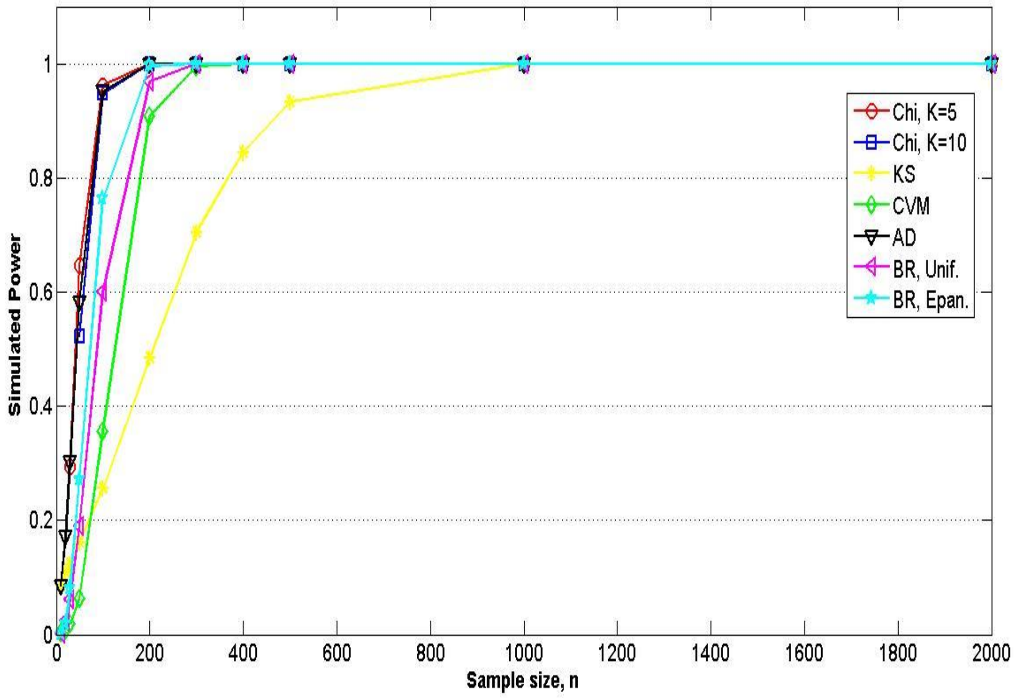


Figure 2(b): Comparison of power for different normality tests against $U(0,1)$

Case 2(b): Testing Normality against $U(0,2)$

Table 2(c): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05 .

	χ^2					**BR**	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	9.488	16.919	0.409	0.212	0.752	1.914676	1.137251
20	9.488	16.919	0.232	0.217	0.752	1.769148	1.051772
30	9.488	16.919	0.19	0.218	0.752	1.695004	1.008222
50	9.488	16.919	0.149	0.220	0.752	1.611681	0.9592811
100	9.488	16.919	0.107	0.220	0.752	1.51436	0.9021182
200	9.488	16.919	0.075	0.220	0.752	1.432524	0.8540501
300	9.488	16.919	0.061	0.220	0.752	1.390829	0.8295602
400	9.488	16.919	0.053	0.220	0.752	1.363708	0.8136298
500	9.488	16.919	0.047	0.220	0.752	1.343973	0.8020386

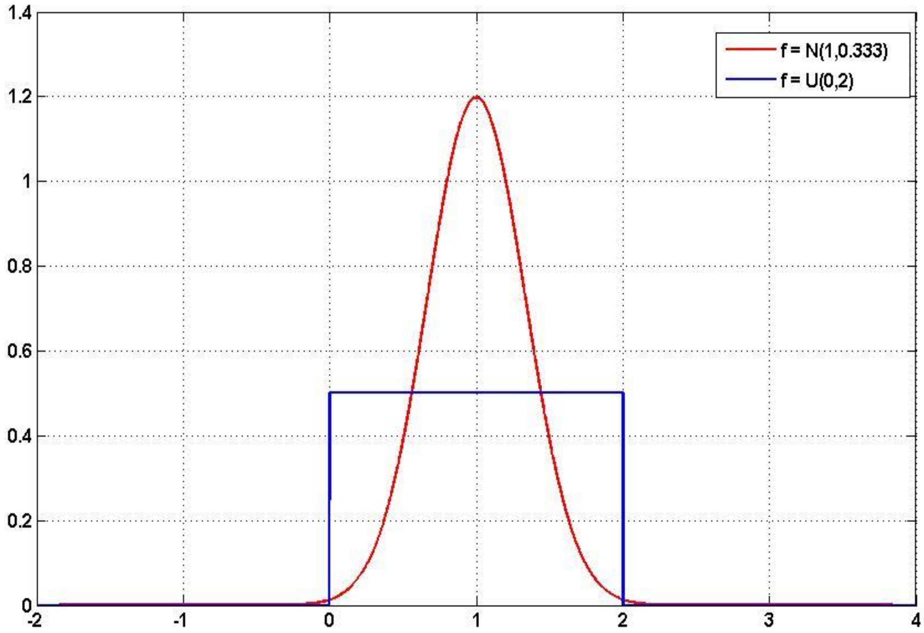


Figure 2(c): The graph of the null distribution H_0: $f = N(1,0.333)$ and the alternative distribution H_1: $f = U(0,2)$

Table 2(d): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes(n) and number of intervals(K)and different kernel functions(Uniform,Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $U(0,2)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	-	-	0.0001	0.0022	0.0793	0.0098	0.0097
20	-	-	0.0063	0.0072	0.1718	0.0379	0.0665
30	0.3001	-	0.0133	0.0210	0.2931	0.0729	0.1412
50	0.6466	0.5234	0.0267	0.0734	0.5750	0.1693	0.2998
100	0.9620	0.9492	0.1417	0.3556	0.9489	0.4806	0.7196
200	1.0000	0.9999	0.6097	0.9108	1.0000	0.9065	0.9869
300	1.0000	1.0000	0.9161	0.9908	1.0000	0.9916	0.9998
400	1.0000	1.0000	0.9912	1.0000	1.0000	0.9995	1.0000
500	1.0000	1.0000	0.9994	1.0000	1.0000	1.0000	1.0000

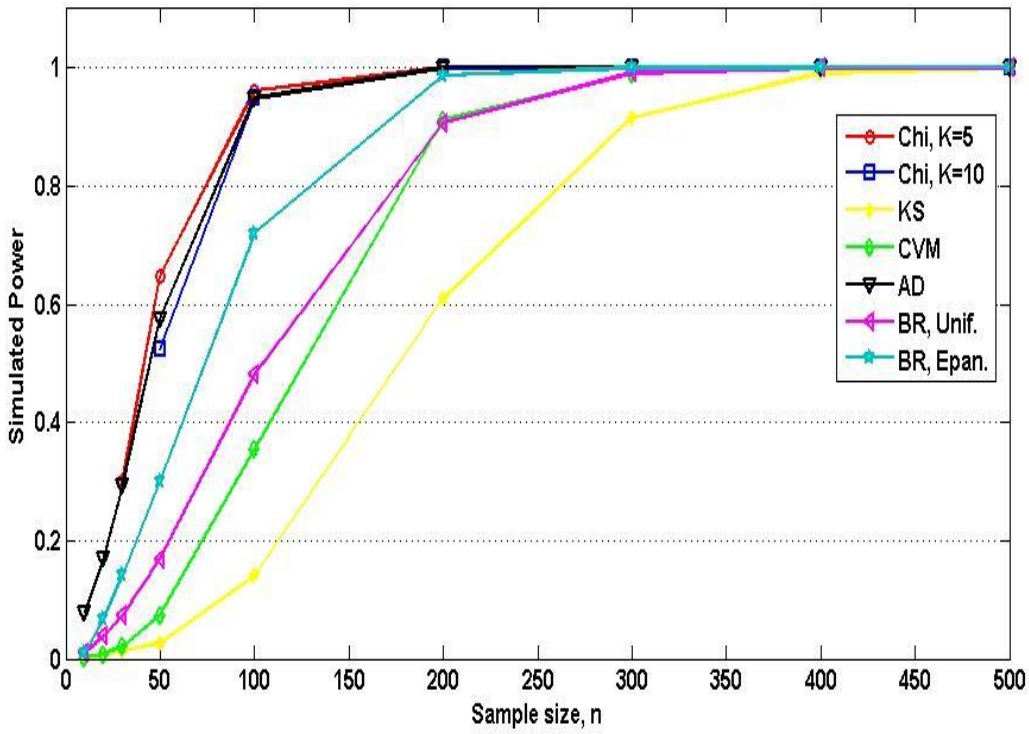


Figure 2(d): Comparison of power for different normality tests against $U(0,2)$

Summary

Tables 2(b), 2(d) and Figures 2(b), 2(d) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS, CVM,\ AD,\ BR$) for testing $H_0\colon Normal$ against different alternatives $H_1\colon U(0,1)$ and $H_1\colon U(0,2)$ at significance level $\alpha = 0.05$ and at different sample sizes n . We can conclude that BR test has a good power at sample sizes greater than or equal 400, and it is able to detect the null hypothesis at sample sizes greater than or equal 400 in both cases. BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel at sample sizes greater than or equal 30. BR test has a higher power than KS and CVM tests for n = 50, 100, 200, 300. In addition, all tests have good powers at sample sizes greater than or equal 200 except the KS test.

One important thing is that BR test has a lower power when testing normality and the data generated from $U(0,2)$ than that of $U(0,1)$ for sample sizes n = 100, 200, 300, 400. In addition, the power of KS and AD tests decrease when the data generated from the $U(0,2)$ than that of $U(0,1)$ for sample sizes n = 10, 30, 50, 100 .

Case 3(a): Testing normality against $t(7)$

Table 3(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05.

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	9.488	16.919	0.409	0.212	0.752	1.638773	0.9751941
20	9.488	16.919	0.232	0.217	0.752	1.537142	0.9154994
30	9.488	16.919	0.19	0.218	0.752	1.485362	0.8850858
50	9.488	16.919	0.149	0.220	0.752	1.427173	0.8509074
100	9.488	16.919	0.107	0.220	0.752	1.359208	0.8109871
200	9.488	16.919	0.075	0.220	0.752	1.302057	0.7774183
300	9.488	16.919	0.061	0.220	0.752	1.272939	0.7603156
400	9.488	16.919	0.053	0.220	0.752	1.253999	0.7491904
500	9.488	16.919	0.047	0.220	0.752	1.240217	0.7410956
1000	9.488	16.919	0.034	0.220	0.752	1.201998	0.7186468
2000	9.488	16.919	0.024	0.220	0.752	1.169859	0.6997697

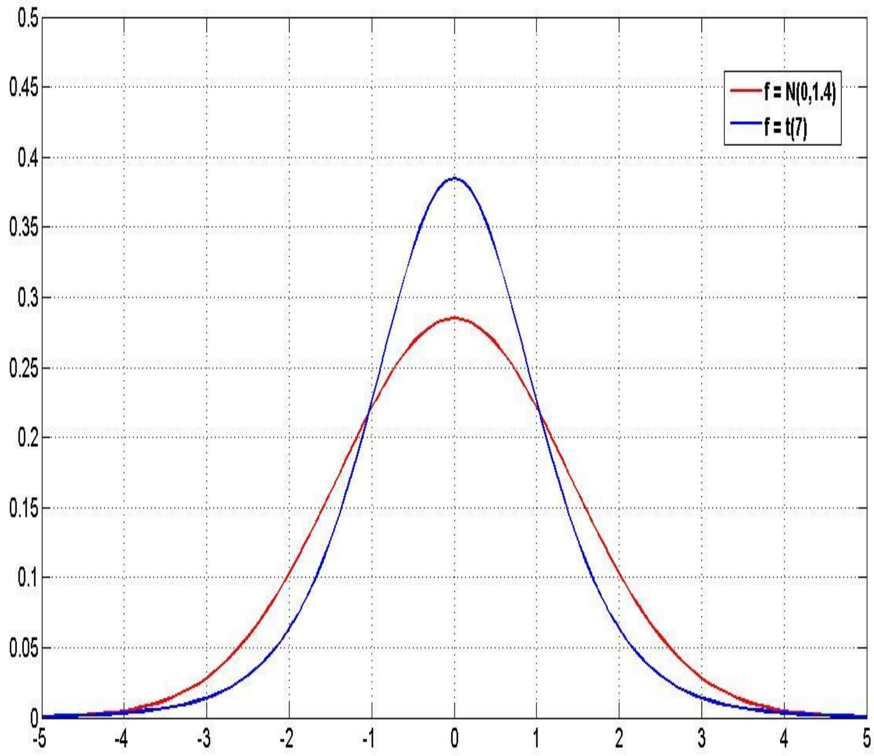


Figure 3(a): The graph of the null distribution H_0: $f = N(0,1.4)$ and the alternative distribution H_1: $f = t(7)$

Table 3(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $t(7)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10			0.0421	0.0085	0.0862	0.0129	0.0217
20	-	-	0.0437	0.0169	0.1177	0.0175	0.0355
30	0.035	-	0.0467	0.0215	0.1431	0.0228	0.0395
50	0.0554	0.1271	0.0529	0.0355	0.1785	0.0281	0.0585
100	0.1057	0.2253	0.0593	0.0595	0.2781	0.0380	0.0767
200	0.2113	0.3927	0.0935	0.1194	0.4496	0.0552	0.1053
300	0.3035	0.5484	0.1280	0.2133	0.5984	0.0642	0.1339
400	0.4111	0.6598	0.1625	0.2968	0.7115	0.0821	0.1597
500	0.5107	0.7531	0.2009	0.4064	0.8065	0.0966	0.1864
1000	0.8395	0.9521	0.4248	0.7958	0.9794	0.1767	0.3148
2000	0.9844	0.9898	0.8106	0.9918	0.9999	0.3481	0.5782

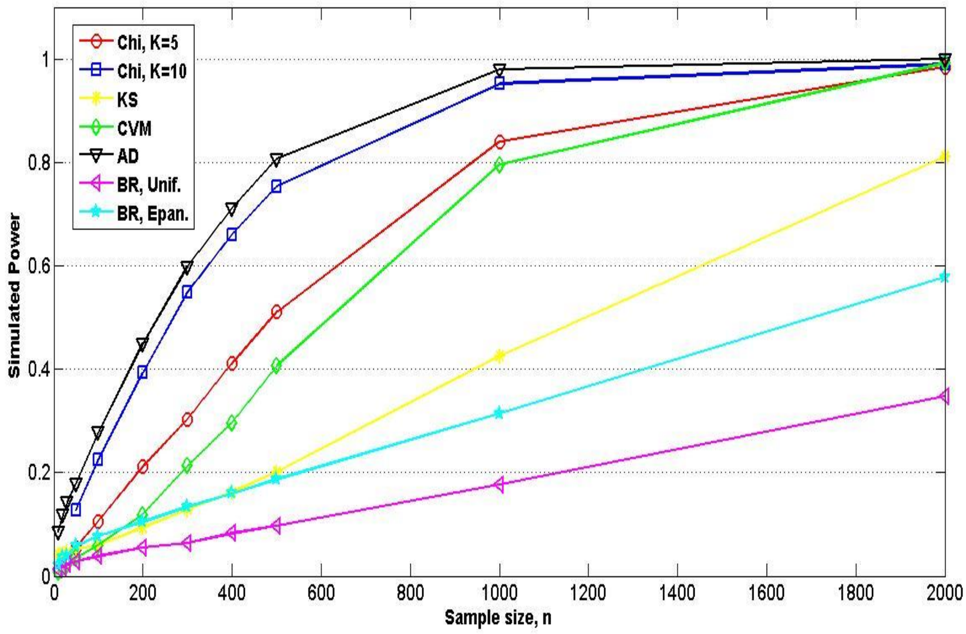


Figure 3(b): Comparison of power for different normality tests against $t(7)$

Case 3(b): Testing normality against $t(15)$

Table 3(c): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05 .

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	9.488	16.919	0.409	0.212	0.752	1.670389	0.9937645
20	9.488	16.919	0.232	0.217	0.752	1.563728	0.9311151
30	9.488	16.919	0.19	0.218	0.752	1.509386	0.8991963
50	9.488	16.919	0.149	0.220	0.752	1.448316	0.8633262
100	9.488	16.919	0.107	0.220	0.752	1.376988	0.82143
200	9.488	16.919	0.075	0.220	0.752	1.317007	0.7861997
300	9.488	16.919	0.061	0.220	0.752	1.286449	0.7682504
400	9.488	16.919	0.053	0.220	0.752	1.26657	0.7565747
500	9.488	16.919	0.047	0.220	0.752	1.252107	0.7480792

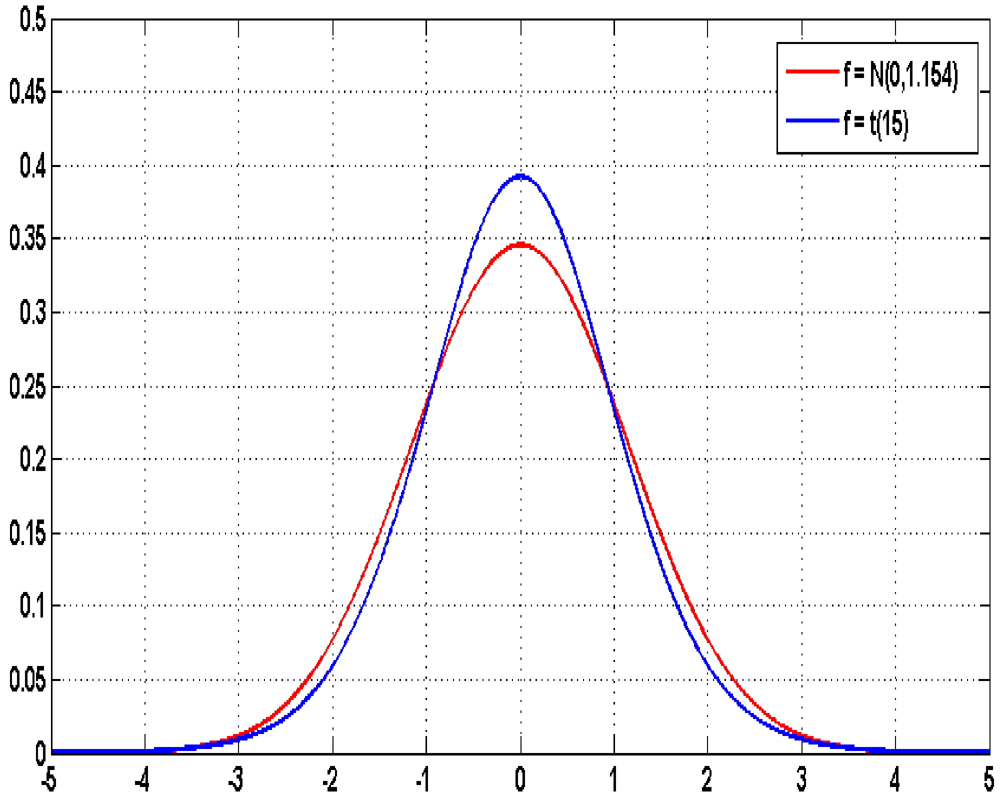


Figure 3(c): The graph of the null distribution H_0: $f = N(0,1.154)$ and the alternative distribution H_1: $f = t(15)$

Table 3(d): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes(n) and number of intervals(K)and different kernel functions(Uniform,Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $t(15)$

	χ^2					**BR**	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	-	-	0.0001	0.0030	0.0603	0.0116	0.0139
20	-	-	0.0051	0.0049	0.0710	0.0134	0.0270
30	0.0236	-	0.0056	0.0065	0.0780	0.0179	0.0303
50	0.0259	0.0583	0.0073	0.0083	0.0846	0.0181	0.0385
100	0.0314	0.0869	0.0089	0.0097	0.0996	0.0237	0.0443
200	0.0470	0.1271	0.0125	0.0127	0.1340	0.0305	0.0587
300	0.0621	0.1615	0.0150	0.0155	0.1551	0.0309	0.0614
400	0.0772	0.2021	0.0193	0.0185	0.1897	0.0375	0.0646
500	0.0934	0.2365	0.0238	0.0260	0.2283	0.0410	0.0671

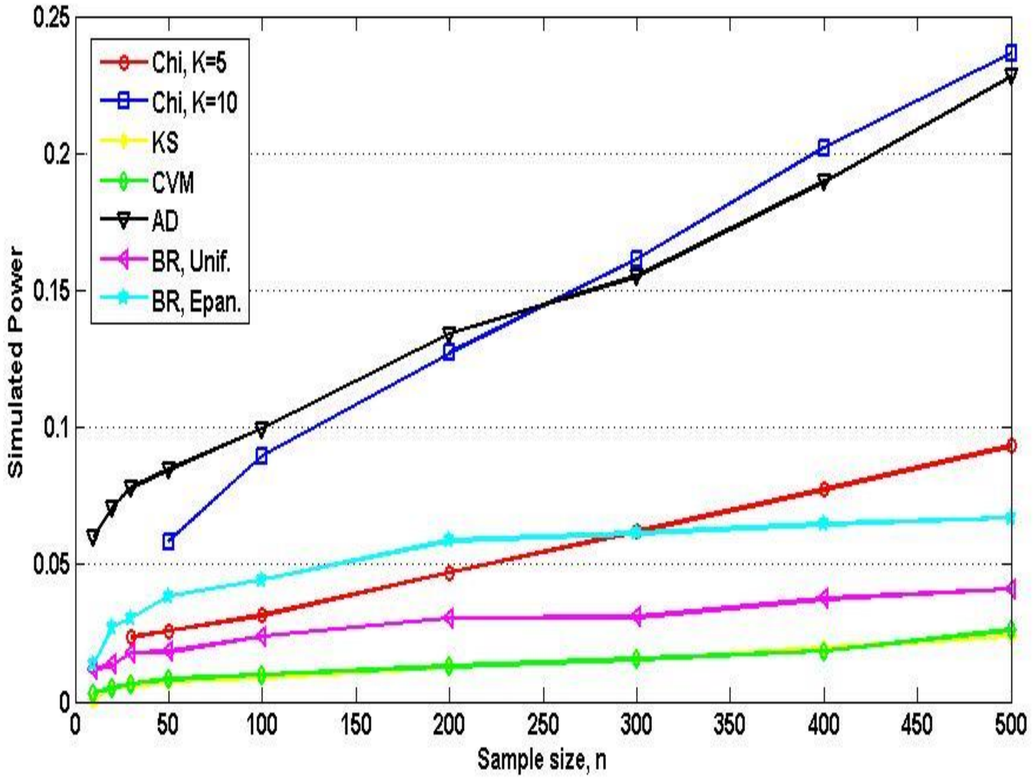


Figure 3(d): Comparison of power for different normality tests against $t(15)$

Case 3(c): Testing Normality against $t(4)$

Table 3(e): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and $\alpha = 0.05$.

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	9.488	16.919	0.409	0.212	0.752	1.58428	0.9431867
20	9.488	16.919	0.232	0.217	0.752	1.491319	0.8885845
30	9.488	16.919	0.19	0.218	0.752	1.443957	0.8607655
50	9.488	16.919	0.149	0.220	0.752	1.390732	0.8295028
100	9.488	16.919	0.107	0.220	0.752	1.328565	0.7929881
200	9.488	16.919	0.075	0.220	0.752	1.276289	0.762283
300	9.488	16.919	0.061	0.220	0.752	1.249655	0.7466392
400	9.488	16.919	0.053	0.220	0.752	1.23233	0.7364632
500	9.488	16.919	0.047	0.220	0.752	1.219724	0.7290589

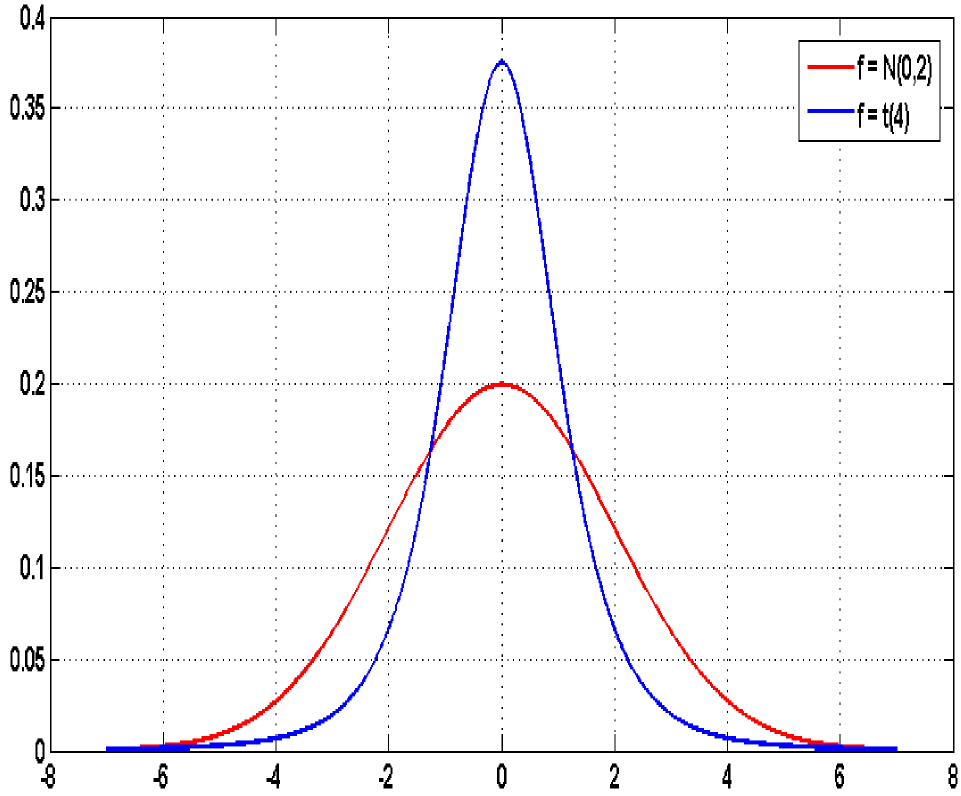


Figure 3(e): The graph of the null distribution H_0: $f = N(0,2)$ and the alternative distribution H_1: $f = t(4)$

Table 3(f): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes(n) and number of intervals(K)and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $t(4)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	-	-	0.0010	0.0242	0.1362	0.0128	0.0250
20	-	-	0.0331	0.0673	0.2284	0.0252	0.0486
30	0.0931	-	0.0628	0.0930	0.3049	0.0326	0.0685
50	0.1703	0.2839	0.1003	0.1584	0.4123	0.0476	0.0970
100	0.3396	0.5283	0.2143	0.3192	0.6582	0.0827	0.1738
200	0.6148	0.8049	0.4663	0.6276	0.8892	0.1592	0.3096
300	0.7879	0.9166	0.6932	0.8272	0.9704	0.2327	0.4430
400	0.8903	0.9559	0.8284	0.9304	0.9947	0.3202	0.5741
500	0.9338	0.9673	0.9158	0.9757	0.9983	0.4065	0.6616

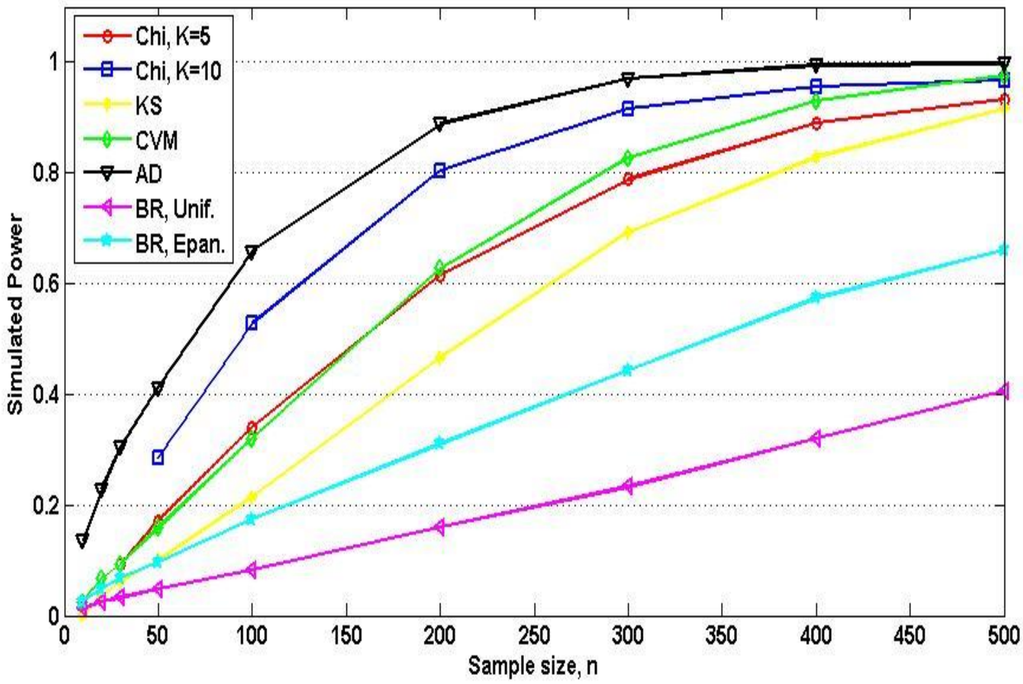


Figure 3(f): Comparison of power for different normality tests against $t(4)$

Summary

Table 3(b), 3(d) and 3(f) and Figure 3(b), 3(d), 3(f) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD, BR$) for testing H_0: $Normal$ against different alternatives H_1: $t(7),\ t(15), t(4)$ at significance level $\alpha = 0.05$ and at different sample sizes n. we can conclude that:

1. The BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel at different sample sizes n and for all cases.

2. The BR test has a higher power when using the Epanechnikov kernel than that of KS test at sample sizes n = 50, 100, 200, 300 when data generate from $t(7)$ and it's always has higher power than that of KS for all sample sizes n when the data generated from $t(15)$.

3. The BR test has a higher power when using the Epanechnikov kernel than that of CVM test at sample sizes n = 10, 20, 30, 50, 100 when the data generated from $t(7)$ and BR test has higher power than that KS and CVM tests for all sample sizes in the case data were generated from $t(15)$.

4. In addition, when the data are generated from $t(4)$ then AD test has a higher power followed by CVM and KS tests.

Remarks:

There are general remarks about the behavior of the power of GOF tests when data are generated from t-distribution with different degrees of freedom as in the following :

1. The power of GOF tests decrease when the data are generated from $t(15)$ than that data generated from $t(7)$, because when the number of degrees of freedom of t-distribution is large enough then the t-distribution is close to the standard normal distribution and the variance of the t-distributions close to 1 .

2. The power of GOF tests increase when testing normality and the data generated from $t(4)$ than that the data are generated from $t(7)$, this is because when the number of degrees of freedom is small then the t-distribution doesn't has the same variance of the standard normal distribution and then the two distribution doesn't have the same shape.

3.2.2 Comparison of power of GOF tests for testing symmetric distributions against skewed parametric alternative distributions.

Case 4: Testing Normality against $Exp(5)$

Table 4(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05 .

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	7.815	15.507	0.232	0.217	0.752	2.30649	1.367389
30	7.815	15.507	0.19	0.218	0.752	2.180547	1.293414
50	7.815	15.507	0.149	0.220	0.752	2.039013	1.210282
100	7.815	15.507	0.107	0.220	0.752	1.873702	1.113184
200	7.815	15.507	0.075	0.220	0.752	1.734693	1.031534
300	7.815	15.507	0.061	0.220	0.752	1.66387	0.9899354
400	7.815	15.507	0.053	0.220	0.752	1.617801	0.9628758
500	7.815	15.507	0.047	0.220	0.752	1.58428	0.9431867

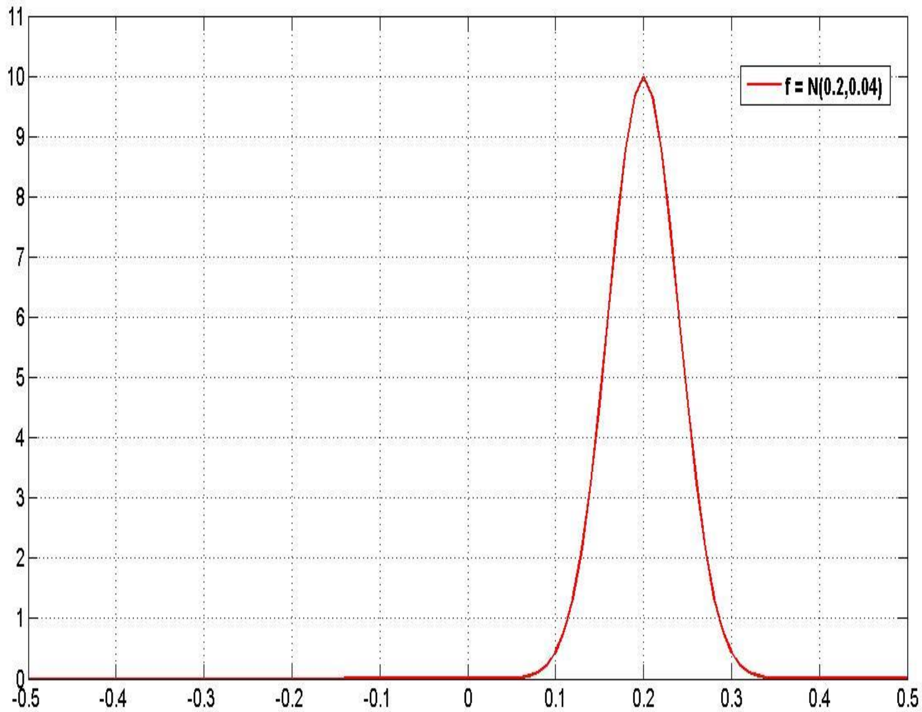


Figure 4(a): The graph of the null distribution H_0: $f = N(0.2,0.04)$

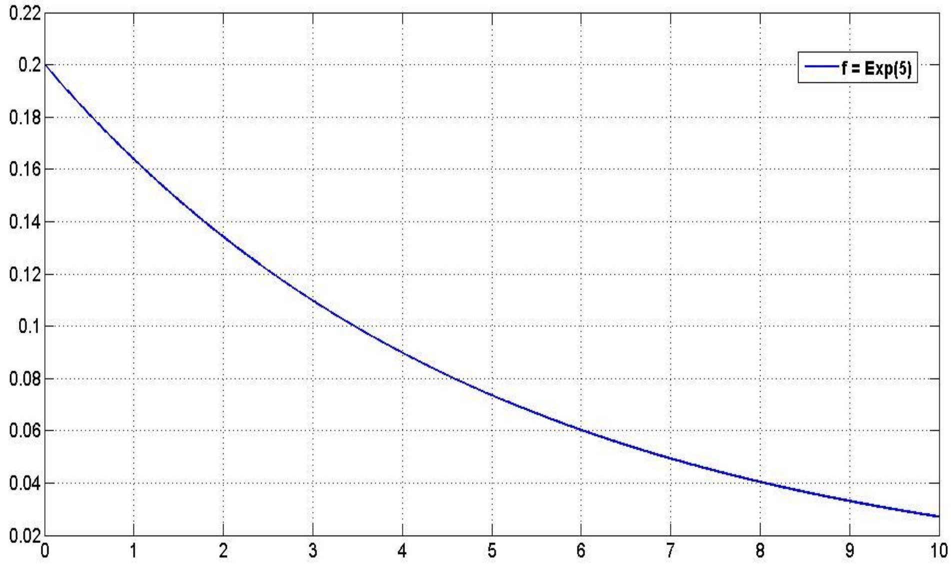


Figure 4(b): The graph of the alternative distribution $H_1: f = Exp(5)$

Table 4(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanechnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $Exp(5)$

	χ^2					BR	
$\boldsymbol{n}$	$(\boldsymbol{K = 5})$	$(\boldsymbol{K = 10})$	$\boldsymbol{KS}$	$\boldsymbol{CVM}$	$\boldsymbol{AD}$	$\boldsymbol{Uniform}$	$\boldsymbol{Epan.}$
20	-	-	0.2756	0.3695	0.7643	0.1474	0.0847
30	0.5678	-	0.5025	0.6365	0.9349	0.4240	0.2913
50	0.7269	0.9832	0.8201	0.9152	0.9958	0.8708	0.8112
100	0.9398	1.0000	0.9972	0.9997	1.0000	0.9996	0.9998
200	0.9987	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
300	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

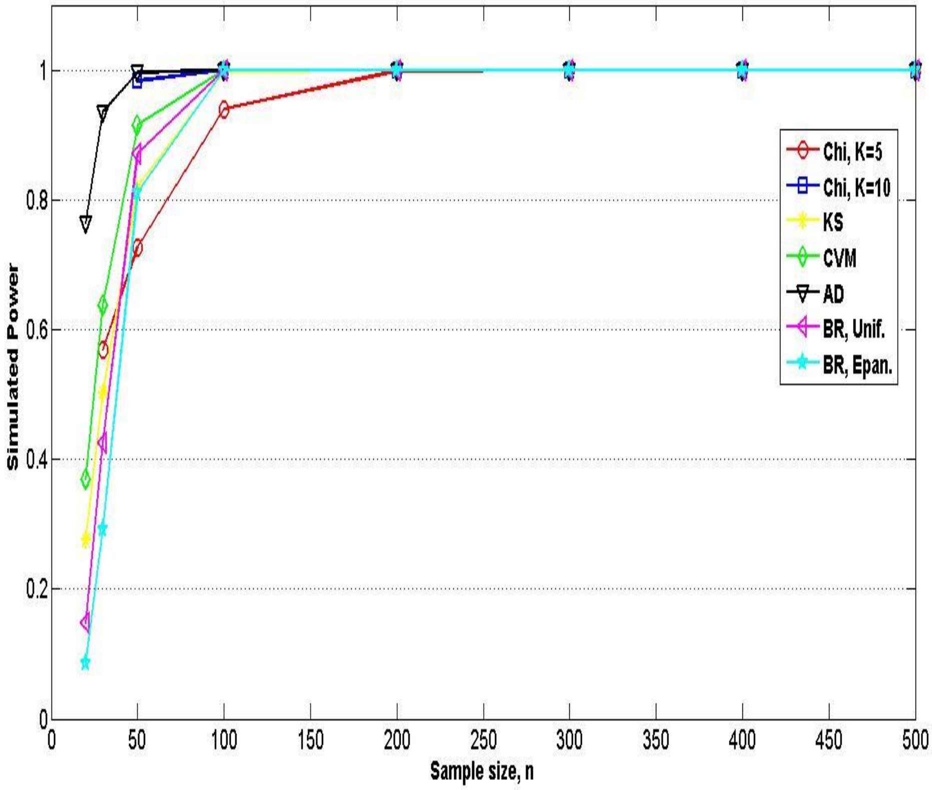


Figure 4(c): Comparison of power for different normality tests against $Exp(5)$

Summary

Table 4(b) and Figure 4(c) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD,\ BR$) for testing $H_0: N(0.5,1)$ against $H_1: Exp(5)$ at significance level $\alpha = 0.05$ and at different sample sizes n. We observe the following main findings:

1. AD test has a good power compared with all other tests, and it is able to detect the null hypothesis at small sample sizes $n = 30$, 50.

2. BR test has a higher power under uniform kernel than χ^2 at (K =5) and KS tests at sample sizes $n = 50$, 100.

3. In addition, all tests have good powers when the sample size is greater than or equal 100.

Case 5: Testing Normality against $Weibull(10, 2)$

Table 5(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05.

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Unifom$	**Epan.**
20	5.991	14.067	0.232	0.217	0.752	2.221204	1.310609
30	5.991	14.067	0.19	0.218	0.752	2.103482	1.242108
50	5.991	14.067	0.149	0.220	0.752	1.971188	1.165127
100	5.991	14.067	0.107	0.220	0.752	1.816668	1.075213
200	5.991	14.067	0.075	0.220	0.752	1.686733	0.9996049
300	5.991	14.067	0.061	0.220	0.752	1.620534	0.9610837
400	5.991	14.067	0.053	0.220	0.752	1.577472	0.9360263
500	5.991	14.067	0.047	0.220	0.752	1.546139	0.917794

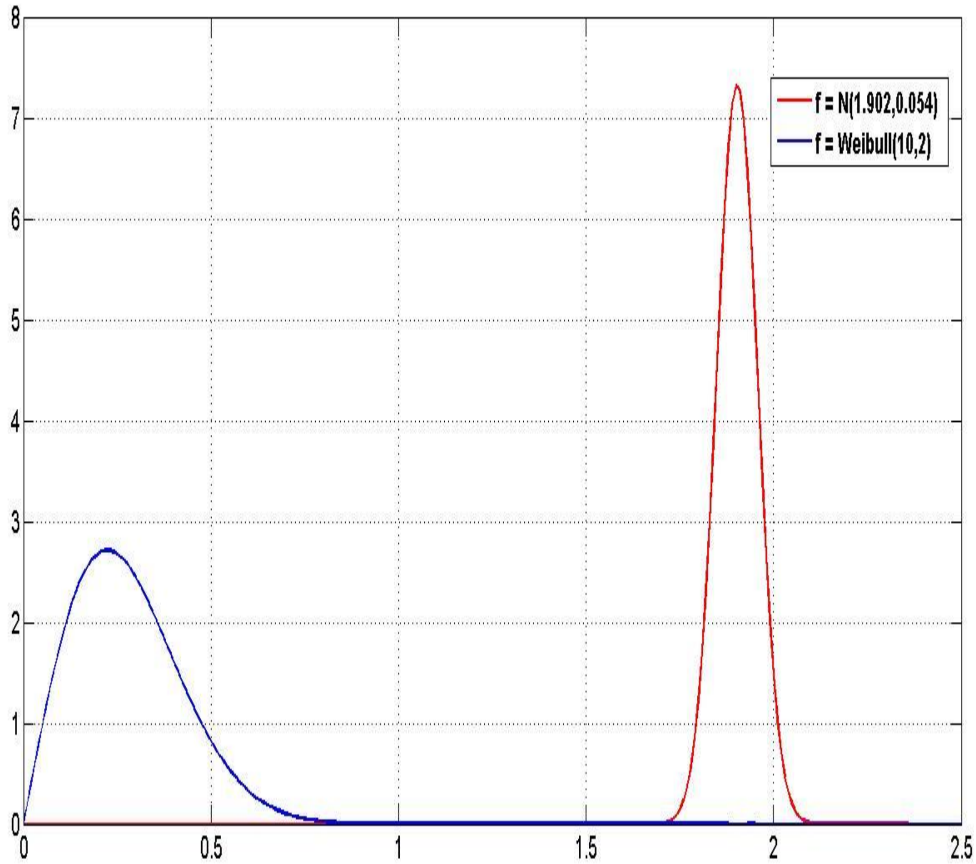


Figure 5(a): The graph of the null distribution H_0: $f = N(1.902,0.054)$ and the alternative distribution H_1: $f = Weibull(10,2)$

Table 5(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanechnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $Weibull(10,2)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	-	-	0.0209	0.0142	0.1358	0.0033	0.0019
30	0.1091	-	0.0313	0.0226	0.1826	0.0098	0.0085
50	0.1387	0.1221	0.0599	0.0521	0.2752	0.0233	0.0344
100	0.1923	0.2007	0.1338	0.1367	0.5156	0.0631	0.1119
200	0.3032	0.3748	0.3424	0.4135	0.8305	0.1622	0.2928
300	0.4197	0.5356	0.5629	0.6699	0.9543	0.2690	0.4652
400	0.5436	0.6757	0.7333	0.8373	0.9897	0.3932	0.6177
500	0.6600	0.7927	0.8569	0.9339	0.9971	0.4985	0.7472

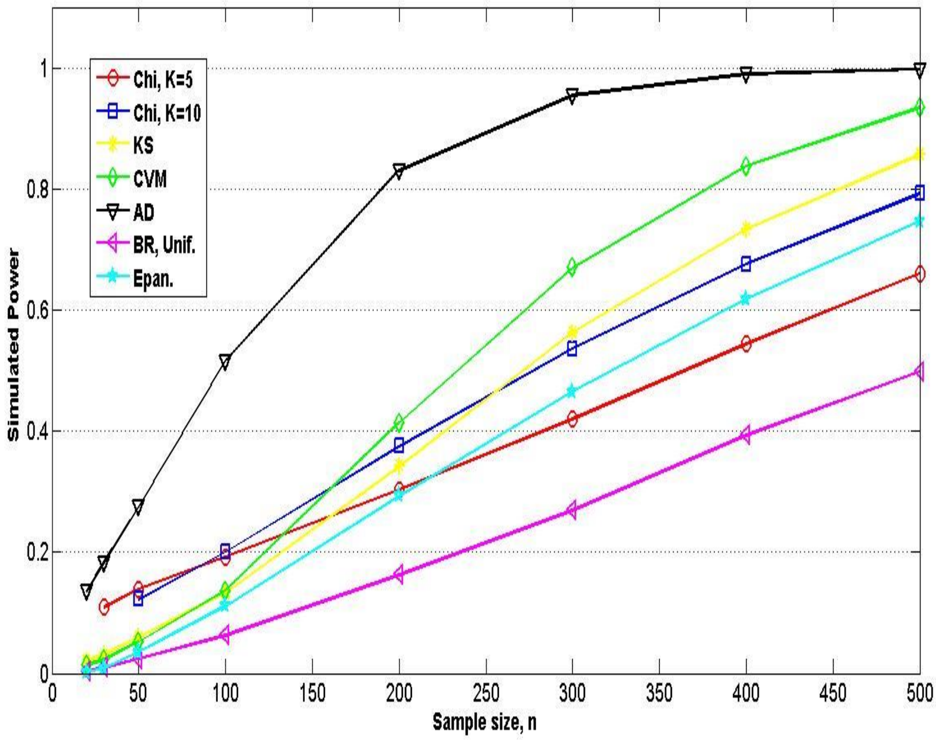


Figure 5(b): Comparison of power for different normality tests against $Weibull(10,2)$

Summary

Table 5(b) and Figure 5(b) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD, BR$) for testing $H_0\colon Normal$ against $H_1\colon Weibull(10,2)$ at significance level $\alpha = 0.05$ and at different sample sizes n. We observe the following main findings:

1. AD test has a higher power compared with the other 3 tests, and it is able to detect the null hypothesis at sample sizes greater than or equal 300.

2. BR test has a higher power at Epanechnikov kernel than uniform kernel at sample sizes greater than or equal 50.

3. In addition, the power of Chi-square test is larger when we grouped the data into ten intervals than that of five intervals at sample sizes $n \geq 100$.

<u>Case 6: Testing Normality against $\boldsymbol{Gamma(4,5)}$</u>

Table 6(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05 .

	$\boldsymbol{\chi^2}$					**BR**	
$\boldsymbol{n}$	$\boldsymbol{K=5}$	$\boldsymbol{K=10}$	$\boldsymbol{KS}$	$\boldsymbol{CVM}$	$\boldsymbol{AD}$	$\boldsymbol{Uniform}$	$\boldsymbol{Epan.}$
10	9.488	16.919	0.4092	0.212	0.752	2.098622	1.245295
20	9.488	16.919	0.2940	0.217	0.752	1.923828	1.142626
30	9.488	16.919	0.2417	0.218	0.752	1.834772	1.090318
50	9.488	16.919	0.189	0.220	0.752	1.734693	1.031534
100	9.488	16.919	0.1360	0.220	0.752	1.617801	0.9628758
200	9.488	16.919	0.0961	0.220	0.752	1.519506	0.9051409
300	9.488	16.919	0.0785	0.220	0.752	1.469427	0.875726
400	9.488	16.919	0.0680	0.220	0.752	1.436851	0.8565919
500	9.488	16.919	0.0608	0.220	0.752	1.413148	0.8426697
1000	9.488	16.919	0.0430	0.220	0.752	1.347415	0.80406
2000	9.488	16.919	0.0304	0.220	0.752	1.29214	0.7715934

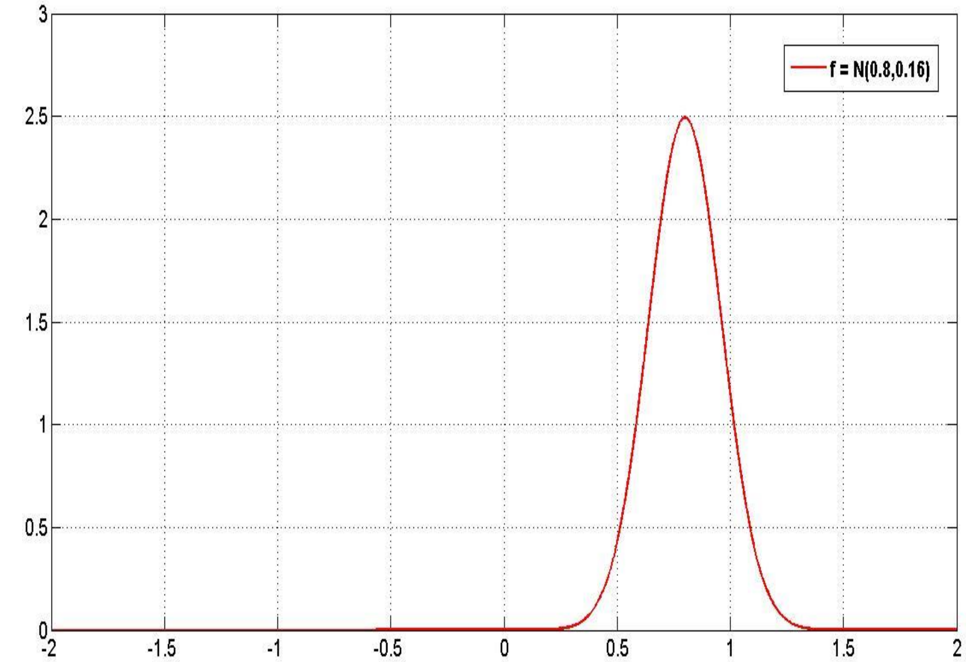


Figure 6(a): The graph of the null distribution H_0: $f = N(0.8{,}0.16)$

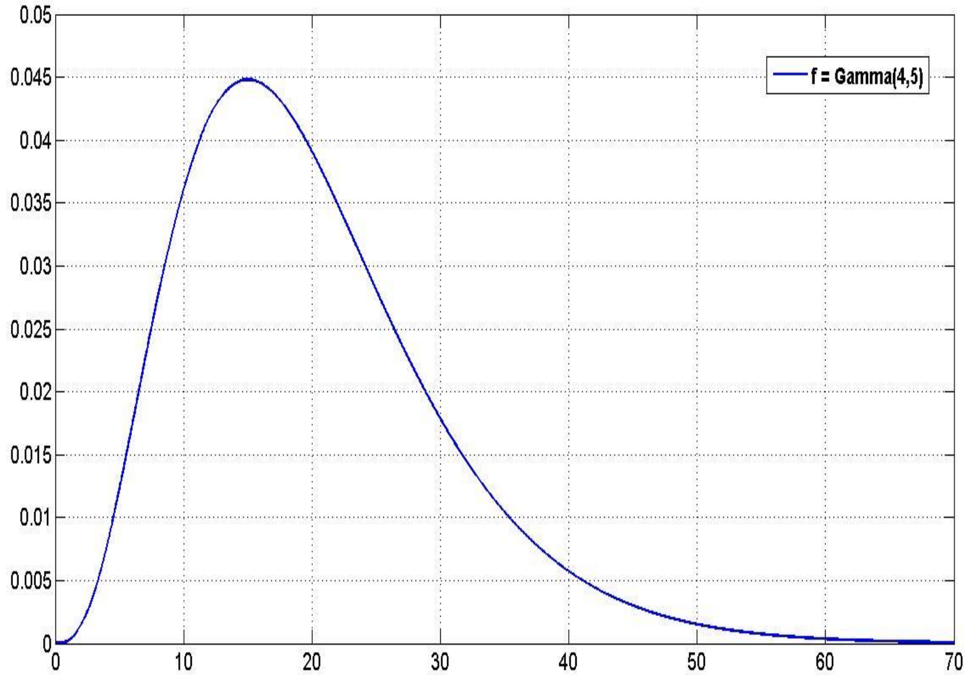


Figure 6(b): The graph of the alternative distribution H_1: $f = Gamma(4{,}5)$

Table 6(b): The power for tests (χ^2, KS, CVM , AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $Gamma(4,5)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	-	-	0.0669	0.0137	0.1285	0.0049	0.0052
20	-	-	0.0861	0.0407	0.2469	0.0233	0.0349
30	0.1265	-	0.1078	0.0812	0.3765	0.0427	0.0669
50	0.2493	0.2771	0.1495	0.1733	0.5908	0.0836	0.1501
100	0.5270	0.5699	0.2423	0.4925	0.8925	0.2069	0.3851
200	0.8658	0.8934	0.4424	0.9058	0.9970	0.5020	0.7376
300	0.9779	0.9831	0.6233	0.9914	1.0000	0.7315	0.9201
400	0.9965	0.9981	0.7568	0.9993	1.0000	0.8614	0.9813
500	0.9994	0.9995	0.8738	1.0000	1.0000	0.9367	0.9937
1000	0.9999	1.0000	0.9999	1.0000	1.0000	0.9997	1.0000
2000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

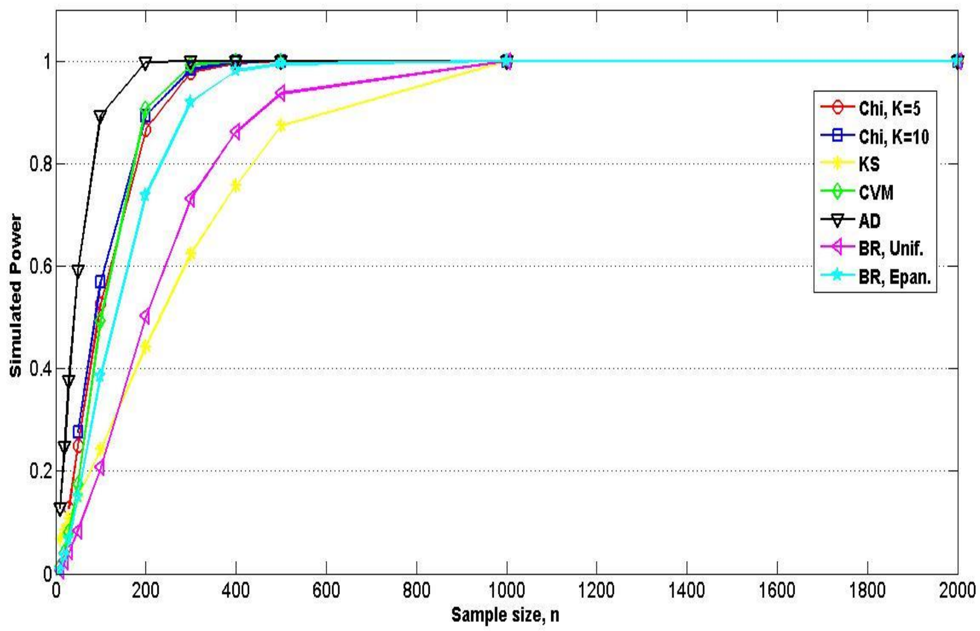


Figure 6(c): Comparison of power for different normality tests against $Gamma(4,5)$

Summary

If we look at the Table 6(b) and Figure 6(c) that summarize the simulated power of Goodness-of-fit tests(χ^2, KS, CVM, AD, BR) for testing H_0: $Normal$ against H_1 : $Gamma(4,5)$ at significance level $\alpha = 0.05$ and at different sample sizes n. We can conclude that: The AD test has a higher power compared with other tests at different sample sizes n followed by CVM test at sample sizes greater than or equal 200. And BR test has a higher power than KS test at sample sizes greater than or equal 50. In addition, BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel at different sample sizes n.

Case 7: Testing Normality against $\chi^2(4)$

Table 7(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under normal distribution tests with different sample sizes n and α = 0.05.

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	9.488	16.919	0.409	0.212	0.752	1.413148	0.8426697
20	9.488	16.919	0.232	0.217	0.752	1.347415	0.80406
30	9.488	16.919	0.19	0.218	0.752	1.313925	0.7843891
50	9.488	16.919	0.149	0.220	0.752	1.276289	0.762283
100	9.488	16.919	0.107	0.220	0.752	1.23233	0.7364632
200	9.488	16.919	0.075	0.220	0.752	1.195366	0.7147514
300	9.488	16.919	0.061	0.220	0.752	1.176533	0.7036896
400	9.488	16.919	0.053	0.220	0.752	1.164282	0.696494
500	9.488	16.919	0.047	0.220	0.752	1.155369	0.6912584
1000	9.488	16.919	0.034	0.220	0.752	1.130649	0.6767389
2000	9.488	16.919	0.024	0.220	0.752	1.109862	0.6645295

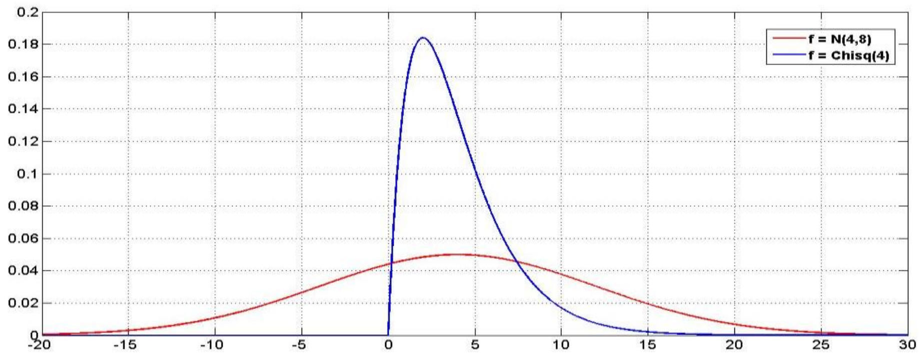


Figure 7(a): The graph of the null distribution H_0: $f = N(4,8)$ and the alternative distribution H_1: $f = \chi^2(4)$

Table 7(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: Normal having the same mean and variance as in H_1

H_1: $\chi^2(4)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
10	-	-	0.0801	0.0367	0.2196	0.0406	0.0673
20	-	-	0.1205	0.1196	0.4620	0.0598	0.1247
30	0.3472	-	0.1584	0.2331	0.6617	0.0804	0.1675
50	0.6305	0.6014	0.2402	0.4905	0.8891	0.1262	0.2747
100	0.9379	0.9358	0.4391	0.9110	0.9971	0.2709	0.5164
200	0.9994	0.9982	0.8417	0.9992	1.0000	0.5726	0.8499
300	0.9998	1.0000	1.0000	1.0000	1.0000	0.8028	0.9703
400	1.0000	1.0000	1.0000	1.0000	1.0000	0.9180	1.0000
500	1.0000	1.0000	1.0000	1.0000	1.0000	0.9674	1.0000
1000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

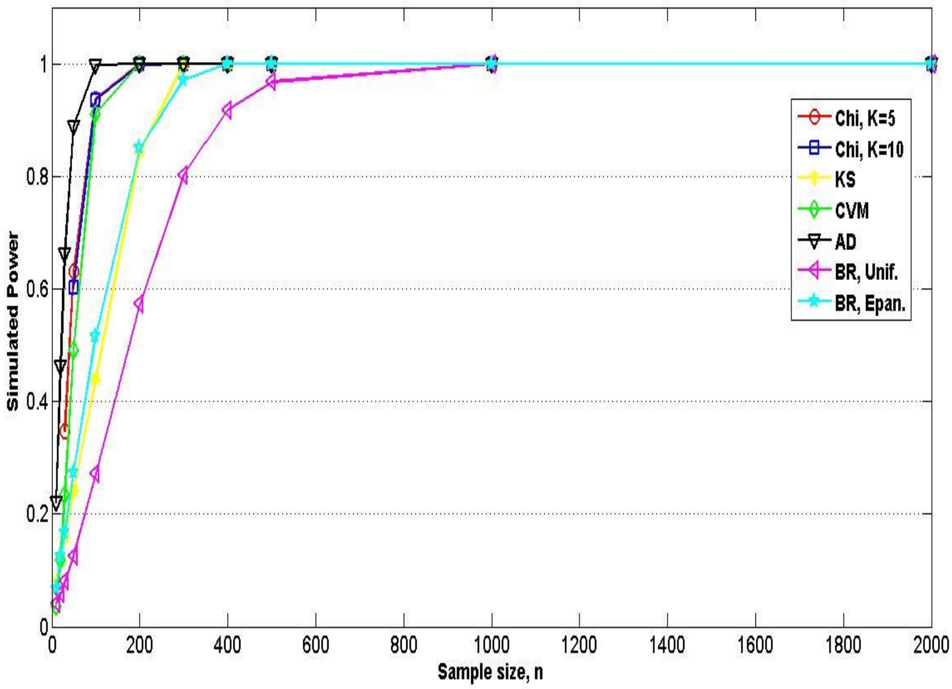


Figure 7(b): Comparison of power for different normality tests against $\chi^2(4)$

Summary

Table 7(b) and Figure 7(b) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD, BR$) for testing $H_0\!: Normal$ against $H_1\!: \chi^2(4)$ at significance level $\alpha = 0.05$ and at different sample sizes n. We observe the following main findings:

1. The AD test has a higher power compared with other all tests at different sample sizes n.

2. BR test has a higher power when using the Epanechnikov kernel than KS test at sample sizes $n = 20, 30, 50, 100, 200$.

3. BR test has a higher power when using the Epanechnikov kernel than CVM test at sample sizes $n = 10, 20$.

Case 8: Testing T-distribution against $Exp(0.9)$

Table 8(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under T distribution tests with different sample sizes n and α = 0.05.

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	9.488	16.919	0.232	0.217	0.752	1.566328.	0.9326425
30	9.488	16.919	0.19	0.218	0.752	1.511735	0.9005765
50	9.488	16.919	0.149	0.220	0.752	1.450384	0.8645409
100	9.488	16.919	0.107	0.220	0.752	1.378727	0.8224515
200	9.488	16.919	0.075	0.220	0.752	1.31847	0.7870586
300	9.488	16.919	0.061	0.220	0.752	1.28777	0.7690266
400	9.488	16.919	0.053	0.220	0.752	1.2678	0.7572969
500	9.488	16.919	0.047	0.220	0.752	1.25327	0.7487623

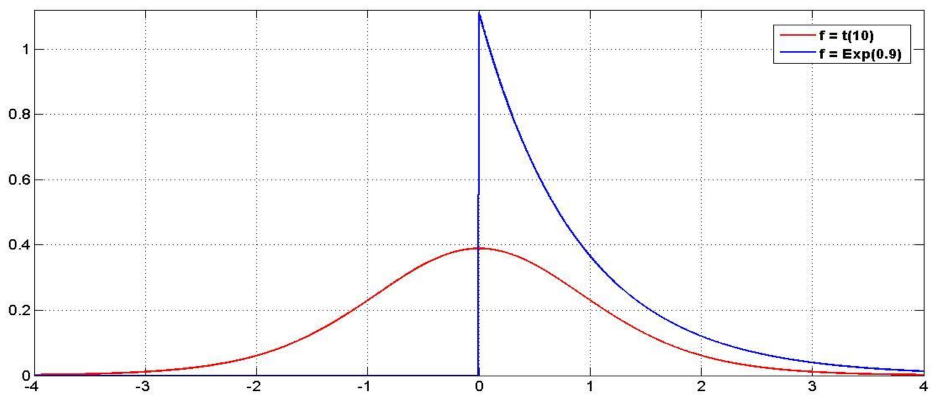


Figure 8(a): The graph of the null distribution $H_0\!: f = t(10)$ and the alternative distribution $H_1\!: f = Exp(0.9)$.

Table 8(b): The power for tests (χ^2, KS, CVM , AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: T distribution having the same variance as in H_1

H_1: $Exp(0.9)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	-	-	1.0000	1.0000	1.0000	0.7496	0.9917
30	1.0000	-	1.0000	1.0000	1.0000	0.9848	1.0000
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
300	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

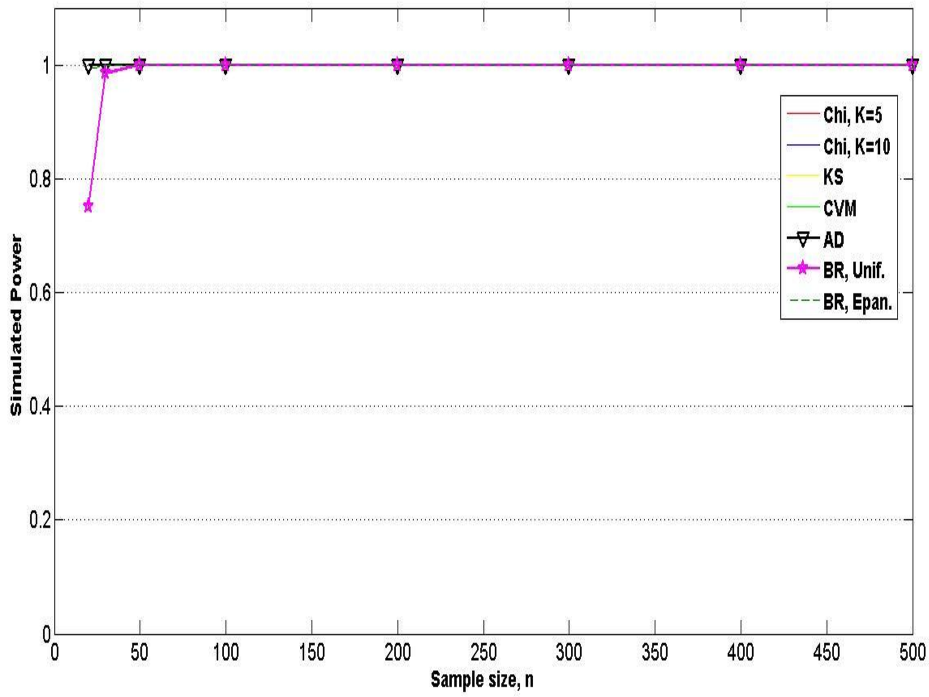


Figure 8(b): Comparison of power of GOF tests for testing $t(10)$ against $Exp(0.9)$

Case 9: Testing Uniformity against $Exp(0.9)$

Table 9(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under uniform distribution tests with different sample sizes n and α = 0.05.

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	9.488	16.919	0.232	0.217	2.5020	1.668319	0.9925484
30	9.488	16.919	0.19	0.218	2.5130	1.603894	0.9547075
50	9.488	16.919	0.149	0.220	2.4941	1.531494	0.9121822
100	9.488	16.919	0.107	0.220	2.4901	1.446932	0.8625129
200	9.488	16.919	0.075	0.220	2.4901	1.375823	0.8207462
300	9.488	16.919	0.061	0.220	2.4901	1.339595	0.7994667
400	9.488	16.919	0.053	0.220	2.4901	1.316028	0.7856247
500	9.488	16.919	0.047	0.220	2.4901	1.298881	0.775553

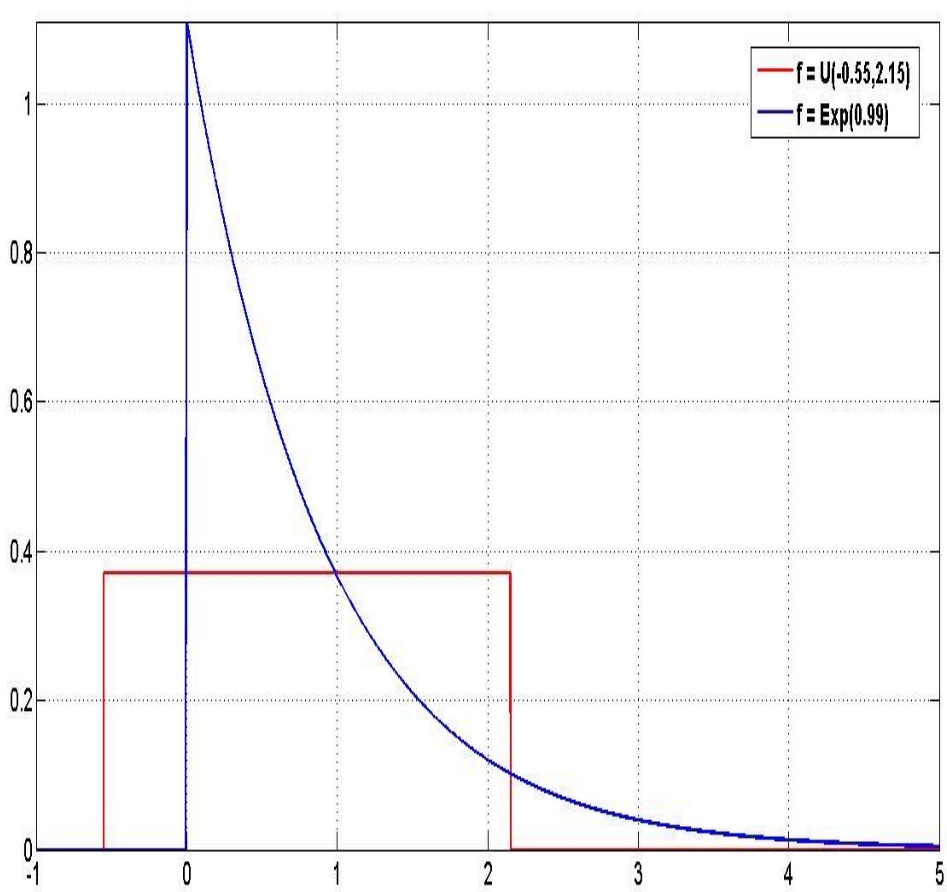


Figure 9(a): The graph of the null distribution $H_0: f = U(-0.55,2.15)$ and the alternative distribution $H_1: f = Exp(0.9)$

Table 9(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.05$.

Data were generated from the Alternative Hypothesis with different sample sizes (n) and number of intervals (K) and different kernel functions(Uniform, Epanchnikov) for 10000 simulations.

H_0: Uniform having the same mean and variance as in H_1

H_1: $Exp(0.9)$

	χ^2					BR	
n	$K = 5$	$K = 10$	KS	CVM	AD	$Uniform$	$Epan.$
20	-	-	0.2602	0.5452	0.9740	0.9885	0.9995
30	0.9453	-	0.5406	0.7927	0.9985	0.9998	1.0000
50	0.9951	0.9979	0.9053	0.9711	1.0000	1.0000	1.0000
100	1.0000	1.0000	0.9997	0.9999	1.0000	1.0000	1.0000
200	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
300	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
400	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

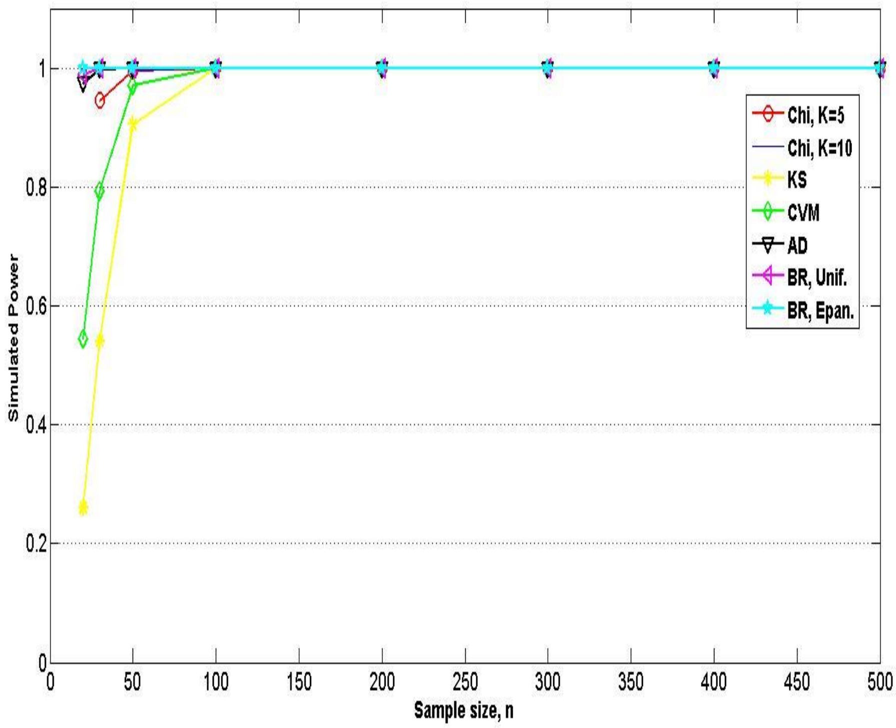


Figure 9(b): Comparison of power of GOF tests for testing $U(-0.55,2.15)$ against $Exp(0.9)$

Summary

Table 9(b) and Figure 9(b) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD, BR$) for testing H_0: $Uniform$ against H_1: $Exp(0.9)$ at significance level $\alpha = 0.05$ and at different sample sizes n. One can say that: The BR test has a higher power compared with other tests. The BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel at sample sizes n = 20, 30. And, all tests are able to detect the null hypothesis at small sample sizes n = 20, 30, 50 except the KS and CVM tests that have a small power at n = 20, 30.

3.2.3 Comparison of power of GOF tests for testing symmetric distributions against non-parametric alternative distributions.

Case 10: Testing Uniformity against Non-parametric alternative $A(\theta)$

Table 10(a): Critical values, $CV(\alpha, n)$ of the (χ^2, KS, CVM, AD, BR) under uniform distribution tests with different sample sizes n and α = 0.1.

					BR	
n	χ^2	KS	CVM	AD	$Uniform$	$Epan.$
10	-	0.368	0.167	1.9518	2.017947	1.197908
20	6.251	0.264	0.172	1.9385	1.855988	1.102779
40	12.017	0.193	0.173	1.9362	1.719797	1.022785

Remark: Concerning χ^2 test We used in Table 10(a) the number of degrees of freedom 3 for n = 20 and 7 for n = 40.

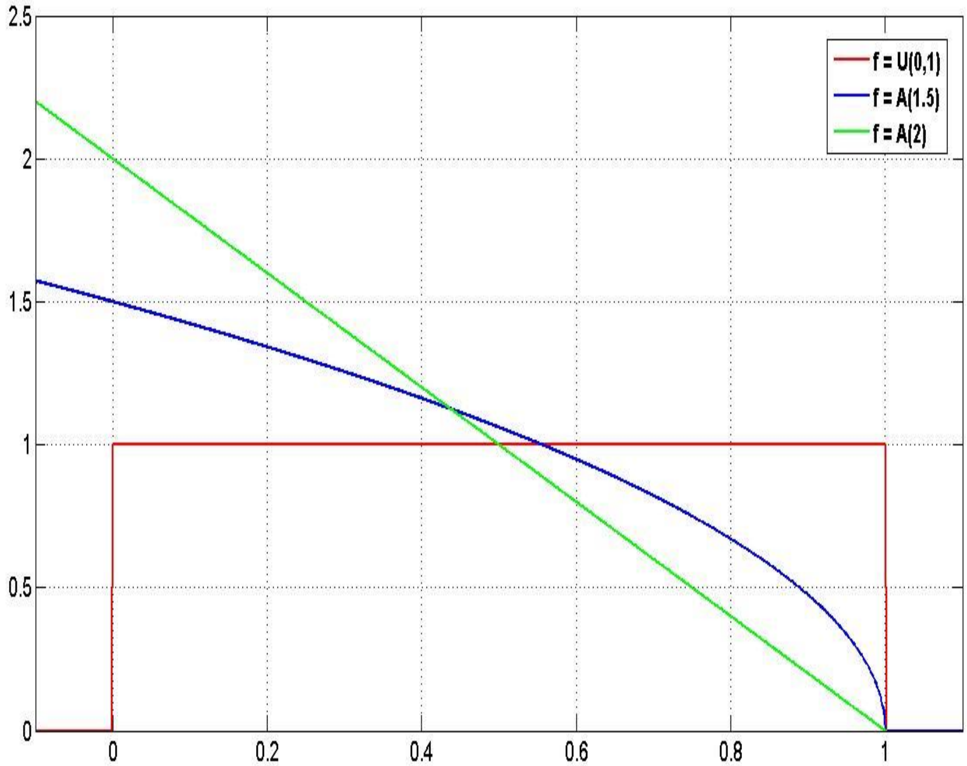


Figure 10(a): The graph of the null distribution $H_0: f = U(0{,}1)$and the nonparametric alternative distribution $H_1: f = A(\theta)\ at\ \theta = 1.5, 2$

Table 10(b): The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis (H_0) at $\alpha = 0.1$.

Data were generated from the Nonparametric Alternative Hypothesis $A(\theta)$ with different sample sizes(n) and number of intervals(K) and different kernel functions(Uniform, Epanchnikov) for 1000 simulations.

$$H_0: U(0,1)$$

$$H_1: A(\theta) = \theta(1-x)^{\theta-1}, 0 \leq x \leq 1, \qquad \theta \in \mathcal{R}$$

						BR	
N	n	χ^2	KS	CVM	AD	$Uniform$	$Epan.$
$A, \theta = 1.5$	10	-	0.23	0.27	0.24	0.06	0.09
	20	-	0.38	0.46	0.46	0.16	0.24
	40	0.40	0.60	0.70	0.74	0.35	0.54
$A, \theta = 2.0$	10	-	0.54	0.60	0.58	0.14	0.23
	20	0.59	0.78	0.87	0.87	0.43	0.63
	40	0.89	0.98	0.99	0.99	0.87	0.95

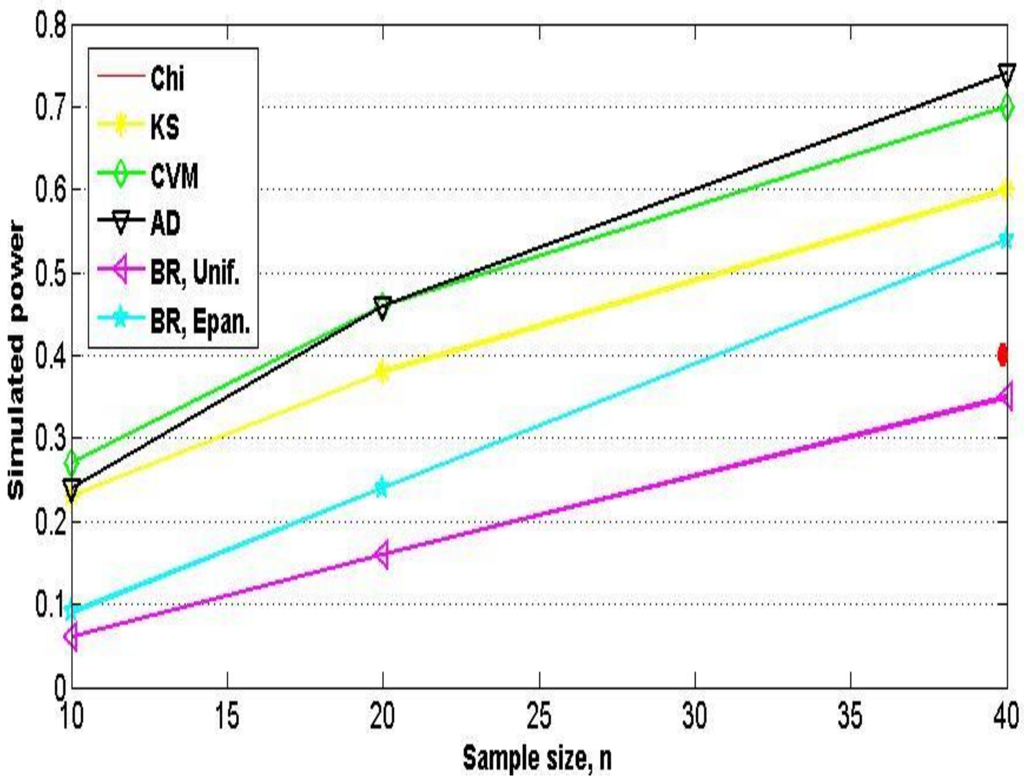


Figure 10(b): Comparison of power for different uniformity tests against $A(\theta)$, $\theta = 1.5$

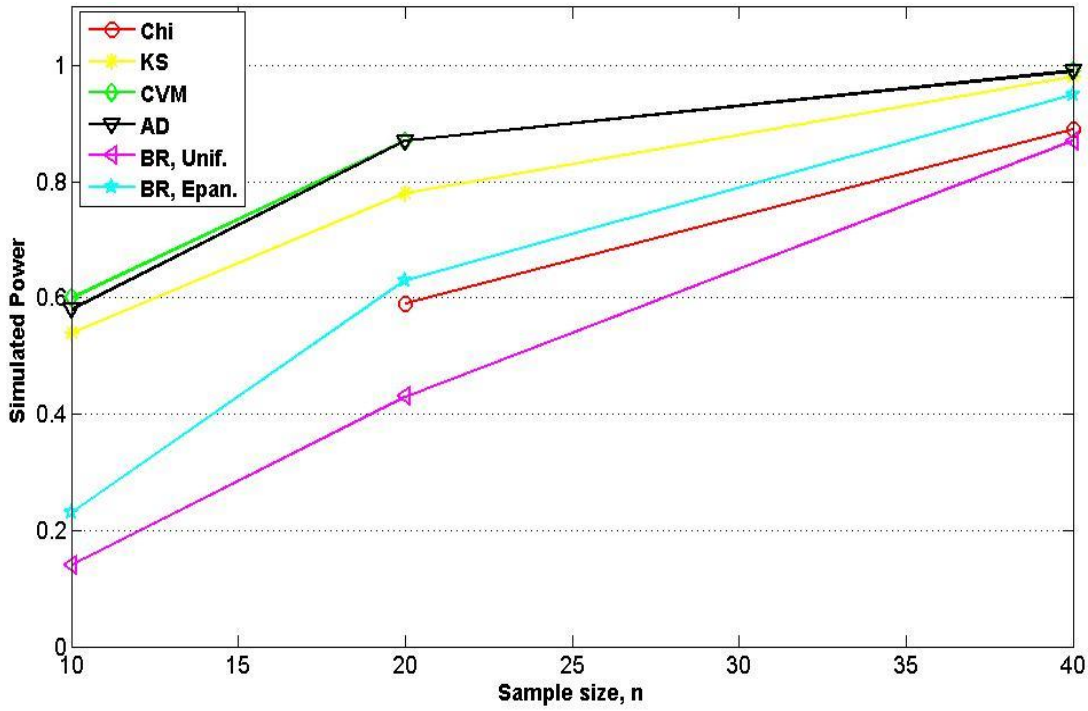


Figure 10(c): Comparison of power for different uniformity tests against $A(\theta),\ \theta = 2$

Summary

Table 10(b) and Figures 10(b) and 10(c) summarize the simulated power of Goodness-of-fit tests $(\chi^2, KS,\ CVM,\ AD,\ BR)$ for testing $H_0\!: U(0,1)$ against nonparametric alternatives $H_1\!: A(\theta)$ at significance level $\alpha = 0.1$ and at sample sizes $n =$ 10, 20, 40. We observe the following main findings: The AD test has a higher power compared with all other tests at sample sizes $n =$ 10, 20 under nonparametric alternatives $A(\theta)$, $\theta =$ 1.5, 2. BR test has a higher power when using the Epanechnikov kernel than χ^2 test at sample size $n =$ 40 when using $A(1.5)$ and sample sizes $n =$ 20, 40 when using $A(2)$. In addition, BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel for all sample sizes n.

Case 11: Testing Uniformity against Non-parametric alternative $B(\theta)$

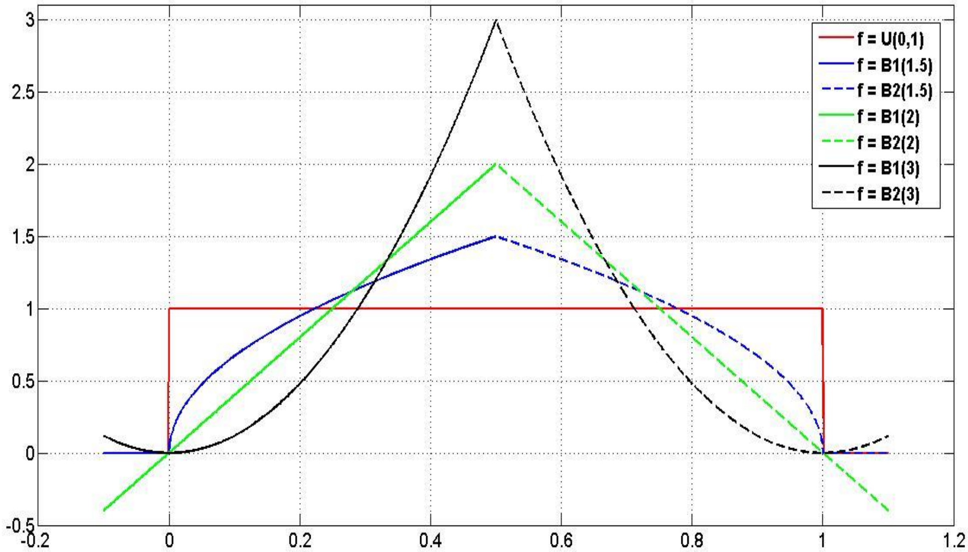


Figure 11(a): The graph of the null distribution $H_0: f = U(0,1)$ and the nonparametric alternative distribution $H_1: f = B(\theta)\ at\ \theta = 1.5, 2, 3$

Table 11: The power for tests (χ^2, KS, CVM, AD, BR) under the Null Hypothesis(H_0) at $\alpha = 0.1$.

Data were generated from the Nonparametric Alternative Hypothesis $B(\theta)$ with different sample sizes (n) and different kernel functions (Uniform, Epanchnikov) for 1000 simulations.

$$H_0: U(0,1)$$

$$H_1: B(\theta) = \begin{cases} B_1(\theta) = \theta(2x)^{\theta-1} & , \quad 0 \le x < 1/2 \\ B_2(\theta) = \theta(2-2x)^{\theta-1} & , \quad 1/2 \le x \le 1 \end{cases} , \theta \in \mathcal{R}$$

N	n	χ^2	KS	CVM	AD	BR $Uniform$	BR $Epan.$
$B, \theta = 1.5$	10	-	0.09	0.07	0.06	0.09	0.07
	20	-	0.13	0.11	0.10	0.20	0.32
	40	0.39	0.19	0.22	0.25	0.47	0.65
$B, \theta = 2.0$	10	-	0.09	0.7	0.06	0.16	0.17
	20	-	0.25	0.25	0.28	0.51	0.65
	40	0.85	0.56	0.72	0.78	0.89	0.97

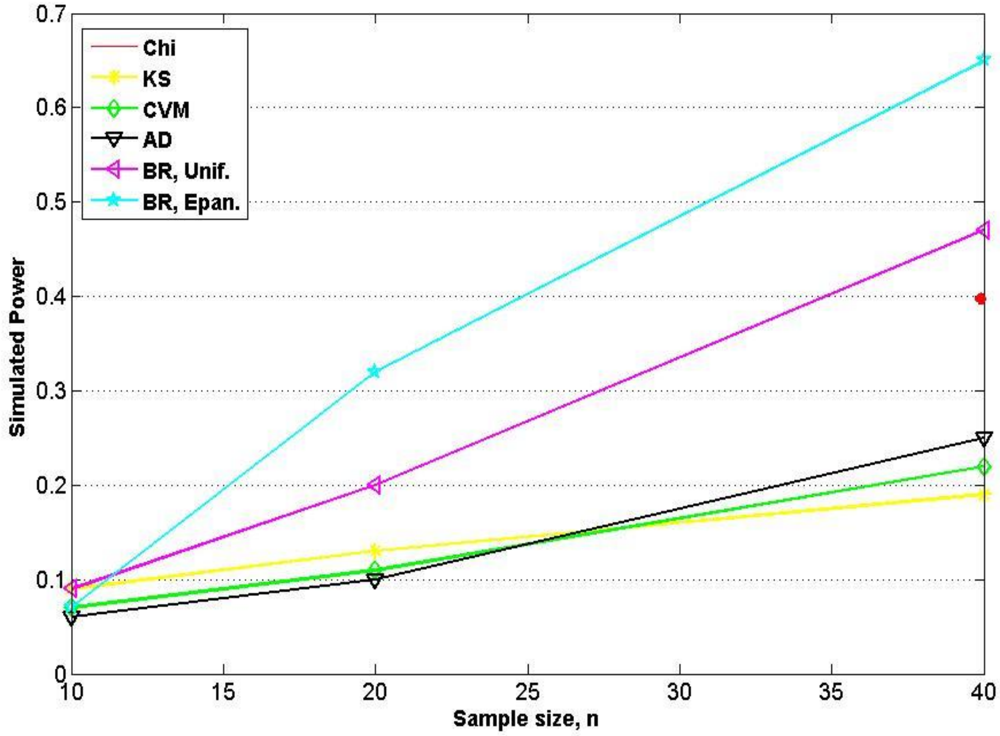


Figure 11(b): Comparison of power for different uniformity tests against $B(\theta),\ \theta = 1.5$

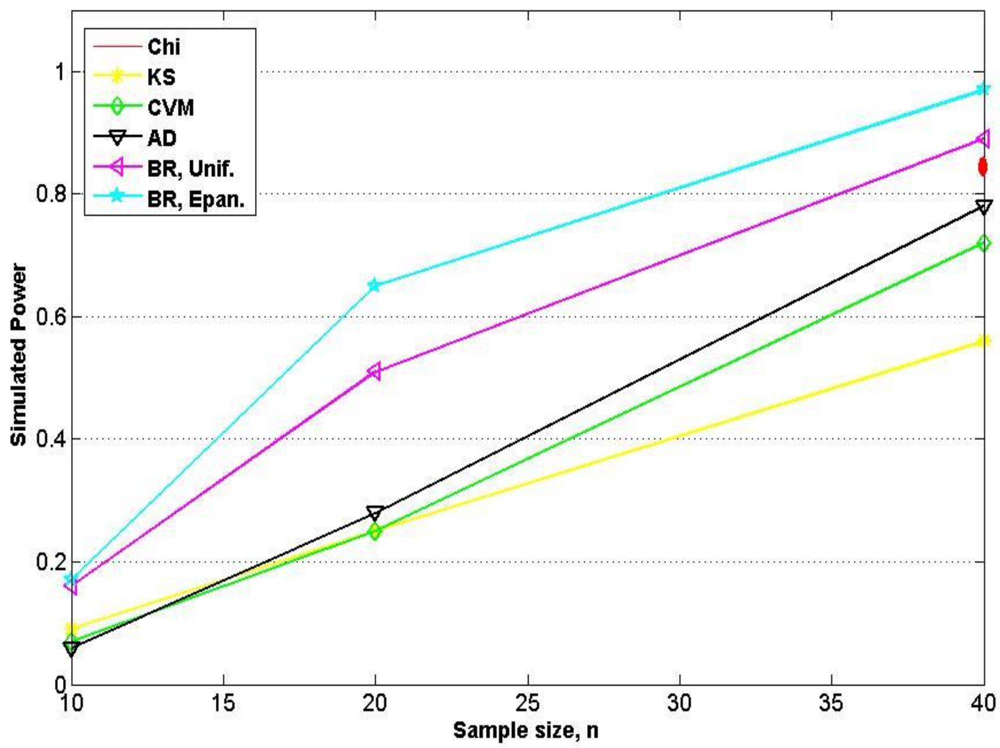


Figure 11(c): Comparison of power for different uniformity tests against $B(\theta),\ \theta = 2$

Summary

Table 11 and Figures 11(b) and 11(c) summarize the simulated power of Goodness-of-fit tests ($\chi^2, KS,\ CVM,\ AD,\ BR$) for testing $H_0: U(0,1)$ against nonparametric alternatives $H_1: B(\theta)$ at significance level $\alpha = 0.1$ and at sample sizes $n =$ 10, 20, 40. We observe the following main findings:

1. The BR test has a higher power compared with all other tests at sample sizes $n =$ 10, 20, 40 under nonparametric alternatives $B(\theta),\ \theta =$ 1.5, 2, 3.

2. In addition, BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel when using nonparametric alternatives $B(\theta),\ \theta =$ 1.5, 2, 3 for sample sizes $n =$ 10, 20, 40 except for $B(\theta)\ at\ \theta =$ 1.5 and $n =$ 10.

<u>**Case 12: Testing Uniformity against Non-parametric alternative $C(\theta)$**</u>

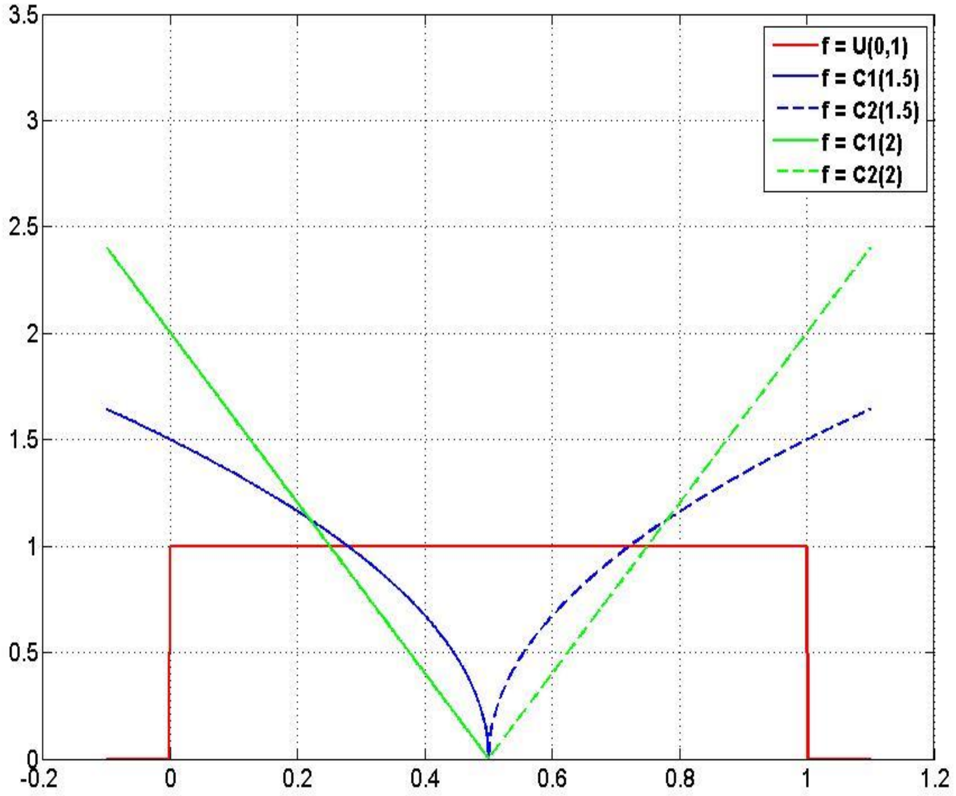


Figure 12(a): The graph of the null distribution $H_0: f = U(0,1)$ and the nonparametric alternative distribution $H_1: f = C(\theta)\ at\ \theta = 1.5, 2$

Table 12: The power for tests (KS, CVM, AD, BR) under the Null Hypothesis(H_0) at $\alpha = 0.1$.

Data were generated from the Nonparametric Alternative Hypothesis $C(\theta)$with different sample sizes (n) and different kernel functions (Uniform, Epanchnikov) for 1000 simulations.

$$H_0: U(0,1)$$

$$H_1: C(\theta) = \begin{cases} C_1(\theta) = \theta(1-2x)^{\theta-1} & , \quad 0 \le x < 1/2 \\ C_2(\theta) = \theta(2x-1)^{\theta-1} & , \quad 1/2 \le x \le 1 \end{cases} \quad , \theta \in \mathcal{R}$$

					BR	
N	$\boldsymbol{n}$	$\boldsymbol{KS}$	$\boldsymbol{CVM}$	$\boldsymbol{AD}$	$\boldsymbol{Uniform}$	$\boldsymbol{Epan.}$
$\boldsymbol{C, \theta = 1.5}$	20	0.25	0.20	0.28	0.10	0.11
	40	0.36	0.32	0.39	0.23	0.30
$\boldsymbol{C, \theta = 2.0}$	20	0.47	0.44	0.54	0.26	0.28
	40	0.71	0.80	0.85	0.75	0.83

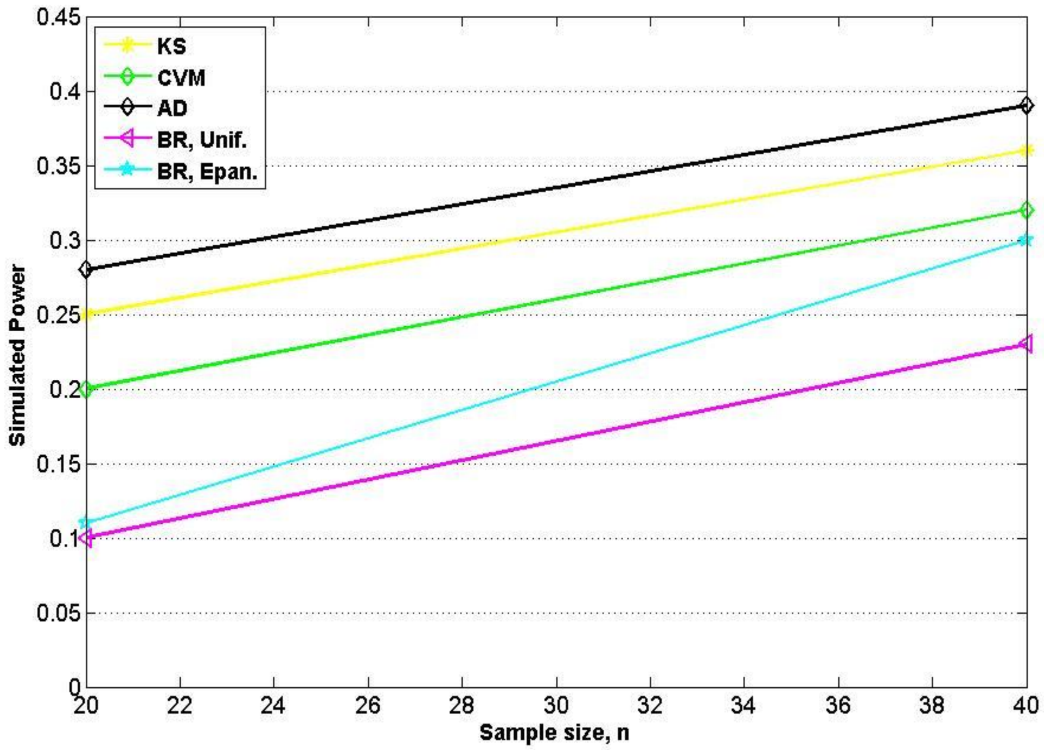


Figure 12(b): Comparison of power for different uniformity tests against $C(\theta), \theta = 1.5$

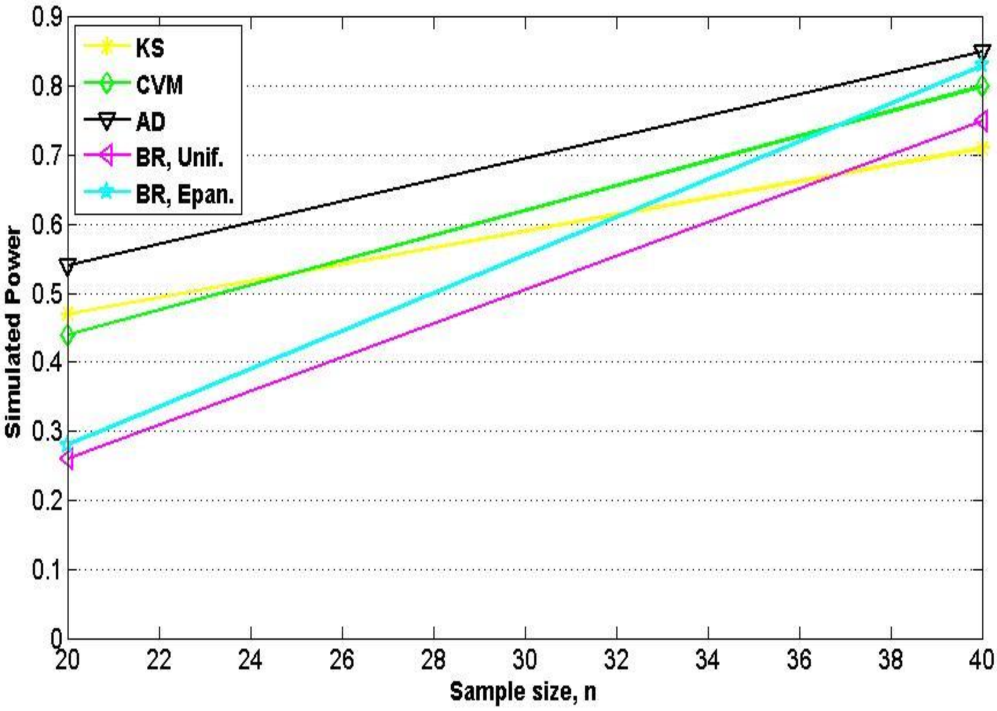


Figure 12(c): Comparison of power for different uniformity tests against $C(\theta),\ \theta = 2$

Summary

Table 12 and Figures 12(b) and 12(c) summarize the simulated power of Goodness-of-fit tests(χ^2, KS, CVM, AD, BR) for testing $H_0: U(0,1)$ against nonparametric alternatives $H_1: C(\theta)$ at significance level $\alpha = 0.1$ and at sample sizes $n =$ 20, 40. We can conclude that: The AD test has a higher power compared with all other tests at sample sizes $n =$ 20, 40 under nonparametric alternatives $C(\theta),\ \theta =$ 1.5, 2. In addition, BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel when using nonparametric alternatives $C(\theta),\ \theta =$ 1.5, 2 for sample sizes $n =$ 20, 40.

3.3 Conclusions and A Future work

In the real world, the parameters of the distribution are usually unknown and need to be estimated. When the parameters are estimated from the data, it affects the power of the GOF tests. In general, there is no most powerful test i.e. there is no test has a higher power for all cases, because the power of tests are dramatically changed with sample sizes n, the significance level α, the type of the distribution being tested, and also it depends on the alternative distribution. In addition, simulated power for all tests increased as the sample size and significance level increased.

In general, it can be concluded that from comparison of power of GOF tests the following:

1) For testing the symmetric distributions against parametric alternative distributions (Sec 3.2.1, Sec 3.2.2) among the five tests considered that the AD test has a higher power compared with all other tests, whereas the χ^2 test has lower power, and the power of AD test is still low for small sample sizes.

2) For testing the symmetric distributions and the data that are generated from non-parametric alternative distribution (Sec 3.2.3), we can conclude that from comparison of power of GOF tests among the five tests considered that the power of tests is dramatically changed with the type of non-parametric distribution $A(\theta), B(\theta), C(\theta)$, $\theta = 1.5, 2$. The AD test has a higher power compared with all other tests, whereas the χ^2 test has lower power when testing $U(0,1)$, and the data generated from alternative $A(\theta), C(\theta), \theta = 1.5, 2$, and the BR test has a higher power compared with all other tests when data generated from alternative $B(\theta), \theta = 1.5, 2$. Also, we can conclude that the BR test has a higher power when using the Epanechnikov kernel than that of uniform kernel.

As a future work, we are planning to study the power of the Bickel-Rosenblatt (BR) test by using different distributions as Cauchy distribution, Laplace distribution and compare it with four well-known Goodness-of-fit tests: Chi-square test, Kolmogorov-Smirnov (KS) test, Cramér-von Mises (CVM) test and Anderson-Darling (AD) test. Moreover, we want to study the power of the BR test under other choices of the kernel functions as Gaussian kernel, Biweight kernel and Triangular kernel. Finally, as we have seen from the previous results the BR test has a higher power when the data are generated from non-parametric alternative distributions, for this purpose we want to study the power of BR test when the data are generated from different non-parametric alternative distributions.

Bibliography

[1] Pearson, K., "Contributions to the Mathematical Theory of Evolution, II: Skew Variation Homogeneous Material", Transactions of the Royal Philosophical Society, Series A, 186, 343-414, 1895.

[2] Althouse, L.A., Ware, W.B. and Ferron, J.M., Detecting Departures from Normality: A Monte Carlo Simulation of A New Omnibus Test based on Moments. Paper presented at the Annual Meeting of the American Educational Research Association, San Diego, CA., 1998.

[3] Conover, W.J., Practical Nonparametric Statistics . Third Edition, John Wiley & Sons, Inc. New York, pp.428-433 (6.1), 1999.

[4] Cramér, H., On the composition of elementary errors. Skand. Akt., 11:141-180, 1928.

[5] Von Mises, R. E., Wahrscheinlichkeitsrechnung und Ihre Anwendung in der Statistik und Theoretischen Physik. Leipzig: Deuticke, 1931.

[6] Farrel, P.J. and Stewart, K.R., Comprehensive Study Of Tests For Normality And Symmetry: Extending The Spiegelhalter Test. Journal of Statistical Computation and Simulation , Vol. 76, No. 9, pp. 803–816, 2006.

[7] FIX, E. and HODGES, J. L. JR., Discriminatory Analysis, Nonparametric Discrimination: Consistency Properties. Report No. 4, Project No. 21-49-004, USAF School of Aviation Medicine, 1951.

[8] Rosenblatt, M., "Remarks on some nonparametric estimates of a density function". Annals of Mathematical Statistics 27: 832-837, 1956.

[9] Parzen, E., "On estimation of a probability density function and mode". Annals of Mathematical Statistics 33: 1065-1076, 1962.

[10] CENCOV, N. N., Evaluation of an unknown distribution density from observations. Soviet. Math. 3 1559-1562, 1962.

[11] WATSON, G. S. and LEADBETTER, M. R., On the estimation of the probability density, I. Ann. Math. Statist. 34 480-491, 1963.

[12] Loftsgaarden, D. O. and Quesenberry, C. P., A Nonparametric Estimate of a Multivariate Density Function, Annals of Mathematical Statistics, Volume 36, Issue 3, 1049-1051, 1965.

[13] SCHWARTZ, STUART C., Estimation of probability density by an orthogonal series. Ann. Math. Statist. 38 1261-1265, 1967.

[14] Epanechnikov, V. A., "Non-Parametric Estimation of a multivariate Probability Density". Theory Probab. Appl. 14 (1): 153-158, 1969.

[15] KRONMAL, R. and TARTER, M., The estimation of probability densities and cumulatives by Fourier series methods. J. Amer. Statist. Assoc. 63 925-952, 1968.

[16] WAHBA, G., A polynomial algorithm for density estimation. Ann. Math. Statist. 42 1870-1886, 1971.

[17] Scott, David W., "On optimal and data-based histograms". Biometrika 66 (3): 605–610, 1979.

[18] Freedman, David, Diaconis, P., "On the histogram as a density estimator: L_2 theory". Zeitschrift für Wahrscheinlichkeits theorie und verwandte Gebiete 57(4): 453 - 476, 1981.

[19] Shimazaki, H., Shinomoto, S., "A method for selecting the bin size of a time histogram". Neural Computation 19 (6): 1503-1527, 2007.

[20] Silverman, B.W., Density Estimation for Statistics and Data Analysis. Chapman & Hall/CRC. pp. 7-11, 1986.

[21] J. Gao and I. Gijbels. Bandwidth selection in nonparametric kernel testing. Journal of the American Statistical Association, 103(484), 1584-1594, 2008.

[22] Anderson, T.W., Darling, D.A., A test of goodness of fit. Journal of the American Statistical Association 49, 765-769, 1954.

[23] Stephens, M.A., EDF statistics for goodness of fit and some comparisons. J. Am. Stat. Assoc. 69, 730-737, 1974.

[24] R. B. D'Agostino and M. A. Stephens. Goodness-of-Fit Techniques (Eds.), Marcel Dekker, New York, 1986.

[25] Read, T. R. C. and Cressie, N. A. C., Goodness-of-Fit Statistics for Discrete Multivariate Data. Springer, New York, 1988.

[26] Thode, H.C., Jr., Testing for Normality, Marcel Dekker: New York, 2002.

[27] E.L. Lehmann, Joseph P. Romano. Testing Statistical Hypothesis, Third Edition, pp.583-587.

[28] Akritas, M.G., Pearson-type goodness-of-fit tests: The Univariate case. Journal of the American Statistical Association 83, 222-230, 1988.

[29] Dahiya, R.C., Gurland, J., How many classes in Pearson chi-squared test? Journal of the American Statistical Association 68, 707-712, 1973.

[30] Abdel-Aty, S.H., Approximate formulae for the percentage points and the probability integral of the non-central χ^2 distribution. Biometrika 41, 538–540, 1954.

[31] Ahmad, I.A., Alwasel, I.A., A goodness-of-fit test for exponentiality based on the memoryless property. Journal of the Royal Statistical Society, Series B 61, 681-689, 1999.

[32] Akritas, M.G., Torbeyns, A.F., Pearson-type goodness-of-fit tests for regression. The Canadian Journal of Statistics 25, 359-374, 1997.

[33] W. G. Cochran.“ The χ^2 test of goodness-of-fit”, Annals of Mathematical Statistics, 23, pp.315-345, 1952.

[34] Damico, J., A new one-sample test for goodness-of-fit. Communications in Statistics-Theory and Methods 33, 181-193, 2004.

[35] Massey, F. J., Jr., "The Kolmogorov-Smirnov Test for Goodness of Fit", J. Amer. Stat. Ass., 46, pp. 68-78, 1951.

[36] Drew, J.H., Glen, A.G. and Leemis, L.M., Computing the Cumulative Distribution Function of the Kolmogorov-Smirnov Statistic, Computational Statistics and Data Analysis 34,1-15, 2000.

[37] G. Marsaglia, Wai Wan Tsang and Jingbo Wang. Evaluating Kolmogorov's distribution, Journal Statistical Software 8, Issue 18, 2003.

[38] J. L. Romeu, Kolmogorov-Smirnov. A Goodness-of-Fit Test for Small Samples. RAC START, volume 10, number 6, 2003.

[39] Feller, W., On the Kolmogorov-Smirnov Limit Theorems for Empirical Distributions. Ann. Math. Stat., 19, pp. 177-189, 1948.

[40] Anderson, T.W., and D.A. Darling. “Asymptotic theory of certain ‘goodness-of-fit’ criteria based on stochastic processes,” Ann. Math, Statistics, Vol. 23, 193-212, 1952.

[41] Anderson, G., Simple tests of distributional form. Journal of Econometrics 62, 265-276, 1994.

[42] J. H. Pollard. A Handbook of Numerical and Statistical Techniques, pp. 151-156.

[43] M. Rahman and S. Chakrobartty. Tests for uniformity: a comparative study, J. Korean Data & Information Sci. Soc. 15(1), 211-218, 2004.

[44] Anderson, G., Nonparametric tests of stochastic dominance in income distributions. Econometrica 64, pp.1183-1193, 1996.

[45] Stephens, M. A., "Test of fit for the logistic distribution based on the empirical distribution function". Biometrika 66 (3): 591-595, 1979.

[46] Lockhart, R.A., and M.A., Stephens“, Estimation and tests of fit for the three-parameter Weibull distribution,” J. Roy. Statist. Soc. B., Vol. 56, 491-500, 1994.

[47] C. D. Sinclair and B. D. Spurr. Approximations to the Distribution Function of The Anderson-Darling Test Statistic, Journal American Statistical Association, 83, 1190-1191, 1988.

[48] D. E. A. Giles. A Saddle Point Approximation to the Distribution Function of the Anderson-Darling Test Statistic, SBN 1485-6441, 2000.

[49] Durbin, J., Knott, M., Taylor, CC., Components of Cramér-Von Mises statistics, II. J. Roy. Stat. Soc. Ser. B 37, 216-237, 1975.

[50] Lockhart, R.A. and M.A. Stephens. “Tests of fit for Cramér-Von Mises distribution,” Biometrika, Vol. 72, 647-652, 1985.

[51] D. Bachmann and H. Dette. A note on the Bickel - Rosenblatt test in autoregressive time series. Stat. Probab. Lett., 74(3):221-234, 2005.

[52] C. Tenreiro. On the role played by the selected bandwidth in the Bickel-Rosenblatt goodness-of-fit test. SORT, 29(2), 201-216, 2005.

[53] B.K. Ghosh, W.M. Huang. "The Power and Optimal Kernel of the Bickel-Rosenblatt Test for Goodness-of-Fit", The Annals of Statistics, Vol. 19, No. 2, 999-1009, 1991.

[54] P. J. Bickel and M. Rosenblatt. On some global measures of the deviations of density function estimates. The Annals of Statistics, Vol. 1, No. 1071-1095, 1973.

[55] Yazici, B., Yolacan, S., A comparison of various tests of normality. Journal of Statistical Computation and Simulation, 77(2), 175-183, 2007

[56] Hall, P.J., Central limit theorem for integrated square error of multivariate nonparametric density estimators. J. Multivariate Anal., 1984.

[57] G. Marsaglia and Wai Wan Tsang. The ziggurat method for generating random variables, Journal Statistical Software, 5, Issue 8, 2000.

[58] W. C. M. Kallenberg and T. Ledwina. Consistency and Monte Carlo simulation of a data driven version of smooth goodness-of-fit tests. Annals of Statistics, 23(5):1594-1608, 1995.

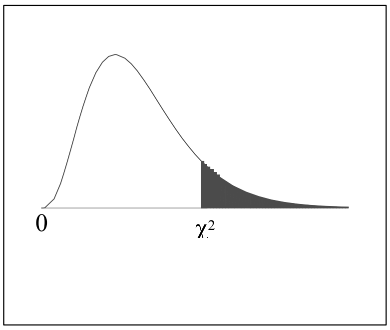
ملخص

الهدف الرئيسي من هذه الأطروحة هو دراسة ومقارنة قوة جودة المطابقة لخمسة اختبارات وهي: اختبار كاي تربيع، اختبار (Kolmogorov-Smirnov) ، اختبار (Cramér-von Mises) ، اختبار (Anderson-Darling) واختبار (Bickel-Rosenblatt) تحت تأثير البدائل المعلمية وغير المعلمية. تم الحصول على مقارنات قوى جودة المطابقة لهذه الاختبارات الخمسة باستخدام طريقة مونتي كارلو لسحب عينة من المشاهدات من بدائل معلمية وغير المعلمية والبدائل المعلمية تتبع توزيعات متماثلة وغير متماثلة. تم استخدام برنامج الإحصاء R لسحب عينات لأغراض المحاكاة. اثنتان من مستويات الدلالة تم استخدامهما: ٥٪ و١٠٪ والقيم الحرجة لمقارنات القوى التي تم الحصول علها بالاعتماد على ١٠٠٠٠ عينة مسحوبة من توزيعات صفرية مختلفة. ١٠٠٠٠عينة وحجم كل عينة:١٠، ٢٠، ٣٠، ٤٠ ، ٥٠، ١٠٠، ٢٠٠، ٣٠٠، ٤٠٠، ٥٠٠، ١٠٠٠، ٢٠٠٠ تم سحبها من توزيعات بديلة معطاة. قوة كل اختبار تم الحصول عليه بواسطة مقارنة اختبارات جودة المطابقة مع القيم الحرجة منها. تبين نتائج المحاكاة أن اختبار (Anderson-Darling) له أعلى قوة في حالة اختبار التوزيعات المتماثلة والعينات مسحوبة من توزيعات البدائل المعلمية متبوعة باختبار (Cramér-von Mises) واختبار (Kolmogorov-Smirnov). اختبار (Bickel-Rosenblatt) له أعلى قوة في حالة اختبار التوزيعات المتماثلة والعينات مسحوبة من توزيعات بعض البدائل غير المعلمية واختبار (Anderson-Darling) له أعلى قوة تحت تأثير البدائل غير المعلمية الأخرى. هذه الدراسة أيضا بينت أن اختبار (Bickel-Rosenblatt) له قوة أكبر في حالة استخدام اقتران (Epanechnikov) عن استخدام الاقتران المنتظم.

Appendices

Appendix A: Critical Values of Goodness-of-Fit Tests

A.1 Chi-Square Distribution Table



The shaded are is equal to α for $\chi^2 = \chi^2{}_{\alpha}$

df	$\chi^2{}_{0.995}$	$\chi^2{}_{0.990}$	$\chi^2{}_{0.975}$	$\chi^2{}_{0.950}$	$\chi^2{}_{0.900}$	$\chi^2{}_{0.100}$	$\chi^2{}_{0.050}$	$\chi^2{}_{0.025}$	$\chi^2{}_{0.010}$	$\chi^2{}_{0.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672

A.2 Kolmogorov-Smirnov Table

Critical values, $d_{alpha};(n)^{a}$, of the maximum absolute difference between sample $F_n(x)$ and population $F(x)$ cumulative distribution.

Number of trials, n	Level of significance, α			
	0.10	0.05	0.02	0.01
1	0.95000	0.97500	0.99000	0.99500
2	0.77639	0.84189	0.90000	0.92929
3	0.63604	0.70760	0.78456	0.82900
4	0.56522	0.62394	0.68887	0.73424
5	0.50945	0.56328	0.62718	0.66853
6	0.46799	0.51926	0.57741	0.61661
7	0.43607	0.48342	0.53844	0.57581
8	0.40962	0.45427	0.50654	0.54179
9	0.38746	0.43001	0.47960	0.51332
10	0.36866	0.40925	0.45662	0.48893
11	0.35242	0.39122	0.43670	0.46770
12	0.33815	0.37543	0.41918	0.44905
13	0.32549	0.36143	0.40362	0.43247
14	0.31417	0.34890	0.38970	0.41762
15	0.30397	0.33760	0.37713	0.40420
16	0.29472	0.32733	0.36571	0.39201
17	0.28627	0.31796	0.35528	0.38086
18	0.27851	0.30936	0.34569	0.37062
19	0.27136	0.30143	0.33685	0.36117
20	0.26473	0.29408	0.32866	0.35241
21	0.25858	0.28724	0.32104	0.34427
22	0.25283	0.28087	0.31394	0.33666
23	0.24746	0.27490	0.30728	0.32954
24	0.24242	0.26931	0.30104	0.32286
25	0.23768	0.26404	0.29516	0.31657
26	0.23320	0.25907	0.28962	0.31064
27	0.22898	0.25438	0.28438	0.30502
28	0.22497	0.24993	0.27942	0.29971
29	0.22117	0.24571	0.27471	0.29466
30	0.21756	0.24170	0.27023	0.28987
31	0.21412	0.23788	0.26596	0.28530
32	0.21085	0.23424	0.26189	0.28094
33	0.20771	0.23076	0.25801	0.27677
34	0.20472	0.22743	0.25429	0.27279
35	0.20185	0.22425	0.26073	0.26897
36	0.19910	0.22119	0.24732	0.26532
37	0.19646	0.21826	0.24404	0.26180
38	0.19392	0.21544	0.24089	0.25843
39	0.19148	0.21273	0.23786	0.25518
40[b]	0.18913	0.21012	0.23494	0.25205

Values of $d_\alpha(n)$ such that $p(\max)|F^{n}(x) - F(x)| d^{\alpha}(n) = \alpha$

[b] $n > 40 \approx \frac{1.22}{n^{1/2}}, \frac{1.36}{n^{1/2}}, \frac{1.51}{n^{1/2}}$ and $\frac{1.63}{n^{1/2}}$ for the four levels of significance.

A.3.1 Critical values for the AD test under uniform distribution at different significance levels.

Number of samples n	Significance level, $\alpha = 0.1$	Significance level, $\alpha = 0.05$
10	1.9518	2.5121
20	1.9385	2.5020
30	1.9313	2.5130
40	1.9362	2.5042
50	1.9277	2.4941
60	1.9367	2.5044
70	1.9304	2.4959
80	1.9235	2.4951
90	1.9326	2.5064
100	1.9325	2.4901

A.3.2 Critical values for the AD test under normal distribution at different significance levels.

Significance level	0.05	0.025	0.01
Critical values	0.752	0.873	1.035

A.4 Critical Values for Cramér-von Mises Test-Statistic

	Significance level, α				
Sample size, n	0.20	0.15	0.10	0.05	0.01
2	0.138	0.149	0.162	0.175	0.186
3	0.121	0.135	0.154	0.184	0.23
4	0.121	0.134	0.155	0.191	0.28
5	0.121	0.137	0.160	0.199	0.30
6	0.123	0.139	0.162	0.204	0.31
7	0.124	0.140	0.165	0.208	0.32
8	0.124	0.141	0.165	0.210	0.32
9	0.125	0.142	0.167	0.212	0.32
10	0.125	0.142	0.167	0.212	0.32
11	0.126	0.143	0.169	0.214	0.32
12	0.126	0.144	0.169	0.214	0.32
13	0.126	0.144	0.169	0.214	0.33
14	0.126	0.144	0.169	0.214	0.33
15	0.126	0.144	0.169	0.215	0.33
16	0.127	0.145	0.171	0.216	0.33
17	0.127	0.145	0.171	0.217	0.33
18	0.127	0.146	0.171	0.217	0.33
19	0.127	0.146	0.171	0.217	0.33
20	0.128	0.146	0.172	0.217	0.33
30	0.128	0.146	0.172	0.218	0.33
60	0.128	0.147	0.173	0.220	0.33
100	0.129	0.147	0.173	0.220	0.34

Appendix B: R codes

B.1 Algorithm for calculating the power of Anderson-Darling test, testing Normality against $Exp(5)$

```
for (n in c(20,30,50,100,200,300,400,500))
  {
AD <- c(0,1)
F <- c(0,1)
power <- 0
number_of_samples <- 10000
partion_number <- 330
H0 <- function(x){(1/sqrt(2*pi*vari))*exp(-((x-mu)^2)/(2*vari))}
# for estimated parameters
#cv<-0.752
# for not estimated parameters
cv<-2.492
y[1] <-(-5)
j<-1
repeat {
y[j+1] <- (y[j]+0.03)
if(j==partion_number)break
j<-j+1
   }
kh<-1
repeat
{
sum <- 0
set.seed(123)
x = sort(rexp(n,5))
mu = mean(x)
vari = var(x)
cat('x=',x,'\n')
s <- 1
j <- 1
repeat
{
F[j] = 0
k <- 1
repeat
{
if (x[k] <= y[j])
{
F[j]=s/n
```

```
s = s+1
{
if(k==n)break
k<-k+1
  }
cat('F[j]=',F[j],'\n")
cat('y[j]=',y[j],'\n')
if(j==partion_number)break
j <- j+1
s = 1
  }
j<-1
repeat
{
p = integrate(H0,-Inf,y[j])$val
cat('p=',p,'\n')
if(p<=1 && p>= 0.9999999){p = 0.9999999}
if(p>=0 && p<=0.000001){p = 0.000001}
sum = sum + (F[j]-p)^2*(H0(y[j])*0.03)*(1/(p*(1-p)))
sum = sum + (2*i-1)*log(pnorm(x[i],0.5,1))+(2*n+1-2*i)*log(1-pnorm(x[i],0.5,1))
if(j==partion_number)break
j<-j+1
  }
#AD[kh]=n*sum
AD[kh]=n*(1+(0.75/n)+(2.25/(n^2)))*sum
if(AD[kh]>cv)
{
power <- power + 1/number_of_samples
#cat('k=',k,' ', 'power=',power,'\n')
}
if(kh==number_of_samples)break
kh <- kh +1
}
cat('n=',n,' ', 'power=',power,'\n')
power = 0
}
```

B.2 Algorithm for calculating the power of Bickel-Rosenbaltt test, testing Uniformity against non-parametric alternatives.

```
for (n in c(10,20,40))
{
number_of_samples <- 1000
partion_number <- 100
I <- c(0,1)
```

```
z <- c(0,1)
d <- c(0,1)
x <- c(0,1)
BR <- c(0,1)
power <- 0
sum2<-0
y <- c(0,1)
FN <- c(0,1)
honenorm <- c(0,1)
sum <- 0
K <- c(0,1)
zalpha=1.28
mu<-0
vari<-0
mean <- c(0,1)
variance <- c(0,1)
mu <- c(0,1)
vari <- c(0,1)
bn=1/sqrt(n)
H0 <- function(x){(1+(x-x))}
H0int <- function(x){(1+(x-x))}
# Calculation the Critical value of BR test
kernel <- function(x){(3/4)*(1-(x^2))}
# kernel <- function(x){(1)}
kernel2 <- function(x,y)sapply(x,function(z,y){kernel(z)},y=y)
multiply=function(a,b)
{
force(a)
force(b)
function(x){a(x)*b(x)}
}
(multiply2=function(c,d)
{
force(c)
force(d)
function(x,y)sapply(x,function(z,y){c(z)*d(z+y)},y=y)
}
mult=multiply(kernel,kernel)
normal2=multiply(H0int,H0int)
I = integrate(Vectorize(mult),-1,1)$val)
fvec=multiply2(kernel,kernel
gvec = function(x) sapply(x, function(y) integrate(fvec, lower=-1, upper=1, y=y)$val)
gvec2=multiply(gvec,gvec)
J = integrate(Vectorize(gvec2), lower=-1, upper=1)$val
mu = I*(integrate(H0int,0,1)$val)
var=2*J*(integrate(normal2,0,1)$val)
```

```
mu = I*1
var=2*J*1
segma=sqrt(var)
cv=mu+(zalpha*sqrt(bn)*segma
k<-1
repeat {
# Generate data from non-parametric alternative distributons
# Alternative A
I <- function(t){(1-(1-t)^(2/3))}
# I <- function(t){(1-sqrt(1-t))}
set.seed(123)
z = sort(runif(n,0,1))
mean = mean(z)
variance = var(z)
m <- 1
repeat
{
d[1]=I(z[1])
d[m]=sort(I(z[m]))
x[m]=sort(d[m])
if(m==n)break
m<-m+1
}
mu = mean(x)
vari = var(x)
y[1] <-(0)
j<-1
repeat {
y[j+1] <- (y[j]+0.01)
if(j==partion_number)break
j<-j+1
}
j <- 1
repeat {
honenorm[j]=H0(y[j])
if(j==partion_number)break
j<-j+1
}
j<-1
}repeat
i<-1
repeat {
if(((y[j]-x[i])/bn)>=(-1)&&((y[j]-x[i])/bn)<=(1))
K[i]<-kernel(((y[j]-x[i])/bn ))
else
K[i]<-0
```

```
sum = sum + K[i ]
if(i==n )break
i<-i+1
}
FN[j] = sum * (1/(n*bn))
sum =0
if (j==partion_number)break
j<-j+1
}
j<-1
repeat{
diff = FN[j]-honenorm[j]
sum2 = sum2 + (diff^2)*0.01
if(j==partion_number)break
j <- j+1
}
BR[k] = sum2 * n * bn
#cat('BR=','[',k,']',BR[k],'\n ')
sum2 = 0;
if(BR[k]>cv)
power = power + 1/number_of_samples
if(k==number_of_samples)break
k<- k +1
}
cat('n=',n,' ','cv=',cv,' ' ,'power=',power,'\n')
power = 0
}
```