Arab American University

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Non-Commuting Graph of the Special Linear Groups $SL(2, q)$ and $SL(3, q)$

By

Duaa Nader Abu-Zaina

Supervisor

Dr. Amina Afaneh

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Committee Decision

Non-Commuting Graph of the Special Linear Groups

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This thesis was defended successfully on September 2018 and

approved by

Committee Member	**Signature**
Dr. Amina Afaneh (Supervisor)	....
Dr. Mahmoud Almanasra (Internal Examiner)	....
Dr. Ali Altawaiha (External Examiner)	....

Declaration

I, Duaa Nader Abu-Zaina declare this work provided in this thesis, unless otherwise referenced is my original work and that it has not been presented, and will not be presented, to any university for similar or any other degree award.

Signature ....

Date ....

Dedication

I dedicate this thesis to:

My great parents, who never stop giving themselves in countless way.

My dearest husband Fadi Abualrob, who supports me all the ways.

My kindest sisters and brothers.

My beloved kids Layan and Razan.

My doctor Amina Afaneh.

My second home Arab American University.

Abstract

If G is a non-abelian group and Z_G is the center of G. The non-commuting graph of G is defined to be the graph $\varGamma_G$ where $G - Z_G$ is set of vertices such that any two vertices x and y are adjacent if and only if $xy \neq yx$. In this work, we will investigate the non-commuting graph of the special linear group $SL(n, q)$ where $n = 2, 3$ over the Galois filed of order q. We will find the clique number $\omega(\varGamma_{SL(n,q)})$, the independent number $\alpha(\varGamma_{SL(n,q)})$, the minimum size of vertex cover $\beta(\varGamma_{SL(n,q)})$ and the vertex chromatic number $\chi(\varGamma_{SL(n,q)})$.

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Symbols and Notation

Γ	The graph.
Γ_G	The non-commuting graph of group G.
$V(\Gamma_G)$	The vertices set of non-commuting graph Γ_G.
$E(\Gamma_G)$	The edges set of non-commuting graph Γ_G.
$deg(x)$	The degree of a vertex x in Γ_G.
$\omega(\Gamma_G)$	Clique number of Γ_G.
$\chi(\Gamma_G)$	Chromatic number of Γ_G.
$\alpha(\Gamma_G)$	Independent number of Γ_G.
$\beta(\Gamma_G)$	Vertex cover of Γ_G.
$\gamma(\Gamma_G)$	Dominating number of Γ_G
$diam(\Gamma_G)$	Diameter of Γ_G.
$girth(\Gamma_G)$	Girth of Γ_G.
Z_G	The center of a group G.
$C_G(g)$	The centralizer of an element $g \in G$.
$C_s(g)$	The conjugacy class of g.
$k(G)$	The class number.

Introduction

Many researches were published on assigning a graph to a group, and use properties of graphs to study algebraic properties of groups [2].

The non-commuting graph of a group G is the graph Γ_G where the vertex set is $G - Z_G$, denoted by $V(\Gamma_G)$, such that any two vertices $x, y \in V(\Gamma_G)$ are adjacent if and only if $xy \neq yx$. And we denote the edge set by $E(\Gamma_G)$.

In chapter one, we give preliminaries of graphs, groups and finite fields. First, we give some important definitions and properties of graphs, such as connected graph, distance, clique number, independent number, covering number and chromatic number. Then we introduce some definitions and theorems on groups and finite fields.

In part one of chapter two, we define non-commuting graph and consider some examples by using sagemath program, then we give some general properties and theorems of Γ_G, we also find $diam(\Gamma_G)$ and $girth(\Gamma_G)$. In second part, we consider some properties and theorems of Γ_G that we need in the last chapter.

Chapter three is the aim of this research. First, we define the general linear group $GL(n, q)$, special linear group $SL(n, q)$ and projective special linear group $PSL(n, q)$, and we give some properties of these groups. In the second section of this chapter, we consider $\Gamma_{SL(2,q)}$. At the beginning we use sagemath program to draw the graph $\Gamma_{SL(2,q)}$ where $q = 2, 3, 4, 5$ and 7, after that we find some properties of these graphs, such as the clique number $\omega(\Gamma_{SL(2,q)})$, chromatic number $\chi(\Gamma_{SL(2,q)})$, the independent number $\alpha(\Gamma_{SL(2,q)}$ and the vertex cover $\beta(\Gamma_{SL(2,q)})$.

In the last section, we draw the graph $\Gamma_{SL(3,2)}$ and find some properties of this graph. We prove if $\Gamma_G \cong \Gamma_{SL(3,q)}$ then $|G| = |SL(3, q)|$. Also we find the clique number of $\Gamma_{SL(3,3)}$, an upper bound of the independent number of $\Gamma_{SL(3,q)}$, a lower bound of $\beta(\Gamma_{SL(3,q)})$ and we conjecture the exact formula of the independent number of $\Gamma_{SL(3,q)}$ and number of vertex cover of $\Gamma_{SL(3,q)}$. Finally, we prove if $\Gamma_G \cong \Gamma_{SL(3,q)}$ and $|Z_{SL(3,q)}| = 1$, then $G \cong SL(3,q)$.

Finally, the sagemath programs that we use in this research are presented. In program 1, we find the non-commuting graph of some groups such as symmetric group S_3, Dihedral group D_4 and Quaternion group Q_8. Program 2 determine wither a given group is an AC-group or not. In program 3, we draw the non-commuting graph of $SL(2,q)$ for $q = 2,3,4,5$ and 7, we also find some properties of $\Gamma_{SL(2,q)}$ such as independent number, chromatic number, number of vertex cover and clique number. In program 4, we find the set of order of centralizers of groups. In program 5, we find the independent number and the number of vertex cover of $SL(3,q)$ where $q = 2,3,4,5,7,8,9$ and 11 without finding the graphs. Finally, in program 6, we find the clique number of $SL(3,2)$.

Chapter 1

Preliminaries

1.1 Graphs

In this section we give some important definitions and properties of graph.

***Definition* 1.1.** [20] A **graph** Γ consists of a non-empty finite set $V(\Gamma)$ of elements called **vertices of** Γ, and a finite family $E(\Gamma)$ of unordered pairs of elements in $V(\Gamma)$ called **edges of** Γ.

If v and w are two vertices in Γ, and if the unordered pair $\{v, w\}$ is an edge denoted by e, we say e **joins** v and w, and we say that v and w are **adjacent**. We say two edges are adjacent, if they have a common vertex.

***Definition* 1.2.** [20] A **loop** in Γ is an edge, that joins a vertex to itself.

***Definition* 1.3.** [20] Two or more edges that join the same pair of distinct vertices are called **multiple edges**.

***Definition* 1.4.** [20] A path in Γ is a finite sequence of distinct edges of the form $v_1v_2,\quad v_2v_3,\quad v_3v_4,\quad \ldots,\quad v_{n-1}v_n$, in which any two consecutive edges are adjacent, v_1 is called initial vertex and v_n terminal vertex, such that all vertices of these edges are distinct except possibly the initial vertex and terminal one.

A path in which the initial vertex is equal to the terminal vertex is called a cyclic.

Definition **1.5.** [20] A graph Γ is said to be a **connected** graph if and only if there is a path between each pair of vertices.

Definition **1.6.** [20] The **distance** between two vertices v and w in a graph is defined to be the length of the shortest path from v to w, and it is denoted by $d(v, w)$.

Definition **1.7.** [20] If Γ is a graph without loops, then the number of vertices adjacent to a vertex $v \in V(\Gamma)$ is called the **degree** of v, denoted by $deg(v)$. And the sum of degrees of all vertices is:

$$\rho(\Gamma) := \sum_{g \in V(\Gamma)} deg(g)$$

Definition **1.8.** [9] The number of vertices in Γ is called the **order** of Γ. While the number of edges is its **size**.

Theorem **1.1.** [20] **(Hand-Shaking Lemma)** In any graph Γ, the sum of all the vertex-degrees is equal to twice the number of edges, if $|E(\Gamma)|$ is the number of edges in Γ, then

$$\sum_{x \in V(\Gamma)} deg(x) = 2|E(\Gamma)|$$

Definition **1.9.** [20] If all vertices in a graph have the same degree, then we say the graph is **regular**.

***Definition* 1.10.** [20] A graph Γ is called **simple** if and only if it contains no loops or multiple edges.

***Definition* 1.11.** [20] A simple graph Γ is called **complete**, if any two vertices in Γ are adjacent.

***Definition* 1.12.** [9] If a subgraph of a graph Γ has the same vertex set as Γ, then it is a **spanning subgraph** of Γ.

***Definition* 1.13.** [9] A cycle in a graph Γ that contains every vertex of Γ is called a **Hamiltonian cycle** of Γ. Thus a Hamiltonian cycle of Γ is a spanning cycle of Γ. A **Hamiltonian graph** is a graph that contains a Hamiltonian cycle.

Theorem* 1.2.** [20] ***(Dirac's theorem) If Γ is a simple graph on n vertices $(n \geq 3)$ in which $deg(x) \geq \dfrac{n}{2}$ for all $x \in V(\Gamma)$. Then Γ is Hamiltonian.

***Definition* 1.14.** [12] A **diameter** of a graph Γ is defined to be the largest distance between any two of the vertices in Γ, we denote it by $diam(\Gamma)$.

***Definition* 1.15.** [12] The shortest cycle in a graph Γ is called a **girth** of Γ, we denote it by $girth(\Gamma)$.

***Definition* 1.16.** [20] A graph Γ is called **planar graph** if and only if it can be drawn without intersection of edges.

***Definition* 1.17.** [12] A **clique** of a graph Γ is defined to be a complete subgraph of Γ. The clique number of Γ is defined to be the maximum size of clique of Γ, we denote it by $\omega(\Gamma)$.

***Definition* 1.18.** [12] Two distinct vertices are said to be **independent** if they are not adjacent.

***Definition* 1.19.** [12] In a graph Γ, if $I \subseteq V(\Gamma)$ and any two distinct vertices in I are independent, then I is called **an independent set.**

***Definition* 1.20.** [12] An independent set I in a graph $\Gamma = (V, E)$ is called **a maximum independent set** provided that no other independent set in Γ has larger cardinality, we say an independent set I is maximal if it is contained in no larger independent set. We say $V - I$ is a **covering** set of Γ.

***Definition* 1.21.** [12] The number of vertices in a maximum independent set in Γ is called the **independent number** of Γ, and it is denoted by $\alpha(\Gamma)$.

***Definition* 1.22.** [12] A covering of a graph Γ is called minimum covering of Γ if and only if no other covering of Γ with less number of vertices. The number of vertices in a minimum covering of a graph Γ is called the **covering number** of Γ and is denoted by $\beta(\Gamma)$.

***Definition* 1.23.** [20] If Γ is a graph without loops, then Γ is k-colourable if we can assign one of k colours to each vertex so that adjacent vertices have different colours. If Γ is k-colourable, but not $(k-1)$-colourable, we say that the **chromatic number** of Γ is k, the chromatic number of Γ is denoted by $\chi(\Gamma) = k$.

***Theorem* 1.3.** [20] The graph K_5 is not a planar where K_5 is a complete graph with 5 vertices.

***Theorem* 1.4.** [9] If Γ is any graph of order n, then

$$\chi(\Gamma) \geq \omega(\Gamma).$$

***Definition* 1.24.** [20] Let Γ_1 and Γ_2 be two graphs. If there exists a bijection

$$\phi : V(\Gamma_1) \longrightarrow V(\Gamma_2)$$

such that for any two vertices v, w that are adjacent in Γ_1, then $\phi(v)$, $\phi(w)$ are adjacent in Γ_2, then these two graphs are said to be isomorphic.

***Proposition* 1.1.** [12] In any graph Γ on m vertices, we have

$$\alpha(\Gamma) + \beta(\Gamma) = m.$$

***Definition* 1.25.** [12] Let $\Gamma = (V, E)$ be a graph, a subset H of V such that every vertex not in H is adjacent to at least one vertex in H, is called a **dominating set**. The number of vertices in a smallest dominating set for Γ is called a **dominating number**, and it is denoted by $\gamma(\Gamma)$.

***Theorem* 1.5.** [20] Let Γ be a simple planar graph, then there exists $x \in V(\Gamma)$ such that $deg(x) \leq 5$.

1.2 Groups

In this section, we introduce some definitions, properties and theorems about groups.

***Definition* 1.26.** [11] The **order** of a group G is defined to be the number of elements in G if G finite, and if G is infinite then G has infinite order. We will use $|G|$ to denote the order of a group G.

***Definition* 1.27.** [11] The center of a group G is denoted by Z_G, and it is defined as follows:

$$Z_G = \{g \in G| \quad gx = xg \quad \text{for all} \quad x \in G\}.$$

***Definition* 1.28.** [11] Let $H \leq G$. The left coset of H containing an element $a \in G$ is denoted by aH and defined as:

$$aH = \{ah| \quad h \in H\}.$$

We call the number of left cosets of H in G, the **index of** G over H, and it is denoted by $[G : H]$.

It can easily be proved that, if G is finite then $[G : H] = |G|/|H|$.

***Definition* 1.29.** [11] Let H be a subgroup of a group G. If $xH = Hx$ for all $x \in G$, then H is called **normal subgroup**. It is denoted by $H \trianglelefteq G$.

It can be easily shown that, the center of G is normal subgroup of G.

Theorem* 1.6.** [11] (First Isomorphism Theorem***) Let G and H be two groups. If $\phi : G \longrightarrow H$ is homomorphism with kernel K then $G/K \cong \phi(H)$, where kernel K of ϕ is defined as follows:

$$K = \{x \in G| \quad \phi(x) = e_H\},$$

where e_H is the identity in H.

Theorem* 1.7.** [11] (Second Isomorphism Theorem***) If G is a group and $H \leqslant G$, $N \trianglelefteq G$ then $\dfrac{|H|}{|H \cap N|} = \dfrac{|HN|}{|N|}$.

***Theorem* 1.8.** [11] If G is a group with prime order, then it is a cyclic group, that is G is generated by one element.

***Definition* 1.30.** [11] If a group G has exactly two normal subgroups, itself and the trivial subgroup, then it is called **simple** group.

***Theorem* 1.9.** [5] Every simple finite non-abelian group can be generated by two elements.

***Definition* 1.31.** [11] Let G be a group and a an element in G. The order of a, denoted by $|a|$, is defined to be the least positive integer m such that $a^m = e$, where e is the identity of G. We say that a has infinite order if no such integer exists.

***Proposition* 1.2.** [11] If G is a finite group and $x \in G$, then $|x|$ is a divisor of $|G|$.

***Theorem* 1.10.** [11] If p is a prime factor of the order of a finite abelian group G, then G contains a nontrivial element of order p.

***Definition* 1.32.** [11] **(P-Group)** If p is a prime number, and G is a group of order p^n for some integer $n > 0$. Then we say that G is a p–group.

***Definition* 1.33.** [11] Let g_1 and g_2 be two elements of a group G. We say that g_1 and g_2 are conjugates in G (and call g_2 a conjugate of g_1) if $xg_1x^{-1} = g_2$ for some $x \in G$. The conjugacy class of g_1 is the set given as follows:

$$C_s(g_1) = \{xg_1x^{-1} | \quad x \in G\}$$

.

***Theorem* 1.11.** [11] If G is a finite group and x is an element of G. Then,

$$|C_s(x)| = [G : C_G(x)].$$

***Corollary* 1.1.** [11] In a finite group, $|C_s(x)|$ divides $|G|$.

***Proposition* 1.3.** [11] If G is a group and A subgroup of G, then the number of conjugates of A is $[G : N_G(A)]$ where $N_G(A)$ is the normalizer of A in G, that is

$$N_G(A) = \{x \in G| \quad xA = Ax\}.$$

Note that Conjugacy is an equivalence relation. We call the equivalence classes of this relation conjugacy classes.

Corollary* 1.2.** [11] (Class Equation***) If G is a finite group. Then

$$|G| = \sum [G : C_G(x)] = \sum |C_s(x)|,$$

where the sum runs over one element x from each conjugacy class of G.

***Definition* 1.34.** The **class number** of a group G is the number of distinct conjugacy classes, and it is denoted by $k(G)$.

***Definition* 1.35.** [11] If G is a group, and $a, b \in G$, we define the commutator $[a, b]$ as follows:

$$[a, b] = aba^{-1}b^{-1}.$$

***Proposition* 1.4.** [11] If G is a group. Let G' be the subgroup of G which is generated by the set

$$\{xyx^{-1}y^{-1} | \quad x, y \in G\}$$

Then G' is called the commutator subgroup of G.

***Theorem* 1.12.** [11] If the order of a group G is equal to $2p$, where p is a prime number and $p > 2$. Then $G \cong Z_{2p}$ or D_p, where D_p is the p-dihedral group.

***Theorem* 1.13.** [11] If G is a group of order p^2, where p is a prime number. Then $G \cong Z_{p^2}$ or $Z_p \oplus Z_p$.

***Theorem* 1.14.** [8] Any non-abelian group of order 8 is isomorphic to D_4 or Q_8.

***Definition* 1.36.** [11] Suppose that G is a finite group and let p be a prime. If p^k divides $|G|$ and p^{k+1} does not divide $|G|$, then any subgroup of G of order p^k is called a **Sylow p-subgroup** of G.

***Definition* 1.37.** [8] A **normal series** for a group G is a finite chain of subgroups A_i beginning with G and ending with the identity subgroup $\{e\}$

$$G \supset A_0 \supset A_1 \supset A_2 \supset \cdots \supset \{e\}$$

in which each A_{i+1} is a proper normal subgroup of A_i. The factor groups A_i/A_{i+1} are called the factors of the series.

***Definition* 1.38.** [8] A group G is called **solvable** if it has a normal series with abelian factors, that is A_i/A_{i+1} is abelian for all $i = 0, 1, \ldots n$.

***Theorem* 1.15.** [8] A group G is solvable if and only if $G^{(n)} = \{e\}$ for some $n \geq 0$, where $G^0 = G$, $G^{(i)} = [G, G]$, $G^{i+1} = [G, G^{(i)}]$.

1.3 Finite Fields

The Galois field is a finite field with p^n elements, where p is a prime number and n is a positive integer, we denote it by F_q, where $q = p^n$.

***Proposition* 1.5.** [15] For every element β of a finite field F with q elements, we have $\beta^q = \beta$. That is $\beta^{q-1} = 1$.

Proof. If $\beta = 0$ then $\beta^q = 0$.

Let F^* be the nonzero elements of F; that is

$$F^* = \{\beta_1, \beta_2, \ldots, \beta_{q-1}\}$$

Now, we need to prove $\beta^q = \beta$. Assume $\beta \neq 0$ where β is an element in F^*. Thus

$$F^* = \{\beta\beta_1, \beta\beta_2, \ldots, \beta\beta_{q-1}\}$$

So,

$$\begin{aligned} \beta_1\beta_2 \ldots \beta_{q-1} &= \beta\beta_1\beta\beta_2 \ldots \beta\beta_{q-1} \\ &= \beta^{q-1}(\beta_1\beta_2 \ldots \beta_{q-1}) \end{aligned}$$

This shows that $\beta^{q-1} = 1$. □

***Theorem* 1.16.** [15] Given any prime p and integer $n \geq 1$, then there exist a unique finite field of p^n elements.

***Definition* 1.39.** [15] An element γ in a finite field F_q is called a primitive element (or generator) of F_q if $F_q = \{0, \gamma, \gamma^2, \ldots, \gamma^{q-1}\}$.

Chapter 2

Definition and General Properties of Non-commuting Graph

2.1 Definition, Examples and Some Properties of Non-commuting Graphs

In this section, we introduce the definition of non-commuting graph(Γ_G), and we give some examples. Then we give some general properties and theorems of non-commuting graphs.

***Definition* 2.1.** [2] Non-commuting graph of a group G is defined to be the graph Γ_G where the vertex set is $V_{\Gamma_G} = G - Z_G$, such that two distinct vertices x and y are adjacent if and only if $xy \neq yx$.

***Example* 2.1.** The non-commuting graph of the symmetric group S_3 (done by program 1 in sagemath):

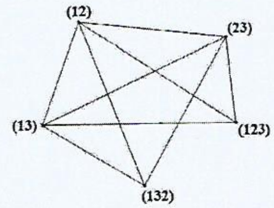

Figure 2.1: Non-commuting graph of S_3.

***Example* 2.2.** The non-commuting graph of dihedral group D_4 (done by program 1 in sagemath):

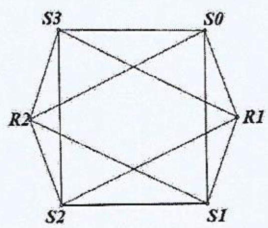

Figure 2.2: Non-commuting graph of D_4

Where R_0 denotes the identity, R_1 and R_2 denote counterclockwise rotations by 120° and 240° respectively, and S_0, S_1 and S_2 denote reflections.

***Example* 2.3.** The non-commuting graph of Quaternion group Q_8 (done by program 1 in sagemath):

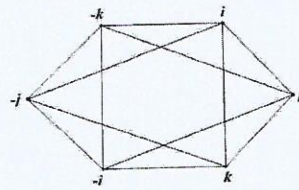

Figure 2.3: Non-commuting graph of Q_8

***Proposition* 2.1.** [16] The non-commuting graph Γ_G is connected for any group G.

Proof. Let g, h be any two non-adjacent vertices in $V(\Gamma_G)$, that is $gh = hg$ and $g, h \notin Z_G$ then

$$g \in C_G(h) = \{x \in G| \quad hx = xh\}$$

and

$$h \in C_G(g) = \{x \in G| \quad gx = xg\}.$$

Since $g, h \notin Z_G$, there exist $a, b \in V(\Gamma_G)$ such that $ga \neq ag$ and $hb \neq bh$.

If g, b are adjacent then there exist a path $g - b - h$.

If h, a are adjacent then there exist a path $h - a - g$.

In these two cases g and h are connected by a path of length 2.

If $gb = bg$ and $ha = ah$ then ab is adjacent to both g and h, otherwise, $gab = abg$, $gab = agb$. Hence, $ga = ag$ so we get a contradiction. Thus $g - ab - h$. □

***Proposition* 2.2.** [2] If G is any non-abelian group, then $diam(\Gamma_G) = 2$ and $girth(\Gamma_G) = 3$.

Proof. Let a, b be two distinct vertices in Γ_G.

If a, b adjacent then $d(a, b) = 1$. If not, that is $ab = ba$. Since $a, b \in G - Z_G$, there exist $c, d \in G - Z_G$, such that $ac \neq ca$ and $bd \neq db$. If a, d are adjacent, then

$ad \neq da$ and by assumption $bd \neq db$.

So, there is a path $a - d - b$, then $d(a, b) = 2$.

If c, b are adjacent, then $cb \neq bc$ and by assumption $ac \neq ca$.

So, $diam(\Gamma_G) = 2$.

If a, d is not adjacent and a, b not adjacent then cd adjacent to both a and b. Then $d(a, b) = 2$. Hence, $dim(\Gamma_G) = 2$.

For the second part, let a, b be two adjacent vertices in Γ_G, then $ab \neq ba$.

Hence,

$$a(ba) \neq (ab)a \quad \text{and} \quad b(ab) \neq (ab)b$$

Thus $\{a, b, ab\}$ is a triangle in Γ_G. Thus the girth of Γ_G is 3. □

***Lemma* 2.1.** If G is any group then the degree of any vertex in Γ_G is

$$deg(x) = |G| - |C_G(x)|.$$

Proof. By definition, $deg(x)$ is the number of vertices adjacent to x.

So, if $y \in V(\Gamma_G) = G - Z_G$, then

$$xy \in E(\Gamma_G) \iff xy \neq yx \quad \text{and} \quad y \in V(\Gamma_G) = G - Z_G$$

Then $y \notin C_G(x)$.

So the $deg(x)$ is the number of all vertices in G that are not in $C_G(x)$. Then

$$deg(x) = |G| - |C_G(x)|.$$

□

***Proposition* 2.3.** [2] The non-commuting graph of a group G is Hamiltonian.

Proof. By previous lemma, if $x \in V(\Gamma_G)$ where G is non-abelian group, then

$$deg(x) = |G| - |C_G(x)|,$$

since $x \in V(\Gamma_G) = G - Z_G$, and we know that $Z_G = \bigcap_{x \in G} C_G(x)$ then $Z_G \leq C_G(x)$, and $C_G(x) < G$ but $C_G(x) \neq G$ since Γ_G is connected so $[G : C_G(x)] \geq 2$, then $|G| \geq 2|C_G(x)|$, so $|G| - |C_G(x)| \geq |C_G(x)|$

$$|G| - |Z_G| = |G| - |C_G(x)| + |C_G(x)| - |Z_G|$$

but

$$|C_G(x)| - |Z_G| \leq |C_G(x)| \leq |G| - |C_G(x)$$

then

$$|G| - |Z_G| \leq \Big(|G| - |C_G(x)|\Big) + \Big(|G| - |C_G(x)|\Big) = 2\Big(|G| - |C_G(x)|\Big)$$

so

$$\frac{|G| - |Z_G|}{2} \leq |G| - |C_G(x)| = deg(x) \tag{2.1}$$

By Dirac's theorem 1.2, Γ_G is Hamiltonian. □

***Proposition* 2.4.** [16] If G is a group, then

$$\rho(G) = |G|(|G| - k(G))$$

Also, we have

$$|E(\Gamma_G)| = \frac{|G|(|G| - k(G))}{2}$$

Proof.

$$\begin{aligned}
\rho(\Gamma_G) &= \sum_{a\in V(\Gamma_G)} deg(a) \\
&= \sum_{a\in V(\Gamma_G)} |G| - |C_G(a)| \\
&= \sum_{a\in V(\Gamma_G)} |G| - \sum_{a\in V(\Gamma_G)} |C_G(a)| \\
&= |G| \sum_{a\in V(\Gamma_G)} 1 - \sum_{a\in V(\Gamma_G)} \frac{|G|}{|C_s(a)|} \\
&= |G|^2 - |G| \sum_{a\in V(\Gamma_G)} \frac{1}{|C_s(a)|} \\
&= |G|^2 - |G|k(G)
\end{aligned}$$

and by Hand-Shaking Lemma we get:

$$|E(\Gamma_G)| = \frac{\rho(G)}{2} = \frac{|G|(|G| - k(G))}{2}$$

□

***Proposition* 2.5.** [16] There is no non-abelian finite group G with a normal subgroup $N \neq \{e\}$ such that $\Gamma_G \cong \Gamma_{G/N}$.

Proof. We use contradiction. Let G be a non abelian group and $N \trianglelefteq G$, $|N| \neq \{1\}$ such that $\Gamma_G \cong \Gamma_{G/N}$. Let $r = |G|$, $f = |N|$, $e = |Z_G|$, $s = |N \cap Z_G|$

If

$$\Gamma_G \cong \Gamma_{G/N} \quad \text{Then} \quad |V_{\Gamma_G}| = |V_{\Gamma_{G/N}}|$$

that is

$$|G - Z_G| = |G/N - Z_{G/N}|$$

So

$$|G| - |Z_G| = \frac{|G|}{|N|} - |Z_{G/N}|$$

$$|Z_{G/N}| = \frac{|N||Z_G|}{|N|} + \frac{|G| - |N||G|}{|N|}$$

As

$$|Z_{G/N}| \geq \frac{|N||Z_G|}{|N|} \geq \frac{|NZ_G|}{|N|},$$

then by second isomorphism theorem 1.7

$$\frac{|NZ_G|}{|N|} = \frac{|Z_G|}{|Z_G \cap N|} = \frac{e}{s}$$

and

$$|G - Z_G| = |G/N - Z_{G/N}| \leq \frac{|G|}{|N|} - \frac{|Z_G N|}{|N|} = \frac{|G|}{|N|} - \frac{|Z_G|}{|Z_G \cap N|}$$

$$r - e \leq \frac{r}{f} - \frac{e}{s}$$

$$r - \frac{r}{f} \leq e - \frac{e}{s}$$

$$r\left(1 - \frac{1}{f}\right) \leq e\left(1 - \frac{1}{s}\right)$$

$$\frac{r}{e} \leq \frac{1 - \frac{1}{s}}{1 - \frac{1}{f}}$$

From the assumption that $|N| \neq \{1\}$ then we can assume that $f \geq 2$ so,

$$\frac{r}{e} \leq \frac{1 - \frac{1}{s}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{s}\right) = 2 - \frac{2}{s} < 2$$

Thus $|G|/|Z_G| = 1$, this implies that G is abelian, which contradicts the assumption.

□

***Proposition* 2.6.** [16] No finite non-abelian group G exists with $H < G$ and $\Gamma_G \cong \Gamma_H$.

Proof. We use contradiction. Given a finite non-abelian group G and $H < G$ (proper subgroup) such that $\Gamma_G \cong \Gamma_H$ then $|G| - |Z_G| = |H| - |Z_H|$.

By assumption $Z_G \lneqq G$, if $[G : Z_G] = n$, then

$$|Z_G| = \frac{|G|}{n} \leq \frac{|G|}{2}, \quad n \neq 1$$

Since H is a proper subgroup of G, we get $|H| = \dfrac{|G|}{m}$ for some $m \neq 1$.

Hence, $|H| \leq \dfrac{|G|}{2}$, so we have

$$|G| = |H| - |Z_H| + |Z_G| < \frac{1}{2}|G| - |Z_H| + \frac{1}{2}|G| < |G|,$$

So we get a contradiction. □

***Proposition* 2.7.** [2] Let G be a finite non-abelian group. Then Γ_G is planar if and only if $G \cong S_3, D_4$ or Q_8.

Proof. ($\Longrightarrow$) The non-commuting graphs of S_3, D_4 and Q_8 are all planar as shown respectively below, where program 1 was used to draw them.

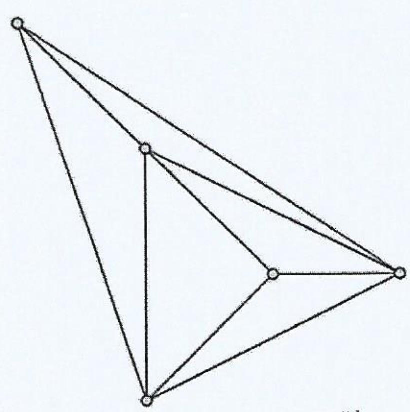

Figure 2.4: Γ_{S_3}

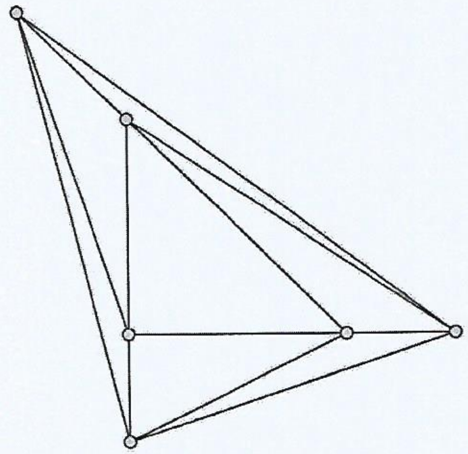

Figure 2.5: Γ_{Q_8}

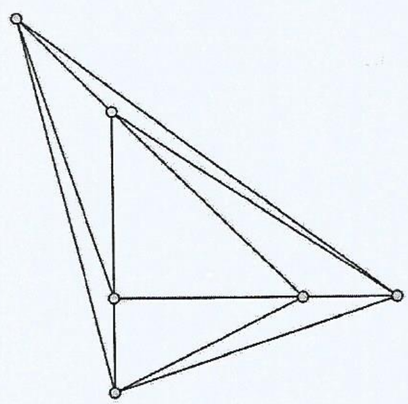

Figure 2.6: Γ_{D_4}

($\Longleftarrow$) Now if Γ_G is planar. By theorem 1.3 the complete graph of order 5 is not planar, thus we have $\omega(\Gamma_G) < 5$. Now we prove $|Z_G| \leq 5$.

By contradiction suppose that $|Z_G| > 5$. Since G is not abelian, there exist $x, y \in G$ such that $xy \neq yx$. Let

$$T = xZ_G \cup yZ_G$$

where xZ_G, yZ_G are two distinct left cosets and hence disjoint. The induced subgraph

Γ_T of Γ_G by T is a planar graph. Since Γ_T is planar graph, by theorem 1.5 there exist a vertex $v \in T$ such that $deg_{\Gamma_T}(v) \leq 5$. But for every vertex $w \in T$, $deg_{\Gamma_T}(w) > 5$, which is contradiction.

Since Γ_G is planar, there exist $x \in G - Z_G$ such that $deg(x) \leq 5$. We know that $|C_G(x)| \leq \frac{|G|}{2}$ and $deg(x) = |G| - |C_G(x)|$, so

$$5 \geq |G| - |C_G(x)| \geq |G| - \frac{|G|}{2} = \frac{|G|}{2}$$

$$\frac{|G|}{2} \leq 5$$

$$|G| \leq 10$$

If $|G| = 10$ or 6, that is of the form $2p$ where p is prime then by Theorem 1.12 is isomorphic to D_{10} and D_6 respectively, and the non-commuting graph of D_{10} is not planar, and $\Gamma_{D_6} \cong \Gamma_{S_3}$ is planer. Then G isomorphic to S_3 .

If $|G| = 8$, then $G \cong Q_8$ or $G \cong D_8$ by theorem 1.14.

If $|G| = 9$ or 4, then by Theorem1.13, G is abelian. Hence $|G| \neq 9, 4$

If $|G| = 7, 5, 3$ or 2. By Theorem1.8 G is cyclic group, then G abelian, hence $|G| \neq 7, 5, 3$ or 2. □

***Proposition* 2.8.** [2] If G is a non-abelian group and $\{g\}$ is a dominating set for Γ_G, where $g \in G$, then Z_G is the trivial subgroup, $g^2 = 1$ and $< g >= C_G(g)$.

Proof. By contradiction, suppose that $Z_G \neq \{e\}$. So there exists a non-trivial element $z \in Z_G$ then $zg \in Z_G$. Otherwise, if $zg \in Z_G$, then $zgx = xzg$ for all $x \in G$. Then

$$(zg)x = zxg \implies gx = xg,$$

hence $g \in Z_G$ which is a contradiction, then zg is a vertex in Γ_G, since $zg \neq g$, otherwise $z = e$. Hence $zgg = gzg$, then zg also is not adjacent to g, which is a contradiction to the definition of dominating set.

Now, we prove $g^2 = 1$, also by contradiction. Let $g^2 \neq 1$, then $g \neq g^{-1}$. So g^{-1} is not adjacent to g. Which is a contradiction, since g is adjacent to all vertices in Γ_G. Finally, since $Z_G = \{e\}$ and g adjacent to all vertices in Γ_G, then by definition of centralizer $C_G(g) = \{e, g\}$. □

***Proposition* 2.9.** [2] Let G be a non-abelian group. Then the dominating set for Γ_G is $X - Z_G$ where X is a generating set for G.

Proof. Let $G =< X >$ then $Y = X - Z_G \neq \phi$ since G is non-abelian. From the definition of centralizer and that X is a generating set of G, then $C_G(Y) = Z_G$. Now we claim that Y is a dominating set. Let $a \notin Z_G$ then $a \notin C_G(Y)$. So, there exists $y \in Y$ such that $ay \neq ya$, hence there exists $y \in Y$ such that a is adjacent to y. □

2.2 Further Properties of Non-commuting Graphs of Groups

Now, we introduce some properties of non-commuting graphs that we need in chapter 3. We consider two non-abelian groups G and H such that $\Gamma_G \cong \Gamma_H$. Then we study the properties of the groups that are preserved by this isomorphism.

***Proposition* 2.10.** [2] If G and H are two finite non-abelian groups, where $\Gamma_G \cong \Gamma_H$.

Then $|Z_H|$ is a divisor of $gcd\Big(|G|-|Z_G|, \quad |G|-|C_G(v)|, \quad |C_G(v)|-|Z_G|\Big)$. That is

$$|Z_H|\Big|gcd\Big(|G|-|Z_G|, |G|-|C_G(v)|, |C_G(v)|-|Z_G|\Big),$$

for all $v \in V(\Gamma_G)$.

Proof. Since $\Gamma_G \cong \Gamma_H$, $|G - Z_G| = |H - Z_H|$. Now, since $\Gamma_G \cong \Gamma_H$, there exists a bijection

$$\phi : V(\Gamma_G) \longrightarrow V(\Gamma_H)$$

Such that, if $v \in V(\Gamma_G)$ and $\phi(v) := h_v$

$$deg(v) = deg(h_v)$$

$$|G| - |C_G(v)| = |H| - |C_H(h_v)|$$

Since $|Z_H|\Big||H|$ and $|Z_H|\Big|C_H(h_v)|$ then $|Z_H|\Big|deg(h_v)$, thus $|Z_H|\Big|deg(v)$, where $deg(v) = |G| - |C_G(v)|$. That is

$$|Z_H|\Big|(|G| - |C_G(v)|) \tag{2.2}$$

and since

$$|G| - |Z_G| = |H| - |Z_H|$$

Then

$$|Z_H|\Big|(|G| - |Z_G|) \tag{2.3}$$

Then by 2.2 and 2.3,

$$|Z_H|\Big|(|G| - |Z_G|) - (|G| - |C_G(v)|).$$

Then

$$|Z_H|\Big|(|C_G(v) - |Z_G|), \quad \forall v \in G$$

Hence, by the definition of greatest common divisor

$$|Z_H| \quad \text{is a divisor of} \quad gcd\big(|G|-|Z_G|, |G|-|C_G(v)|, |C_G(v)|-|Z_G|\big).$$

□

***Definition* 2.2.** [2] If $C_G(a)$ for all a in a group G is abelian then this group is called an **AC-group**.

***Proposition* 2.11.** [1] If $\Gamma_G \cong \Gamma_H$ where G and H are two finite non-abelian groups. Then

1. If $|G| = |H|$ then $\{|C_G(g)| : g \in G - Z_G\}$ and $\{|C_H(h)| : h \in H - Z_H\}$ are equal.

2. H is an AC-group, if G is an AC-group.

Proof. Let $\Gamma_G \overset{\phi}{\cong} \Gamma_H$, hence $\phi : V(\Gamma_G) \to V(\Gamma_H)$ is a bijection, such that for any two distinct vertices $x, y \in V(\Gamma_G)$ are adjacent if and only if $\phi(x), \phi(y)$ are adjacent in Γ_H.

If $\phi(g) = h$ where $g \in G$ and $h \in H$, then

1. $deg(g) = deg(h)$, that is $|G| - |C_G(g)| = |H| - |C_H(h)|$. Since $|G| = |H|$ then

$$|C_G(g)| = |C_H(h)|.$$

2. Let $h \in H - Z_H$, then there exist $g \in G$ such that $\phi(g) = h$

$$C_H(h) - Z_H = \phi\big(C_G(g) - Z_G\big) \tag{2.4}$$

Now since G is an AC-group, $C_G(g)$ is abelian for all $g \in G - Z_G$. To show $C_H(h)$ is abelian let $x, y \in C_H(h)$, $x = \phi(x')$ and $y = \phi(y')$, where $x', y' \in$

$C_G(g) - Z_G$. Then $x'y' = y'x'$.

Hence, x',y' are not adjacent in Γ_G, then $\phi(y')$, $\phi(x')$ are not adjacent in Γ_H.

Hence,

$$\phi(x')\phi(y') = \phi(y')\phi(x')$$

$$xy = yx$$

Then x, y are not adjacent in Γ_H that is $xy = yx$. Hence H is also an AC-group.

□

***Proposition* 2.12.** [2] If G is an AC-group and $x, y \in G - Z_G$ with $xy = yx$ then $C_G(x) = C_G(y)$.

***Proposition* 2.13.** [2] If G is a non-abelian group and $\Gamma_G \cong \Gamma_{S_3}$. Then $G \cong S_3$.

Proof. Since Γ_{S_3} is planar, then Γ_G is also planar. Hence by proposition 2.7, $G \cong S_3$.

□

***Lemma* 2.2.** Every finite non-abelian group has order greater than or equal to 6.

Proof. If $|G| = 1$ then the group is the identity so it is abelian groups.

If $|G| = 2$, 3 or 5, then by theorem 1.8, it is abelian group.

If $|G| = 4$ by theorem 1.13 $G \cong Z_4$ or $G \cong Z_2 \oplus Z_2$, so in both cases G is abelian group. □

***Theorem* 2.3.** [2] If G and H are two finite non-abelian groups, with $\Gamma_G \cong \Gamma_H$, then $|G| = |H|$.

Proof. By induction on $|G|$.

Since G is finite non-abelian by lemma 2.2, $|G| \geq 6$.

If $|G| = 6$ then $G \cong S_3$ and by proposition 2.13, $|G| = |H|$.

If $|G| > 6$, and A is a finite non-abelian subgroup of G, then $|A| < |G|$. If H is a finite non-abelian group such that $\Gamma_A \cong \Gamma_H$, then $|A| = |H|$.

Now, if ϕ is a graph isomorphism where

$$\phi : V(\Gamma_G) \longrightarrow V(\Gamma_H)$$

Then, for all $x \in G - Z_G$,

$$|C_G(x) - Z_{C_G(x)}| = |C_H(\phi(x)) - Z_{C_H(\phi(x))}|.$$

If $C_G(x)$ is non-abelian and $\Gamma_{C_G(x)} \cong \Gamma_{C_H(\phi(x))}$, by last induction step

$$|C_G(x)| = |C_H(\phi(x))|.$$

By definition of graph isomorphism, $|C_G(x)| - |Z_G| = |C_H(\phi(x))| - |Z_H|$, so

$$|Z_G| = |Z_H|.$$

But we know $\Gamma_G \cong \Gamma_H$, Then we have

$$|G| - |Z_G| = |H| - |Z_H|$$

So,

$$|G| = |H|.$$

□

***Lemma* 2.4.** [3] If G is a finite non-abelian group and H is non-abelian subgroup of G. Then $\omega(\Gamma_H) \leq \omega(\Gamma_G)$.

Proof. Since H is a subgroup of G, then Γ_H is also a subgraph of Γ_G.

Hence,

$$\omega(\Gamma_H) \leq \omega(\Gamma_G).$$

□

***Proposition* 2.14.** [3] If G is a finite non-abelian group and $H_1, \dots, H_n$ are non-trivial subgroups such that $G = \bigcup_{i=1}^{n} H_i$ and $H_i \cap H_j = Z_G$ for $i \neq j$. If, in addition $C_G(g) \leq H_i$ for all $g \in H_i - Z_G$, then $\omega(\Gamma_G) = \sum_{i=1}^{n} \omega(\Gamma_{H_i})$.

Proof. If C is any clique of Γ_G, then

$$C = \bigcup_{i=1}^{n} C_i \quad \text{where} \quad C_i \subseteq H_i - Z_G \quad \text{for all} \quad i \in \{1, \dots, n\}$$

By hypothesis,

$$|C| = \sum_{i=1}^{n} |C_i|$$

and since $|C_i| \leq \omega(\Gamma_{H_i})$, it follows that

$$|C| \leq \sum_{i=1}^{n} \omega(\Gamma_{H_i})$$

Now, if M_i is a maximum clique of Γ_{H_i} for all $i \in \{1, \dots, n\}$. Let

$$M = \bigcup_{i=1}^{n} M_i$$

We show that M is a maximum clique for Γ_G.

First, we show that M is a clique for G.

We use contradiction, assume that there exist a, b two commuting elements where

$$a, b \in \bigcup_{i=1}^{n} M_i$$

such that $ab = ba$. Thus there exist $i \neq j$ where

$$a \in H_i \qquad \text{and} \qquad b \in H_j$$

Therefore

$$a \in C_G(b) \leq H_j \qquad \text{and so} \qquad a \in H_i \cap H_j = Z_G,$$

which is a contradiction.

Second, we show that the clique M is a maximum clique for Γ_G.

If not, then there exists $x \notin M$ such that $xa \neq ax$ for all $a \in M$. Then

$$xa \neq ax \quad \forall a \in M_i \quad \forall i$$

Then M_i is not a maximum clique for H_i, we get a contradiction. So

$$|M| = \sum_{i=1}^{n} \omega(\Gamma_{H_i}).$$

That is,

$$\omega(\Gamma_G) = \sum_{i=1}^{n} \omega(\Gamma_{H_i}).$$

□

***Proposition* 2.15.** [2] If G is a finite non-abelian group. Then $\chi(\Gamma_G) =$ minimum number of abelian subgroups of G, where their union is G.

Also, $\omega(\Gamma_G) \leq \chi(\Gamma_G) \leq [G : Z_G]$.

Proof. Let $X_1, X_2, \ldots, X_n$ be abelian subgroups of G such that $G = \bigcup_{i=1}^{n} X_i$ where n is the minimum number of abelian subgroups with such property. Then the vertices of Γ_G in X_i are independent. Then $\chi(\Gamma_G) \leq n$.

Let $k = \chi(\Gamma_G)$, so there exist k independent subsets of $V(\Gamma_G)$. Say, $N_1, N_2, \ldots, N_k$

such that $V(\Gamma_G) = \bigcup_{i=1}^{k} N_i$. The subgroup generated by $< N_j, Z_G >$ is an abelian subgroup of G for each j. So, $k \geq n$. Then

$$\chi(\Gamma_G) = n$$

By theorem 1.4, $\chi(\Gamma_G) \geq \omega(\Gamma_G)$. Now, let $[G : Z_G] = m$ then $G = \bigcup_{i=1}^{m} g_i Z_G$ for $g_i \in G$ and $< g_i, Z_G >$ for $1 \leq i \leq m$ is an abelian subgroup of G so by part one of the proof $\chi(\Gamma_G) \leq m$. ☐

***Theorem* 2.5.** [2] If G is a finite non-abelian AC-group, then $\chi(\Gamma_G) = \omega(\Gamma_G)$

Proof. Let

$$w = \omega(\Gamma_G) \quad \text{and} \quad k = \chi(\Gamma_G)$$

and let $\{a_1, a_2, \ldots, a_w\}$ be a maximal clique in Γ_G. So each $a_i \notin Z_G$ and $C_G(a_i)$ is abelian. Then $G = \bigcup_{i=1}^{w} C_G(a_i)$, and $k \leq w$. By last proposition $\chi(\Gamma_G) \geq \omega(\Gamma_G)$. Hence,

$$\chi(\Gamma_G) = \omega(\Gamma_G)$$

☐

***Proposition* 2.16.** [3] If G is a finite non-abelian group. Then $\omega(\Gamma_{G/N}) \leq \omega(\Gamma_G)$ where G/N is any non-abelian factor group.

Proof. Let C/N be a maximum clique of $\Gamma_{G/N}$. Then for all cN, $c^*N \in C/N$, we have $cNc^*N \neq c^*NcN$. So $(c^*)^{-1}c^{-1}cc^* \notin N$, then $(c^*)^{-1}c^{-1}cc^* \neq e$. Then $cc^* \neq c^*c$, thus C is a clique of Γ_G. So $|C| \leq \omega(\Gamma_G)$. Hence

$$\omega(\Gamma_{G/N}) = |C/N| \leq |C| \leq \omega(\Gamma_G).$$

☐

***Proposition* 2.17.** [2] If G is finite non-abelian simple group. Then $\gamma(\Gamma_G) = 2$.

Proof. Since G is a finite simple non-abelian group, G is generated by two elements, by theorem 1.9, $X - Z_G$ is a dominating set of Γ_G. So $\gamma(\Gamma_G) = 2$. □

***Theorem* 2.6.** [18] Let G be a finite group and S a finite non-abelian simple group with $\Gamma_G \cong \Gamma_S$. Then $G \cong S$.

The proof is long and done by Salomon and Woldar in [18].

Chapter 3

Non-commuting Graph of $SL(n, q)$ where $n = 2, 3$

In this chapter we present the main goal of this research. In section 1, we give the definition and the structure of the special linear group $SL(n, q)$ and some generic properties. In section two, we consider the non-commuting graph of $SL(2, q)$, we find the clique number $\omega(\Gamma_{SL(2,q)})$, chromatic number $\chi(\Gamma_{SL(2,q)})$, independent number $\alpha(\Gamma_{SL(2,q)})$ and the minimum size of vertex cover $\beta(\Gamma_{SL(2,q)})$.

In last section, we prove $|G| = |SL(3, q)|$ if $\Gamma_G \cong \Gamma_{SL(3,q)}$. We also prove if $\Gamma_G \cong \Gamma_{SL(3,q)}$ and $|Z_{SL(3,q)}| = 1$, then $G \cong SL(3, q)$. Finally, we find an upper bound of the independent number $\alpha(\Gamma_{SL(3,q)})$ and the minimum size of vertex cover $\beta(\Gamma_{SL(3,q)})$.

3.1 Definition and Some Properties of $SL(n, q)$

The general linear group $GL(n, q)$ is a non-abelian group of all invertible $n \times n$ matrices over the Galois field F_q.

We consider the determinant function :

$$det : GL(n, q) \to F_q^*,$$

that assign to each matrix in $GL(n, q)$ its determinant. It is a group homomorphism. The kernel of this homomorphism is the **special linear group** $SL(n, q)$. That is

$$SL(n, q) = \{M \in GL(n, q) | \quad det(M) = 1\}$$

The factor group $PSL(n, q) = SL(n, q)/Z_{SL(n,q)}$ is called the **projective special linear group**.

***Theorem* 3.1.** [17] The center of $SL(n, q)$, denoted by $Z_{SL(n,q)}$, is given by

$$Z_{SL(n,q)} = \{mI_n | \quad m^n = 1 \text{ in } \quad F_q\}.$$

Proof. ($\subseteq$) If $m^n = 1$ then any matrix of the form mI_n belongs to $Z_{SL(n,q)}$.

($\supseteq$) Let $M = (m_{ij}) \in Z_{SL(n,q)}$ and consider $E_{1,2}(\lambda)$ where $E_{i,j}(\lambda)$ is the $n \times n$ matrix formed from the $n \times n$ identity matrix I_n by replacing 0 with λ at the $(i, j)-th$

location [17], $\lambda \neq 0$.

$$E_{1,2}(\lambda)M = \begin{bmatrix} 1 & \lambda & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2n} \\ m_{31} & m_{32} & m_{33} & \dots & m_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \dots & m_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} m_{11}+\lambda m_{21} & m_{12}+\lambda m_{22} & m_{13}+\lambda m_{23} & \dots & m_{1n}+\lambda m_{2n} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2n} \\ m_{31} & m_{32} & m_{33} & \dots & m_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n1} & m_{n2} & m_{n3} & \dots & m_{nn} \end{bmatrix}$$

and

$$ME_{1,2}(\lambda) = \begin{bmatrix} m_{11} & \lambda m_{11}+m_{12} & m_{13} & \dots & m_{1n} \\ m_{21} & \lambda m_{21}+m_{22} & m_{23} & \dots & m_{2n} \\ m_{31} & \lambda m_{31}+m_{32} & m_{33} & \dots & m_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ m_{n1} & \lambda m_{n1}+m_{n2} & m_{n3} & \dots & m_{nn} \end{bmatrix}$$

If $E_{1,2}(\lambda)M = ME_{1,2}(\lambda)$. Then we have

$$m_{21} = m_{23} = m_{24} = \cdots = m_{2n} = 0$$

and

$$m_{21} = m_{31} = m_{41} = \cdots = m_{n1} = 0$$

and we have also

$$m_{12} + rm_{22} = \lambda m_{11} + m_{12} \quad \text{which lead to} \quad m_{11} = m_{22}$$

Apply for $E_{1,j}(\lambda)$ for $2 \leq j \leq n$. Then we get

$$M = \begin{bmatrix} m_{11} & 0 & 0 & \dots & 0 \\ 0 & m_{22} & 0 & \dots & 0 \\ 0 & 0 & m_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & m_{nn} \end{bmatrix}$$

and

$$m_{11} = m_{22} = \dots = m_{nn} = m$$

So, $M = mI_n$ □

***Corollary* 3.1.** $|Z_{SL(n,q)}| = gcd(n, q-1)$

Proof. Let $F_q = \{0, \alpha, \dots, \alpha^{q-1}\}$, then $Z_{SL(n,q)} = \{\alpha^t I_n \quad | \quad (\alpha^t)^n = 1\}$.

$$(\alpha^t)^n = 1 \iff \alpha^{tn} = 1$$

Then, by proposition 1.5

$$\alpha^{q-1} = 1 = \alpha^{tn}$$

Hence,

$$(q-1)\Big|tn$$

By theorem on linear congruences ([6] Let $g = gcd(n, q-1)$. Then the linear congruence $cx \equiv d \ mod \ m$ has a solution if and only if $g\Big|d$. If $g\Big|d$ then $cx \equiv d$ $mod \ m$ has exactly g incongruent solutions.), if $nt \equiv 0 \ mod \ (q-1)$, then it has d solutions where $d = gcd(n, q-1)$.

□

Theorem **3.2.** [14]

1. $|GL(n,q)| = q^{\frac{n(n-1)}{2}}(q^n-1)(q^{n-1}-1)...(q^2-1)(q-1)$.

2. $|SL(n,q)| = q^{\frac{n(n-1)}{2}}(q^n-1)(q^{n-1}-1)...(q^2-1)$.

Proof. 1. Let $A \in GL(2,q)$. So,

there exist (q^n-1) choices for first row,

there exist (q^n-q) choices for second row,

there exist (q^n-q^2) choices for third row,

$\vdots$

there exist (q^n-q^{n-2}) choices for (n-1)-th row,

there exist (q^n-q^{n-1}) choices for (n)-th row.

Second row is independent from row 1. Third row is independent from row 1 and row 2.(That means row 3 $\notin$ span{row 1, row 2}).

On the other side, if A is a matrix with linearly independent rows, then A is invertible matrix. Thus

$$\begin{aligned}
|GL(n,q)| &= (q^n-1)(q^n-q)(q^n-q^2)\dots(q^n-q^{n-2})(q^n-q^{n-1}) \\
&= (q^n-1)q(q^{n-1}-1)q^2(q^{n-2}-1)\dots q^{n-2}(q^2-1)q^{n-1}(q-1) \\
&= (q^n-1)(q^{n-1}-1)(q^{n-2}-1)\dots(q^2-1)(q-1)q^{1+2+\dots+(n-2)+(n-1)} \\
&= (q^n-1)(q^{n-1}-1)(q^{n-2}-1)\dots(q^2-1)(q-1)q^{\frac{n(n-1)}{2}}
\end{aligned}$$

2. Define the function

$$\phi : GL(n,q) \longrightarrow F_q^* \quad \text{such that} \quad \phi(A) =: det(A)$$

Then ϕ is a group homomorphism and onto. Since $ker\phi = SL(n,q)$, by first isomorphism theorem $GL(n,q)/SL(n,q) \cong F_q^*$.

Then we have the following:

$$|GL(n,q)/SL(n,q)| = |F_q^*| = q-1$$
$$\frac{|GL(n,q)|}{|SL(n,q)|} = q-1$$

Then,

$$|SL(n,q)| = \frac{q^{\frac{n(n-1)}{2}}(q^n-1)(q^{n-1}-1)...(q-1)}{q-1}$$
$$= q^{\frac{n(n-1)}{2}}(q^n-1)(q^{n-1}-1)...(q^2-1).$$

□

***Lemma* 3.3.** [14] Let $E_{ij}(\lambda)$ be the elementary matrix with 1 on main diagonal and λ at the (i,j)-th position and zero elsewhere. Then $E_{ij}(\lambda)$ generates $SL(n,q)$, for $n \geq 2$, $\{E_{ij}(\lambda)| \quad i,j \in 1,2,\dots n, \quad \lambda \in F_q^*\}$.

Proof. Let A be any matrix of $SL(n,q)$ for $n \geq 2$, it is sufficient to reduce A to the identity matrix by elementary matrix operations. Assume $a_{11} = 1$ So,

$$A = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{2n} & \dots & a_{nn} \end{bmatrix}$$

If $a_{21} = 0$, multiply A from left by $E_{21}(a)$ to get a matrix with nonzero element in the (2,1)-position. If $a_{21} \neq 0$, we multiply A by $E_{21}(-a_{21})$ from left and $E_{12}(-a_{12})$ from right, we get

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ -a_{21} & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{2n} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & -a_{21} & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$$

where $B \in SL(n-1, q)$ is a product of elementary matrices by induction on n. □

***Proposition* 3.1.** [19]

1. The generators of $SL(2, q)$ for $q = 2$ or 3 are:

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2. The generators of $SL(2, q)$ for $q > 3$ are:

$$X = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$$

3. The generators of $SL(3, q)$ for $q \leq 3$ are:

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

4. The generators of $SL(3, q)$ for $q > 3$ are :

$$X = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proof. By previous lemma enough to show that each $E_{ij}(\lambda)$ is generated by $\{X, W\}$.

1. For $SL(2,2)$. Let

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   $E_{12}(1) = X$ and $E_{21}(1) = X^{-1}WX$.

   So, $E_{12}(1)$ and $E_{21}(1)$ generate $SL(2,2)$ by lemma 3.3.

   For $SL(2,3)$. Let

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

   $E_{12}(1) = X$ and $E_{12}(2) = E_{12}^{-1}(1)$.

   $E_{21}(1) = X^{-1}WX$ and $E_{21}(2) = E_{21}^{-1}(1)$.

   So, $E_{12}(1)$, $E_{12}(2)$, $E_{21}(1)$ and $E_{21}(2)$ generate $SL(2,3)$ by lemma 3.3.

2. For $SL(2,q)$ where $q > 3$. we may proceed in the same manner to show that

$$X = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$$

   generate all elementary matrix of $SL(2,q)$ and hence generate $SL(2,q)$ by lemma 3.3.

3. For $SL(3,2)$.

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

   The elementary matrices are generated by X and W as follows:

   $E_{12}(1) = X$.

$E_{23}(1) = WXW^{-1}$.

$E_{31}(1) = W^{-1}XW$.

$E_{21}(1) = E_{23}(1)E_{31}(1)E_{23}^{-1}(1)E_{31}^{-1}(1)$.

$E_{32}(1) = WE_{21}(1)X^{-1}$.

$E_{13}(1) = W^{-1}E_{21}(1)W$.

Now for $SL(3,3)$.

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

The elementary matrices $E_{ij}(\lambda)$ are generated by X and W as follows:

$E_{12}(1) = X$ and $E_{12}(2) = X^{-1}$.

$E_{23}(1) = WXW^{-1}$ and $E_{23}(2) = E_{23}^{-1}(1)$.

$E_{31}(2) = W^{-1}XW$and $E_{31}(1) = XE_{23}(1)X^{-1}$.

$E_{21}(1) = E_{23}(1)E_{31}(1)E_{23}^{-1}(1)E_{31}^{-1}(1)$ and $E_{21}(2) = E_{21}^{-1}(1)$.

$E_{32}(1) = WE_{21}(1)W^{-1}$ and $E_{32}(2) = E_{32}^{-1}(1)$.

$E_{13}(1) = W^{-1}E_{21}(2)W$ and $E_{13}(2) = E_{13}^{-1}(1)$.

4. For $SL(3,q)$ where $q > 3$ we may proceed in the same manner to show that

$$X = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

generate all elementary matrix $E_{ij}(\lambda)$.

□

***Definition* 3.1.** [14] If a group G is equal to its commutator subgroup i.e $G = G'$. Then it is called **perfect**.

***Lemma* 3.4.** [14] The elementary matrices $\{E_{ij}(\lambda)| \quad \lambda \in F_q\}$ are commutators in $SL(n, q)$ except with $n = 2$ and $q = 2$ or 3.

***Theorem* 3.5.** [14] The group $SL(n, q)$ is *perfect*. That means, it is equal to its commutator subgroup, except in the cases $SL(2, 2)$ and $SL(2, 3)$.

Proof. We use lemma 3.4 and proposition 3.1 to get the conclusion. □

***Corollary* 3.2.** [14] $SL(n, q)$ is not solvable for $n \geq 2$, except in the cases of $SL(2, 2)$ and $SL(2, 3)$.

Proof. Since $SL(n, q)$ is perfect, except in the cases $SL(2, 2)$ and $SL(2, 3)$. Then by theorem 1.15, $SL(n, q)$ is not solvable except for $n = 2$ and $q = 2$ or 3. □

***Proposition* 3.2.** [7] If $n > 2$ and $q > 3$ then $PSL(n, q)$ is simple.

Proof. The proof is long, and stated in theorem 2.10 in [7]. □

***Proposition* 3.3.** If $|Z_{SL(n,q)}| = 1$, then the special linear group $SL(n, q)$ is simple group except $SL(2, 2)$ and $SL(2, 3)$.

Proof. By definition of $PSL(2, q)$

$$PSL(2, q) = SL(2, q)/Z_{SL(2,q)}$$

If $|Z_{SL(n,q)}| = 1$ then $SL(n, q) \cong PSL(n, q)$, and by proposition 3.2 the group $PSL(n, q)$ is simple group for every $q > 3$. So, $SL(n, q)$ is simple for every $q > 3$ and $|Z_{SL(n,q)}| = 1$. □

***Proposition* 3.4.** [2] If $G = GL(2, q)$, then G is an AC-group.

Proof. The proof of this proposition is done by Abdollahi, lemma 3.1 in [2]. □

***Lemma* 3.6.** [1] Let G be a finite non-solvable group. If $G/Z_G \cong PSL(2, q)$ and $G' \cong SL(2, q)$, where $q > 3$, then G is an AC-group.

Proof. The proof of this lemma is done by Huppert [13], satz 3.9. □

***Theorem* 3.7.** $SL(2, q)$ is an AC-group.

Proof. For $SL(2, 2)$ and $SL(2, 3)$ we use program 2 in sagemath to show that these groups are AC-groups. For $SL(2, q)$ where $q > 3$, it is non-solvable perfect group and by lemma 3.6, it is an AC-group. □

***Proposition* 3.5.** [2] Let $G = GL(2, q)$ and $x \notin G - Z_G$. Then $C_G(x)$ is conjugate to exactly one of the following subgroups of G:

1. $\mathcal{D} = \{D \in G \mid$ where D is diagonal matrix$\}$, $|\mathcal{D}| = (q-1)^2$ and $\mathcal{D} = C_G(D)$ for every $D \in \mathcal{D} - Z_G$. Also the number of conjugates of $\mathcal{D}$ is $\dfrac{q(q+1)}{2}$

2. A cyclic subgroup $\mathcal{I}$ of order $q^2 - 1$, $\mathcal{I} = C_G(X)$ or any generator X of I, and the number of conjugates of $\mathcal{I}$ is $\dfrac{q(q-1)}{2}$.

3. SZ_G, where S is the Sylow p-subgroup, where $q = p^m$ such that for all $a \in F_q$,

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \in S,$$

$|SZ_G| = q^2 - q$, and $SZ_G = C_G(X)$, where X is any non-trivialelement in SZ_G. Also, the number of conjugates of SZ_G is $q + 1$.

Moreover, each of the above subgroups is equal to the centralizer of an element in G and the union of all conjugates of the above subgroups is G.

Abdollahi proved this proposition in [2] by using satz 7.2 in [13].

3.2 The Non-commuting Graph of $SL(2, q)$

In this section, we give some examples of the non-commuting graph of $SL(2, q)$. We give the independent number $\alpha(\Gamma_{SL(2,q)})$, vertex chromatic number $\chi(\Gamma_{SL(2,q)})$, clique number $\omega(\Gamma_{SL(2,q)})$, minimum size of vertex cover $\beta(\Gamma_{SL(2,q)})$ of the non-commuting graph of $SL(2, q)$.

***Example* 3.1.** The non-commuting graph of $SL(2, 2)$:

By sagemath program 3 in appendix we get:

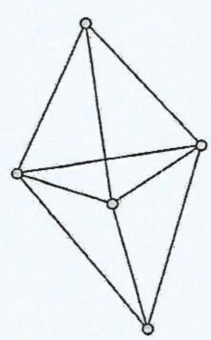

Figure 3.1: Non-commuting graph of $SL(2, 2)$

The group $SL(2, 2)$ is finite non-abelian group of order 6. $|Z_{SL(2,2)}| = 1$. The Non-commuting graph of $SL(2, 2)$, as shown in figure 3.1, is connected graph and diameter 2 and girth 3. It is Hamiltonian graph, planar and not regular graph. We find also by using sagemath that:

$\alpha(\Gamma_{SL(2,2)}) = 2$, $\omega(\Gamma_{SL(2,2)}) = 4$, $\beta(\Gamma_{SL(2,2)}) = 3$, $\chi(\Gamma_{SL(2,2)}) = 4$.

***Example* 3.2.** The non-commuting graph of $SL(2,3)$:

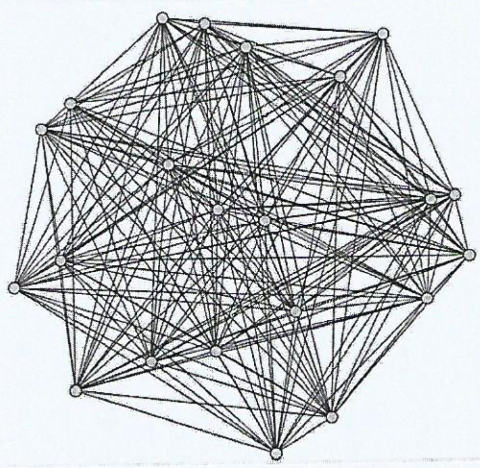

Figure 3.2: Non-commuting graph of $SL(2,3)$

The group $SL(2,3)$ is finite non-abelian group of order 24. $|Z_{SL(2,3)}| = 2$. The Non-commuting graph of $SL(2,3)$, as shown in figure 3.2, is connected graph, diameter 2 and girth 3. It is Hamiltonian graph but not planar and not regular graph. By using sagemath program 3 we get that:

$\alpha(\Gamma_{SL(2,3)}) = 4$, $\omega(\Gamma_{SL(2,3)}) = 7$, $\beta(\Gamma_{SL(2,3)}) = 18$, $\chi(\Gamma_{SL(2,3)}) = 7$.

***Example* 3.3.** The non-commuting graph of $SL(2,4)$:

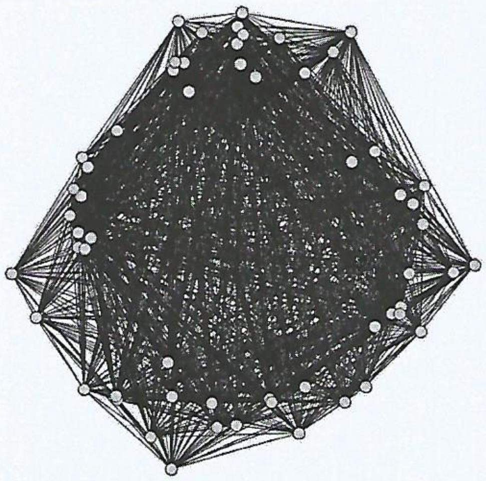

Figure 3.3: Non-commuting graph of $SL(2,4)$

The group $SL(2,4)$ is finite non-abelian group of order 60. $|Z_{SL(2,4)}| = 1$. The Non-commuting graph of $SL(2,4)$, as shown in figure 3.3, is connected graph, diameter 2 and girth3. It is Hamiltonian graph but not planar and not regular graph. By using sagemath program 3 we get that:

$\alpha(\Gamma_{SL(2,4)}) = 4$, $\omega(\Gamma_{SL(2,4)}) = 21$, $\beta(\Gamma_{SL(2,4)}) = 55$, $\chi(\Gamma_{SL(2,4)}) = 21$.

***Example* 3.4.** The non-commuting graph of $SL(2,5)$:

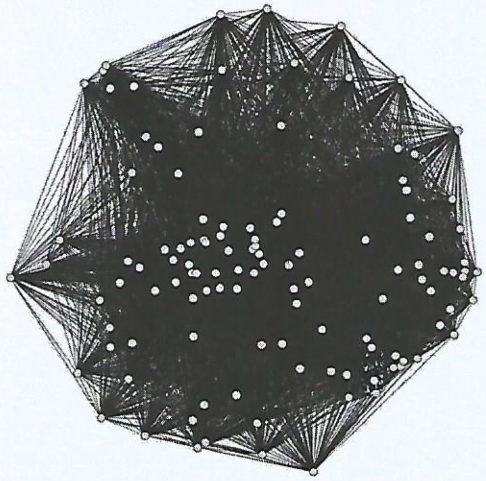

Figure 3.4: Non-commuting graph of $SL(2,5)$

The group $SL(2,5)$ is finite non-abelian group of order 120. $|Z_{SL(2,5)}| = 2$. The Non-commuting graph of $SL(2,5)$, as shown in figure 3.4, is connected graph, diameter 2 and girth 3. It is Hamiltonian graph but not planar and not regular graph. By using sagemath program 3 we get that:

$\alpha(\Gamma_{SL(2,5)}) = 8$, $\omega(\Gamma_{SL(2,5)}) = 31$, $\beta(\Gamma_{SL(2,5)}) = 110$, $\chi(\Gamma_{SL(2,5)}) = 31$.

***Example* 3.5.** The non-commuting graph of $SL(2,7)$:

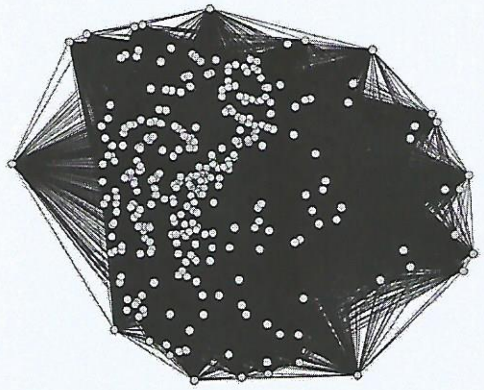

Figure 3.5: Non-commuting graph of $SL(2, 7)$

The group $SL(2, 7)$ is finite non-abelian group of order 336. $|Z_{SL(2,7)}| = 2$. The Non-commuting graph of $SL(2, 7)$, as shown in figure 3.5, is connected graph, diameter 2 and girth 3. It is Hamiltonian graph but not planar and not regular graph. By using sagemath program 3 we get that:

$\alpha(\Gamma_{SL(2,7)}) = 12$, $\omega(\Gamma_{SL(2,7)}) = 57$, $\beta(\Gamma_{SL(2,7)}) = 322$, $\chi(\Gamma_{SL(2,7)}) = 57$.

Note that, from corollary 3.1, the order of the center of $SL(2, q)$ is given by:

$$|Z_{SL(2,q)}| = \begin{cases} 1 & \text{if q even} \\ 2 & \text{if q odd} \end{cases}$$

Hence, if $q = 2^n$ is even then $PSL(2, q) \cong SL(2, q)$

***Proposition* 3.6.** [2] The dominating number of $\Gamma_{SL(2,q)}$, where q is a power of prime, is equal to 2.

Proof. $SL(2, q)$ is a finite non-abelian group.

Case 1: If $|Z_{SL(2,q)}| = 1$ then $SL(2, q) \cong PSL(2, q)$, thus $SL(2, q)$ is simple group, so by proposition 2.17, $\gamma(\Gamma_{SL(n,q)}) = 2$.

Case 2: If $|Z_{SL(2,q)}| = 2$, let X be the generating set for $SL(2,q)$, then by proposition 2.9, $X - Z_{SL(2,q)}$ is the dominating set, also by proposition 3.1 we know that $|X| = 2$ and $X \cap Z_{SL(2,q)} = \phi$. Then $\gamma(\Gamma_{SL(2,q)}) = 2$

□

Lemma **3.8.** [2] If $q > 2$. Then

$$\omega(\Gamma_{GL(2,q)}) = \chi(\Gamma_{GL(2,q)}) = q^2 + q + 1$$

Proof. By proposition 3.5,

1. The number of conjugates of $D = \dfrac{q^2+q}{2}$.
2. The number of conjugates of any cyclic group is $\dfrac{q^2-q}{2}$.
3. The number of conjugates of $SZ_G = (q+1)$.

Then the class number of G is

$$\frac{q(q+1)}{2} + \frac{q(q-1)}{2} + (q+1) = q^2 + q + 1,$$

Thus,

$$G = \bigcup_{i=1}^{q^2+q+1} C_G(g_i) \quad \text{and} \quad C_G(g_i) \cap C_G(g_j) = Z_G \quad \text{for} \quad i \neq j$$

Since G is an AC-group, then $g_1, \ldots, g_{q^2+q+1}$ are pairwise non-commutative. So $\omega(\Gamma_G) \geq q^2 + q + 1$. On the other hand, since G is covered by $q^2 + q + 1$ abelian subgroups,

$$\omega(\Gamma_G) \leq q^2 + q + 1.$$

Hence,

$$\omega(\Gamma_{GL(2,q)}) = \chi(\Gamma_{GL(2,q)}) = q^2 + q + 1$$

□

Lemma **3.9.** [2] The clique number of the non-commuting graph $\Gamma_{PSl(2,q)}$ is given as follows:

$$\omega(\Gamma_{PSL(2,q)}) = \begin{cases} q^2+q+1 & if \quad q>5 \\ 21 & if \quad q=4 \text{ or } 5 \\ 5 & if \quad q=3 \\ 4 & if \quad q=2 \end{cases}$$

Proof. We consider the following cases:

1. If $q=2$. Using program 3 in sagemath, we find that, $\omega(\Gamma_{PSL(2,2)})=4$.

2. If $q=3$. Using program 3 in sagemath, we find that, $\omega(\Gamma_{PSL(2,3)})=5$.

3. If $q=4$ or 5. Using program 3 in sagemath, we find that, $\omega(\Gamma_{PSL(2,4)}) = \omega(\Gamma_{PSL(2,5)})=21$.

4. Now, if $q>5$. By the definition of the clique number and the partition of $PSL(2,q)$, the size of the partition set is equal to $\omega(\Gamma_{PSL(2,q)})$ as given in ([13], Theorem 8.5). Then, if $k := |Z_{PSL(2,q)}|$

$$\begin{aligned}\omega(\Gamma_{PSL(2,q)}) &= (q+1)+\frac{(q+1)(q-1)q/k}{2(q-1)/k}+\frac{(q+1)(q-1)q/k}{2(q+1)/k}\\ &= (q+1)+\frac{(q+1)q}{2}+\frac{(q-1)q}{2}\\ &= q+1+q^2\end{aligned}$$

□

Theorem **3.10.** The clique number of the non-commuting graph $\Gamma_{SL(2,q)}$ is given as follows:

$$\omega(\Gamma_{SL(2,q)}) = \begin{cases} q^2+q+1 & if \quad q>3 \\ 4 & if \quad q=2 \\ 7 & if \quad q=3 \end{cases}$$

Proof. If $q = 2$ or 3 we get the clique number by program 3 in sagemath.

If $q > 3$, by lemma 2.4 $\omega(\Gamma_H) \leq \omega(\Gamma_G)$, where $H \leq G$. We know that $SL(2,q) \leq GL(2,q)$, so

$$\omega(\Gamma_{SL(2,q)}) \leq \omega(\Gamma_{GL(2,q)}) = q^2+q+1$$

$$\omega(\Gamma_{SL(2,q)}) \leq q^2+q+1 \tag{3.1}$$

Now, $PSL(2,q)$ is the factor group of $SL(2,q)$ and by lemma 2.16,

$$\omega(\Gamma_{PSL(2,q)}) \leq \omega(\Gamma_{SL(2,q)})$$

$$\omega(\Gamma_{SL(2,q)}) \geq q^2+q+1 \tag{3.2}$$

From equation (3.1) and (3.2), we get

$$\omega(\Gamma_{SL(2,q)}) = q^2+q+1$$

□

***Theorem* 3.11.** The chromatic number of non-commuting graph $\Gamma_{SL(2,q)}$ is given by:

$$\chi(\Gamma_{SL(2,q)}) = \begin{cases} q^2+q+1 & if \quad q>3 \\ 4 & if \quad q=2 \\ 7 & if \quad q=3 \end{cases}$$

Proof. By theorem 3.7 the group $SL(2,q)$ is an AC-group, then by proposition 2.5, $\chi(\Gamma_{SL(2,q)}) = \omega(\Gamma_{SL(2,q)})$. □

***Lemma* 3.12.** [10] The centralizers of $SL(2,q)$ have the following orders:

1. If q is even, then

$$|C_{SL(2,q)}(x)| = \begin{cases} q, \quad \text{or } q+1 & \text{if} \quad q=2 \\ q, \quad \text{or } q+1, \quad \text{or } q-1 & \text{if} \quad q \neq 2 \end{cases}$$

2. If q is odd, then

$$|C_{SL(2,q)}(x)| = \begin{cases} 2q, \quad \text{or } q+1 & \text{if} \quad q=3 \\ 2q, \quad \text{or } q+1, \quad \text{or } q-1 & \text{if} \quad q \neq 3 \end{cases}$$

***Lemma* 3.13.** [4]

1. If S is maximal independent set of Γ_G, then $S \cup Z_G$ is a maximal abelian subgroup of G.

2. If X is maximal abelian subgroup of G, then $X - Z_G$ is a maximal independent set of Γ_G.

The proof of the lemma is easily done by definition of maximal independent set and definition of maximal abelian subgroup.

***Lemma* 3.14.** Let $S \in V(\Gamma_G)$, then S is the maximal independent set in Γ_G if and only if

$$S = C_G(S) - Z_G$$

That means

$$C_G(S) = S \cup Z_G$$

Proof. ($\subseteq$) Let $x \in S$, so for all $s \in S$, s is not adjacent to x.

$$xs = sx \qquad \text{for all} \quad s \in S$$

So,

$$x \in C_G(S) - Z_G$$

Hence,

$$S \subseteq C_G(S) - Z_G.$$

($\supseteq$) Let $g \in C_G(S) - Z_G$, then $g \notin Z_G$. And,

$$gs = sg \qquad \forall s \in S$$

Since $g \in V(\Gamma_G)$, then g is not adjacent to all $s \in S$. That is $S \cup \{g\}$ is independent set. Since S is the maximal independent set, then $g \in S$. □

***Theorem* 3.15.** The independent number of the non-commuting graph $\Gamma_{SL(2,q)}$ is :

$$\alpha(\Gamma_{SL(2,q)}) = \begin{cases} q & \text{if} \quad q \quad \text{even} \\ 2(q-1) & \text{if} \quad q \quad \text{odd} \end{cases}$$

Proof. Let S be a maximum independent set of $\Gamma_{SL(2,q)}$. By lemma 3.13, $S \cup Z_{SL(2,q)}$ is maximal abelian subgroup of $SL(2,q)$. Since $SL(2,q)$ is an AC-group, then for all $x \in SL(2,q)$ the centralizer $C_{SL(2,q)}(x)$ is abelian. So,

$$|C_{SL(2,q)}(x)| \leq |S \cup Z_{SL(2,q)}| \qquad \text{for all} \quad x \in SL(2,q) \tag{3.3}$$

On the other hand. From lemma 3.14

$$S \cup Z_{SL(2,q)} = C_{SL(2,q)}(S) = \{g \in SL(2,q), \quad gx = xg, \quad \text{for all } x \in S\}$$

$$S \cup Z_{SL(2,q)} = \bigcap_{x \in S} C_{SL(2,q)}(x) \leq C_{SL(2,q)}(x) \quad \text{for all } x \in S$$

Then,

$$|S \cup Z_{SL(2,q)}| \leq |C_{SL(2,q)}(x)| \quad \text{for all } x \in S \tag{3.4}$$

Hence, by equation 3.3 and equation 3.4 we have:

$$|S \cup Z_{SL(2,q)}| = n \quad \text{where} \quad n \quad \text{is the maximum among all orders of centralizers.}$$

By lemma 3.12 we have two cases:

Case 1: If q is odd, then n is equal to one of the followings:

$$\begin{cases} 2(q-1) \\ q-1 \\ q-3 \end{cases}$$

Case 2: If q is even, then n is equal to one of the followings:

$$\begin{cases} q-1 \\ q \\ q-2 \end{cases}$$

Since S is the maximum independent set, n is equal to maximum number. Then,

$$n = \begin{cases} q \quad \text{if} \quad q \quad \text{is even} \\ 2(q-1) \quad \text{if} \quad q \quad \text{is odd} \end{cases}$$

□

***Corollary* 3.3.** The vertex cover of the non-commuting graph $\Gamma_{SL(2,q)}$ is given as follows:

$$\beta(\Gamma_{SL(2,q)}) = \begin{cases} q^3 - 2q - 1 & \text{if} \quad q \quad \text{is even} \\ q(q^2-3) & \text{if} \quad q \quad \text{is odd} \end{cases}$$

Proof. By proposition 1.1,

$$\beta(\Gamma_{SL(2,q)}) = |V(\Gamma_{SL(2,q)})| - \alpha(\Gamma_{SL(2,q)})$$

and we know that

$$|V(\Gamma_{SL(2,q)})| = \begin{cases} q^3 - q - 1 & \text{if} \quad q \quad \textit{even} \\ q^3 - q - 2 & \text{if} \quad q \quad \textit{odd} \end{cases}$$

Therefore, we have the following cases:

Case 1: If q even:

$$\beta(\Gamma_{SL(2,q)}) = q^3 - q - 1 - q = q^3 - 2q - 1$$

Case 2: If q odd:

$$\beta(\Gamma_{SL(2,q)}) = q^3 - q - 2 - 2q + 2 = q^3 - 3q = q(q^2-3).$$

□

***Proposition* 3.7.** Let G be a group such that $\Gamma_G \cong \Gamma_{SL(2,q)}$. Then $|G| = |SL(2,q)|$.

Proof. By Theorem 3.7, $SL(2,q)$ is an AC-group, and $\Gamma_G \cong \Gamma_{SL(2,q)}$, since the clique number $\omega(\Gamma_{SL(2,q)}) \leq n$ where $n = |V(\Gamma_{SL(2,q)})|$. Then by proposition 2.3,

$$|G| = |SL(2,q)|.$$

□

The following theorem is proved by Abdollahi in [1].

***Theorem* 3.16.** [1] If $q \geq 2$, G is a group, and $\Gamma_G \cong \Gamma_{SL(2,q)}$. Then G is isomorphic to $SL(2, q)$.

3.3 The Non-commuting Graph of $SL(3, q)$

In this section, we consider the non-commuting graph of $SL(3, q)$, we introduce some properties of this graph.

***Example* 3.6.** The non-commuting graph of the $SL(3, 2)$ is done by program 3 in sagemath.

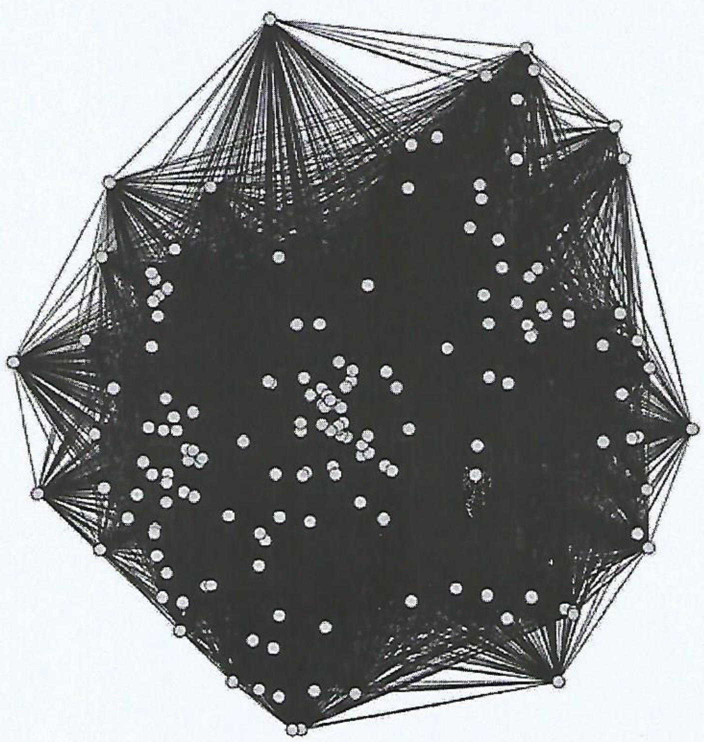

Figure 3.6:

The group $SL(3,2)$ is finite non-abelian group of order 168. $|Z_{SL(3,q)}| = 1$. The non-commuting graph of $SL(3,2)$ as shown in figure 3.6 is connected graph, diameter 2 and girth 3. It is Hamiltonian graph but not planar and not regular graph. We used a program 3 in sagemath to find that:

$\alpha(\Gamma_{SL(3,2)}) = 6$, $\omega(\Gamma_{SL(3,2)}) = 57$, $\beta(\Gamma_{SL(3,2)}) = 161$, $\chi(\Gamma_{SL(3,2)}) = 57$.

Note that by corollary 3.1, $|Z_{SL(3,q)}| = 1$ or 3, since :

$$|Z_{SL(3,q)}| = gcd(3, q-1) = \begin{cases} 1 & \text{if} \quad 3 \nmid (q-1) \\ 3 & \text{if} \quad 3 \mid (q-1) \end{cases}$$

***Lemma* 3.17.** [?] The table of the order of centralizers of $SL(3,q)$ is given as follows:

$$q^3(q-1)^2(q+1)(q^2+q+1)$$

$$q^3(q-1)$$

$$q(q-1)^2(q+1)$$

$$q(q-1)$$

$$(q-1)^2$$

$$q^2-1$$

$$q^2+q+1$$

$$q^2 \quad \text{if} \quad gcd(3, q-1) = 1$$

$$3q^2 \quad \text{if} \quad gcd(3, q-1) = 3$$

***Proposition* 3.8.** [2] The dominating number of $\Gamma_{SL(3,q)}$ where q is a power of prime is equal to 2.

Proof. $SL(3,q)$ is a finite non-abelian group.

Case 1: If $|Z_{SL(3,q)}| = 1$ then $SL(3,q) \cong PSL(3,q)$ and hence, it is the simple group, so by proposition 2.17, $\gamma(\Gamma_{SL(3,q)}) = 2$.

Case 2: If $|Z_{SL(3,q)}| = 3$, let A be the generating set for $SL(3,q)$ then by proposition 2.9, $A - Z_{SL(3,q)}$ is the dominating set for $\Gamma_{SL(3,q)}$, also by proposition 3.1 we know that $|A| = 2$ and $A \cap Z_{SL(3,q)} = \phi$. Then $\gamma(\Gamma_{SL(3,q)}) = 2$. □

***Theorem* 3.18.** The upper bound of independent number of the non-commuting graph $\Gamma_{SL(3,q)}$ is given as follows:

$$\alpha(\Gamma_{SL(3,q)}) \leq \begin{cases} q^4 - q^3 - 1 & \text{if} \quad |Z_{SL(3,q)}| = 1 \\ q^4 - q^3 - 3 & \text{if} \quad |Z_{SL(3,q)}| = 3 \end{cases}$$

Proof. Let S be a subset of $V(SL(3,q))$, where $S = \{s_1, s_2, \ldots, s_n\}$ maximum independent set of $\Gamma_{SL(3,q)}$ such that $|S| = n$.

$$x \in S \iff xs = sx \quad \forall \quad s \in S$$

So the degree of x is

$$deg(x) \leq |V(\Gamma_{SL(3,q)})| - |S|$$

$$= |SL(3,q)| - |Z_{SL(3,q)}| - n \tag{3.5}$$

But from lemma 2.1 , we know that

$$deg(x) = |SL(3,q)| - |C_{SL(3,q)}(x)| \tag{3.6}$$

From 3.5 and 3.6, we have

$$|SL(3,q)| - |Z_{SL(3,q)}| - n \geq |SL(3,q)| - |C_{SL(3,q)}(x)|$$

$$|Z_{SL(3,q)}| + n \leq |C_{SL(3,q)}(x)|$$

So,

$$n \leq |C_{SL(3,q)}(x)| - |Z_{SL(3,q)}|$$

By lemma 3.17 there are two cases as the following:

Case 1: If $|Z_{SL(3,q)}| = 1$. Then the independent number is less than the maximum number of the following:

$$\begin{cases} q^3(q-1) - 1 = q^4 - q^3 - 1 \\ q(q-1)^2(q+1) - 1 = q^4 - q^3 - q^2 + q - 1 \\ q(q-1) - 1 = q^2 - q - 1 \\ (q-1)^2 - 1 = q^2 - q \\ q^2 - 1 - 1 = q^2 - 2 \\ q^2 + q + 1 - 1 = q^2 + q \\ q^2 - 1 \end{cases}$$

The maximum number is $q^4 - q^3 - 1$

Case 2: If $|Z_{SL(3,q)}| = 3$. Then the independent number is less than the maximum number of the following:

$$\begin{cases} q^3(q-1)-3 = q^4-q^3-3 \\ q(q-1)^2(q+1)-3 = q^4-q^3-q^2+q-3 \\ q(q-1)-3 = q^2-q-3 \\ (q-1)^2-3 = q^2-q-2 \\ q^2-1-3 = q^2-4 \\ q^2+q+1-3 = q^2+q-2 \\ 3q^2-3 = 3(q^2-1) \end{cases}$$

The maximum number is $q^4 - q^3 - 3$ □

***Corollary* 3.4.** The lower bound of vertex cover of the non-commuting graph $\Gamma_{SL(3,q)}$ is given as follows:

$$\beta(\Gamma_{SL(3,q)}) \geq q^8 - q^6 - q^5 - q^4 + 2q^3$$

Proof. By proposition 1.1,

$$\beta(\Gamma_{SL(3,q)}) = |V(\Gamma_{SL(3,q)})| - \alpha(\Gamma_{SL(3,q)})$$

and we know that

$$|V(\Gamma_{SL(3,q)})| = \begin{cases} q^8 - q^6 - q^5 + q^3 - 1 & \text{if} \quad Z_{SL(3,q)} = 1 \\ q^8 - q^6 - q^5 + q^3 - 3 & \text{if} \quad Z_{SL(3,q)} = 3 \end{cases}$$

Therefore we have the following cases:

Case 1: If $|Z_{SL(3,q)}| = 1$:

$$\beta(\Gamma_{SL(3,q)}) \geq q^8 - q^6 - q^5 + q^3 - 1 - (q^4 - q^3 - 1)$$

$$\geq q^8 - q^6 - 2q^5 - q^4 + 2q^3$$

Case 2: If $|Z_{SL(3,q)}| = 3$:

$$\beta(\Gamma_{SL(3,q)}) \geq q^8 - q^6 - q^5 + q^3 - 3 - (q^4 - q^3 - 3)$$

$$\geq q^8 - q^6 - q^5 - q^4 + 2q^3$$

□

The following table gives the independent number and number of vertex cover for $SL(3, q)$ by using program 5 in sagemath as follows:

q	$\alpha(\Gamma_{SL(3,q)})$	$\beta(\Gamma_{SL(3,q)})$
2	6	161
3	12	5603
4	45	60432
5	30	371969
7	144	5630541
8	72	16482743
9	90	42456869
11	132	212427467

From the table, we may expect that for $q = 2, 3, 4, 5, 7, 8, 9, 11$:

1. The independent number of the non-commuting graph $(\Gamma_{SL(3,q)})$ is given as follows:

$$\alpha(\Gamma_{SL(3,q)}) = \begin{cases} q^2+q & \text{if} \quad |Z_{SL(3,q)}| = 1 \\ 3(q^2-1) & \text{if} \quad |Z_{SL(3,q)}| = 3 \end{cases}$$

.

2. The vertex cover of the non-commuting graph $(\Gamma_{SL(3,q)})$ is given as follows:

$$\beta(\Gamma_{SL(3,q)}) = \begin{cases} q^8-q^6-q^5+q^3-q^2-q-1 & \text{if} \quad |Z_{SL(3,q)}| = 1 \\ q^8-q^6-q^5+q^3-3q^2 & \text{if} \quad |Z_{SL(3,q)}| = 3 \end{cases}$$

.

***Theorem* 3.19.** [3] The clique number of the non-commuting graph of $SL(3,3)$ is $\omega(\Gamma_{SL(3,3)}) = 1067$.

Proof. Let A_1, A_2, A_3, A_4 be subsets of $SL(3,3)$ such that $A_i \cap A_j = Z_{SL(3,3)}$ where $i \neq j$, be given as follows:

$A_1 = \{C_{SL(3,3)}(g) \quad | \quad g \in SL(3,3), \quad |C_{SL(3,3)}(g)| = 6\}$ and $|A_1| = 468$.

$A_2 = \{C_{SL(3,3)}(g) \quad | \quad g \in SL(3,3), \quad |C_{SL(3,3)}(g)| = 8\}$ and $|A_2| = 351$.

$A_3 = \{C_{SL(3,3)}(g) \quad | \quad g \in SL(3,3), \quad |C_{SL(3,3)}(g)| = 9\}$ and $|A_3| = 104$.

$A_4 = \{C_{SL(3,3)}(g) \quad | \quad g \in SL(3,3), \quad |C_{SL(3,3)}(g)| = 13\}$ and $|A_4| = 144$.

Where the orders of centralizers of $SL(3,3)$ are calculated in program 4, {5616, 54, 48, 13, 9, 8, 6}.

Let $X = \{a_i, b_j, c_k, d_f\}$ such that:

$$a_i \in A_1 \quad 1 \leq i \leq 468$$

$b_j \in A_2 \quad 1 \leq i \leq 351$

$c_k \in A_3 \quad 1 \leq i \leq 104$

$d_f \in A_4 \quad 1 \leq i \leq 144$

Then,

$$SL(3,3) = \bigcup_{x \in X} C_{SL(3,3)}(x)$$

Now, by proposition 2.14 we have

$$\omega(\Gamma_{SL(3,3)}) = |X| = |A_1| + |A_2| + |A_3| + |A_4| = 468 + 351 + 104 + 144 = 1067$$

□

***Theorem* 3.20.** Let G be a non-abelian group such that $\Gamma_G \cong \Gamma_M$ where $M = SL(3,q)$. Then $|G| = |M|$.

Proof. Since G, M are two non-abelian groups and $\Gamma_G \cong \Gamma_M$. Then we have by theorem 2.3, $|G| = |M|$. □

Note That: If G is any non-abelian group and $\Gamma_G \cong \Gamma_{SL(n,q)}$, then $|G| = |SL(n,q)|$.

***Theorem* 3.21.** If G is a group such that $\Gamma_G \cong \Gamma_{SL(3,q)}$, where $|Z_G| = 1$. Then $G \cong SL(3,q)$.

Proof. Since $|Z_{SL(3,q)}| = 1$, then it is simple group (see proposition 3.3). By theorem 2.6, $G \cong SL(3,q)$ □

Appendix

Sagemath Programs

Program **1.** We find Γ_G use the following program:

```
# 1- G=Symmetric Group (S3).

sage: G=SymmetricGroup(3)


# 2- element=Set of all elements.

sage: element=G.list()


# 3- Z=Center of Group G.

sage: Z=G.center()


# 4- AM=adjacent matrix.

sage: AM=matrix(G.order())

sage: for i in range(G.order()):

          for j in range(G.order()):

              if element[i]*element[j]<>element[j*element[i]]:
```

```
                AM[i,j]=1


# 5- graph= Non-commuting graph of a group G.

sage: graph=Graph(AM)

sage: for i in range(G.order()):

          for j in range(Z.order()):

              if element[i].matrix()==Z[j].matrix():

                  graph.delete_vertex(i)

sage: graph.show(vertex_size=50, vertex_labels=False)


# Now, the planar graph of a Symmetric Group (S3).

sage: void= graph.layout(layout="planar",save_pos=True)

graph.plot(vertex_size=50, vertex_labels=False)


# Now, if G=Dihedral Group (D4).

sage: G=DihedralGroup(4)


# element=Set of all elements.

sage: element=G.list()


# Z=Center of Group G.

sage: Z=G.center()


# AM=adjacent matrix.
```

```
sage: AM=matrix(G.order())

sage: for i in range(G.order()):

          for j in range(G.order()):

             if element[i]*element[j]<>element[j*element[i]]:

                 AM[i,j]=1


# graph= Non-commuting graph of a group G.

sage: graph=Graph(AM)

sage: for i in range(G.order()):

          for j in range(Z.order()):

             if element[i].matrix()==Z[j].matrix():

                 graph.delete_vertex(i)

sage: graph.show(vertex_size=50, vertex_labels=False)


# Now, the planar graph of a Dihedral Group (D4).

sage: void= graph.layout(layout="planar",save_pos=True)

graph.plot(vertex_size=50, vertex_labels=False)


# Now, if G=Quaternion Group (Q8).

sage: G=QuaternionGroup()


# element=Set of all elements.

sage: E=G.list()
```

```
# Z=Center of Group G.
sage: Z=G.center()


# AM=adjacent matrix.
sage: AM=matrix(G.order())
sage: for i in range(G.order()):
          for j in range(G.order()):
              if element[i]*element[j]<>element[j*element[i]]:
                      AM[i,j]=1


# graph= Non-commuting graph of a group G.
sage: graph=Graph(AM)
sage: for i in range(G.order()):
         for j in range(Z.order()):
             if element[i].matrix()==Z[j].matrix():
                  graph.delete_vertex(i)
sage: graph.show( vertex_size=50, vertex_labels=False)


# Now, the planar graph of a Quaternion Group (Q8)
sage: void= graph.layout(layout="planar",save_pos=True)
graph.plot(vertex_size=50, vertex_labels=False)
```

***Program* 2.** To verify if the special linear group $SL(2,2)$ and $SL(2,3)$ are AC-groups in sagemath use the following program:

```
# First: Define The group G=SL(2,2).

sage: G=SL(2,GF(2))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()


# E= Set of all elements in G.

E=G2.list()


# Z = Center of a group G.

Z=G2.center()


# Second: Define V equal the number of vertices in a graph.

sage: V=G.order()-Z.order();V

5


# Finally: We check if C_G(x) for all x in G-Z(G) is abelian to

know if the group is an AC-group.

sage: C=[]

for i in E:

    if G2.centralizer(i).is_abelian()==True:

        C.append(i)

if len(C)==V:
```

```
    print'G is an AC-group'

G is an AC_group


# First: Define The group G=SL(2,3).

sage: G=SL(2,GF(3))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()


# E is the set of all elements in G.

E=G2.list()


#Z is  the center of a group G.

 Z=G2.center()


# Second: Define V to be the number of vertices in a graph.

sage: V=G.order()-Z.order();V

22


# Finally: We check if C_G(x) for all x in G-Z(G) is abelian to

know if the group is an AC-group.

sage: C=[]

for i in E:

    if G2.centralizer(i).is_abelian()==True:
```

```
        C.append(i)
if len(C)==V:
     print'G is an AC-group'
G is an AC_group
```

***Program* 3.** The following program is to find $\Gamma_{SL(n,q)}$ where q is the power of prime.

```
# First, Define the group G=SL(2,2).
sage: G=SL(2,GF(2))

# Element=The set of all elements in G.
sage: Element=G.list()

# The center Z(G) is Z.
sage: Z=G.center()

# Find the order of G.
sage: G.order()
6

# Define adjacent matrix of a graph G as follows:
sage: A=matrix(G.order())
sage: for i in range (G.order()):
          for j in range (G.order()):
```

```
            if Element[i]*Element[j]<>Element[j]*Element[i]:

                A[i,j]=1


# The non-commuting graph of a group G is:

sage: graph= Graph(A)

sage: for i in range(G.order()):

          for j in range(Z.order()):

              if Element[i].matrix()==Z[j].matrix():

                  graph.delete_vertex(i)
```

```
# Now, we find the properties of this graph as follows:


#  I is the independent set of graph.

I= graph.independent_set(); print'Independent number=',len(I)

Independent number= 2


#  CV is the set of vertex cover of graph.

CV=graph.vertex_cover();print'Minimum size of vertex cover=',len(CV)

Minimum size of vertex cover= 3


#  W is the clique number of graph.

W=graph.clique_number();print'Clique number=',W
```

```
Clique number= 4


#  X is the chromatic number

X=graph.chromatic_number();print'Chromaric number=',X

Chromaric number= 4
```

We can use this program to find the non-commuting graph for $SL(2,3)$, $SL(2,4)$, $SL(2,5)$ and $SL(2,7)$

***Program* 4.** We can find the set of order of all $C_{SL(3,q)}(x)$ by the following program.

```
# Defind the group G=SL(3,2).

sage: G=SL(3,GF(2))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()

E=G2.list()

Z=G2.center()


# Find order of the C_G(x).

sage: C=[]

for i in E:

    C.append (G2.centralizer(i).order())

set(C)

set([168, 8, 3, 4, 7])
```

```
# Defind the group G=SL(3,3).

sage: G=SL(3,GF(3))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()

E=G2.list()

Z=G2.center()


# Find order of the C_G(x).

sage: C=[]

for i in E:

    C.append (G2.centralizer(i).order())

set(C)

set([48, 6, 8, 9, 13, 5616, 54])
```

***Program* 5.** To find the independent number and minimum vertex cover of a group $G = SL(3, q)$ where q is a power of prime.

```
# G=SL(3,2)

sage: G=SL(3,GF(2))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()


# Z the center of a group G.

Z=G2.center()
```

```
# Also, let A be the maximum number of order of conjugacy classes

subgroup such that is commutative.

A= max([H.order() for H in G2.conjugacy_classes_subgroups()

if H.is_commutative()])


# I be the number of independent set.

I= A-Z.order(); print'The independent number=',I

The independent number= 6


# B be the number of vertex cover.

B= G2.order()-A; print'The minimum size of vertex cover=',B

The minimum size of vertex cover= 161



# G=SL(3,3)

sage: G=SL(3,GF(3))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()


# Z the center of a group G.

Z=G2.center()


# Also, let A be the maximum number of order of conjugacy classes
```

```
subgroup such that is commutative.

A= max([H.order() for H in G2.conjugacy_classes_subgroups()
if H.is_commutative()])


# I be the number of independent set.

I= A-Z.order(); print'The independent number=',I

The independent number= 12


# B be the number of vertex cover.

B= G2.order()-A; print'The minimum size of vertex cover=',B

The minimum size of vertex cover= 5603


# complete in the same maner for q=4,5,7,8,...
```

***Program* 6.** This program find the $\omega(\Gamma_G)$ $G = SL(3, 2)$.

```
# G=SL(3,2).

sage: G=SL(3,GF(2))

G1=G.as_matrix_group()

G2=G1.as_permutation_group()

E=G2.list()

Z=G2.center()


# First, If H is the set of all subgroups in G.

sage: H=G2.subgroups();print' number of subgroups in G is equal
```

```
 to ',len(H)

number of subgroups in G is equal to  179


# If H1 is the set of all cyclic subgroups in G.

sage: H1=[]

for i in H:

if i.is_cyclic()==True:

H1.append(i)

print'Number of cyclic subgroups in G is equal to ',len(H1)

Number of cyclic subgroups in G is equal to  79


# H2 be the number of cyclic subgroups, whose the intersection between

any two of them is the center of group.

sage: H2=[]

for i in H1:

    for j in H1:

        if i.intersection(j)==Z:

            H2.append(j)

len(H2)

H3=set(H2),len(H3)

6121

79
```

```
# If C is the set of all centralizers of a group G.

sage: C=[]

for i in E:

    C.append(G2.centralizer(i))


# If C_G(x) subgroup of i where i in H2 put it subgroup in H4.

sage: H4=[]

for i in H3:

    for j in C:

        if j.is_subgroup(i)==True:

            H4.append(i)

print'number of subgroup such that C_G(x)subgroup of i is equal to',

len(set(H4))

number of subgroup such that C_G(x)subgroup of i is equal to 57


# Let A be the set of all C_G(x) such that |C_G(x)|=8

sage: A=[]

for i in range(len(H4)):

    if G2.centralizer(H4[i]).order()==8:

        A.append(G2.centralizer(H4[i]))

print'|A|=',len(set(A))

|A|= 0
```

```
# Let B be the set of all C_G(x) such that |C_G(x)|=3

sage: B=[]

for i in range(len(H4)):

    if G2.centralizer(H4[i]).order()==3:

        B.append(G2.centralizer(H4[i]))

print'|B|=', len(set(B))

|B|= 28


# Let D be the set of all C_G(x) such that |C_G(x)|=4

sage: D=[]

for i in H4:

    if G2.centralizer(i).order()==4:

        D.append(G2.centralizer(i))

print'|D|=',len(set(D))

|D|= 21


# Let X be the set of all C_G(x) such that |C_G(x)|=7

sage: X=[]

for i in H4:

    if G2.centralizer(i).order()==7:

        X.append(G2.centralizer(i))

print'|X|=',len(set(X))

|X|= 8
```

```
# Finaly, the clique number of non-commuting graph of G is:

sage: print'cliqu number=' ,

len(set(A))+len(set(B))+len(set(D))+len(set(X))

clique number= 57
```

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الرسم الغير تبديلي للزمر الخطية الخاصة $SL(2,q)$ و $SL(3,q)$

اذا كانت G زمرة غير تبديلية و Z_G مركز هذه الزمرة فان الرسم الغير تبديلي لهذه الزمرة Γ_G هو الرسم الذي يكون رؤوسه مجموعة العناصر الموجودة في $G - Z_G$، يكون xy ضلع في Γ_G اذا وفقط اذا كانت $xy \neq yx$ ، في هذه الرسالة سنبحث في الرسم الغير تبديلي للزمرة الخطية $SL(n,q)$ بحيث $n = 2, 3$ على حقل جالويس F_q، ثم سنعمل على ايجاد عدد العصبة $\omega(\Gamma_{SL(n,q)})$ ، العدد المستقل $\alpha(\Gamma_{SL(n,q)})$، و العدد الاصغر لغطاء الرأس $\beta(\Gamma_{SL(n,q)})$، و العدد الصبغي $\chi(\Gamma_{SL(n,q)})$.