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Weighted Estimators for the Survival and Cumulative Hazard Functions of Quality Adjusted Lifetime

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Committee Decision

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Declaration

I, the undersigned, the author of the Master thesis entitled "Weighted Estimators for the Survival and Cumulative Hazard Functions of Quality Adjusted Lifetime ".

Where I submitted to the Arab American University for a master's degree and that it is the result of my own research, except ad indicated, of which none has been offered for a higher degree to any university or other educational institution.

Signature..........................

Date: 12.3.2018

Dedication

To my dear father and mother who supported me in my educational career and encouraged me to continue my studies and help me in all aspects, including moral and material.

Acknowledgements

I cordially dedicate this work to my loving family who were always there for me, my sincere friends with whom I shared every single moment of happiness, and who always went the extra mile for me to make me feel special, my great Dr. Mahmoud Almanassra , and Dr. Elias Dabeet whom have generously helped me throughout my research, offering guidance and various support with everything I needed.

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Abstract

The survival function of restricted quality adjusted lifetime (RQAL) has become more important in studies nowadays than survival function of overall lifetime. The reason is due to the fact that the researcher needs real-time for the observations under the conditions of his or her life.

In this work we will estimate efficient estimators for the survival function of restricted quality adjusted lifetime if the data has left and right censorship. Also, we derived a class of estimators for the cumulative hazard function based on estimations of the survival function of (RQAL). Simulation study using R-programming has been conducted to compare the efficiency of the estimators of the true survival function.

Contents

1 INTRODUCTION

Survival analysis is used to analyze data in which the time until the event is of interest [4]. The response is often referred to as a failure time, survival time, or event time.

Examples:

- Time until tumor recurrence.
- Time until cardiovascular death after some treatment. intervention.
- Time until AIDS for HIV patients.
- Time until a machine part fails.

Estimation of survival functions and hazard function, have been studied by several researchers under several assumptions. In simplest type, it is assumed that there is no censored data of a random sample are a available. In this case we use an empirical distribution function.

Definition 1.1. Let $X_1, X_2, ..., X_n$, be an independent nonnegative with distribution function $F(x) = P(X \leq x)$. An empirical distribution function $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$

Where $I(X_i \leq x)$ is the indicator function which is equal to one if $X_i \leq x$, zero otherwise. The properties of the empirical distribution as an estimation of survival function have been studied by many researchers. In general, studies contain censoring data and there is three types of censored data [8] [9] [20] [14].

Definition 1.2. Censoring:

Censoring occurred when missing data in a certain period of the study about the item due to some uncontrolled circumstances [5] [15] [19].

Types of censored data:

- Left censoring : It occurs upon allowing new item into the study, though inadequate data about the item from the beginning of the study until the time of entry into the study.

- Interval censoring : It occurs when a person exists the study and then comes back in, with no data to have about their absence period.

- Right censoring : It occurs when the item takes part in the study, but then gets out and never be back in .

We note that here, when there is no censored data, the empirical distribution function is the best estimate function for the survival function. But it is not good enough if there is censoring data. On the other hand, the most popular estimators estimate survival function taking into account missing data, Kaplan-Meier estimator and the Nelson-Aalen estimator. Kaplan-Meier (1958) derived the estimator through the nonparametric maximum likelihood approach. Efron (1967) was the first one recognize the self consistency of the Kaplan-Meier estimator. Breslow and Crowley (1974) showed that the Kaplan-Meier estimator is weakly consistent. The asymptotic normality of this estimator was established by Meier(1975). Peterson(1977) showed that the Kaplan-Meier estimator can be expressed as a function of empirical subsurvival functions. Aalen (1978) established the strong consistency of the estimator. Gillespie and Fisher (1979) derived confidence bands, and Gill (1983) studied some of the large sample properties of this estimator. The small sample size properties of the Kaplan-Meier estimator have been studied by Guerts (1987). Fleming and Harrington (1991) studied the properties of this estimator using martingale theory. The strong and weak representations of the Kaplan-Meier estimator of survival function, which are valid up to a given order statistics of the observations, are derived by Stute (1996). Cai (1998) established the asymptotic properties of this estimator for the censored data. Satten and Datta (2001) showed that the Kaplan-Meier estimator can be represented as a weighted average of identically distribution terms.

The second popular estimator used to analyze censored data is the Nelson-Aalen estimator which is an estimator for the cumulative hazard function. The first one who suggested this estimator, is Nelson (1972). Aalen (1978) derived this estimator using modern counting process techniques. He also found an estimator of the variance of this estimator. Klein (1991) suggested an alternative estimator of the variance and studied the small sample properties of the variance estimator. The derivation of the Nelson-Aalen estimator using counting processes can be found in Anderson et al (1993) and in Fleming an Harrington (1991). Kaplan-Meier and Nelson-Aalen estimators are considered to be the best estimators when the parameters of interest are the survival function and the cumulative hazard functions for the overall lifetime respectively.

Cox(1972) pointed out that in the evaluation of treatments for chronic diseases expanding overall survival time is not the only goal of the therapy. We want to have a new measure which combines both the quality and the quantity of the patient's life. One such measure is called quality-adjusted lifetime. Studying the quality adjusted survival time, in the recent days, has received much attention because of its ability to take in to consideration both the quantity and the quality of the patient's life. The researchers who studied quality adjusted lifetime include Van der Laan and Hubbard(1999), Gelbre, Gelman and Goldhirsch (1989), Glasziou, Gelbre and Simes (1990) and Gelbre et al (1995).

Gelbre et al (1989) pointed out that the use of the Kaplan-Meier estimator with censored quality adjusted life data will lead to biased and inconsistent estimation. A number of different estimators have been proposed to solve this difficulty. One such estimator was studied by Korn (1993). Zhao and Tsiatis (1997) proposed a clever method to find an estimator of the survival function of the restricted quality adjusted lifetime for censored data. This estimator is a member of a class of inverse probability of censoring weighted (IPCW) estimators. They also showed that their estimator is consistent and asymptotically normal. Huang and Louis (1998) proposed an estimator of the distribution function of the unrestricted quality adjusted lifetime using (IPCW) technique. Zhao and Tsiatis (1999) discussed the efficiency of the weighted estimators for the survival function of the quality adjusted lifetime. They derived a modified estimator, which is more efficient than the one proposed before. Zhao and Tsiatis (2000) considered the problem of estimating the mean of the quality adjusted survival time.

Wang (2001), proposed an improved version of the Zhao and Tsiatis estimator. He showed that his estimator is more efficient than the original Zhao and Tsiatis estimator in terms of asymptotic variance.

An estimator of survival function is not considered to be a true survival function if it assigns negative mass to some points. Survival function assign negative mass to a point b, if $S(b+) > S(b-)$. Dabrowska (1988) pointed out that the bivariate Kaplan-Meier estimator may not be survival function because this estimator may fail to be monotonic. Pruitt (1991) studied the Dabrowska estimator in details and found all points assigns negative mass, under the assumption that the observations follow an absolutely continuous distribution. He pointed out that the number of points assigned negative mass does not vanish as n goes to infinity [21] [22] [13] [1].

Mean Squared Error

The Mean Squared Error (MSE) or Mean Squared Deviation (MSD) of an estimator (of a procedure for estimating an unobserved quantity) measures the average of the squares of the errors or deviations—that is, the difference between the estimator and what is estimated.

The MSE is a measure of the quality of an estimator—it is always non-negative, and values closer to zero are better.

Definition 1.3. If $\hat{Y}$ is a vector of n predictions, and Y is the vector of observed values corresponding to the inputs to the function which generated the predictions, then the MSE of the predictor can be estimated by

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(\hat{Y}_i - Y_i)^2$$

Biasness

The bias of an estimator is the difference between the expected value of the estimator and the true value of the parameter being estimated. An estimator is said to be unbiased if its bias is equal to zero for all values of parameter γ. Otherwise, the estimator is said to be biased [16] [10]..

Definition 1.4. The bias of $\hat{\gamma}$ which is an estimator to the parameter γ is defined as

$$Bias[\,\hat{\gamma}\,] = E[\,\hat{\gamma}\,] - \gamma = E[\,\hat{\gamma} - \gamma\,]$$

Monotonic Function

A function f defined on a subset of the real numbers with real values is called monotonic if and only if it is either entirely non-increasing, or entirely non-decreasing.

Definition 1.5. A function is called monotonically increasing if for all x and y such that $x \leq y$ then, $f(x) \leq f(y)$.

Definition 1.6. A function is called monotonically decreasing (also decreasing or non-increasing) if, whenever $x \leq y$ then, $f(x) \geq f(y)$.

2 PRELIMINARIES

Almanassra [1] [2], considered two types of survival time. The first one which is over all lifetime, the second which is with restricted quality adjusted lifetime(RQAL). In this part, we will review survival functions in the case of right censored data, and how to find a jump point of the two types of survival time, and which of these points assigns negative mass. Then, we will also consider the way of Almanassra et al (2005) on how to get a monotonic function to the second type of survival time. Also, in this chapter we will introduce the Kaplan Meir estimator(1958), Nelson Aalen estimator (1978), Zhao-Tsiatis estimator (1997), Wang estimator(2001), and the monotonized Wang and Zhao-tsitis estimators (2005).

2.1 Notations

Let $R = (R_1, R_2, ..., R_n)$ be a continuous failure time which is overall life time random variables with hazard function λ_R and survival function $S(.)$. Let $D_i = R_i \wedge L$ be a truncated failure times, where L is an artificial end point of studies and $\wedge$ is the minimization operator.

Definition 2.1. Zhao and Tsiatis(1997) defined restricted quality adjusted lifetime

$$N(D_i) = \int_0^{D_i} U(TH_i(t))dt$$

,

where TH_i is the health status function of the ith item at time t, and $U(.)$ is a quality function which maps $TH_i(t)$ to the interval $[0, 1]$.

Definition 2.2. The survival function of restricted quality adjusted lifetime(RQAL) is

$$S_r(b) = Pr(N(D_i) > b),$$

where $0 \leq b < L$

Now, let censoring time be denoted by $C = (C_1, C_2, ..., C_n)$ which are continuous random variables with hazard function $\lambda_C(.)$ and survival function $K(.)$

In the case of being a censored data in the study, we will consider the following assumption on the data. Assume that $\{D_i, C_i, N(r), r \in [0, D_i], i = 0, 1, ...\}$. The observed data with sample size n is given by

$$\{X_i = D_i \wedge C_i, \Gamma_i = I(D_i \leq C_i), N_i(r), r \in [0, X_i], i = 1, 2, ..., n.\}$$

Definition 2.3. The possible censored RQAL is

$$N(X_i) = \int_0^{X_i} U(TH_i(t))dt$$

The purpose is to find consistent estimators for the survival function of restricted quality adjusted lifetime. In the following sections, we will consider some estimators for the survival function $S_r(b)$

- Kaplan Meire Estimator
- Korn Estimator
- Simple weighted estimator
- Zhao and Tsiatis estimator
- Wang estimator

also, we will consider Nelson–Aalen Estimator.

2.2 Kaplan Meire Estimator

In all fields of studies; viz medical , economical, etc., researchers interested in medical studies or others, predict a certain occurrence to happen to the targeted sample; that is, death. As observed through long-term studies, a loss/damage may occur to the polled items/individuals; thus causing a problem in estimating the survival estimation.

The time starting a certain point to the occurrence of a given event, for instance, the effectiveness of a certain drug on patients, death, or machine malfunction, is called a survival time.

Studies are generally influenced by some things that are normally out of researcher's control such as loss of one or more in the elements of the study during the course of study.

With the above in mind, the simplest way to estimate the survival over time is Kaplan-Meir estimator [11] [7].

Definition 2.4. The estimator is given by:

$$\hat{S}(t) = \prod_{t_i \leq t} (1 - \frac{d_i}{n_i}),$$

where $\hat{S}(t)$ is the estimated survival probability for any particular one of the t time periods; n_i is the number of subjects at risk at the beginning of time period t_i; and d_i is the number of subjects who die during time period t_i.

Definition 2.5. The Kaplan–Meier estimator is statistic, and several estimators are used to approximate its variance. One of the most common estimators is Greenwood's formula

$$\widehat{Var}(\hat{S}(t)) = \hat{S}(t)^2 \sum_{t_i \leq t} \frac{d_i}{n_i(n_i - d_i)}$$

Example 2.1. Consider the following data in table with 100 subjects enrolled at beginning of study.

In this table, we will calculate the Kaplan-Meier estimator and Standard Error

Time	t_1	t_2	t_3	t_4	t_5
At Risk	100	95	90	81	75
Censored	2	1	3	4	1
Died	3	4	6	2	4
$\hat{S}(t)$	0.97	0.9291	0.8672	0.8458	0.8007
Std.Error	0.0171	0.0258	0.0338	0.0366	0.0407

2.3 Nelson–Aalen Estimator

The Nelson–Aalen estimator is a nonparametric estimator which may be used to estimate the cumulative hazard rate function from censored survival data. Independently of Nelson, Altshuler (1970) [3] derived the same estimator in the context of competing risks animal experiments. Later, by adopting a counting process formulation, Aalen extended its use beyond the survival data and competing risks setups, and studied its small and large sample properties using martingale methods. The estimator is nowadays denoted by the Nelson–Aalen estimator, although other names (the Nelson estimator, the Altshuler estimator, the Aalen–Nelson estimator, the empirical cumulative hazard estimator) are sometimes used as well [12].

Definition 2.6. The Nelson–Aalen estimator for the cumulative hazard rate function then takes the form

$$\hat{A}(t) = \sum_{t_j \leq t} \frac{d_j}{n_j}$$

where n_j is the number of individuals at risk (alive and not censored) just prior to time t_j . Thus the Nelson–Aalen estimator is an increasing right continuous step function with increments $\frac{d_j}{n_j}$ at the observed failure times.

Definition 2.7. The variance of the Nelson–Aalen estimator may be estimated by

$$v\hat{a}r(\hat{A}(t)) = \sum_{t_j \leq t} \frac{(n_j - d_j)d_j}{(n_j - 1)n_j^2}$$

Example 2.2. In this example, we will calculate the Nelson-Aalen cumulative hazard estimator and Standard Error for Nelson-Aalen.

Consider the following data such that:

Total: Number of observations.

Died: Number of observations that died coses the disease.

Censored : Number of people lost during the study.

Time	Total	Died	Censored	Nelson-Aalen	Std. error
t_1	20	1	0	0.05	0.05
t_2	19	2	0	0.1552	0.0879
t_3	17	1	1	0.2140	0.1057
t_4	15	1	0	0.2807	0.1250
t_5	14	1	0	0.3521	0.1440
t_6	13	0	1	0.3521	0.1440
t_7	12	2	0	0.5188	0.1826
t_8	10	0	1	0.5188	0.1826
t_9	9	3	0	0.8521	0.2472
t_{10}	6	0	1	0.8521	0.2472
t_{11}	5	1	1	1.0521	0.3180
t_{12}	3	0	1	1.0521	0.3180
t_{13}	2	1	0	1.5521	0.5926
t_{14}	1	0	1	1.5521	0.5926

Relationship Between Nelson-Aalen and Kaplan-Meire Estimators

The relationship between the two estimators is given by the following function

$$\hat{S}(t) = \exp^{-\hat{A}(t)}$$

2.4 Korn Estimator

Korn(1993) [13] suggested another method to calculate a survival function, but the method he suggested with quality of life for item. Korn's method(1993) depends on area under the curve (AUC)for every item.

Definition 2.8. (Korn Estimator)

Let $0 = t_0 < t_1 < ... < t_J = \infty$, and let $U(t_j) \geq 0$ be the quality of life measured at time $t_j(j < J)$.

Let R be the death time. Assume that $U(t_j) \equiv 0$ for all $t_j \geq R$. Let

$$U(t_j) \equiv (U(t_0), U(t_1), ..., U(t_j))$$

Korn(1993) defined the area under the quality of life curve up to time z by

$$A(z) = U(t_{j_{(z)}})(z - t_{j_{(z)}}) + \sum_{j=1}^{j(z)} \frac{U(t_j) + U(t_j - 1)}{2}(t_j - t_{j-1})$$

where $j(z)$ is such that $t_{j_{(z)}} < z < t_{j_{(z)}+1}$. The AUC quality of life is then $U \equiv A(R)$. Because of administrative censoring, we do not always get the observe R or U.

Let C be the potentially unobserved censoring time. We observe $X = minimum(R, C)$ and the indicator ζ that equals 1 if the observation was a death, 0 if the observation was a censored. We also observe the potentially censored quality of life AUC, $U \equiv A(X)$. We assume that C is jointly independent of $(R, U(t_{J-1}))$. Let $S_R(.)$ and $S_C(.)$ be the survival functions of R and C which are assumed to be continuous.

$$\begin{aligned} P(U > x) &= \sum_{j=0}^{J-1} pr(U > x, R \in [t_j, t_{j+1}]) \\ &= \sum_{j=0}^{J-1} \frac{\hat{A}_j + \hat{B}_j}{\hat{S}_C(t_j)} \end{aligned}$$

where,

$$\hat{A}_j = \frac{1}{n}\sum_{i=1}^{n} I\{X_i \in (t_j, t_{j+1}), U_i > x, \zeta_i = 1\}$$

$$\begin{aligned}\hat{B}_j = & \frac{1}{n}\sum_{i=1}^{n}[I\{X_i \in (t_j, t_{j+1}), U_i > x, \zeta_i = 0\}\frac{\hat{S}_R(X_i) - \hat{S}_R(t_{j+1})}{\hat{S}_R(X_i)}] \\ & + \frac{1}{n}\sum_{i=1}^{n}[I\{X_i \in (t_j, t_{j+1}), U_i < x, \zeta_i = 0, t_{j+1} > T_i\}\frac{\hat{S}_R(T_i) - \hat{S}_R(t_{j+1})}{\hat{S}_R(X_i)}]\end{aligned}$$

$T_i = X_i + (x - U_i)/U_i(t_j)$, $\hat{S}$ is the product-limit estimator. Also, define $0/0 = 0$

2.5 Some Estimators of the Survival Function of Restricted Quality Adjustd Lifetime

A single item from observations whether censored or uncensored can be sorted into one of the exclusive classes. We can determine if this item belong to one of these classes if we answer these two questions:

1) Can the value of $I(N(D) > b)$ be determined from observation?

2)If the value of $I(N(D) > b)$ can be determined, what is the first time point at which the value can be determined?

Now, let us consider the four cases. In every case we will define a new variable. These variables will be used later in this chapter.

Case (1): If $I(N(X) > b) = 1$
since $D \geq X$, $N(D) \geq N(X)$. Thus, the value of $I(N(D) > b) = 1$. Now, to find the first time point at which this value can be determine, consider the following

$$m(b) = inf\{e : N(X \wedge e) > b\}$$

Where, $m(b)$ is the first time t at which $N(t)$ exceeds b. Notice that here, $N(t)$ is a nondecreasing function, thus $I(N(t) > b)$ is equal to 1 for all t greater than or equal to $m(b)$.
Now, let us define the new variables, if $N(X) > b$, $D(b) = m(b)$ and $\hat{D}(b) = m(b)$. Let $X(b) = D(b) \wedge C$ and $\hat{X}(b) = \hat{D}(b) \wedge C$. We noticed from this case, if $C > D(b)$, we know that from the above $I(N(D) > b) = 1$

Case(2): $I(N(X) > b) = 0$ and $N(X) + L - X < b$.
Since $U(.) \leq 1$, $D \geq X$ and

$$N(D) = \int_0^D U(.)dt$$

We can be sure that $I(N(D) > b) = 0$.

Now, to find the first time point at which this value can be determined, consider the following formula

$$z(b) = inf\{z : N(z) + (L - z) \leq b\}$$

Where, $z(b)$ is the first point t at which $N(t)+(L-t) \leq b$. And since $N(t)+(L-t)$ is a non-increasing function of t, $N(t) + (L - t)$ is less than or equal to b for all $t \geq z(b)$.

Now, if $N(X) < b$ and $N(X) + (L - X) < b$, define

$D(b) = D$ and $\hat{D}(b) = z(b)$.

$X(b) = D(b) \wedge C$ and $\hat{X}(b) = \hat{D}(b) \wedge C$.

In this case, for an observation if $C > \hat{D}(b)$, we can be sure that $I(N(D) > b)$ is equal to zero.

Case (3) : $I(N(X) > b) = 0$, $(N(X) + (L - X) \geq b$ and $C < D$. Here, we can not determine the value of $I(N(D) > b)$, because the value of the expression $N(D) + (L - D)$ could be greater than b.

Now, define

$D(b) = D$, and $\hat{D}(b) = D$.

Also, let $X(b) = D(b) \wedge C$ and $\hat{X}(b) = \hat{D}(b) \wedge C$.

Case(4): $I(N(X) > b) = 0$, $(N(X) + (L - X) \geq b$ and $D \leq C$.

In this case, $D = X$ therefore, $I(N(D) > b) = 0$.

Define $D(b) = D$, and $\hat{D}(b) = D$ Also, let $X(b) = D(b) \wedge C$ and $\hat{X}(b) = \hat{D}(b) \wedge C$.

The indicator functions for a complete observation are:

$$\Gamma(b) = I\{D(b) \leq C\}$$

$$\hat{\Gamma}(b) = I\{\hat{D(b)} \leq C\}$$

Almanassra(2005) extended a class of estimators to include the Wang estimator. The new class of estimators is given by:

$$\hat{\delta}_b(s,k) = \begin{cases} \frac{1}{n}\sum_{j=1}^{n} \beta_{s,j}(b) I(N_j(X_j) > b) & \text{if } k = 1 \\ \frac{1}{n}\sum_{j=1}^{n} \frac{\beta_{s,j}(b)}{\beta_s(b)} I(N_j(X_j) > b) & \text{if } k = 2 \end{cases}$$

Where s is used to explain which method of estimation is used to estimate the Kaplan-Meier estimator $K(.)$.

Also, the index k is used to indicate whether the estimator renormalized $k = 1$ or not $k = 2$.

The weights function $\beta_{s,j}(b)$ are defined by:

$$\beta_{s,j}(b) = \begin{cases} \Gamma_j/\hat{K}(X_j) & \text{if } s = 1 \\ \Gamma_j(b)/\hat{K}(X_j(b)) & \text{if } s = 2 \\ \Gamma_j(b)/\hat{K}_b(X_j(b)) & \text{if } s = 3 \\ \hat{\Gamma}_j(b)/\hat{K}'_b(\hat{X}_j(b)) & \text{if } s = 4 \end{cases}$$

Let $\beta_s(b) = \frac{1}{n}\sum_{j=1}^{n} \beta_{s,j}(b)$ be the mean of the estimated weights.

Note that, $\beta_1(b) = \beta_2(b) = \beta_3(b) = \beta_4(b) = 1$ for each b as long as the largest observation is not censored.

Hence, we have $\hat{\delta}_b(1;1) = \hat{\delta}_b(1;2), \hat{\delta}_b(3;1) = \hat{\delta}_b(3;2)$ and $\hat{\delta}_b(4;1) = \hat{\delta}_b(4;2)$.

Let, $\hat{A}^{(C)}(.)$, $\hat{A}^{(C,b)}(.)$ and $\hat{A'}^{(C,b)}(.)$ be the Nelson and Aalen estimators for the cumulative hazard functions obtained from the data $(X_i, 1-\Gamma_i)$, $(X_i(b), 1-\Gamma_i(b))$ and $(\hat{X}_i(b), 1-\hat{\Gamma}_i(b))$ for $i = 1, ..., n$. We can estimate Kaplan-Meier estimator $K(.)$

by using Nelson and Aalen estimator by:

$$\hat{K}(t) = \prod_{u \le t}(1 - d\hat{A}^{(C)}(u))$$

$$\hat{K}_b(t) = \prod_{u \le t}(1 - d\hat{A}^{(C,b)}(u))$$

$$\hat{K}'_b(t) = \prod_{u \le t}(1 - d\hat{A'}^{(C,b)}(u))$$

Definition 2.9. When we use the data (X_i, Γ_i) to find $\hat{K}(X_i)$, we get

$$\hat{\delta}_b(1;1) = \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_j}{\hat{K}(X_j)}I(N_j(X_j) > b)$$

This is a definition of simple weighted estimator for the survival function of restricted quality adjusted lifetime (RQAL).

Definition 2.10. When we use the data $(X_i(b), \Gamma_i(b))$ to find $\hat{K}_b(X_i(b))$, we get

$$\hat{\delta}_b(3;1) = \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_j(b)}{\hat{K}_b(X_j(b))}I(N_j(X_j) > b)$$

This is a definition of Zhao and Tsiatis (1997) estimator for the survival function of restricted quality adjusted lifetime (RQAL).

Definition 2.11. When we use the data $(\hat{X}_i(b), \hat{\Gamma}_i(b))$ to find $\hat{K'}_b(\hat{X}_i(b))$, we get

$$\hat{\delta}_b(4;1) = \frac{1}{n}\sum_{j=1}^{n}\frac{\hat{\Gamma}_j(b)}{\hat{K'}_b(\hat{X}_j(b))}I(N_j(X_j) > b)$$

This is a definition of Wang estimator for the survival function of restricted quality adjusted lifetime (RQAL).

Strawderman (2000) [18] mentioned that the estimator $\hat{\delta}_b(2;1)$ sometime exceeds one, which means that it may not be a proper survival function.

2.6 Jump Points

Pruit(1991) discussed the jump point of the bivariate survival estimator and which of these points assigned a negative mass. Almanssra(2005) identified the jump points of univariate survival estimators for the simple weighted estimator, Zhao- Tsiatis estimator and the Wang estimator. Also he identified and discussed which of these points are assigned a negative mass.

Definition 2.12. Points assigned negative mass

The survival function $S(.)$ assigns negative mass at the point b if and only if $S(b-) - S(b+) < 0$

2.6.1 Jump Points Of The Simple Weighted Estimator.

The simple weighted estimator has a jump point at a point b if and only if there exist an item i such that $b = N(X_i)$ and $\Gamma_i = 1$, for $i = 1, 2, ..., n$. Which means that, this estimator has jump points just at points of death.

These jump points are not assigned a negative mass. It is a monotonically decreasing estimator.

2.6.2 Jump Points Of The Zhao And Tsiatis Estimator.

Zhao and Tsiatis estimator may not be monotonic. This estimator has three types of jump points.

(i) If there exists an index i such that $b = N(X_i)$ with $\Gamma_i = 1$, for $i = 1, 2, ..., n$, then b is a jump point.

This kind of jump point is not assigned a negative mass.

(ii) If there exists an index i such that $b = N(X_i)$ with $\Gamma_i = 0$, for $i = 1, 2, ..., n$, then b could be a jump point.

Also, these jump points are not assigned a negative mass.

(iii) If there exists an index i such that $\Gamma_i = 0$, for $i = 1, 2, ..., n$, $N(X_i) < b$ and there exists another index j such that

$$b = \int_0^{X_i} U_j(TH(t))dt$$

and $X_j > X_i$ then b is a jump point.

These jump points are assigned a negative mass.

2.6.3 Jump Points Of The Wang Estimator.

In general, Wang estimator has more jump points than Zhao and Tsiatis estimator. This estimator is a modified version of Zhao and Tsiatis estimator, and it sometimes reduces the number of the negative mass jump points of the third kind in Zhao and Tsiatis estimator. The Wang estimator has four types of jump points.

(i) If there exists an index i such that $b = N(X_i)$ with $\Gamma_i = 1$, for $i = 1, 2, ..., n$, then b is a jump point.

This kind of jump point is not assigned a negative mass.

(ii) If there exists an index i such that $b = N(X_i)$ or $b = N(X_i) + L - X_i$ with $\Gamma_i = 0$, for $i = 1, 2, ..., n$, then b may be a jump point.

This kind of jump point is not assigned a negative mass.

(iii) If there exists an index i such that $\Gamma_i = 0$, for $i = 1, 2, ..., n$,

$N(X_i) < b < N(X_i) + L - X_i$.

And there exists another index j such that

$$b = \int_0^{X_i} U_j(TH(t))dt$$

and $X_j > X_i$. Then, b is a jump point.

This kind of jump point is assigned a negative mass.

(iv) If there exists an index i such that $\Gamma_i = 0$, for $i = 1, 2, ..., n$,

$N(X_i) < b < N(X_i) + L - X_i$.

And there exists another index j such that

$$b = \int_0^{X_i} U_j(TH(t))dt + (L - X_i)$$

$X_j > X_i$ and $N(X_j) + L - X_j < b$. Then b is a jump point.

This kind of jump point is assigned a negative mass.

2.7 Cumulative Hazard Function

In this section we will talk about the relationship between hazard function and the simple weighted estimator, Zhao-Tsiatis estimator and the Wang estimator. We mention in 2.3 the relationship between the survival function and the hazard function which is given by

$$\hat{S}(t) = \exp^{-\hat{A}(t)},$$

then

$$\hat{A}(t) = -\ln(\hat{S}(t))$$

Definition 2.13. When we use the data (X_i, Γ_i) to find $\hat{K}(X_i)$, we get

$$\hat{A}(t)_b(1;1) = -\ln(\frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_j}{\hat{K}(X_j)}I(N_j(X_j) > b))$$

This is a definition of hazard function for the simple weighted estimator .

Definition 2.14. When we use the data $(X_i(b), \Gamma_i(b))$ to find $\hat{K}_b(X_i(b))$, we get

$$\hat{A}(t)_b(3;1) = -\ln(\frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_j(b)}{\hat{K}_b(X_j(b))}I(N_j(X_j) > b))$$

This is a definition of hazard function for the Zhao and Tsiatis estimator.

Definition 2.15. When we use the data $(\hat{X}_i(b), \hat{\Gamma}_i(b))$ to find $\hat{K'}_b(\hat{X}_i(b))$, we get

$$\hat{A}(t)_b(4;1) = -\ln(\frac{1}{n}\sum_{j=1}^{n}\frac{\hat{\Gamma}_j(b)}{\hat{K'}_b(\hat{X}_j(b))}I(N_j(X_j) > b))$$

This is a definition of hazard function for the Wang estimator.

2.8 Applications

Example 1

In this example, we will illustrate the differences among the simple weighted estimator, the Zhao-Tsiatis estimator, and the Wang estimator [1].

Consider the following table:

Table 1: Quality of life

i	1	2	3	4	5	6
$U(TH_i(t))$	0.04	0.9	0.04	0.04	0.9	0.9

Let $L = 8$ which is an artificial endpoint. We want to estimate the survival function at $b = 4$.

Now, consider the following table which shows the data that will be used to show the differences among the simple weighted estimator, Zhao-Tsiatis estimator and the Wang estimator. First, let us evaluate the possibly censored quality adjusted

i	X_i	Γ_i
1	3	0
2	4.2	0
3	4.5	0
4	5	1
5	5	0
6	6	1

lifetime for each item, which is defined by

$$N(X_i) = \int_0^{X_i} U(TH_i(t))dt$$

depending on the data in the tables above, we get:

$N(X_1) = 0.12$, $N(X_2) = 3.78$, $N(X_3) = 0.18$, $N(X_4) = 0.2$, $N(X_5) = 4.5$ and $N(X_6) = 5.4$

Now, we will find the estimators of the survival function by using three different estimators.

A) The Simple Weighted Estimator.

To evaluate the simple weighted estimator

$$\hat{\delta}_4(1;1) = \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_j}{\hat{K}(X_j)}I(N_j(X_j) > 4)$$

i	X_i	Γ_i	$N(X_i)$	$I(N_i(X_i) > 4)$
1	3	0	0.12	0
2	4.2	0	3.78	0
3	4.5	0	0.18	0
4	5	1	0.2	0
5	5	0	4.5	1
6	6	1	5.4	1

Table 2: Data summary for the simple weighted estimator

Now, Both $I(N(X_i) > 4)$ and Γ_i are equal one only for $i = 6$. So, we need to evaluate Kaplan-Meier estimator $\hat{K}$ at the point X_6 only. Then, we have

$$\hat{K}(X_6) = (1 - \frac{1}{6})(1 - \frac{1}{5})(1 - \frac{1}{4})(1 - \frac{1}{2}) = \frac{1}{4}$$

Then,

$$\hat{\delta}_4(1;1) = \frac{1}{6}(0 + 0 + 0 + 0 + 0 + 1/\frac{1}{4}) = \frac{2}{3}$$

B) The Zhao and Tsiatis estimator.

To evaluate the Zhao and Tsiatis estimator

$$\hat{\delta}_4(3;1) = \frac{1}{n}\sum_{j=1}^{n} \frac{\Gamma_j(4)}{\hat{K}_4(X_j(4))} I(N_j(X_j) > 4)$$

Now, we need to find the value of $X_i(4)$.

Since $I(N_i(X_i) > 4) = 1$ only for X_5 and X_6, the value of $X_i(4) = X_i$ for $i = 1, 2, 3, 4$.

$U(TH_i) = 0.9$ for X_5 and X_6 so we have :

$X_5(4) = X_6(4) = 4.45$

i	X_i	Γ_i	$N(X_i)$	$I(N_i(X_i) > 4)$
1	3	0	0.12	0
2	4.2	0	3.78	0
3	4.5	0	0.18	0
4	5	1	0.2	0
5	4.45	1	4.5	1
6	4.45	1	5.4	1

Table 3: Data summary for the Zhao and Tsiatis estimator

The Kaplan-Meier estimator for $X_5(4)$ and $X_6(4)$ is

$$\hat{K}_4(X_5(4)) = \hat{K}_4(X_6(4)) = (1 - \frac{1}{6})(1 - \frac{1}{5}) = \frac{2}{3}$$

Therefore,

$$\hat{\delta}_4(3;1) = \frac{1}{6}(0 + 0 + 1/\frac{2}{3} + 1/\frac{2}{3} + 0 + 0) = \frac{1}{2}$$

C) The Wang estimator.

To evaluate The Wang estimator:

$$\hat{\delta}_4(4;1) = \frac{1}{n}\sum_{j=1}^{n} \frac{\hat{\Gamma}_j(b)}{\hat{K'}_b(\hat{X}_j(b))} I(N_j(X_j) > b)$$

Now, we want to find the set of values $\hat{X}_i(b)$.

$I(N(X_i) > 4) = 1$ for X_5 and X_6 then, $\hat{X}_5(4) = \hat{X}_6(4) = 4.45$.

$I(N(X_i) > 4) = 0$ for X_1, X_2, X_3 and X_4. The value of $\hat{X}_i(b)$ for $i = 1, 2, 3, 4$ are given in the table.

i	$\hat{X}_i(4)$	$\hat{\Gamma}_i(4)$	$N(X_i)$	$I(N_i(X_i) > 4)$
1	3	0	0.12	0
2	4.2	0	3.78	0
3	4.17	1	0.18	0
4	4.17	1	0.2	0
5	4.45	1	4.5	1
6	4.45	1	5.4	1

Table 4: Data summary for the Wang estimator

The Kaplan-Meier estimator for $\hat{X}_5(4)$ and $\hat{X}_6(4)$ is

$$\hat{K'}_4(\hat{X}_5(4)) = \hat{K'}_4(\hat{X}_6(4)) = (1 - \frac{1}{6})(1 - \frac{1}{3}) = \frac{5}{9}$$

Therefore,

$$\hat{\delta}_4(4;1) = \frac{1}{6}(0 + 0 + +0 + 0 + 1/\frac{5}{9} + 1/\frac{5}{9}) = \frac{3}{5}$$

Example 2.

In this example, we will investigate the negative mass jump point of the Zhao and Tsiatis estimator and of the Wang estimator.

We will follow the way in 2.6 to find a jump point. Also, we will find the value of each estimator just before and just after each point [1].

Now, Consider the following data given in table

i	X_i	Γ_i	$U(.)$	$N_i(X_i)$
1	9	0	1	9
2	38	0	$\frac{1}{4}$	9.5
3	45	1	$\frac{1}{2}$	22.5
4	50	1	1	50

Table 5: Data used to investigate the jump points and which of these jump points assigns a negative mass for the Zhao and Tsiatis estimator and The Wang estimator

(I) The Simple Weighted Estimator Jump Points.

This estimator has jump points only when $b = 22.5$ and $b = 50$ Now, the value of the simple weighted estimator just before and after these jump points is shown in the following table.

(II) The Zhao-Tsiatis estimator jump points. As mentioned in 2.6.2 this estimator has three kind of jump points.

In the first kind, the jump points when $b = 22.5$ and $b = 50$.

In the second kind, the jump points when $b = 9$ and $b = 9.5$.

In the third kind, the jump points when $b = 19$ and $b = 38$.

Now, the value of the simple weighted estimator just before and after these jump points is shown in the following table.

b	$\hat{\delta}_b(1;1)$
22.5−	1
22.5+	$\frac{1}{2}$
50−	$\frac{1}{2}$
50+	0

Table 6: The value of the simple weighted estimator just before and after jump points

b	$\hat{\delta}_b(3;1)$
9−	1
9+	1
9.5−	1
9.5+	$\frac{2}{3}$
19−	$\frac{2}{3}$
19+	1
22.5−	1
22.5+	$\frac{1}{3}$
38−	$\frac{1}{3}$
38+	$\frac{1}{2}$
50−	$\frac{1}{2}$
50+	0

Table 7: The value of the Zhao-Tsiatis estimator just before and after jump points

(III) The Wang Estimator jump points.

As mentioned in 2.6.3, this estimator has four kind of jump points.

In the first kind, the jump point is when $b = 22.5$ and $b = 50$.

In the second kind, the jump point is when $b = 9$, $b = 9.5$, $b = 21.5$ and $b = 50$.

In the third kind, the jump point is when $b = 21.5$.

Finally, the jump points of this kind are when $b = 43.25$, $b = 45.5$. Now, the value of the simple weighted estimator just before and after these jump points is shown in the following table.

.

b	$\hat{\delta}_b(3;1)$
$9-$	1
$9+$	1
$9.5-$	1
$9.5+$	$\frac{2}{3}$
$19-$	$\frac{2}{3}$
$19+$	1
$21.5-$	1
$21.5+$	$\frac{2}{3}$
$22.5-$	$\frac{2}{3}$
$22.5+$	$\frac{1}{3}$
$43.25-$	$\frac{1}{3}$
$43.25+$	$\frac{3}{8}$
$45.5-$	$\frac{3}{8}$
$45.5+$	$\frac{1}{2}$
$50-$	$\frac{1}{2}$
$50+$	0

Table 8: The value of the Wang estimator just before and after jump points. The Wang estimator increases the number of jump points with negative mass. For example the points 45.5, and 43.25 are jump points only for the Wang estimator. Both of them assigned negative mass.

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3 SOME ESTIMATORS WITH LEFT AND RIGHT-CENSORSHIP DATA

3.1 Introduction

In this chapter, we will consider estimation of three types of survival time. The first one is a simple weighted estimator. This estimator is overall times. The second estimator is Zhao-Tsiats estimator. Finally, Wang estimator. The two estimators have restricted quality adjusted lifetime (RQAL).

In this chapter, one of our purposes is to identify the three estimators (Simple, Zhai-Tsiats, Wang) in left and right-censorship case. Also, we will give two numeric example. The first one is to interpret the difference through the simple weighted estimator, the Zhao-Tsiats estimator and the Wang estimator.

In chapter four, we will define the jump point of the simple weighted estimator, Zhao-Tsiats estimator and the Wang estimator. Also, we will illustrate when these points assign negative mass to the estimate. We use the second example to explain the calculation of these jump point and to identify the jump points that assigned negative mass.

3.2 Notations

Here we will introduce some notations for left and right-censorship data. Let $R_l = (R_{l1}, R_{l2}, ..., R_{ln})$ be a continuous failure time (over all time) random variables with hazard function λ_{R_l} and survival function $S(.)$.

Also, let $D_{li} = R_{li} \wedge L$ be truncated failure times , where $\wedge$ is the minimization operator and L is an artificial end point.

Now, let censoring time, be donate by $C_l = (C_{l1}, C_{l2}, ..., C_{ln})$ which is continuous random variables with hazard function λ_{C_l} and survival function $K_l(.)$.

Assume that

$$\{D_{li}, C_{li}, N_l(r), r \in [\alpha_i, D_{li}], i = 1, 2, ...\}$$

where $\alpha_i = 0 \vee sp_i$, $\vee$ is a maximization operator and sp_i is a start point.

The observed data with sample size n is given by

$$\{X_i = D_{li}, \Gamma_{li} = I(\alpha_i = 0), N_l(r), r \in [\alpha_i, X_i], i = 1, 2, ..., n.\}$$

Definition 3.1. The possible censored restricted quality adjusted lifetime (RQAL) is

$$N(X_i) = \int_{\alpha_i}^{X_i} U(TH_i(t))dt$$

The target is to find consistent estimators for the survival function of restricted quality adjusted lifetime in left censorship. In the next section we will consider three estimators for $S_l(b)$. The three estimators are

- Simple weighted estimator
- Zhao and Tsiatis estimator
- Wang estimator

3.3 Some Weighted Estimators Of The Survival Function Of RQALT For Left and Right-Censorship

In this section, we will introduce a class of estimators for $S_u(a)$.Three important members of this class of estimators are the simple weighted estimator, the Zhao-Tsiatis estimator and the Wang estimator. A single (censored or uncensored) observation can be classified into four categories or cases. The cases depend on the answers to the following two questions:

A) Can the value of $I(N(D_l) > b)$ be determined from observation?

B) If the value of $I(N(D_l) > b)$ can be determined, what is the first time point at which the value can be determined?

We now consider each of the four cases in turn. For each case, we will define variables $D_l(b), D_l'(b), X(b)$ and $X'(b)$.

These variables will be used later in this chapter to define new estimators.

Case(1):

$I(N(X) > b) = 1$

Since $D_l \geq X$, $N(D_l) \geq N(X)$. Therefor, the value of $I(N(D_l) > b)$ is 1.

To find out the first time point at which this value can be determined, consider

$$m_l(b) = inf\{e : N(X \wedge e) > b\}$$

$m_l(b)$ is the first point time t when $N(t)$ exceeds b.

Since $N(t)$ is a nondecreasing function, $I(N(t) > b) = 1$ for all $t \geq m_l(b)$.

If $N(X) > b$,

Define $D_l(b) = m_l(b)$ and $D_l'(b) = m_l(b)$.

Let $X(b) = D_l(b) \wedge C_l$ and $X'(b) = m_l'(b) \wedge C_l$

Notice that for an observation in this case, if $C_l > D_l(b)$, we know that $I(N(D_l) > b)$ has to be 1

Case (2):

$I(N(X) > b) = 0$ and $N(X) + (sp - 0) + (L - X) < b$.

Since $U(.) \leq 1$, $D_l \geq X$ and

$$N(D_l) = \int_{\alpha_i}^{D_l} U(.)dt,$$

here we can be sure that $I(N(D_l) > b) = 0$

Now, to find out the first time at which this value can be determined, consider

$$z_l(b) = inf\{z : N(z) + sp + (L - z) \leq b\}$$

$z_l(b)$ is the first time point t when $N(t) + (sp - 0) + (L - X) \leq b$.

Since $N(t)+(sp-0)+(L-X)$ is a decreasing function of t, $N(t)+(sp-0)+(L-X) \leq b$ for all $t \geq z_l(b)$.

If $N(X) < b$ and $N(X) + (sp - 0) + (L - X) < b$, define

$$D_l(b) = D_l \text{ , } D_l'(b) = z_l(b)$$

$$\text{Let } X(b) = D_l(b) \wedge C_l \text{ and } X'(a) = D_l'(b) \wedge C_l$$

For an observation in this case if $C_l > D_l'(b)$, we can be sure that $I(N(D_l) > b) = 0$

Case (3):

$I(N(X) > b) = 0$, $N(X) + (sp - 0) + (L - X) \geq b$ and $C_l < D_l$. In this case, we can't determine the value of $I(N(D_l) > b)$ because $N(D_l) + (sp - 0) + (L - D_l)$ can be greater than b. Let

$$D_l(b) = D_l, \qquad D_l'(b) = D_l$$

$$X(b) = D_l(b) \wedge C_l, \qquad X'(b) = D_l'(b) \wedge C_l$$

Case (4): $I(N(X) > b) = 0$, and $N(X) + sp + (L - X) \geq b$ and $D_l \leq C_l$, in this case $D_l = X$ and hence $I(N(D_l) > b) = 0$

$$D_l(b) = D_l, \qquad D_l'(b) = D_l$$

$$X(b) = D_l(b) \wedge C_l, \qquad X'(b) = D_l'(b) \wedge C_l$$

$$N(X_i) = \int_{sp_i}^{X_i} U(TH_i(t))dt$$

3.4 Example

This example illustrates the difference among the simple weighted estimator, the Zhao-Tsiatis estimator, and the wang estimator. Also, we will find the value of cumulative hazard function. Here, we will depending on the definitions of these estimators in the 2.5 and 2.7 to calculate it.

Now, consider the following table:

Table 9: Quality of life for the items

i	1	2	3	4	5	6
$U(TH_i(t))$	0.1	0.9	0.5	0.6	0.8	0.7

Now, consider the following table which shows the data that will be used to show the differences among the simple weighted estimator, Zhao-Tsiatis estimator and the Wang estimator,

i	X_i	Γ_{li}
1	$[1-6]$	0
2	$[1.5-7.5]$	0
3	$[0.5-7]$	0
4	$[0-7]$	1
5	$[0-7.5]$	1
6	$[1-9]$	0

First, let us evaluate the possibly censored quality adjusted lifetime for each item, which is defined by

$$N(X_i) = \int_{\alpha_i}^{X_i} U(TH_i(t))dt$$

depending on the data in the tables above, we get:

$N(X_1) = 0.5$, $N(X_2) = 5.4$, $N(X_3) = 3.25$, $N(X_4) = 4.2$, $N(X_5) = 6$ and $N(X_6) = 5.6$

The artificial endpoint $L = 9+$, here we want to estimate the survival function at $b = 5$

1. Simple weighted estimator:

To evaluate simple weighted estimator

$$\hat{\delta_{lb}}(1;1) = \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma_{lj}}{\hat{K}(X_j)}I(N_j(X_j) > b)$$

First, we have to calculate the values of $N(X_i)$ and $I(N(X_i) > b)$. These values are given in Table.

i	X_i	Γ_{li}	$U(THi(t))$	$N(X_i)$	$I(N(X_i) > b)$
1	[1-6]	0	0.1	0.5	0
2	[1.5-7.5]	0	0.9	5.4	1
3	[0.5-7]	0	0.5	3.25	0
4	[0-7]	1	0.6	4.2	0
5	[0-7.5]	1	0.8	6	1
6	[1-9]	0	0.7	5.6	1

Both $I(N(X_i) > 5)$ and Γ_{li} are equal to 1 is only for $i = 5$.

Therefore, we need the value of the function $\hat{K(.)}$ at the point X_5 only. We have

$$\hat{K}(X_5) = (1 - \frac{1}{6})(1 - \frac{1}{5})(1 - \frac{1}{4}) = \frac{1}{2}$$

Then,

$$\hat{\delta_{l5}}(1;1) = \frac{1}{6}(0+0+0+0+\frac{1}{\frac{1}{2}}+0) = \frac{1}{3}$$

The cumulative hazard function

$$\hat{A}(t)_5(1;1) = 1.09$$

2. Zhao-Tsiatis estimator:

To evaluate the Zhao and Tsiatis estimator

$$\hat{\delta_l b}(3;1) = \frac{1}{n}\sum_{j=1}^{n} \frac{\Gamma_{lj}(b)}{\hat{K}_b(X_j(b))} I(N_j(X_j) > b)$$

We have to find the set of values $X_i(b)$.

Since $N(X_i) > 5$ only for X_2, X_5 and for X_6, we have $X_i(b) = X_i$, for $i = 1, 3, 4$.

Note that :

$$m_l(b) = inf\{e : N(X_i \wedge e) > 5\}$$

Now want to find $X_i(5)$ for $i = 2, 5$ and 6.

$X_2(5) = [1.5 - 7.1]$,

$X_5(5) = [0 - 6.25]$,

$X_6(5) = [1 - 8.2]$.

Now, we need the value of the function $\hat{K(.)}$ at the point X_2, X_5 and X_6.

Then,

$$\hat{K_a}(X_2(b)) = \hat{K_a}(X_5(b)) = (1 - \frac{1}{6}) = \frac{5}{6}$$

$$\hat{K_a}(X_6(b)) = (1 - \frac{1}{6})(1 - \frac{1}{3}) = \frac{5}{9}$$

Table 10: Data for Zhao-Tsiatis Estimator

i	$X_i(b)$	$\Gamma_{li}(b)$	$U(TH_i(t))$	$N(X_i)$	$I(N(X_i) > b)$
1	[1-6]	0	0.1	0.5	0
2	[1.5-7.1]	1	0.9	5.4	1
3	[0.5-7]	0	0.5	3.25	0
4	[0-7]	1	0.6	4.2	0
5	[0-6.25]	1	0.8	6	1
6	[1-8.2]	1	0.7	5.6	1

$$\hat{\delta_{l5}}(3;1) = \frac{1}{6}[0 + \frac{1}{\frac{5}{6}} + 0 + 0 + \frac{1}{\frac{5}{6}} + \frac{1}{\frac{5}{9}}] = 0.7$$

The cumulative hazard function

$$\hat{A}(t)_5(;1) = 0.36$$

3. Wang Estimator :

To evaluate the Wang estimator

We have to find the set of value of values $X_i'(b)$.

Here we need to study a cases of :if $N(X_i) < b$ and $N(X_i) + (sp_i - 0) + (L - X_i) < b$

$N(X_i) < b = 5$ for X_1, X_3 and X_4

$N(X_1) + (sp_1 - 0) + (L - X_1) = 0.5 + (1 - 0) + (9 - 6) = 4.5 < 5$; So $\Gamma_{l1}'(b) = 1$

$N(X_3) + (sp_3 - 0) + (L - X_3) = 3.25 + (0.5 - 0) + (9 - 7) = 5.75 > 5$; Here we can not be sure

$I(N(X_i) > b) = 0$; So $\Gamma_{l3}'(b) = 0$

$N(X_4) + (sp_4 - 0) + (L - X_4) = 4.2 + (0 - 0) + (9 - 7) = 6.2 > 5$; Here $sp_i = 0$, so it stays the same $\Gamma_{l4}'(b)$.

Table 11: Data for Wang Estimator

i	$X'_i(b)$	$\Gamma'_{li}(b)$	$U(TH_i(t))$	$N(X_i)$	$I(N(X_i) > b)$
1	[1-5.4]	1	0.1	0.5	0
2	[1.5-7.1]	1	0.9	5.4	1
3	[0.5-7]	0	0.5	3.25	0
4	[0-7]	1	0.6	4.2	0
5	[0-6.25]	1	0.8	6	1
6	[1-8.2]	1	0.7	5.6	1

We need to find Kappa estimator for Wang estimator

$$\hat{K_b}'(X'_2(b)) = 1$$

$$\hat{K_b}'(X'_5(b)) = 1$$

$$\hat{K_b}'(X'_6(b)) = (1 - \frac{1}{3}) = \frac{2}{3}$$

$$\hat{\delta}_b(4;1) = \frac{1}{6}(0 + 1 + 0 + 0 + 1 + \frac{1}{\frac{2}{3}}) = 0.6$$

The cumulative hazard function

$$\hat{A}(t)_5(4;1) = 0.5$$

3.5 Jump Point for Left and Right-Censorship:

In this section, we will define a jump point in simple weighted estimator, the Zhao-Tsiatis estimator and the Wang estimator. Also, we will define which of these point (jump point) is assigned negative mass.

3.5.1 Jump Point Of The Simple Weighted Estimator In Left And Right-Censorship

Simple weighed estimator has a jump point at b if and only if there exists an index i such that $b = N(X_i)$ and $\Gamma_{li} = 1, \;\; i = 1, 2, ..., n$.

In this estimator, the jump point is only at points of deaths. No negative mass is assigned to any point by this estimator. It is monotonically non- decreasing estimator.

3.5.2 Jump Points of the Zhao-Tsiatis Estimator in Left And Right-Censorship

Almanassra (2005) [1] mentioned that Zhao-Tsiatis estimator may not be monotone. In this section, we will identify the jump points in left and right censorship of the Zhao-Tsiatis. Also, we will investigate which of these points are assigned negative mass.

The Zhao-Tsiatis has three kinds of jump points:

A) Suppose that there exists an item i such that $b = N(X_i)$ and $\Gamma_{li} = 1$, $i = 1, 2, ..., n$, then b is a jump point.

   This kind of jump point is not assigned a negative mass.

B) Suppose that there exists an item i such that $b = N(X_i)$ and $\Gamma_{li} = 0$, $i = 1, 2, ..., n$, then b could be a jump point.

   This kind of jump point is also not assigned a negative mass.

C) Suppose that there exists an item i such that $\Gamma_{li} = 0$, $i = 1, 2, ..., n$, and there exists another item j such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt$$

such that $N(X_i) < b$ and $X_j - sp_j > X_i - sp_i$, then b is a jump point.

This kind of jump point is assigned a negative mass at this point.

3.5.3 Jump Points of the Wang Estimator in Left And Right-Censorship

As mentioned before, this estimator has, in general more jump points than Zhao-Tsaitis estimator.

This estimator has four kind of jump points.

A) If there exist an index i such that $b = N(X_i)$ with $\Gamma_{li} = 1$, for $i = 1, 2, ..., n$, then b is a jump point.

This kind of jump points not assigned a negative mass.

B) If there exist an index i such that $b = N(X_i)$ or $b = N(X_i) + (sp_i - 0) + (L - X_i)$ with $\Gamma_{li} = 0$, for $i = 1, 2, ..., n$, then b may be a jump point.

This kind of jump points not assigned a negative mass.

C) If there exist an index i such that $\Gamma_{li} = 0$, for $i = 1, 2, ..., n$,

$N(X_i) < b < N(X_i) + (sp_i - 0) + (L - X_i)$.

And there exist another index j such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt$$

and $X_j - sp_j > X_i - sp_i$. Then b is a jump point.

This kind of jump point is assigned a negative mass.

D) If there exists an index i such that $\Gamma_{li} = 0$, for $i = 1, 2, ..., n$,

$N(X_i) < b < N(X_i) + (sp_i - 0) + (L - X_i)$.

And there exist another index j such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt + (sp_i - 0) + (L - X_i)$$

$X_j - sp_j > X_i - sp_i$ and $N(X_j) + (sp_j - 0) + (L - X_j) < b$. Then b is a jump point.

This kind of jump point is assigned a negative mass.

3.6 Example

In this section we will illustrate how to find the jump points of simple weighted estimator, Zhao-Tsiatis estimator and for the Wang estimator. We will find the value of estimators just befor and just after each jump point.
Also, we will investigate which of these jump point is assigned negative mass.

Now, consider the following table,

Table 12: Data with quality of life

i	1	2	3	4
X_i	[10 - 19]	[6 - 44]	[0-45]	[0-50]
$U(TH_i(t))$	1	0.25	0.5	1
Γ_{li}	0	0	1	1

(I) Simple weighted estimator jump points

The simple weighted estimator has jump points only at points of death, where $b = N(X_i)$, $\Gamma_{li} = 1$.

The jump points are at $b = 22.5$ and at $b = 50$

Table 13: The simple weighted estimator just before and after $b = 22.5$

i	X_i	Γ_{li}	$N(X_i)$	$I(N(X_i) > 22.5-)$	$I(N(X_i) > 22.5+)$
1	$[10 - 19]$	0	9	0	0
2	$[6 - 44]$	0	9.5	0	0
3	$[0 - 45]$	1	22.5	1	0
4	$[0 - 50]$	1	50	1	1

$$\hat{\delta_{l22.5-}}(1;1) = \frac{1}{4}(0 + 0 + \frac{1}{0.5} + \frac{1}{0.5}) = 1$$

$$\hat{\delta_{l22.5+}}(1;1) = \frac{1}{4}(0 + 0 + 0 + \frac{1}{0.5}) = \frac{1}{2}$$

Table 14: The simple weighted estimator just before and after $b = 50$

i	X_i	Γ_{li}	$N(X_i)$	$I(N(X_i) > 50-)$	$I(N(X_i) > 50+)$
1	$[41 - 50]$	0	9	0	0
2	$[12 - 50]$	0	9.5	0	0
3	$[0 - 45]$	1	22.5	0	0
4	$[0 - 50]$	1	50	1	0

$$\hat{\delta_{l50-}}(1;1) = \frac{1}{4}(0 + 0 + 0 + \frac{1}{0.5}) = \frac{1}{2}$$

$$\hat{\delta_{l50+}}(1;1) = \frac{1}{4}(0 + 0 + 0 + 0) = 0$$

(II) The Zhao-Tsiatis estimator's jump points

This estimator has three kind of jump points.

(i) When $b = N(X_i)$ and $\Gamma_{li} = 1$.

Then, the jump points are $b = N(X_3) = 22.5$ and $b = N(X_4) = 50$

This jump point is not assigned a negative mass.

(ii) When $b = N(X_i)$ and $\Gamma_{li} = 0$

Then, the jump points are $b = N(X_1) = 9$ and $b = N(X_2) = 9.5$

Table 15: The Zhao-Tsitis estimator just before and after $b = 9$

i	$X_i(9-)$	$\Gamma_{li}(9-)$	$I(N(X_i) > 9-)$	$X_i(9+)$	$\Gamma_{li}(9+)$	$I(N(X_i) > 9+)$
1	$[10-19-]$	1	1	$[10-19]$	0	0
2	$[6-42-]$	1	1	$[6-42+]$	1	1
3	$[0-18-]$	1	1	$[0-18+]$	1	1
4	$[0-9-]$	1	1	$[0-9+]$	1	1

$$\hat{\delta_{l9-}}(3;1) = \frac{1}{4}(1+1+1+1) = 1$$

$$\hat{\delta_{l9+}}(3;1) = \frac{1}{4}(0 + \frac{1}{3/4} + \frac{1}{3/4} + \frac{1}{3/4}) = 1$$

(iii) When $\Gamma_{li} = 0$, s.t

$$b = \int_{sp_j}^{x_i - sp_i + sp_j} U_j(TH(t))dt$$

, $b > N(X_i)$ where $X_j - sp_j > X_i - sp_i$.

The jump points for this kind when $b = 19$ and $b = 38$.

Table 16: The Zhao-Tsitis estimator just before and after $b = 19$

i	$X_i(19-)$	$\Gamma_{li}(19-)$	$I(N(X_i) > 19-)$	$X_i(19+)$	$\Gamma_{li}(19+)$	$I(N(X_i) > 19+)$
1	$[10 - 19]$	0	0	$[10 - 19]$	0	0
2	$[6 - 44]$	0	0	$[6 - 44]$	0	0
3	$[0 - 38-]$	1	1	$[0 - 38+]$	1	1
4	$[0 - 19-]$	1	1	$[0 - 19+]$	1	1

$$\hat{\delta_{l19-}}(3;1) = \frac{1}{4}(0 + 0 + \frac{1}{3/4} + \frac{1}{3/4}) = \frac{2}{3}$$

$$\hat{\delta_{l19+}}(3;1) = \frac{1}{4}(0 + 0 + \frac{1}{3/8} + \frac{1}{3/4}) = 1$$

Table 17: The Zhao-Tsitis estimator just before and after $b = 38$

i	$X_i(38-)$	$\Gamma_{li}(38-)$	$I(N(X_i) > 38-)$	$X_i(38+)$	$\Gamma_{li}(38+)$	$I(N(X_i) > 38+)$
1	$[10 - 19]$	0	0	$[10 - 19]$	0	0
2	$[6 - 44]$	0	0	$[6 - 44]$	0	0
3	$[0 - 45]$	1	0	$[0 - 45]$	1	0
4	$[0 - 38-]$	1	1	$[0 - 38+]$	1	1

$$\hat{\delta_{l38-}}(3;1) = \frac{1}{4}(0 + 0 + 0 + \frac{1}{3/4}) = \frac{1}{3}$$

$$\hat{\delta_{l38+}}(3;1) = \frac{1}{4}(0 + 0 + 0 + \frac{1}{1/2}) = \frac{1}{2}$$

(III) The Wang estimator jump points

This estimator has three kinds of jump points.

(i) When $b = N(X_i)$ with $\Gamma_{li} = 1$. The jump points are $b = 22.5$ and $b = 50$

The Wang estimator just before and after $b = 22.5$

$$\hat{\delta_{lb}}(4;1) = \frac{1}{n}\sum_{j=1}^{n}\frac{\Gamma'_{lj}(b)}{\hat{K'}_b(\hat{X}_j(b))} I(N_j(X_j) > b)$$

$$\hat{\delta}_{l22.5-}(4;1) = \frac{1}{4}(0 + 0 + \frac{1}{3/4} + \frac{1}{3/4}) = \frac{2}{3}$$

$$\hat{\delta}_{l22.5+}(4;1) = \frac{1}{4}(0 + 0 + 0 + \frac{1}{3/4}) = \frac{1}{3}$$

The Wang estimator just before and after $b = 50$

$$\hat{\delta}_{l50-}(4;1) = \frac{1}{4}(0 + 0 + 0 + \frac{1}{1/2}) = \frac{1}{2}$$

$$\hat{\delta}_{l50+}(4;1) = \frac{1}{4}(0 + 0 + 0 + 0) = 0$$

(ii) When $b = N(X_i)$ or $b = N(X_i) + (sp_i - 0) + (L - X_i)$ with $\Gamma_{li} = 0$.

The jump points are $b = 9$, 9.5, 21.5, 50.

(iii) When $\Gamma_{li} = 0$, and $N(X_i) < b < N(X_i) + (sp_i - 0) + (L - X_i)$.

And there exist another index j such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt$$

and $X_j - sp_j > X_i - sp_i$. Then, b is a jump point.

The jump point is at $b = 19$.

Table 18: The Wang estimator just before and after $b = 19$

i	$X_i(19-)$	$\Gamma_{li}(19-)$	$I(N(X_i) > 19-)$	$X_i(19+)$	$\Gamma_{li}(19+)$	$I(N(X_i) > 19+)$
1	$[10-19]$	0	0	$[10-19]$	0	0
2	$[6-44]$	0	0	$[6-44]$	0	0
3	$[0-38-]$	1	0	$[0-38+]$	1	1
4	$[0-19-]$	1	1	$[0-19+]$	1	1

$$\hat{\delta}_{l19-}(4;1) = \frac{1}{4}(0+0+\frac{1}{3/4}+\frac{1}{3/4}) = \frac{2}{3}$$

$$\hat{\delta}_{l19+}(4;1) = \frac{1}{4}(0+0+\frac{1}{3/8}+\frac{1}{3/4}) = 1$$

(iv) $\Gamma_{li} = 0$ and $N(X_i) < b < N(X_i) + (sp_i - 0) + (L - X_i)$.

And there exist another index j such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt + (sp_i - 0) + (L - X_i)$$

$X_j - sp_j > X_i - sp_i$ and $N(X_j) + (sp_j - 0) + (L - X_j) < b$. Then, b is a jump point.

The jump points for this kind is at $b = 43.25$ and $b = 45.5$

Table 19: The Wang estimator just before and after $b = 43.25$

i	$X_i(43.25-)$	$\Gamma_{li}(43.25-)$	$I(N(X_i) > 43.25-)$	$X_i(43.25+)$	$\Gamma_{li}(43.25+)$	$I(N(X_i) > 43.25+)$
1	$[10-19]$	0	0	$[10-19]$	0	0
2	$[6-15+]$	1	0	$[6-15-]$	1	0
3	$[0-13.5+]$	1	0	$[0-13.5-]$	1	0
4	$[0-43.25-]$	1	1	$[0-43.25+]$	1	1

$$\hat{\delta}_{l43.25-}(4;1) = \frac{1}{4}(0+0+0+\frac{1}{3/4}) = \frac{1}{3}$$

$$\hat{\delta}_{l43.25+}(4;1) = \frac{1}{4}(0+0+0+\frac{1}{2/3}) = \frac{3}{8}$$

Table 20: The Wang estimator just before and after $b = 45.5$

i	$X_i(45.5-)$	$\Gamma_{li}(45.5-)$	$I(N(X_i) > 45.5-)$	$X_i(45.5+)$	$\Gamma_{li}(45.5+)$	$I(N(X_i) > 45.5+)$
1	$[10-19]$	0	0	$[10-19]$	0	0
2	$[6-12+]$	1	0	$[6-12-]$	1	0
3	$[0-9+]$	1	0	$[0-9-]$	1	0
4	$[0-45.5-]$	1	1	$[0-45.5+]$	1	1

$$\hat{\delta}_{l45.5-}(4;1) = \frac{1}{4}(0+0+0+\frac{1}{2/3}) = \frac{3}{8}$$

$$\hat{\delta}_{l45.5+}(4;1) = \frac{1}{4}(0+0+0+\frac{1}{1/2}) = \frac{1}{2}$$

4 MONOTONIC QUALITY ADJUSTED LIFETIME SURVIVAL FUNCTION

4.1 Introduction

Almanasra et al (2005) [2] noted that the Zhao-Tsiatis estimator for the quality adjusted life data, is not a monotonic estimator, and hence it is not a suitable survival function. Also the Wang estimator, which is a modified version of the Zhao-Tsiatis estimator, is not a monotonic estimator. Both the Zhao-Tsiatis estimator and the Wang estimator are consistent and reasonable efficient estimators. But, we noticed that the simple weighted estimator is monotonic and consistent, but it is less efficient than the Zhao-Tsiatis and Wang estimators.

In this chapter, we will propose two monotonic estimators for the survival function of RQAL , this procedure is similar to the procedure given by Almanassra et al [2] . We will call the first one the monotonized Zhao-Tsiatis estimator, and the second one the monotonized Wang estimator. The two new estimators are linear combinations of other consistent estimators.

4.2 Monotonizing The Zhao-Tsiatis Estimator For The Left and Right Censorship

In this section, we will give a procedure for monotonizing the Zhao-Tsiatis estimator $\hat{\delta_Z}(b)$. The new estimator is a linear combination of the simple weighted estimator $\hat{\delta_S}$ which is a monotonic function, and the Zhao-Tsiatis estimator $\hat{\delta_Z}$. As mentioned above, both the simple weighted estimator and the Zhao-Tsiatis estimator are consistent estimators. The procedure for monotonizing the Zhao-Tsiatis estimator is given in the following steps:

(I) Let $W_0 = 0$.

(II) Find all possible jump points for simple weighted estimator and Zhao-Tsiatis estimator, denote these jump points by $W_1, W_2, W_3, ..., W_{N-1}$.

(III) Let $P_1 = W_0$ and $P_2 = W_1, ..., P_N = W_{N-1}$, be the possible ordered jump points.

(IV) Find the values of the simple weighted estimator and the Zhao-Tsiatis estimator at all of these points.

(V) The suggested estimator is a step function. The value of this function between the jump points which is mentioned before is defined by :

$$\hat{\delta}_{MZi}(b) = K_i \hat{\delta}_Z(P_i) + (1 - K_i)\hat{\delta}_S(P_i).$$

Such that, $i = 1, 2, ..., N$, $P_i \le b < P_{i+1}$, K_i's are random numbers such that

$$0 \le K_i \le 1,$$

and

$$1 \ge K_1 \hat{\delta}_Z(P_1) + (1 - K_1)\hat{\delta}_S(P_1)$$

$$K_1 \hat{\delta}_Z(P_1) + (1 - K_1)\hat{\delta}_S(P_1) \ge K_2 \hat{\delta}_Z(P_2) + (1 - K_2)\hat{\delta}_S(P_2)$$

$$K_2 \hat{\delta}_Z(P_2) + (1 - K_2)\hat{\delta}_S(P_2) \ge K_3 \hat{\delta}_Z(P_3) + (1 - K_3)\hat{\delta}_S(P_3)$$

$$...$$

$$K_{N-1} \hat{\delta}_Z(P_{N-1}) + (1 - K_{N-1})\hat{\delta}_S(P_1) \ge K_N \hat{\delta}_Z(P_N) + (1 - K_N)\hat{\delta}_S(P_N)$$

$$K_N \hat{\delta}_Z(P_N) + (1 - K_N)\hat{\delta}_S(P_N) \ge 0$$

(VI) We expect the Zhao-Tsiatis estimator is more efficient than the simple weighted estimator. Therefore, we want to find a monotonic survival function estimate that is close as possible to the Zhao-Tsiatis estimator. We will get the monotonic survival function estimate by maximizing the objective function

$$\sum_{i=1}^{N} q_i K_i$$

Where q_i's are known weights (positive). The constraints are given by,

$$Q\,Y \geq A,$$

where

$$Y = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix}$$

$$A = \begin{pmatrix} \hat{\delta}_S(P_1) - 1 \\ \hat{\delta}_S(P_2) - \hat{\delta}_S(P_1) \\ \vdots \\ \hat{\delta}_S(P_N) - \hat{\delta}_S(P_{N-1}) \\ -\hat{\delta}_S(P_N) \end{pmatrix}$$

and the matrix Q is as given as follows,

$$
\begin{pmatrix}
-(\hat{\delta}_Z(P_1) - \hat{\delta}_S(P_1)) & 0 & 0 & \dots & 0 & 0 \\
(\hat{\delta}_Z(P_1) - \hat{\delta}_S(P_1)) & -(\hat{\delta}_Z(P_2) - \hat{\delta}_S(P_2)) & 0 & \dots & \dots & 0 \\
0 & (\hat{\delta}_Z(P_2) - \hat{\delta}_S(P_2)) & -(\hat{\delta}_Z(P_3) - \hat{\delta}_S(P_3)) & \ddots & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & 0 & 0 & (\hat{\delta}_Z(P_{N-1}) - \hat{\delta}_S(P_{N-1})) & -(\hat{\delta}_Z(P_N) - \hat{\delta}_S(P_N)) \\
0 & 0 & 0 & 0 & 0 & (\hat{\delta}_Z(P_N) - \hat{\delta}_S(P_N))
\end{pmatrix}
$$

(VII) Use linear programming to find the matrix Y. Note that the linear programming problem has a known feasible solution given by the simple weighted estimator.

The montonized Zhao-Tsiatis estimator is given by:

$$\hat{\delta}_{MZi}(b) = \begin{cases} K_1\hat{\delta}_Z(P_1) + (1-K_1)\hat{\delta}_S(P_1) & P_1 \le b < P_2 \\ K_2\hat{\delta}_Z(P_2) + (1-K_2)\hat{\delta}_S(P_2) & P_2 \le b < P_3 \\ \vdots & \vdots \\ K_{N-1}\hat{\delta}_Z(P_{N-1}) + (1-K_{N-1})\hat{\delta}_S(P_{N-1}) & P_{N-1} \le b < P_N \\ 0 & P_N \le b \end{cases}$$

4.3 Monotonizing The Wang Estimator For The Left and Right Censorship

In this section, we will give a procedure for monotonizing the Wang estimator $\hat{\delta_W}(b)$. The new estimator is a linear combination of the simple weighted estimator $\hat{\delta_S}$ which is a monotonic function, and the Wang estimator $\hat{\delta_W}$. As mention above, both the simple weighted estimator and the Wang estimator are consistent estimators. Now, the procedure for monotonizing the Wang estimator is given in the following steps:

(I) Let $W_0 = 0$.

(II) Find all possible jump points for simple weighted estimator and Wang estimator, denote these jump points by $W_1, W_2, W_3, ..., W_{N-1}$.

(III) Let $P_1 = W_0$ and $P_2 = W_1, ..., P_N$, be the possible ordered jump points.

(IV) Find the values of the simple weighted estimator and the Wang estimator at all of these points.

(V) The suggested estimator is a step function. The value of this function between the jump points which were mentioned before is defined by :

$$\hat{\delta}_{MWi}(b) = K_i\hat{\delta}_W(P_i) + (1 - K_i)\hat{\delta}_S(P_i).$$

Such that, $i = 1, 2, ..., N$, $P_i \leq b < P_{i+1}$, K_i's are random numbers such that

$$0 \leq K_i \leq 1$$

, and

$$1 \geq K_1\hat{\delta}_W(P_1) + (1 - K_1)\hat{\delta}_S(P_1)$$

$$K_1\hat{\delta}_W(P_1) + (1 - K_1)\hat{\delta}_S(P_1) \geq K_2\hat{\delta}_W(P_2) + (1 - K_2)\hat{\delta}_S(P_2)$$

$$K_2\hat{\delta}_W(P_2) + (1 - K_2)\hat{\delta}_S(P_2) \geq K_3\hat{\delta}_W(P_3) + (1 - K_3)\hat{\delta}_S(P_3)$$

$$...$$

$$K_{N-1}\hat{\delta}_W(P_{N-1}) + (1 - K_{N-1})\hat{\delta}_S(P_1) \geq K_N\hat{\delta}_W(P_N) + (1 - K_N)\hat{\delta}_S(P_N)$$

$$K_N\hat{\delta}_W(P_N) + (1 - K_N)\hat{\delta}_S(P_N) \geq 0$$

(VI) We expect the Wang estimator is more efficient than the simple weighted estimator. Therefore, we want to find a monotonic survival function estimate that is close as possible to the Wang estimator. We will get the monotonic survival function estimate by maximizing the objective function

$$\sum_{i=1}^{N} q_i K_i$$

Where q_i's are known weights (positive). The constraints are given by,

$$Q\,Y \geq A$$

Also,

$$Y = \begin{pmatrix} K_1 \\ K_2 \\ \vdots \\ K_N \end{pmatrix}$$

$$A = \begin{pmatrix} \hat{\delta}_S(P_1) - 1 \\ \hat{\delta}_S(P_2) - \hat{\delta}_S(P_1) \\ \vdots \\ \hat{\delta}_S(P_N) - \hat{\delta}_S(P_{N-1}) \\ -\hat{\delta}_S(P_N) \end{pmatrix}$$

and the matrix Q is as given as follows,

$$
\begin{pmatrix}
-(\hat{\delta}_W(P_1) - \hat{\delta}_S(P_1)) & 0 & 0 & \dots & 0 & 0 \\
(\hat{\delta}_W(P_1) - \hat{\delta}_S(P_1)) & -(\hat{\delta}_W(P_2) - \hat{\delta}_S(P_2)) & 0 & \dots & \dots & 0 \\
0 & (\hat{\delta}_W(P_2) - \hat{\delta}_S(P_2)) & -(\hat{\delta}_W(P_3) - \hat{\delta}_S(P_3)) & \ddots & \dots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & 0 & 0 & (\hat{\delta}_W(P_{N-1}) - \hat{\delta}_S(P_{N-1})) & -(\hat{\delta}_W(P_N) - \hat{\delta}_S(P_N)) \\
0 & 0 & 0 & 0 & 0 & (\hat{\delta}_W(P_N) - \hat{\delta}_S(P_N))
\end{pmatrix}
$$

(VII) Using linear programming to find the matrix Y.Note that the linear programming problem has a known feasible solution given by the simple weighted estimator.

The montonized Wang estimator is given by:

$$\hat{\delta}_{MWi}(b) = \begin{cases} K_1\hat{\delta}_W(P_1) + (1 - K_1)\hat{\delta}_S(P_1) & P_1 \leq b < P_2 \\ K_2\hat{\delta}_W(P_2) + (1 - K_2)\hat{\delta}_S(P_2) & P_2 \leq b < P_3 \\ \vdots & \vdots \\ K_{N-1}\hat{\delta}_W(P_{N-1}) + (1 - K_{N-1})\hat{\delta}_S(P_{N-1}) & P_{N-1} \leq b < P_N \\ 0 & P_N \leq b \end{cases}$$

4.4 Some General Result On the consistency Of A Convex Linear Combination Of Estimator

Amanassra(2005) [1] proved the consistency of the monotonic efficient estimator for the Zhao-Tsiatis estimator and Wang estimator.

4.4.1 Weak consistency

Definition 4.1. Let $\hat{\delta_1}$, $\hat{\delta_2}$, ..., $\hat{\delta_n}$, ... be a sequence of estimators of a real valued parameter δ. The sequence $\{\hat{\delta_n}\}$ is defined to be weakly consistent estimator of δ if for every $\epsilon > 0$

$$\lim_{n\to\infty} P[|\hat{\delta}_n - \delta| < \epsilon] = 1$$

$\forall$ δ in the parameter space

Remark.

If $\hat{\delta_n}$ is a weakly consistent estimator of δ the estimators then $\hat{\delta_n}$ converges in probability to the constant parameter δ

$$\hat{\delta_n} \to \delta$$

Definition 4.2. [6] Convergence in distribution:

A sequence of random variables X_1, X_2,..., converges in distribution to a random variable X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

$\forall$ points x such that $F_X(x)$ continuous.

Remark.

The sequence X_n converges in distribution to a constant b $\iff$ converges in probability to b

Theorem 1. [17] (Slutsky's theorem)

If $X_n \to X$ in distribution and $Y_n \to a$ a constant , in probability, then

a) $Y_n X_n \to aX$ in distribution.

b) $X_n + Y_n \to X + a$ in distribution.

Lemma 2. Suppose that $X_n \to 0$ in probability and $\alpha_n' s$ are random variables such that $|\alpha_n| \leq M$ where M is a positive real number. Then, $\alpha_n X_n \to 0$ in probability.

Proof.

$$\begin{aligned} P[|\alpha_n X_n - 0| < \epsilon] &= P[|\alpha_n||X_n - 0| < \epsilon] \\ &\geq P[M|X_n - 0| < \epsilon] \\ &= P[|X_n - 0| < \frac{\epsilon}{M}] \\ &\longrightarrow 1, n \longrightarrow \infty \end{aligned} \tag{1}$$

□

Theorem 3. [1] Let $\hat{\delta}_{1,n}$ and $\hat{\delta}_{2,n}$ be two sequences of weakly estimators of a parameter δ.

Consider the sequence of convex linear combinations defined by

$$\hat{\delta}_n = \alpha_n \hat{\delta}_{1,n} + (1 - \alpha_n)\hat{\delta}_{2,n}$$

Such that, α_n are random variables $\in [0, 1]$. Then, $\hat{\delta_n}$ is a weakly consistent estimator of δ

Proof. Since $\hat{\delta}_{1,n} \xrightarrow{P} \delta$ and $\hat{\delta}_{2,n} \xrightarrow{P} \delta$ by Slutsky's Theorem

$$\hat{\delta}_{1,n} - \hat{\delta}_{2,n} \xrightarrow{P} 0$$

Also, since α_n is bounded, by 2

$$\alpha_n(\hat{\delta}_{1,n} - \hat{\delta}_{2,n}) \xrightarrow{P} 0$$

Now,

$$\begin{aligned} \hat{\delta}_n &= \alpha_n \hat{\delta}_{1,n} + (1-\alpha_n)\hat{\delta}_{2,n} \\ &= \hat{\delta}_{2,n} + \alpha_n(\hat{\delta}_{1,n} - \hat{\delta}_{2,n}) \qquad (2) \\ &\xrightarrow{P} \delta, \quad by\ Slutsky's\ theorem \end{aligned}$$

□

4.4.2 Mean Squared Error Consistency

Definition 4.3. Let $\hat{\delta}_1$, $\hat{\delta}_2$, ..., $\hat{\delta}_n$, .. be a sequence of estimators of a real valued parameter δ, the sequence $\{\hat{\delta}_n\}$ is defined to be a mean squared error consistent sequence of estimators of δ if and only if

$$\lim_{n\to\infty} E[(\hat{\delta}_n - \delta)^2] = 0$$

We say $\hat{\delta}_n$ is a MSE-consistent estimator of δ.

Theorem 4. [2] Let $\hat{\delta}_{1,n}$ and $\hat{\delta}_{2,n}$ be two sequences of MSE consistent estimators of δ. Consider the sequence of convex linear combinations defined by

$$\hat{\delta}_n = \alpha_n \hat{\delta}_{1,n} + (1-\alpha_n)\hat{\delta}_{2,n}$$

Such that,α_n are random variables $\in [0,1]$.

Then, $\hat{\delta}_n$ is MSE consistent estimator of δ.

Proof.

$$
\begin{aligned}
E[(\hat{\delta}_n - \delta)^2] &= E[(\alpha_n(\hat{\delta}_{1,n} - \delta) + (1-\alpha_n)(\hat{\delta}_{2,n} - \delta))^2] \\
&\leq E[2[(\alpha_n)^2(\hat{\delta}_{1,n} - \delta)^2 + (1-\alpha_n)^2(\hat{\delta}_{2,n} - \delta)^2]] \\
&\leq 2E[((\hat{\delta}_{1,n} - \delta)^2 + (\hat{\delta}_{2,n} - \delta)^2] \\
&\longrightarrow 0, \quad as\ n \longrightarrow \infty
\end{aligned}
\tag{3}
$$

So, $\hat{\delta}_n$ is a consistent estimator of δ □

4.5 Example

In this example, first, we will find the jump points of Simple weighted estimator, Zhao-Tsiatis estimator and the Wang estimator. Then, we will find the values of these estimators at the jump points to investigate which of these points assigned negative mass. Finally, we will use these data and the result by applying the procedure given in previous sections to find monotonized the Zhao-Tsiatis estimator and monotonized the Wang estimator.

i	X_i	Γ_{li}	$U(TH_i(t))$	$N(X_i)$
1	$[8-18]$	0	1	10
2	$[10-51]$	0	$\frac{1}{4}$	10.25
3	$[0-48]$	1	$\frac{1}{2}$	24
4	$[0-53]$	1	1	53

(a) The Simple weighted estimator jump points

The simple weighted estimator has jump points only when $\Gamma_{li} = 1$, so b is equal to 24 and 53.

(b) The Zhao-Tsiatis estimator jump points.

(i) When $b = N(X_i)$ and $\Gamma_{li} = 1$, the jump points are 24 and 53.

(ii) When $b = N(X_i)$ and $\Gamma_{li} = 0$, the jump points are 10 and 10.25.

(iii) To find the jump points of the third kind, we want to check if there exists an item i such that $N(X_i) < b$ and $\Gamma_{li} = 0$ and there exists another index j, such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt$$

and $X_j - sp_j > X_i - sp_i$.

The first item, which has $\Gamma_{li} = 0$ is X_1.

Now, X_2, X_3 and X_4 are all greater than X_1. We have to check which of them lead to jump point.

For X_2, we have

$b = \int_{10}^{18-8+10} \frac{1}{4} dt = 2.5$, and since $2.5 < 10$ b is not a jump point.

For X_3, we have

$b = \int_{0}^{18-8} \frac{1}{2} dt = 5$, and since $5 < 10$ b is not a jump point.

For X_4, we have

$b = \int_{0}^{18-8} 1 dt = 10$, and since 10 is not < 10 b is not a jump point.

The second item, which has $\Gamma_{li} = 0$ is X_2.

Now, X_3 and X_4 are all greater than X_2. We have to check which of them lead to jump point.

For X_3, we have

$b = \int_{0}^{51-10} \frac{1}{2} dt = 20.5$, and since $20.5 > 10.25$, b is a jump point.

For X_4, we have

$b = \int_{0}^{51-10} 1 dt = 41$, and since $41 > 10.25$, b is a jump point.

The jump points of Zhao-Tsiatis estimator are

$10, 10.25, 20.5, 24, 41$ and 53.

(c) The Wang estimator jump point

(i) When $b = N(X_i)$ and $\Gamma_{li} = 1$, the jump points are 24 and 53.

(ii) When $b = N(X_i)$ and $\Gamma_{li} = 0$, the jump points are 10 and 10.25.

(iii) $b = N(X_i) + (sp_i - 0) + (L - X_i)$ and $\Gamma_{li} = 0$, the jump points are 53 and 22.25.

(iv) To find the jump points of this kind, we have to check if there exist an item i such that $N(X_i) < b < N(X_i) + (sp_i - 0) + (L - X_i)$ and $\Gamma_{li} = 0$, and there exist another index j, such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt$$

and $X_j - sp_j > X_i - sp_i$.

The first item, which has $\Gamma_{li} = 0$ is X_1.

Now, X_2, X_3 and X_4 are all greater than X_1. We have to check which of them lead to jump point.

For X_2, we have

$$b = \int_{10}^{18-8+10} \frac{1}{4}dt = 2.5$$

, and since 2.5 is not between 10 and 53 b is not a jump point.

For X_3, we have

$$b = \int_{0}^{18-8} \frac{1}{2}dt = 5$$

, and since 5 is not between 10 and 53 b is not a jump point.

For X_4, we have

$$b = \int_{0}^{18-8} 1dt = 10$$

, and since 10 is not between 10 and 53 b is not a jump point.

The second item, which has $\Gamma_{li} = 0$ is X_2.

Now, X_3 and X_4 are all greater than X_2. We have to check which of them lead to jump point.

For X_3, we have

$$b = \int_{0}^{51-10} \frac{1}{2}dt = 20.5$$

, and since 20.5 is between 10.25 and 22.25 b is a jump point.

For X_4, we have

$$b = \int_{0}^{51-10} 1dt = 41$$

, and since 41 is not between 10.25 and 22.25 b is not a jump point.

(v) To find the jump points of this kind, we have to check if there exist an item i such that $N(X_i) < b < N(X_i) + (sp_i - 0) + (L - X_i)$ and $\Gamma_{li} = 0$, and there exist another index j, such that

$$b = \int_{sp_j}^{X_i - sp_i + sp_j} U_j(TH(t))dt + (sp_i - 0) + (L - X_i)$$

$X_j - sp_j > X_i - sp_i$ and $N(X_j) + sp_j + (L - X_j) < b$.

The first item, which has $\Gamma_{li} = 0$ is X_1.

Now, X_2, X_3 and X_4 are all greater than X_1. We have to check which of them leads to a jump point.

For X_2, we have

$$b = \int_{10}^{18-8+10} \frac{1}{4}dt + (8 - 0) + (53 - 18) = 45.5$$

, and since $10.25 + (10 - 0) + (53 - 51) < b$ and $10 < b < 53$, b is a jump point.

For X_3, we have

$$b = \int_{0}^{18-8} \frac{1}{2}dt + (8 - 0) + (53 - 18) = 48$$

, and since $24 + 53 - 48 < b$ and $10 < b < 53$, b is a jump point.

For X_4, we have

$$b = \int_0^{18-8} 1dt + (8-0) + (53-18) = 53$$

, and since 53 is not less than b, b is not a jump point.

The second item, which has $\Gamma_{li} = 0$ is X_2.

Now, X_3 and X_4 are all greater than X_2. We have to check which of them leads to a jump point.

For X_3, we have

$$b = \int_0^{51-10} \frac{1}{2} dt + (10) + (53-51) = 32.5,$$

and since b is not between $10.25 < b < 22.25$, b is not a jump point.

For X_4, we have

$$b = \int_0^{51-10} 1dt + (10) + (53-51) = 53,$$

and since b is not between $10.25 < b < 22.25$, b is not a jump point.

The jump points of Wang estimator are

$24, 53, 22.25, 20.5, 45.5, 48, 10$ and 10.25.

The values of these estimators just before and just after each jump point are given in the next page.

Table 21: Values of estimators just before and after the jump points

b	$\hat{\delta_S}(b)$	$\hat{\delta_Z}(b)$	$\hat{\delta_W}(b)$
10-	1	1	1
10+	1	1	1
10.25-	1	1	1
10.25+	1	$\frac{2}{3}$	$\frac{2}{3}$
20.5-	1	$\frac{2}{3}$	$\frac{2}{3}$
20.5+	1	1	1
22.25-	1	1	1
22.25+	1	1	$\frac{2}{3}$
24-	1	1	$\frac{2}{3}$
24+	0.5	$\frac{1}{3}$	$\frac{1}{3}$
41-	0.5	$\frac{1}{3}$	$\frac{1}{3}$
41+	0.5	$\frac{1}{2}$	$\frac{1}{3}$
45.5-	0.5	$\frac{1}{2}$	$\frac{1}{3}$
45.5+	0.5	$\frac{1}{2}$	0.375
48-	0.5	$\frac{1}{2}$	0.375
48+	0.5	$\frac{1}{2}$	$\frac{1}{2}$
53-	0.5	$\frac{1}{2}$	$\frac{1}{2}$
53+	0	0	0

Now, want to apply the procedure that is given in 4.2 to find the monotonized

Zhao-Tsiatis estimator.

Let $P_1 = W_0 = 0$,

The jump points of Zhao-Tsiatis estimator in 4.5 are $10, 10.25, 20.5, 24, 41$, and 53.

Now, want to find the matrices Q and A

$$
\mathrm{Q} = \begin{pmatrix}
-(1-1) & 0 & 0 & 0 & 0 & 0 & 0 \\
(1-1) & -(1-1) & 0 & 0 & 0 & 0 & 0 \\
0 & (1-1) & -(\frac{2}{3}-1) & 0 & 0 & 0 & 0 \\
0 & 0 & (\frac{2}{3}-1) & -(1-1) & 0 & 0 & 0 \\
0 & 0 & 0 & (1-1) & -(\frac{1}{3}-\frac{1}{2}) & 0 & 0 \\
0 & 0 & 0 & 0 & (\frac{1}{3}-\frac{1}{2}) & -(\frac{1}{3}-\frac{1}{2}) & 0 \\
0 & 0 & 0 & 0 & 0 & (\frac{1}{2}-\frac{1}{2}) & -(0-0) \\
0 & 0 & 0 & 0 & 0 & 0 & (0-0)
\end{pmatrix}
$$

$$
\mathrm{Q} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\text{A} = \begin{pmatrix} (1-1) \\ (1-1) \\ (1-1) \\ (1-1) \\ (\frac{1}{2}-1) \\ (\frac{1}{2}-\frac{1}{2}) \\ (0-\frac{1}{2}) \\ (0) \end{pmatrix}$$

$$\text{A} = \begin{pmatrix} (0) \\ (0) \\ (0) \\ (0) \\ (-\frac{1}{2}) \\ (0) \\ (-\frac{1}{2}) \\ (0) \end{pmatrix}$$

Now, by linear programming problem, we want to maximize

$$\sum_{i=1}^{N} K_i$$

Subject to the constraints

$$QY \geq A$$

Then,

$$\mathrm{Y} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Then, the montonized Zhao-Tsiatis estimator is :

$$\hat{\delta}_{MZ}(b) = \begin{cases} 1 & 0 \le b < 10 \\ 1 & 10 \le b < 10.25 \\ 1 & 10.25 \le b < 20.5 \\ 1 & 20.5 \le b < 24 \\ \frac{1}{2} & 24 \le b < 41 \\ \frac{1}{2} & 41 \le b < 53 \\ 0 & 53 \le b \end{cases}$$

Now, we want to apply the procedure that is given in 4.3 to find the monotonized Wang estimator.

Let $P_1 = W_0 = 0$.

The jump points of Wang estimator in section 4.5 are $10, 10.25, 20.5, 22.25, 24, 45.5, 48,$ and 53

Now, want to find the matrices Q and A

$$
Q = \begin{pmatrix}
-(1-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(1-1) & -(1-1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (1-1) & -(\frac{2}{3}-1) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & (\frac{2}{3}-1) & -(1-1) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (1-1) & -(\frac{2}{3}-1) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (\frac{2}{3}-1) & -(\frac{2}{3}-\frac{1}{2}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & (\frac{2}{3}-\frac{1}{2}) & -(\frac{3}{8}-\frac{1}{2}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & (\frac{3}{8}-\frac{1}{2}) & -(\frac{1}{2}-\frac{1}{2}) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (\frac{1}{2}-\frac{1}{2}) & -(0-0) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (0-0)
\end{pmatrix}
$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and,

$$A = \begin{pmatrix} (1-1) \\ (1-1) \\ (1-1) \\ (1-1) \\ (1-1) \\ (\frac{1}{2}-1) \\ (\frac{1}{2}-\frac{1}{2}) \\ (\frac{1}{2}-\frac{1}{2}) \\ (0-\frac{1}{2}) \\ (0) \end{pmatrix}$$

$$
\text{A} = \begin{pmatrix} (0) \\ (0) \\ (0) \\ (0) \\ (0) \\ (-\frac{1}{2}) \\ (0) \\ (0) \\ (-\frac{1}{2}) \\ (0) \end{pmatrix}
$$

Now, by linear programming problem, we want to maximize

$$
\sum_{i=1}^{N} K_i
$$

Subject to the constraints

$$
QY \geq A
$$

Then,

$$
\text{Y} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}
$$

Then, the montonized Wang estimator is :

$$\hat{\delta}_{MW}(b) = \begin{cases} 1 & 0 \leq b < 10 \\ 1 & 10 \leq b < 10.25 \\ 1 & 10.25 \leq b < 20.5 \\ 1 & 20.5 \leq b < 22.25 \\ \frac{2}{3} & 22.25 \leq b < 24 \\ \frac{1}{2} & 24 \leq b < 45.5 \\ \frac{1}{2} & 45.5 \leq b < 48 \\ \frac{1}{2} & 48 \leq b < 53 \\ 0 & 53 \leq b \end{cases}$$

5 SIMULATION RESULT

In this simulation study, we will estimate and compare the mean squared errors of the simple weighted estimator, the Zhao-Tsiatis estimator, the Wang estimator, the monotonized Zhao-Tsiatis estimator, and the monotonized Wang estimator by R-code for the left and right censorship case. This study is similar to the study by Gelber (1989), Zhao-Tsiatis (1999), Wang (2001) and Almanassra et al(2005). In this simulation we will assume that the time to follow up is uniformly distribution FU $\in [0, 74]$, time to relapse is exponential distribution $TR \sim exp(1/100)$, and time of toxicity is uniform distribution on $[0, TOX2]([0, 50])$, TOX $\sim U[0, 50]$. The true survival function of time without symptoms of disease and toxicity is given by Gelber (1989)

$$Pr(TWiST > b) = \begin{cases} \frac{1}{\lambda TOX2} \exp^{-\lambda b}(1 - \exp^{-\lambda TOX2}) & 0 \leq b < L - TOX2 \\ \frac{1}{\lambda TOX2}(\exp^{-\lambda b} - \exp^{-\lambda L}) & L - TOX2 \leq b < L \end{cases}$$

To compute all estimators, we need

$$T_i = TR_i \wedge L, \Gamma_{li} = I(T_i \leq FU_i), X_i = T_i \wedge FU_i$$

$$N(X_i) = X_i - TOX_i$$

note that, if

$$TOX_i > X_i, \quad then \quad N(X_i) = TR_i$$

To compute simple weighted estimator we need to use

$$X_i, \Gamma_{li}$$

To compute Zhao-Tsiatis and monotonized Zhao-Tsiatis estimators we have to use

$$m_l(b) = TOX_i + b \qquad T_i(b) = m_l(b) \wedge T_i$$

$$\Gamma_{li} = I(T_i(b) \leq FU_i) \qquad X_i(b) = T_i(b) \wedge FU_i$$

To compute Wang and monotonized Wang estimators we have to use if $N(X_i) < b$ and $L - TOX_i \leq b$

$$z_{li}(b) = L - b \qquad T_i'(b) = z_{li} \wedge T_i$$

$$\Gamma_{li}' = I(T_i'(b) \leq FU_i) \qquad X_i'(b) = T_i'(b) \wedge FU_i$$

Also, let the artificial end point $L = 60$.

We consider different sample sizes, $n = 10, 15, 20, 30, 50$. and the number of simulations = 1000

The following table show Comparison of the mean squared error of the simple weighted estimator(MSES), Zhao-Tsiatis estimator (MSEZ), Wang estimator (MSEW), monotonized Zhao-Tsiatis estimator (MSESZ) and monotonized Wang estimator (MSESW) for $n = 10$

b	True sur.	MSES	MSEZ	MSEW	MSESZ	MSESW
0	0.8178	0.07474304	0.02998773	0.02998773	0.0300185	0.02957844
3	0.7976	0.07849751	0.02746582	0.02746582	0.02639179	0.02666184
6	0.7779	0.08158703	0.02961081	0.02961081	0.02836042	0.02869708
9	0.7587	0.09316172	0.03325343	0.03325343	0.03135574	0.03205016
12	0.7159	0.1011616	0.04105563	0.04108205	0.04004778	0.03891603
15	0.6623	0.1075859	0.04536215	0.04407593	0.04418199	0.04355351
18	0.6100	0.1118745	0.05560707	0.05481336	0.05519371	0.0539776
21	0.5590	0.1083168	0.05520278	0.0506974	0.05388446	0.05289561
24	0.5092	0.1067979	0.05860288	0.05526516	0.05893196	0.05605302
27	0.4607	0.1119426	0.06797546	0.05808105	0.06485643	0.06422873
30	0.4134	0.1044183	0.06409174	0.05364102	0.06204535	0.05974146
33	0.3672	0.09364902	0.06749764	0.05200818	0.06098931	0.06134308
36	0.3222	0.09427061	0.06771519	0.05018769	0.06113579	0.06032148
39	0.2783	0.07838869	0.06110788	0.04507448	0.05177513	0.05637079
41	0.2497	0.07351871	0.05830512	0.04169339	0.0471743	0.05340933
45	0.1938	0.05752572	0.0509244	0.03415468	0.03558673	0.03477207
48	0.1530	0.04203092	0.03871217	0.03220218	0.02594359	0.02620825
51	0.1133	0.03529899	0.0344003	0.03048028	0.02326391	0.02398314
54	0.0746	0.03433695	0.02447362	0.02989625	0.01291472	0.02797661
57	0.0368	0.02227016	0.01555561	0.0215349	0.016241666	0.02169442

The following table show Comparison of the mean squared error of the simple weighted estimator(MSES), Zhao-Tsiatis estimator (MSEZ), Wang estimator (MSEW), monotonized Zhao-Tsiatis estimator (MSESZ) and monotonized Wang estimator (MSESW) for $n = 15$

b	True sur.	MSES	MSEZ	MSEW	MSESZ	MSESW
0	0.8178	0.04675011	0.0291764	0.0291764	0.02961733	0.02907438
3	0.7976	0.04809113	0.02387311	0.02387311	0.02325001	0.02371659
6	0.7779	0.05140584	0.02152136	0.02152136	0.02086471	0.02089834
9	0.7587	0.0528319	0.02188569	0.02188569	0.02047801	0.02115797
12	0.7159	0.0653995	0.02760593	0.02753584	0.02601714	0.02630142
15	0.6623	0.06708398	0.03119169	0.03031369	0.03077021	0.02984634
18	0.6100	0.0740463	0.03358691	0.031277	0.03302156	0.03143411
21	0.5590	0.07998414	0.04038658	0.03596618	0.04129995	0.03705215
24	0.5092	0.07598594	0.04118871	0.03453249	0.04141911	0.03603401
27	0.4607	0.07494696	0.04187838	0.03285855	0.04157391	0.03509737
30	0.4134	0.07621592	0.04456445	0.03507865	0.04261589	0.03741765
33	0.3672	0.0741501	0.04682306	0.03737166	0.04697927	0.04167721
36	0.3222	0.07337239	0.04904347	0.0339342	0.04633225	0.04133156
39	0.2783	0.06422287	0.04114563	0.03277678	0.03923355	0.03758155
41	0.2497	0.05872004	0.04095602	0.03147052	0.03734776	0.03568411
45	0.1938	0.04865322	0.03678538	0.03007003	0.03043311	0.03330404
48	0.1530	0.04490944	0.03489118	0.02848593	0.0274471	0.03257862
51	0.1133	0.03007611	0.02606333	0.0246128	0.01491534	0.02555897
54	0.0746	0.02265075	0.01803265	0.01714587	0.019986163	0.01878407
57	0.0368	0.020888565	0.015537954	0.01955409	0.015156999	0.01963702

The following table show Comparison of the mean squared error of the simple weighted estimator(MSES), Zhao-Tsiatis estimator (MSEZ), Wang estimator (MSEW), monotonized Zhao-Tsiatis estimator (MSESZ) and monotonized Wang estimator (MSESW) for $n = 20$

b	True sur.	MSES	MSEZ	MSEW	MSESZ	MSESW
0	0.8178	0.04082473	0.02908122	0.02908122	0.02970224	0.02906971
3	0.7976	0.03714504	0.02267346	0.02267346	0.02258738	0.02270502
6	0.7779	0.03595786	0.01908461	0.01908461	0.01850389	0.01893166
9	0.7587	0.03235296	0.01736576	0.01736576	0.01677299	0.01719787
12	0.7159	0.04223605	0.02073693	0.02066468	0.01976015	0.02020693
15	0.6623	0.04645553	0.0234313	0.02295615	0.02242107	0.02263503
18	0.6100	0.05444374	0.02619495	0.02366995	0.02572323	0.02353724
21	0.5590	0.05709521	0.03063978	0.02882684	0.03102279	0.02878885
24	0.5092	0.05924736	0.03306308	0.02765061	0.03255519	0.02884233
27	0.4607	0.06061138	0.03226736	0.02622823	0.03264801	0.02688132
30	0.4134	0.06106128	0.034067	0.02720181	0.03507911	0.02803506
33	0.3672	0.0551696	0.03444384	0.02561906	0.03355794	0.02689924
36	0.3222	0.05304117	0.03248652	0.02553578	0.03309506	0.02700972
39	0.2783	0.0502455	0.03266887	0.02466695	0.03027919	0.02645267
41	0.2497	0.04835991	0.03069072	0.02475982	0.02893943	0.02581134
45	0.1938	0.04193998	0.03087814	0.02655465	0.02626251	0.02689724
48	0.1530	0.03471768	0.02428041	0.02334133	0.01992209	0.02381572
51	0.1133	0.02513728	0.02120543	0.02426464	0.01525463	0.02395845
54	0.0746	0.01952218	0.01601677	0.02401682	0.01113757	0.02154525
57	0.0368	0.018558461	0.010678318	0.01751105	0.010313428	0.01514554

The following table show Comparison of the mean squared error of the simple weighted estimator(MSES), Zhao-Tsiatis estimator (MSEZ), Wang estimator (MSEW), monotonized Zhao-Tsiatis estimator (MSESZ) and monotonized Wang estimator (MSESW) at $n = 30$

b	True sur.	MSES	MSEZ	MSEW	MSESZ	MSESW
0	0.8178	0.0355016	0.02961865	0.02961865	0.03048181	0.02961865
3	0.7976	0.02922777	0.02297436	0.02297436	0.02328654	0.02293198
6	0.7779	0.02294433	0.01707642	0.01707642	0.01722794	0.0170529
9	0.7587	0.02067176	0.01304471	0.01304471	0.01312092	0.0130018
12	0.7159	0.02091847	0.01431522	0.01390405	0.01384239	0.01368824
15	0.6623	0.02468344	0.01663155	0.01541601	0.01630785	0.01560167
18	0.6100	0.02894441	0.01819369	0.01637553	0.01815756	0.01650236
21	0.5590	0.03552134	0.02187027	0.01870148	0.02196749	0.01853007
24	0.5092	0.04100054	0.02242276	0.01842202	0.02232884	0.01868938
27	0.4607	0.04099474	0.02477909	0.02002884	0.02537755	0.02007539
30	0.4134	0.04054604	0.02349115	0.01779868	0.0238571	0.01807577
33	0.3672	0.03920215	0.02404801	0.01914086	0.02397961	0.01891945
36	0.3222	0.03676045	0.02164463	0.01800792	0.0213065	0.01753236
39	0.2783	0.03656822	0.02407256	0.01940639	0.02305503	0.01880079
41	0.2497	0.03395702	0.02258005	0.01931965	0.02088437	0.01842714
45	0.1938	0.0275423	0.0184743	0.0212554	0.01624212	0.01881472
48	0.1530	0.02665371	0.01948081	0.02137115	0.01645566	0.01941848
51	0.1133	0.0180669	0.01323209	0.02297249	0.01098314	0.0186227
54	0.0746	0.01401586	0.01161476	0.02428067	0.010580321	0.02036471
57	0.0368	0.012207065	0.006107359	0.008943084	0.003696507	0.006126092

The following table show Comparison of the mean squared error of the simple weighted estimator(MSES), Zhao-Tsiatis estimator (MSEZ), Wang estimator (MSEW), monotonized Zhao-Tsiatis estimator (MSESZ) and monotonized Wang estimator (MSESW) at $n = 50$

b	True sur.	MSES	MSEZ	MSEW	MSESZ	MSESW
0	0.8178	0.03337025	0.03137736	0.03137736	0.03177096	0.03137736
3	0.7976	0.02398904	0.02247208	0.02247208	0.02268489	0.02247208
6	0.7779	0.01789944	0.01559886	0.01559886	0.01591248	0.01558766
9	0.7587	0.01271222	0.01087363	0.01087363	0.01097617	0.01085242
12	0.7159	0.01251911	0.01078378	0.0105262	0.01076677	0.0105893
15	0.6623	0.01552099	0.01125635	0.01035131	0.01124896	0.01061229
18	0.6100	0.01630242	0.0116321	0.01046304	0.01198717	0.01066491
21	0.5590	0.01845939	0.01283528	0.01087782	0.01261448	0.01079979
24	0.5092	0.01976126	0.01332439	0.01126718	0.01389321	0.01108469
27	0.4607	0.02157584	0.01464324	0.01173391	0.01498568	0.01153268
30	0.4134	0.02186773	0.01466008	0.01199313	0.01474067	0.01130298
33	0.3672	0.02272172	0.0148191	0.01284899	0.01503883	0.01170523
36	0.3222	0.02093977	0.01361451	0.01251383	0.01314299	0.01152505
39	0.2783	0.02116143	0.01287048	0.01385472	0.01303557	0.01222951
41	0.2497	0.01910735	0.01238812	0.01489535	0.01213889	0.01251581
45	0.1938	0.01697581	0.01115664	0.01883446	0.01041547	0.01368206
48	0.1530	0.01435861	0.010030555	0.01053955	0.010041622	0.01050756
51	0.1133	0.01194778	0.009508963	0.02202022	0.008147243	0.01729751
54	0.0746	0.008020729	0.006685687	0.002351741	0.005395756	0.001913887
57	0.0368	0.004104982	0.003923631	0.002909812	0.002562419	0.002296743

6 Conclusion

From the results of our case study, as expected the new two estimators monotonized Zhao-Tsiatis estimator and monotonized Wang estimator, the MSE is smaller than MSE for simple weighted estimator in general. Also, as expected the Zhao-Tsiatis and Wang estimators have smaller MSE than simple weighted estimator. Therefore, they are better estimators for the survival function of restricted quality adjusted lifetime.

In general Wang estimator has smaller MSE than the Zhao-Tsiatis estimator. The MSE for both the monotonized Wang and Zhao-Tsiatis estimator are very close to the Wang and Zhao-Tsiatis estimator respectively. Also, both the monotonized of Zhao-Tsiatis and monotonized Wang estimators are monotonic functions and hence they are proper survival functions. While, the Wang and Zhao-Tsiatis estimators are not monotonic.

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ملخص

المقدرات الموزونة لاقترانات البقاء والخطر لزمن الحياة المعدل

همام عبد الحليم حافظ (حمد خليل)

أصبح اقتران البقاء بعد اخذ عين الاعتبار عدد السنوات المصححة بجودة الحياة والذي يعني مقياس عبء المريض متضمنا جودة وكمية الحياة، أكثر أهمية في الدراسات التي تهتم فقط بالعمر الكلي من غير اهتمام بجودة الحياة، ويرجع السبب في ذلك الى أن الباحث يحتاج الوقت الحقيقي لحياة الشخص المعني بدراسته. في هذا العمل، سوف نقدر اقتران البقاء المصحح بجودة الحياة اذا كانت البيانات للشخص مفقودة من وقت بدء دراسته الى وقت دخوله. ايضا قدرنا اقتران الخطر المعدل بجودة الحياة استنادا على اقتران البقاء. وقد اجرينا دراسة محاكاة بإستخدام برنامج آر لمقارنة كفاءة المقدرات بالنسبة الى المقدر الحقيقي.