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The Explicit Solution to the Countable Systems of Linear Ordinary Differential Equations with lower Tridiagonal Constant Coefficients Matrix

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Dedication

Thanks and praise to Allah who gave me life

**To the messenger of knowledge and holy religion –prophet Mohammed
To my dear lovely country, Palestine.**

To whom holds me, cares about my tiny inner feelings, let me satisfied at

any time, my mother .

To whom struggles, leaves his rest time to afford every need for me and my sisters and brothers, my father.

To my dear sisters and brothers who help me always remember to hold my hand towards the right way to future.

To whom taught me the best way to increase my knowledge and get me the highest point of education and light my dear teachers and lectures.

To those Who make the world special just by being in it, to my friends.

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Abstract

The Explicit Solution to the Countable Systems of Linear Ordinary Differential Equations with lower tridiagonal Constant Coefficients matrix

By

Haneen Waleed Kamel Zaghloul

In this thesis, we have derived an explicit solution to an infinite countable system of linear ordinary differential equations with constant coefficients. We applied some mathematical techniques and we have found the solution to the system under consideration when the coefficients matrix is lower tridiagonal, and lower triangular. A matlab program is prepared to illustrate results.

Contents

Chapter 1

Introduction

Many problems in our daily life can be formulated by ordinary differential equations (ODEs). Finding the solutions to these ODEs is of interest of many researchers during the last and the present centuries [17].

Initial value problems and boundary value problems are two types of ODEs [23]. Some are infinite or finite systems of ODEs. Going into detail on all of the various types of ODEs and how to solve them would take a large amount of time and research, so the reader is encouraged to view the following texts, as an example, for more information regarding the subject they wish to learn more about: Ascher and Petzold, 1998 [19]; Press et al., 2007 [20]; Hairer et al., 2009; Hairer and Wanner, 2010 [26]; LeVeque, 2007 [14]; Boyes, etc. [12] and [8] are good reference for more information on linear ODE. The author discusses the solutions and their properties of the linear DE in detail. To learn more about Nonlinear ODEs check Peter J. Olver [18]. The solutions and their properties are properly discussed in detail. Finite and Infinite systems of linear ODE are being discussed by many researchers around the world in many different fields of science, such as mathematics, physics, engineering, etc. For example Perdsidski, K.P., 1959; Zautykov, O.A., 1965; and Zbigniew Bartosiewicz and Dorota Mozyrska, 2005. Infinitely many variables are to be determined when infinite system of ODE are involved,

and therefore, the state space of the system is infinitely determined. Finding the solutions for the infinite systems is a very difficult task, and many theoretical and numerical tools have been developed to help in this task. The infinite matrix theory has been used to provide approximations of solutions system of ODEs. Almanassra *et. Al.* (2014) [1] discussed solutions to countable linear systems of ODE with constant coefficients. Infinite system of ODE are also very important in representing many physical models. In the last and present centuries, many analytical and numerical approaches have been and are still being developed to provide solutions of linear systems. [14] Several advanced numerical algorithms which solve DE are available as (open-source) computer codes, written in programming languages such as MATLAB, Fortan, or C. Nowadays, a lot of work is done by using the R-code for solving ODE.(Solving Differential Equations in R, Karline Soetaert, 2010) [22], [21] When linear and non-linear share the same structure of symmetry, linear system can be used to facilitate solving non-linear ones. [10] Infinite-dimensional linear systems appear naturally when studying control problems for systems modelled by linear ODEs. Many problems in dynamical systems can be written in the form of differential equations or infinite differential systems and lead to infinite linear systems. The infinite matrix theory has been used in many branches of classical mathematics such as infinite quadratic forms [15], integral equation, matrix transformations, differential equation, operators between sequence spaces [2].

The object of this thesis is to find a solution to infinite countable systems of linear equations with constant coefficients matrix. The solution of the infinite system is found when the coefficient matrices are diagonal, bi-diagonal, tri-diagonal, and lower diagonal matrices. General information about ODEs are presented in chapter two. In chapter three, an explicit solution to an infinite countable system of linear ODE for diagonal and bi-diagonal is given.

Also, MATLAB codes are presented. In chapter four, a solution of the infinite countable system of linear ODE is given when the coefficient matrix is tri-diagonal and lower diagonal. A MATLAB code is given for tri-diagonal cases.

Chapter 2

Preliminaries

Ordinary Differential Equations (ODEs) are basically used to give mathematical description of many phenomenas in different fields of science such as Physics, Chemistry, Biology, Medicine, Economics, and Engineering. ODEs allow one to study all the various kinds of evolutionary processes with the properties of determinacy, finite-dimensionality, and differentiability. Soon after the invention of differential and integral calculus, the study of ODEs began. The foundation stone for the study of differential equations was laid by Newton in 1671. Afterwards, he was followed by Leibnitz who coined the name differential equation in 1676 to denote the relationship between differentials dx and dy of two variables x and y. The fundamental law of motion in mechanics, known as Newton's second law, is a differential equation which describes the state of a system. The motion of a particle of mass m moving along a straight line under the influence of an external force $F(t, x, \acute{x})$ is described by the ODE [8], [9]

$$m\ddot{u} = F(t, x, \dot{x})$$

2.1 Classification of Ordinary Differential Equation

The general form of the ODE of order n is

$$F(x, y, \acute{y}, \ldots, y^{(n)}) = 0 \tag{2.1}$$

where y (dependent variable) is a function of x (independent variable) and $y^{(k)}$ is the k^{th} derivative of y.

The general linear ODE of order n can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)\acute{y} + a_0(x)y = g(x). \tag{2.2}$$

The coefficients $a_0(x), \ldots, a_{n-1}(x)$ and $g(x)$ can be zero or non-zero functions, constant or non-constant functions, linear or non-linear functions in x. Only the function, $y(x)$, and its derivatives are used in determining if the differential equation is linear.

If a differential equation cannot be written in the form (2.2), then it is called non-linear.

The ODEs $4x^2\ddot{\acute{y}}+12x\acute{y} = x^4$ and $2e^x\ddot{\acute{y}}+y\acute{y} = x^4$ are linear and non-linear ODEs, respectively.

2.2 First Order ODEs

When $n = 1$ Eq (2.1), became

$$f(x, y, \acute{y}) = 0 \tag{2.3}$$

it is called first order ODEs.

The most known general form of the first order linear ODE is

$$\acute{y} + p(x)y = g(x), \tag{2.4}$$

where p **and** g **are given functions of the independent variable** x**. In order to solve Eq (2.4), we multiply both side of the equation** $\mu(x) = e^{\int p(x)dx}$**.** μ **is called integrating factor and satisfies:** $\mu(x)\frac{dy}{dx} + p(x)\mu(x)y = \mu(x)g(x)$

The solution of the Eq (2.4) can be done by elementary integration methods and gives [11]

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)dx + c\right]$$

2.2.1 Separable Linear ODEs

Equation (2.3), first order ODEs, can be always written in the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{2.5}$$

In the case that M **is a function of** x **only and** N **is a function of** y **only, then this equation is said to be separable,**

$$M(x)dx + N(y)dy = 0 \tag{2.6}$$

Therefore, the general solution of the equation is:

$$\int N(y)dy = \int M(x)dx + c$$

and the initial condition $y(x_0) = y_0$ **is used to find** c **[16].**

2.2.2 Exact Linear ODEs

In this section, we consider a first-order ODE in the form

$$M(x,y) + N(x,y)\acute{y} = 0 \tag{2.7}$$

Such an equation is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$**.**

The exact differential equation is provided by the following theorem.

Theorem 2.2.1 *[13]*

Let the functions M, N, M_y, and N_x, where subscripts denote partial derivatives, be continuous in the rectangular region $R : \alpha < x < \beta, \gamma < y < \delta$. Then, Equation 2.7 is an exact differential equation in R, if and only if there exists a function ψ satisfying Equations $\frac{\partial \psi}{\partial x}(x,y) = M(x,y)$, $\qquad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$,

(i.e $\psi_x(x,y) = M(x,y)$, $\psi_y(x,y) = N(x,y)$ if and only if M and N satisfy Eq (2.7))

the general form of ψ **is given by**

$$\psi(x,y) = \int M(x,y)dx + g(y)$$

or

$$\psi(x,y) = \int N(x,y)dy + h(x)$$

2.2.3 Existence and Uniqueness of First Order Linear ODEs

The existence and uniqueness theory appeared nearly 200 years after the development of the theory of differential equations. Due to the lack of a general formula for solving nonlinear ODEs, the study of existence and

uniqueness of solutions became important. We write the IVP of first order ODE as

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0 \tag{2.8}$$

In the following theorem we present the condition of the uniqueness and existence solution of Eq (2.8) when $t_0 = 0$, $y_0 = 0$.

Theorem 2.2.2 *[8]*

If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R : |t| \leq a$, $|y| \leq b$, then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the initial value problem $\acute{y} = f(t, y), \quad y(0) = 0$.

The fundamentally important question of existence and uniqueness of solution for IVP was first answered by Rudolf Lipschitz in 1876. In 1886, Giuseppe Peano discovered that the IVP (2.8) has a solution. It may not be unique if f is a continuous function of (x, y). Peano extended this theorem for system of first order ODE using the method of successive approximation in the year 1890. Also in the same year, Charles Emile Picard and Ernst Leonard Lindelöf presented the existence and uniqueness theorem for the solutions of IVP Eq (2.8).

The existence and uniqueness theorem is given by:

Theorem 2.2.3 *[9] Let $f(x, y)$ in 2.8 be continuous on a rectangle $D = \{(x, y) : x_0 - \delta < x < x_0 + \delta; y_0 - b < y < y_0 + b\}$ then there exists a solution in D. Furthermore, if $f(x, y)$ is Lipschitz continuous with respect to y on a rectangle R (possibly smaller than D) given as $R = \{(x, y) : x_0 - a < x < x_0 + a; y_0 - b < y < y_0 + b, a < \delta\}$, the solution in R shall be unique.*

.

In the following, we present the most important linear ODEs, which are the Bernoulli and Riccati equations:

1. **Bernoulli equation** [4]

   Certain nonlinear ODEs can be reduced to linear form. The practically most famous non-linear equation that can be reduced to linear one is Bernoulli equation, which has the form

$$\acute{y} + p(t) = g(t)y^{\alpha} \tag{2.9}$$

   where $p(t)$ and $g(t)$ are arbitrary functions of t and α is any number. This equation has two elementary cases: when $\alpha = 0$ the equation is linear and when $\alpha = 1$ the equation is separable. For all other values of α, dividing Eq (2.9) through by y^{α} suggests the substitution

   $v = y^{1-\alpha}$. The new differential equation for $v(t)$, $\acute{v} = (1-\alpha)p(t)v + (1-\alpha)q(t)$ is solvable because it is linear in $v(t)$.

2. **Riccati equation** [16]

   A differential equation of Riccati type is

   $\acute{y} = p(t)y + g(t)y^2 + h(t)$. Solving this equation is only possible if a particular solution $y_p(t)$ is known. then the transformation $x = \frac{1}{y - y_p(t)}$ yields the linear equation

$$\acute{y} = -(p(t) + 2y_p(t)g(t1)y - g(t). \tag{2.10}$$

   Eq (2.10) can be solved using the technique presented before

These are only a few of the most important equations which can be explicitly solved using some clever transformations.

2.3 Second - Order Linear Ordinary Differential Equation

A second order ordinary differential equation has the form

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \tag{2.11}$$

where f **is some given function. The Eq (2.11) is said to be linear if the function** f **has the form**

$$f(x, y, \frac{dy}{dx}) = g(x) - p(x)\frac{dy}{dx} - q(x)y \tag{2.12}$$

that is, if f **is linear in** y **and** $\acute{y}$**. In Eq (2.12)** g**,** p**, and** q **are specified functions of the independent variable** t **but do not depend on y. In this case, we usually rewrite Eq (2.11) as**

$$y'' + p(x)\acute{y} + q(x)y = g(x) \tag{2.13}$$

A second order linear equation is said to be homogeneous if the term $g(x)$ **in Eq (2.13), is zero for all** x**. Otherwise, the equation is called non homogeneous.**

Linear equations of second order are of crucial importance in the study of differential equations for two main reasons. The first is that linear equations have a rich theoretical structure that underlies a number of systematic methods of solution. Further, a substantial portion of this structure and these methods are understandable at a fairly elementary mathematical level. In order to present the key ideas in the simplest possible context, we describe them in this section for second order equations. Another reason for the study of second-degree linear equations is that they are vital to any serious

investigation into the classic areas of Mathematical Physics [6].

In the rest of this section, the existence and uniqueness of second - order linear ODEs will be presented in the following theorem.

Theorem 2.3.1 *[3]*

Consider the initial value problem $\acute{\acute{y}}+p(x)\acute{y}+q(x)y=g(x)$ *with* $y(x_0)=y_0$ *and* $\acute{y}(x_0)=\acute{y}_0$*. If the functions* $p(x), q(x)$ *and* $g(t)$ *are continuous for an open interval* $I:\alpha<x<\beta$ *containing the point* $x=x_0$*. Then, there exists a unique solution* $y=\phi(x)$ *of the problem, and that this solution exists throughout the interval I.*

2.3.1 Wronskian Determinant with Second - Order Linear Equations

The representation of the general solution of a second order linear homogeneous differential equation is a linear combination of two solutions whose Wronskian is not zero is intimately related to the concept of linear independence of two functions. We will define the Wronskian determinant in the following argument:

If $y_1(x)$ **and** $y_2(x)$ **satisfy** $y''+p(x)\acute{y}+q(x)y=0$**, then the Wronskian determinant is given by**

$$W(y_1,y_2)=\begin{vmatrix} y_1(x) & y_2(x) \\ \grave{y}_1(x) & \grave{y}_2(x) \end{vmatrix}=y_1(x)\acute{y}_2(t)-y_2(x)\acute{y}_1(x).$$

If Wronskian determinant is not zero, then the general solution of $y''+p(t)\acute{y}+q(x)y=0$ **is given by** $y(x)=c_1y_1(x)+c_2y_2(x)$**. [24]**

2.3.2 Constant Coefficient Equations

The standard form of second order linear ODE, with constant coefficient, is

$$ay'' + b\acute{y} + cy = 0 \tag{2.14}$$

Where a, b and c are given constants. It turns out that this equation can always be solved easily using calculus information.

The equation $ar^2 + br + c = 0$ is called the characteristic equation for the differential equation. The general solution of Eq (2.14) can be obtained by finding the roots of the characteristic equation $ar^2 + br + c = 0$.

There are three possible cases for these roots and the solution depends on these cases.

Case 1 If the characteristic equation has two real and distinct roots r_1 and r_2, then the two linearly independent solutions are e^{r_1x} and e^{r_2x} and the general solution is given by

$$y = c_1e^{r_1x} + c_2e^{r_2x}.$$

Case 2 If the characteristic equation has one real root r, then the two linearly independent solutions are e^{rx} and xe^{rx} and the general solution is given by

$$y = c_1e^{rx} + c_2xe^{rx}$$

.

Case 3 If the characteristic equation has two complex roots $p \mp iq$, then the two linearly independent solutions are $e^{px}cosqx$ and $e^{px}sinqx$ and the general solution is given by

$$y = c_1e^{px}cosqx + c_2e^{px}sinqx$$

.

2.4 Higher Order Linear ODEs

Any linear higher order ODE is of the form

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)\acute{y} + p_0(x)y = g(x). \tag{2.15}$$

We assume that the functions $p_0, ..., p_n$ and g are continuous real-valued functions. Linear ODE of higher order has useful and interesting applications, just as the first-order ODE do. What is important is that we have to appeal to the theory of solving first order ODEs.

Theorem 2.4.1 *[25] If the coefficient functions $p_0(x), p_1(x), \ldots, p_{n-1}(x), p_n(x)$ are continuous and bounded on an open interval I containing t_0, then the equation has a uniqueness with initial condition solution on the interval I.*

Equation (2.15) is called homogeneous when $g(x) = 0$, and there are n linearly independent solution $y_1, y_2, \ldots, y_n$, whose linearly combination gives the general solution

$y_h(t) = c_1y_1 + c_2y_2 + \cdots + c_ny_n$. Eq (2.15) is called non homogeneous when $g(x) \neq 0$ and the solution is given by the sum $y_h(x) + y_p(x)$, where $y_h(x)$ is the solution of the homogeneous equation. The function $y_p(x)$ is the particular solution of the non homogeneous part of the equation.

In order to solve Eq (2.15) we have to solve the homogeneous and non-homogeneous part of the ODEs separately. When solving the homogeneous equation, we need to find the roots of the characteristic equation

$$a_nr^n + a_{n-1}r^{n-1} + \ldots + a_1r + a_0r = 0. \tag{2.16}$$

Its general solution can be determined by finding n linearly independent solutions of its characteristic Eq (2.16).

If $r_1, r_2, ..., r_n$ are different real solutions, then $e^{r_1 x}, e^{r_2 x}, ..., e^{r_n x}$ are linearly independent solutions producing a general solution. $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$. if $r_i = r_j = r$ $i, j = 1, ..., n$, then $e^{rt}, xe^{rt}, ..., x^n e^{rt}$ are linearly independent solutions that generate a general solution.

If $r = \lambda \pm \mu i$ are distinct complex conjugate roots, then $y_1 = e^{\lambda t} cos\mu t$ and $y_2 = e^{\lambda t} sin\mu t$ are solutions

2.5 Systems of ODEs

Differential equations are statements of equality relating a set of functions and their derivatives. We will consider differential equations in which there is just one independent variable.

The n^{th} order ODE is

$$y^{(n)} = f(t, y, \acute{y}, \ldots, y^{(n-1)}) \tag{2.17}$$

If we let $x_1 = y$, $x_2 = \acute{y}$, $\ldots, x_n = y^{(n-1)}$ then Eq (2.17) is converted to a system of n first order ODE:

$$\begin{aligned} \acute{x}_1 &= x_2 \\ \acute{x}_2 &= x_3 \\ &\vdots \\ \acute{x}_{n-1} &= x_n \end{aligned} \tag{2.18}$$

From Eq (2.17)

$$\acute{x}_n = f(t, x_1, x_2, \ldots, x_n) \tag{2.19}$$

Eqs (2.18) and (2.19) are special cases of the more general system

$$
\begin{aligned}
\acute{x}_1 &= f_1(t, x_1, x_2, \ldots, x_n) \\
\acute{x}_2 &= f_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
\acute{x}_n &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}
\tag{2.20}
$$

The system (2.20) is said to have a solution on the interval $I : \alpha < t < \beta$ if there exists a set of n functions

$$x_1 = \phi_1(t), x_2 = \phi_2(t), \ldots, x_n = \phi_n(t).$$

that satisfy Eq (2.20)

In addition to the given system of the ODEs, there may also be given initial conditions of the form

$$
\begin{aligned}
x_1(t_0) &= x_1 \\
x_2(t_0) &= x_2 \\
&\vdots \\
x_n(t_0) &= x_n
\end{aligned}
\tag{2.21}
$$

To solve the system of ODE, it is better to write it in matrix form. In the following section, Basic Theory of Systems of First Order Linear ODEs is presented.

The general theory of a system of n first order ODEs

$$\begin{aligned} \acute{x}_1 &= p_{11}(t)x_1 + \ldots + p_{1n}(t)x_n + g_1(t) \\ &\vdots \\ \acute{x}_n &= p_{n1}(t)x_1 + \ldots + p_{nn}(t)x_n + g_n(t) \end{aligned} \tag{2.22}$$

closely parallels to that of a single linear equation of n^{th} order, In order to write it in matrix form, we let $x_1 = \phi_1(t), \ldots, x_n = \phi_n(t)$ to be components of a vector $x = \phi(t)$; similarly, $g_1(t), \ldots, g_n(t)$ are components of a vector $g(t)$, and $p_{11}(t), ..., p_{nn}(t)$ are elements of an $n \times n$ matrix $P(t)$.

Eq (2.22) takes the form

$$\acute{x} = P(t)x + g(t), \tag{2.23}$$

where

$$x = [x_1, x_{2,} \ldots, x_n]^T$$

$$\acute{x} = [\acute{x}_1, \acute{x}_{2,} \ldots, \acute{x}_n]^T$$

$$\phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_n(t)]^T$$

$$g(t) = [g_1(t), g_2(t), \ldots, g_n(t)]$$

$$P(t) = \begin{bmatrix} P_{11}(t) \ldots\ldots P_{1n}(t) \\ \vdots \\ P_{n1}(t) \ldots\ldots P_{nn}(t) \end{bmatrix}$$

A vector $x = \phi(t)$ is said to be a solution of Eq (2.23) if its components satisfy the system of Eq (2.22). If $g(t) = 0$, then Eq (2.23) is said to be homogeneous. And the set becomes

$$\acute{x} = P(t)x. \tag{2.24}$$

The solution of (2.24) is given by the following theorem.

Theorem 2.5.1 *[5] If the vector functions x_1, $x_2, \dots x_n$ are solutions of the system (2.24), then the linear combination*

$c_1x_1(t) + c_2x_2 + \dots + c_nx_n$ *is also a solution for any constants* $c_1, c_2, \dots, c_n$

.

Consider the matrix $X(t)$ whose columns are the vectors $x_1(t), ..., x_n(t)$:

$$X(t) = \begin{bmatrix} x_{11}(t) \dots\dots x_{1n}(t) \\ \vdots \\ x_{n1}(t) \dots\dots x_{nn}(t) \end{bmatrix}$$

the columns of $X(t)$ are linearly independent for a given value of t if and only if $detX \neq 0$ for that value of t. The Wronskian determinant of the n solutions $x_1, ..., x_n$ is denoted by $W[x_1, ..., x_n]$; and is given $W[x_1, ..., x_n](t) = detX(t)$.

The solutions $x_1, ..., x_n$ are then linearly independent at a point if and only if $W[x_1, ..., x_n]$ is not zero there.

Theorem 2.5.2 *[5] If the vector functions $x_1, ..., x_n$ are linearly independent solutions of the system (2.24), for each point in the interval $\alpha < t < \beta$, then each solution $x = \phi(t)$of the system (2.24) can be expressed as a linear combination of $x_1, ..., x_n$,*

$\phi(t) = c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t)$, *in exactly one way.*

All expressions of this form are solutions of the system (2.24), by theorem (2.5.2) all solutions of Eq (2.24) can be written in the form

$\phi(t) = c_1x_1(t) + c_2x_2(t) + \dots + c_nx_n(t)$.

where $c_1, ..., c_n$ are arbitrary constants.

Theorem 2.5.3 *If $x_1, ..., x_n$are solutions of Eq.(2.24) on the interval $\alpha < t < \beta$, then* *in this interval* $W[x_1, ..., x_n](t)$ *either is identically zero or never vanishes*

The existence and uniqueness theorem of the solution of system is given in the following theorem.

Theorem 2.5.4 *[7] If the functions* $p_{11}, p_{12}, ..., p_{nn},\ g_1, ..., g_n$ *are continuous on an open interval* $I : \alpha < t < \beta$, $\forall t \in (\alpha, \beta)$ *then there exists a unique solution* $x_1 = \phi_1(t), ..., x_n = \phi_n(t)$ *of the system (2.22) that also satisfies the initial conditions (2.21), where* x_0 *is any point in* I *and* $x_1, ..., x_n$ *are any prescribed numbers.*

Chapter 3

Countable System of Ordinary Differential Equations for Constant Diagonal and lower Bi-diagonal Matrix

In this chapter, we will find an explicit solution to an infinite countable system of linear ordinary differential equations with constant coefficients matrix. The solution to the system under consideration when the coefficients matrix is diagonal matrix is given in section one. In section two, the solution is given when the system is lower bidiagonal matrix.

3.1 The Explicit Solution for Countable ODEs with Constant Diagonal Matrix

In this section, we consider the solution to the following system of ODEs:

$$\dot{n}(t) = An(t) \tag{3.1}$$

where A is an infinite diagonal matrix. Now, we will find the solution to system (3.1) when the matrix A is an infinite diagonal matrix given by:

$$A = \begin{bmatrix} -\delta_1 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & -\delta_2 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & -\delta_3 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & 0 & -\delta_k & \ddots & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -\delta_{k+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

and $n(t)$ is an infinite column vector in the form:

$$n(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ \vdots \\ n_k(t) \\ \vdots \end{bmatrix}$$

with $n_k(0)$ given as an initial condition, and $\delta_i \neq \delta_j$, if $i \neq j$.

The solution to this type of problem is trivial and is given by

$$n_k(t) = n_k(0)e^{-\delta_k t}$$

3.2 The Explicit Solution for Countable ODEs with Constant Lower Bi-diagonal Matrix

We will consider the system (3.1) that the case where A is an infinite lower bi-diagonal constant matrix where A is [1]:

$$A = \begin{bmatrix} -\delta_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ a_1 & -\delta_2 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & a_2 & -\delta_3 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & 0 & a_3 & -\delta_4 & \ddots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & 0 & a_k & -\delta_k & \ddots & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_{k+1} & -\delta_{k+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

the solution of the system is:

$$n_k(t) = e^{-\delta_k t} n_k(0) + a_{k-1} n_{k-1}(0) \left[\frac{e^{-\delta_k t}}{\delta_{k-1}-\delta_k} + \frac{e^{-\delta_{k-1} t}}{\delta_k - \delta_{k-1}} \right] +$$

$$a_{k-1} a_{k-2} n_{k-2}(0) \left[\frac{e^{-\delta_k t}}{(\delta_{k-1}-\sigma_k)(\delta_{k-2}-\delta_k)} + \frac{e^{-\delta_{k-1} t}}{(\delta_k-\sigma_{k-1})(\delta_{k-2}-\delta_{k-1})} + \frac{e^{-\delta_{k-2} t}}{(\delta_k-\delta_{k-2})(\delta_{k-1}-\delta_{k-2})} \right]$$

$+...+$

$$a_{k-1} a_{k-2} ... a_1 n_1(0) \left[\begin{array}{c} \frac{e^{-\delta_k t}}{(\delta_{k-1}-\delta_k)(\delta_{k-2}-\delta_k)...(\delta_1-\delta_k)} + \frac{e^{-\delta_{k-1} t}}{(\delta_k-\delta_{k-1})(\delta_{k-2}-\delta_{k-1})...(\delta_1-\delta_{k-1})} \\ +... + \frac{e^{-\delta_1 t}}{(\delta_k-\delta_1)(\delta_{k-1}-\delta_1)...(\delta_2-\delta_1)} \end{array} \right]$$

The prove of this theorem is given in [1].

This solution can be also written as:

$n_k(t) = \Sigma_{m=1}^{k} (\Sigma_{j=1}^{m} (\Pi_{i=j}^{k-1} \frac{a_i}{\delta_i - \delta_m}) n_j(0)) e^{-\delta_m t}$

Note: if $m = i$ make $m = k$.

3.2.1 The Mathlab code

```
function [summ]=coefficients(k)

disp('Enter a')

for i=1:k-1

a(i)=input('')

end

disp('Enter e')

for i=1:k

e(i)=input('')

end

disp('Enter n')

for i=1:k

n(i)=input('')

end

y=1

while (y~=0)

x=input('Enter the number you need n(t) x=')

summ=zeros(1,x);

for j=1:x-1

result=1;

for i=j:x-1

result=result*a(i)/(e(i)-e(x));

end
```

```
summm(1)= summm(1)+n(j)*result;

end

summm(1)= summm(1)+n(x);

for m=2:x

sub=e;

sub(x-m+1)=e(x);

for j=1:x-m+1

result=1;

for i=j:x-1

result=result*a(i)/(sub(i)-e(x-m+1));

end

summm(m)=summm(m)+result*n(j);

end

end

summm

y=input('Enter the number (any number y)=');

end
```

Chapter 4

Countable System of Ordinary Differential Equations for Constant Lower Tri-diagonal and Lower Triangular Matrix

In this chapter, we will find an explicit solution to an infinite countable system of linear ordinary differential equations with constant coefficients. In section (4.1) we will find the solution system (3.1) where A is tri-diagonal matrix. Also, the solution of the same system (3.1) when A is lower diagonal matrix is in section (4.2). We will apply some mathematical techniques and the Mathematical Induction in order to reach these goals.

4.1 The explicit Solution for Countable ODEs for Constant Lower Tri-diagonal Matrix

Consider system (3.1) where A is an infinite lower tri-diagonal matrix given by:

$$A = \begin{bmatrix} -\delta_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ a_1 & -\delta_2 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ b_1 & a_2 & -\delta_3 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & b_2 & a_3 & -\delta_4 & \ddots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & b_k & a_k & -\delta_k & \ddots & 0 & \cdots \\ 0 & 0 & 0 & 0 & b_{k+1} & a_{k+1} & -\delta_{k+1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \ddots & \cdots \end{bmatrix}.$$

The coefficients δ_i, a_j, b_h are constants, and $\delta_i \neq \delta_j$, if $i \neq j$.

In the following, we present and prove a lemma that is used to derive the solution of the system

Lemma 4.1.1 *Let* $A = \{\gamma_i \in R : i = 1, 2, \ldots\}$ *be an infinite sequence, such that for* $i \neq j$ *and* $i, j \in N, \gamma_i \neq \gamma_j$. *Let* $\{\lambda_1, \lambda_2, \ldots\ldots, \lambda_n\}$ *be any finite subset of size* $n \geq 2$ *of* A. *Then, for all* $n \geq 2$,

$$\begin{aligned} &\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\ldots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\ldots(\lambda_n-\lambda_1)} + \\ &\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\ldots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\ldots(\lambda_n-\lambda_2)} + \ldots + \\ &\frac{1}{(\lambda_1-\lambda_{n-1})(\lambda_2-\lambda_{n-1})(\lambda_3-\lambda_{n-1})\ldots(\lambda_i-\lambda_{n-1})(\lambda_{i+1}-\lambda_{n-1})\ldots(\lambda_n-\lambda_{n-1})} \\ = \; &\frac{-1}{(\lambda_1-\lambda_n)(\lambda_2-\lambda_n)(\lambda_3-\lambda_n)\ldots(\lambda_i-\lambda_n)(\lambda_{i+1}-\lambda_n)\ldots(\lambda_{n-1}-\lambda_n)} \end{aligned} \tag{4.1}$$

Proof: We will prove this lemma by mathematical induction.

Let S be the set of all $n \in N$ for which the formula is true

First, we prove lemma (4.1.1) for $n = 2$. for any $i \neq j$.

Then,

$$\frac{1}{\lambda_i - \lambda_j} = \frac{-1}{(\lambda_i - \lambda_j)}$$

.

Assume it is true for $n = m$. That is, let $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ be any subset of size m of S.

We have

$$\begin{aligned}
&\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\ldots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\ldots(\lambda_m-\lambda_1)} + \\
&\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\ldots(\lambda_i-\lambda_2)(\lambda_{i+1}-\mu_2)\ldots(\lambda_m-\lambda_2)} + \ldots + \\
&\frac{1}{(\lambda_1-\lambda_{m-1})(\lambda_2-\lambda_{m-1})(\lambda_3-\lambda_{m-1})\ldots(\lambda_i-\lambda_{m-1})(\lambda_{i+1}-\lambda_{m-1})\ldots(\lambda_m-\lambda_{m-1})} \\
= \; &\frac{-1}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)} \qquad (4.2)
\end{aligned}$$

Multiply both sides of Eq (4.2) by

$(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)$.

We get

$$\begin{aligned}
&\frac{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\ldots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\ldots(\lambda_m-\lambda_1)} \\
&+\frac{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\ldots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\ldots(\lambda_m-\lambda_2)} \\
&+\ldots+ \\
&\frac{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)}{(\lambda_1-\lambda_{m-1})(\lambda_2-\lambda_{m-1})(\lambda_3-\lambda_{m-1})\ldots(\lambda_i-\lambda_{m-1})(\lambda_{i+1}-\lambda_{m-1})\ldots(\lambda_m-\lambda_{m-1})} \\
&= -1 \qquad (4.3)
\end{aligned}$$

After the cancellation of similar terms in Eq(4.3) and multiplying by (-1).

We get

$$\begin{aligned}
&\frac{(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)(\lambda_4-\lambda_m)\dots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\dots(\lambda_{m-1}-\lambda_m)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\dots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\dots(\lambda_{m-1}-\lambda_1)}\\
&+\frac{(\lambda_1-\lambda_m)(\lambda_3-\lambda_m)(\lambda_4-\lambda_m)\dots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\dots(\lambda_{m-1}-\lambda_m)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\dots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\dots(\lambda_{m-1}-\lambda_2)}\\
&+\dots+\\
&\frac{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\dots(\lambda_i-\lambda_m)\dots(\lambda_{m-2}-\lambda_m)}{(\lambda_1-\lambda_{m-1})(\lambda_2-\lambda_{m-1})(\lambda_3-\lambda_{m-1})\dots(\lambda_i-\lambda_{m-1})\dots(\lambda_{m-2}-\lambda_{m-1})}\\
&=1
\end{aligned} \tag{4.4}$$

Now consider $n = m + 1$ we have to prove the identity

$$\begin{aligned}
&\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\dots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\dots(\lambda_{m+1}-\lambda_1)}\\
&+\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\dots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\dots(\lambda_{m+1}-\lambda_2)}\\
&+\dots+\\
&\frac{1}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\dots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\dots(\lambda_{m+1}-\lambda_m)}\\
&=\frac{-1}{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\dots(\lambda_m-\lambda_{m+1})}
\end{aligned} \tag{4.5}$$

Multiply both sides of Eq (4.5) by

$(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\dots(\lambda_m-\lambda_{m+1})$.

We get

$$
\begin{aligned}
&\frac{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\ldots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\ldots(\lambda_{m+1}-\lambda_1)}\\
&+\frac{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\ldots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\ldots(\lambda_{m+1}-\lambda_2)}\\
&+\ldots+\\
&\frac{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m+1}-\lambda_m)}\\
&=\frac{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots.(\lambda_m-\lambda_{m+1})}
\end{aligned}
\tag{4.6}
$$

After the cancellation of similar terms in Eq(4.6) and multiplying by (-1).

We get

$$
\begin{aligned}
&\frac{(\lambda_2-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\ldots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\ldots(\lambda_m-\lambda_1)}\\
&+\frac{(\lambda_1-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\ldots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\ldots(\lambda_m-\lambda_2)}\\
&+\ldots+\\
&\frac{(\lambda_1-\lambda_{m+1})(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_{m-1}-\lambda_{m+1})}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)}\\
&=1
\end{aligned}
\tag{4.7}
$$

Now, we move the right hand side of Eq (4.2) to the left hand side

to get

$$
\begin{aligned}
&\frac{1}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)\dots(\lambda_i-\lambda_1)(\lambda_{i+1}-\lambda_1)\dots(\lambda_m-\lambda_a)}\\
&+\frac{1}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\dots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\dots(\lambda_m-\lambda_2)}\\
&+\dots+\\
&\frac{1}{(\lambda_1-\lambda_{m-1})(\lambda_2-\lambda_{m-1})\dots(\lambda_i-\lambda_{m-1})(\lambda_{i+1}-\lambda_{m-1})\dots(\lambda_m-\lambda_{m-1})}\\
&+\frac{1}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\dots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\dots.(\lambda_{m-1}-\lambda_m)}\\
&=0
\end{aligned}
\tag{4.8}
$$

Then, we multiply Eq (4.8) by

$(\lambda_2-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\dots(\lambda_m-\lambda_{m+1})$.

We get

$$
\begin{aligned}
&\frac{(\lambda_2-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})\dots(\lambda_m-\lambda_{m+1})}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\dots(\lambda_i-\lambda_2)\dots(\lambda_m-\lambda_1)}\\
&=-(\lambda_2-\lambda_{m+1})\frac{(\lambda_3-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})\dots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\dots(\lambda_i-\lambda_2)\dots(\lambda_m-\lambda_2)}\\
&-\dots-\\
&-(\lambda_{m-1}-\lambda_{m+1})\frac{(\lambda_2-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})\dots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_{m-1})(\lambda_2-\lambda_{m-1})\dots(\lambda_i-\lambda_{m-1})\dots(\lambda_m-\lambda_{m-1})}\\
&-(\lambda_m-\lambda_{m+1})\frac{(\lambda_2-\lambda_{m+1})\dots(\lambda_i-\lambda_{m+1})\dots(\lambda_{m-1}-\lambda_{m+1})}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)\dots(\lambda_i-\lambda_m)\dots(\lambda_{m-1}-\lambda_m)}
\end{aligned}
\tag{4.9}
$$

Now, the left hand side of Eq (4.9) is the first term of left hand side of Eq (4.7). We replace the first term of left hand side of Eq (4.7) by the right hande side of Eq (4.9). Then, the left hand side of Eq (4.7) becomes:

$$
\begin{aligned}
&[-(\lambda_2-\lambda_{m+1})+(\lambda_1-\lambda_{m+1})]\,\frac{(\lambda_3-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)\ldots(\lambda_i-\lambda_2)\ldots(\lambda_m-\lambda_2)}\\
&+[-(\lambda_3-\lambda_{m+1})+(\lambda_1-\lambda_{m+1})]\,\frac{(\lambda_3-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})\ldots.(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\ldots(\lambda_i-\lambda_3)\ldots.(\lambda_m-\lambda_3)}\\
&+\cdots+\\
&[-(\lambda_{m-1}-\lambda_{m+1})+(\lambda_1-\lambda_{m+1})]\,\frac{(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_1-\lambda_{m-1})\ldots(\lambda_i-\lambda_{m-1})\ldots(\lambda_m-\lambda_{m-1})}\\
&+[-(\lambda_m-\lambda_{m+1})+(\lambda_1-\lambda_{m+1})]\,\frac{(\lambda_2-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})\ldots(\lambda_{m-1}-\lambda_{m+1})}{(\lambda_1-\lambda_m)(\lambda_2-\lambda_m)\ldots(\lambda_i-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)}\\
&=1
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{(\lambda_3-\lambda_{m+1})(\lambda_4-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)\ldots(\lambda_i-\lambda_2)(\lambda_{i+1}-\lambda_2)\ldots(\lambda_m-\lambda_2)}\\
&+\frac{(\lambda_3-\lambda_{m+1})(\lambda_4-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)\ldots(\lambda_i-\lambda_3)(\lambda_{i+1}-\lambda_3)\ldots(\lambda_m-\lambda_3)}\\
&+\ldots+\\
&\frac{(\lambda_2-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_m-\lambda_{m+1})}{(\lambda_2-\lambda_{m-1})(\lambda_3-\lambda_{m-1})\ldots(\lambda_i-\lambda_{m-1})(\lambda_{i+1}-\lambda_{m-1})\ldots(\lambda_m-\lambda_{m-1})}\\
&+\frac{(\lambda_2-\lambda_{m+1})(\lambda_3-\lambda_{m+1})\ldots(\lambda_i-\lambda_{m+1})(\lambda_{i+1}-\lambda_{m+1})\ldots(\lambda_{m-1}-\lambda_{m+1})}{(\lambda_2-\lambda_m)(\lambda_3-\lambda_m)\ldots(\lambda_i-\lambda_m)(\lambda_{i+1}-\lambda_m)\ldots(\lambda_{m-1}-\lambda_m)}\\
&=1
\end{aligned}
\tag{4.10}
$$

Since the induction hypothesis is assumed to be true for any subset of A of size m, and Eq (4.10) deals with $\{\lambda_a, \lambda_b, \lambda_e...., \lambda_{(m+1)}\}$ which has size m, therefore by Eq (4.4), Eq (4.10) equals Eq (4.1). This proves that Eq 4.7 holds. Therefore, $m+1 \in S$. This is the end of the proof. □

Theorem 4.1.2 *The solution of system 3.1 is*

$n_k(t) =$

$$\sum_{m=1}^{k} \frac{e^{-\delta mt}}{\delta_k-\delta_m} \left[\begin{array}{c} \displaystyle\sum_{j=1}^{m} n_j(0)\alpha_{i=j}^{k-1} + \\[2em] \displaystyle\sum_{y=1}^{m-2} \frac{b_y}{\delta_y-\delta_m}\alpha_{i=j}^{k-1} \left[\begin{array}{c} \displaystyle\sum_{e=1}^{y} n_e(0)\alpha_{d=e}^{y-1} + \\[2em] \displaystyle\sum_{z=1}^{y-2} \frac{b_z}{\delta_z-\delta_m}\alpha_{w=z+2}^{y-1} \left[\begin{array}{c} \displaystyle\sum_{g=1}^{z} n_g(0)\alpha_{f=g}^{z-1} + \\[2em] \ldots + \end{array}\right] \end{array}\right] \\[2em] \displaystyle\sum_{l=m}^{k-2} \frac{b_l}{\delta_l-\delta_m}\alpha_{h=l+2}^{k-1} \left[\begin{array}{c} \displaystyle\sum_{e=1}^{m} n_e(0)\alpha_{d=e}^{l-1} + \\[2em] \displaystyle\sum_{z=1}^{m-2} \frac{b_z}{\delta_z-\delta_m}\alpha_{w=z+2}^{m-1} \left[\begin{array}{c} \displaystyle\sum_{g=1}^{z} n_g(0)\alpha_{f=g}^{z-1} + \\[2em] \ldots + \end{array}\right] + \\[2em] \displaystyle\sum_{p=m}^{l-2} \frac{b_p}{\delta_p-\delta_m}\alpha_{p=q+2}^{l-1} \left[\begin{array}{c} \displaystyle\sum_{R=1}^{p} n_R(0)\alpha_{s=R}^{p-1} + \\[2em] \displaystyle\sum_{z=1}^{p-2} \frac{b_z}{\delta_z-\delta_m}\alpha_{w=z+2}^{p-1} \left[\begin{array}{c} \displaystyle\sum_{g=1}^{z} n_g(0)\alpha_{f=g}^{z-1} + \\ \ldots + \end{array}\right] + \\ \ldots \end{array}\right] \end{array}\right] \end{array}\right]$$

where $\alpha_{i=j}^{k-1} = \prod_{i=j}^{k-1} \frac{a_i}{\delta_i-\delta_m}$*. And note: if* $m = i$ *make* $m = k$*.*

Proof: We will also prove this by Mathematical Induction.

To find $n_1(t)$, from the first equation of the system, we get

$\dot{n}_1(t) = -\delta_1 n_1(t).$

Solve for $n_1(t)$. We get

$$n_1(t) = n_1(0)e^{-\delta_1 t}.$$

To find $n_1(t)$, apply the theorem (4.1.2) to get

$n_1(t) = \sum_{m=1}^{1} \frac{e^{-\delta_m t}}{\delta_k - \delta_m} [\sum_{j=1}^{m} (\prod_{i=j}^{0} \frac{a_i}{\delta_i - \delta_m}) n_j(0) \ ]$

$\Rightarrow n_1(t) = n_1(0)e^{-\delta_1 t}.$

Solve for $n_2(t)$. From the equation of the system, we get

$$\dot{n}_2(t) = a_1 n_1(t) - \delta_2 n_2(t).$$

Since

$$\frac{d}{dt}(n_2(t)e^{\delta_2 t}) = a_1 n_1(t) e^{\delta_2 t}.$$

We get

$$n_2(t) = n_2(0)e^{-\delta_2 t} + a_1 e^{\delta_2 t} \int_0^t n_1(0)e^{-\delta_1 t} e^{\delta_2 t} d\tau$$

$= n_2(0)e^{-\delta_2 t} + a_1 n_1(0) e^{-\delta_2 t} \left[\frac{1 - e^{-(\delta_1 - \delta_2)t}}{(\delta_1 - \delta_2)} \right]$

$= n_2(0)e^{-\delta_2 t} + a_1 n_1(0)\left[\frac{e^{-\delta_2 t}}{(\delta_1-\delta_2)} + \frac{e^{-\delta_1 t}}{(\delta_2-\delta_1)}\right].$

To find $n_2(t)$, we use theorem (4.1.2)

$$n_2(t) = \sum_{m=1}^{2} \frac{e^{-\delta m t}}{\delta_k-\delta_m}[\sum_{j=1}^{m}(\prod_{i=j}^{1} \frac{a_i}{\delta_i-\delta_m})n_j(0).\ ]$$

Then $n_2(t) = e^{\delta_1 t}\left[\frac{a_1 n_1(0)}{\delta_2-\delta_1}\right] + e^{\delta_2 t}\left[n_2(0) + \frac{a_1 n_1(0)}{\delta_1-\delta_2}\right]$

To solve for $n_3(t)$, from the third equation of the system, we get:

$$\dot{n}_3(t) = b_1 n_1(t) + a_2 n_2(t) - \delta_3 n_3(t).$$

Since

$$\frac{d}{dt}(n_3(t)e^{\delta_3 t}) = e^{\delta_3 t}(b_1 n_1(t) + a_2 n_2(t))$$

$$= b_1 e^{\delta_3 t}\left[n_1(0)e^{-\delta_1(t)}\right]$$
$$+a_2 e^{\delta_3 t}\left[n_2(0)e^{-\delta_2 t} + a_1 n_1(0)\left[\frac{e^{-\delta_2 t}}{(\delta_1-\delta_2)} + \frac{e^{-\delta_1 t}}{(\delta_2-\delta_1)}\right]\right].$$

Then $\quad n_3(t)e^{\delta_3 t} \;= n_3(0) + b_1 n_1(0)\int_0^t e^{(\delta_3-\delta_1)\tau}d\tau + a_2 n_2(0)\int_0^t e^{(\delta_3-\delta_2)\tau}d\tau$

$$+a_2 a_1 n_1(0)\int_0^t(\frac{e^{(\delta_3-\delta_2)\tau}}{(\delta_1-\delta_2)} + \frac{e^{(\delta_3-\delta_1)\tau}}{(\delta_2-\delta_1)})d\tau$$

$$e^{\delta_3 t}n_3(t) = n_3(0) + b_1 n_1(0)\left[\frac{e^{(\delta_3-\delta_1)t}-1}{(\delta_3-\delta_1)}\right] + a_2 n_2(0)\left[\frac{e^{(\delta_3-\delta_2)t}-1}{(\delta_3-\delta_2)}\right]$$

$$+a_2a_1n_1(0)\left[\tfrac{e^{(\delta_3-\delta_2)t}-1}{(\delta_1-\delta_2)(\delta_3-\delta_2)}+\tfrac{e^{(\delta_3-\delta_1)t}-1}{(\delta_2-\delta_1)(\delta_3-\delta_1)}\right]$$

$$n_3(t)=e^{-\delta_3 t}n_3(0)+b_1n_1(0)\left[\tfrac{e^{-\delta_1 t}}{(\delta_3-\delta_1)}+\tfrac{e^{-\delta_3 t}}{(\delta_1-\delta_3)}\right]$$

$$+a_2n_2(0)\left[\tfrac{e^{-\delta_2 t}}{(\delta_3-\delta_2)}+\tfrac{e^{-\delta_3 t}}{(\delta_2-\delta_3)}\right]+a_2a_1n_1(0)\begin{bmatrix}\tfrac{e^{-\delta_1 t}}{(\delta_2-\delta_1)(\delta_3-\delta_1)}+\tfrac{e^{-\delta_2 t}}{(\delta_1-\delta_2)(\delta_3-\delta_2)}\\ \\ +\tfrac{e^{-\delta_3 t}}{(\delta_1-\delta_3)(\delta_2-\delta_3)}\end{bmatrix}.$$

To find $n_3(t)$, from the theorem (4.1.2)of the system, we have

$$n_3(t)=e^{\delta_1 t}\left[\tfrac{a_1a_2n_1(0)}{(\delta_2-\delta_1)(\delta_3-\delta_1)}+\tfrac{b_1n_1(0)}{(\delta_3-\delta_1)}\right]+e^{\delta_2 t}\left[\tfrac{a_1a_2n_1(0)}{(\delta_1-\delta_2)(\delta_3-\delta_2)}+\tfrac{a_2n_2(0)}{\delta_3-\delta_2}\right]$$

$$+e^{\delta_3 t}\left[\tfrac{a_1a_2n_1(0)}{(\delta_1-\delta_3)(\delta_2-\delta_3)}+\tfrac{a_2n_2(0)}{\delta_2-\delta_3}+n_3(0)+\tfrac{b_1n_1(0)}{(\delta_1-\delta_3)}\right].$$

Now, we assume that the theorem (4.1.2) is true for $n_k(t)$.

Then, we are going to prove it is true for $n_{k+1}(t)$. From the $(k+1)^{th}$.

Equation of the system we have:

$$\dot{n}_{k+1}(t)=a_kn_k(t)+b_{k-1}n_{k-1}(t)-\delta_{k+1}n_{k+1}(t)$$

$$n_{k+1}(t)=n_{k+1}(0)e^{-\delta_{k+1}t}+a_ke^{-\delta_{k+1}t}\int_0^t n_k(\tau)e^{\delta_{k+1}\tau}d\tau+b_{k-1}e^{-\delta_{k+1}t}\int_0^t n_{k-1}(\tau)e^{\delta_{k+1}\tau}d\tau$$

$$n_{k+1}(t) = n_{k+1}(0)e^{-\delta_{k+1}t} +$$

$$a_k e^{-\delta_{k+1}t} \int_0^t e^{\delta_{k+1}\tau} \left[\sum_{m=1}^{k} \frac{e^{-\delta m t}}{\delta_k - \delta_m} \left[\begin{array}{c} \sum\limits_{j=1}^{m} n_j(0)\alpha_{i=j}^{k-1} + \\ \\ [\sum\limits_{y=1}^{m-2} \alpha_{x=y+2}^{k-1} \frac{b_y}{\delta_y - \delta_m} (\sum_{e=1}^{y} n_e(0)\alpha_{d=e}^{y-1} + \\ \ldots +) \\ \\ \sum\limits_{l=m}^{k-2} \alpha_{h=l+2}^{k-1} \frac{b_l}{\delta_l - \delta_m} (\sum\limits_{j=1}^{m} n_j(0)\alpha_{i=j}^{l-1} + \\ \\ \ldots +) \end{array} \right] \right] d\tau +$$

$$b_{k-1} e^{-\delta_{k+1}t} \int_0^t e^{\delta_{k+1}\tau} \left[\sum_{m=1}^{k-1} \frac{e^{-\delta m t}}{\delta_{k-1} - \delta_m} \left[\begin{array}{c} \sum\limits_{j=1}^{m} (\alpha_{i=j}^{k-2}) n_j(0) + \\ \\ [\sum\limits_{y=1}^{m-2} (\alpha_{x=y+2}^{k-2}) \frac{b_y}{\delta_y - \delta_m} (\sum\limits_{e=1}^{y} (\alpha_{d=e}^{y-1}) n_e(0) + \\ \ldots +) \\ \\ \sum\limits_{l=m}^{k-3} (\alpha_{h=l+2}^{k-2}) \frac{b_l}{\delta_l - \delta_m} (\sum\limits_{j=1}^{m} (\alpha_{i=j}^{l-1}) n_j(0) + \\ \\ \ldots +) \end{array} \right] \right] d\tau$$

After simplification we get

$n_{k+1}(t) = n_{k+1}(0)e^{-\delta_{k+1}t} +$

$$a_k e^{-\delta_{k+1}t}\int\limits_0^t \left[\sum_{m=1}^{k}\frac{e^{(\delta_{k+1}-\delta_m)\tau}}{(\delta_k-\delta_m)}\left[\begin{array}{c} \sum_{j=1}^{m}(\alpha_{i=j}^{k-1})n_j(0)+ \\ [\sum\limits_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\frac{b_y}{\delta_y-\delta_m}(\sum\limits_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \ldots +) \\ \sum\limits_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\frac{b_l}{\delta_l-\delta_m}(\sum\limits_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots +) \end{array}\right]\right] d\tau +$$

$$b_{k-1} e^{-\delta_{k+1}t}\int\limits_0^t \left[\sum_{m=1}^{k-1}\frac{e^{(\delta_{k+1}-\delta_m)\tau}}{\delta_k-\delta_m}\left[\begin{array}{c} \sum\limits_{j=1}^{m}(\alpha_{i=j}^{k-2})n_j(0)+ \\ [\sum\limits_{y=1}^{m-2}(\alpha_{x=y+2}^{k-2})\frac{b_y}{\delta_y-\delta_m}(\sum\limits_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+\cdots+) \\ \sum\limits_{l=m}^{k-3}(\alpha_{h=l+2}^{k-2})\frac{b_l}{\delta_l-\delta_m}(\sum\limits_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots +) \end{array}\right]\right] d\tau$$

we integrate

$$n_{k+1}(t) = n_{k+1}(0)e^{-\delta_{k+1}t}+$$

$$a_k e^{-\delta_{k+1}t} \left[\begin{array}{c} \displaystyle\sum_{m=1}^{k} \frac{e^{(\delta_{k+1}-\delta_m)t}}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)} \left[\begin{array}{c} \sum_{j=1}^{m}(\alpha_{i=j}^{k-1})n_j(0)+ \\ [\displaystyle\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \ldots+) \\ \displaystyle\sum_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots+) \end{array} \right] + \\ \displaystyle\sum_{m=1}^{k} \frac{1}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)} \left[\begin{array}{c} \displaystyle\sum_{j=1}^{m}(\alpha_{i=j}^{k-1})n_j+ \\ [\displaystyle\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \ldots+) \\ \displaystyle\sum_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots+) \end{array} \right] \end{array} \right] +$$

$$b_{k-1}e^{-\delta_{k+1}t}\left[\begin{array}{c} \sum_{m=1}^{k-1}\frac{e^{(\delta_{k+1}-\delta_m)t}}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)}\left[\begin{array}{c} \sum_{j=1}^{m}(\alpha_{i=j}^{k-2})n_j(0)+ \\ \\ [\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-2})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \\ \ldots+) \\ \\ \sum_{l=m}^{k-3}(\alpha_{h=l+2}^{k-2})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots+) \end{array}\right]+ \\ \\ \sum_{m=1}^{k-1}\frac{1}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)}\left[\begin{array}{c} \sum_{j=1}^{m}(\alpha_{i=j}^{k-2})n_j(0)+ \\ \\ [\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-2})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \\ \ldots+) \\ \\ \sum_{l=m}^{k-3}(\alpha_{h=l+2}^{k-2})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots+) \end{array}\right] \end{array}\right]$$

$$
n_{k+1}(t) = n_{k+1}(0)e^{-\delta_{k+1}t} + a_k \left[\begin{array}{c} \sum_{m=1}^{k} \frac{e^{-\delta_m t}}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)} \left[\begin{array}{c} \sum_{j=1}^{m}(\alpha_{i=j}^{k-1})n_j(0)+ \\ \\ [\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \\ \ldots +) \\ \sum_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \\ \ldots +) \end{array} \right] \\ -\sum_{m=1}^{k} \frac{e^{-\delta_{k+1} t}}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)} \left[\begin{array}{c} \sum_{j=1}^{m}(\alpha_{i=j}^{k-1})n_j(0)+ \\ \\ [\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \\ \ldots +) \\ \\ \sum_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \ldots +) \end{array} \right] \end{array} \right] +
$$

$$
b_{k-1}\left[\begin{array}{c}
\sum_{m=1}^{k-1}\frac{e^{-\delta_m t}}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)}\left[\begin{array}{c}
\sum_{j=1}^{m}(\alpha_{i=j}^{k-2})n_j(0)+ \\
\\
[\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-2})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\
\ldots+) \\
\\
\sum_{l=m}^{k-3}(\alpha_{h=l+2}^{k-2})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\
\ldots+)
\end{array}\right] \\
\\
-\sum_{m=1}^{k-1}\frac{e^{-\delta_{k+1} t}}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)}\left[\begin{array}{c}
\sum_{j=1}^{m}(\alpha_{i=j}^{k-2})n_j(0)+ \\
\\
[\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-2})\frac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\
\ldots+) \\
\\
\sum_{l=m}^{k-3}(\alpha_{h=l+2}^{k-2})\frac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\
\ldots+)
\end{array}\right]
\end{array}\right]
$$

Apply lemma for the two terms

$$e^{-\delta_{k+1}t}\sum_{m=1}^{k}\tfrac{1}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)}\left[\begin{array}{c}\sum_{j=1}^{m}(\alpha_{i=j}^{k-1})n_j(0)+\\ \\ [\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\tfrac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+\\ \ldots+)\\ \\ \sum_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\tfrac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+\\ \ldots+)\end{array}\right]$$

and the term

$$e^{-\delta_{k+1}t}\sum_{m=1}^{k-1}\tfrac{1}{(\delta_{k+1}-\delta_m)(\delta_k-\delta_m)}\left[\begin{array}{c}\sum_{j=1}^{m}(\alpha_{i=j}^{k-2})n_j(0)+\\ \\ [\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-2})\tfrac{b_y}{\delta_y-\delta_m}(\sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+\\ \ldots+)\\ \\ \sum_{l=m}^{k-3}(\alpha_{h=l+2}^{k-2})\tfrac{b_l}{\delta_l-\delta_m}(\sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+\\ \\ \ldots+)\end{array}\right]$$

$$
a_k e^{-\delta_{k+1} t}
\left[
\begin{array}{c}
\frac{1}{(\delta_{k+1}-\delta_1)(\delta_k-\delta_1)}
\left[
\begin{array}{c}
\frac{a_1 a_2 ... a_{k-1} n_1(0)}{(\delta_2-\delta_1)(\delta_3-\delta_1)...(\delta_{k-1}-\delta_1)} \\
+\frac{a_3 a_4 ... a_{k-1} b_1 n_3(0)}{(\delta_3-\delta_1)(\delta_4-\delta_1)...(\delta_{k-1}-\delta_1)} \\
+\frac{a_4 a_5 ... a_{k-1} b_2 a_1 n_1(0)}{(\delta_3-\delta_1)(\delta_4-\delta_1)...(\delta_{k-1}-\delta_1)(\delta_2-\delta_1)} \\
+\ldots+
\end{array}
\right] \\
\frac{1}{(\delta_{k+1}-\delta_2)(\delta_k-\delta_2)}
\left[
\begin{array}{c}
\frac{a_1 a_2 ... a_{k-1} n_1(0)}{(\delta_1-\delta_2)(\delta_3-\delta_2)...(\delta_{k-1}-\delta_2)} \\
+\frac{a_2 a_3 ... a_{k-1} n_2(0)}{(\delta_3-\delta_2)(\delta_4-\delta_2)...(\delta_{k-1}-\delta_2)} \\
+\frac{a_3 a_4 ... a_{k-1} b_1 n_3(0)}{(\delta_3-\delta_2)(\delta_4-\delta_2)...(\delta_{k-1}-\delta_2)(\delta_1-\delta_2)} \\
+\frac{a_4 a_5 ... a_{k-1} b_2 a_1 n_1(0)}{(\delta_3-\delta_2)(\delta_4-\delta_2)...(\delta_{k-1}-\delta_2)} \\
+\ldots+
\end{array}
\right] \\
+\ldots+ \\
\frac{1}{(\delta_{k+1}-\delta_k)}
\left[
\begin{array}{c}
\frac{a_1 a_2 ... a_{k-1} n_1(0)}{(\delta_1-\delta_k)(\delta_2-\delta_k)...(\delta_{k-1}-\delta_k)} \\
+\frac{a_2 a_3 ... a_{k-1} n_2(0)}{(\delta_2-\delta_k)(\delta_3-\delta_k)...(\delta_{k-1}-\delta_k)} \\
+\ldots+n_k(0) \\
+\frac{a_3 a_4 ... a_{k-1} b_1 n_1(0)}{(\delta_3-\delta_k)(\delta_4-\delta_k)...(\delta_{k-1}-\delta_k)(\delta_1-\delta_k)} \\
+\ldots
\end{array}
\right]
\end{array}
\right]
$$

Take the term that are similar in the extend

$$\frac{a_1a_2...a_{k-1}n_1(0)}{(\delta_2-\delta_1)(\delta_3-\delta_1)...(\delta_{k-1}-\delta_1)(\delta_k-\delta_1)} + \frac{a_1a_2...a_{k-1}n_1(0)}{(\delta_1-\delta_2)(\delta_3-\delta_2)...(\delta_{k-1}-\delta_2)(\delta_k-\delta_2)} + \cdots$$

$$+\frac{a_1a_2...a_{k-1}n_1(0)}{(\delta_1-\delta_k)(\delta_2-\delta_k)...(\delta_{k-1}-\delta_k)} = -\frac{a_1a_2...a_{k-1}n_1(0)}{(\delta_1-\delta_{k+1})(\delta_2-\delta_{k+1})...(\delta_k-\delta_{k+1})}$$

Thus, we get

$$a_k e^{-\delta_{k+1}t}\left[-\frac{a_1a_2...a_{k-1}n_1(0)}{(\delta_1-\delta_{k+1})(\delta_2-\delta_{k+1})...(\delta_k-\delta_{k+1})}\right]$$

$$= e^{-\delta_{k+1}t}\left[-\frac{a_1a_2...a_{k-1}a_k n_1(0)}{(\delta_1-\delta_{k+1})(\delta_2-\delta_{k+1})...(\delta_k-\delta_{k+1})}\right]$$

we do the same for the other terms.

the $n_{k+1}(t)$

$$
n_{k+1}(t) = \sum_{m=1}^{k+1} \frac{e^{-\delta_m t}}{\delta_{k+1}-\delta_m}
\left[\begin{array}{c}
\displaystyle\sum_{j=1}^{m} (\alpha^{k}_{i=j}) n_j(0) + \\[2ex]
\displaystyle\sum_{y=1}^{m-2} (\alpha^{k}_{x=y+2}) \frac{b_y}{\delta_y-\delta_m}
\left[\begin{array}{c}
\displaystyle\sum_{e=1}^{y} (\alpha^{y-1}_{d=e}) n_e(0) + \\[2ex]
\sum_{z=1}^{y-2} (\alpha^{y-1}_{w=z+2}) \frac{b_z}{\delta_z-\delta_m}
\left[\begin{array}{c}
\displaystyle\sum_{g=1}^{z} (\alpha^{z-1}_{f=g}) n_g(0) + \\[2ex]
\ldots +
\end{array} \right]
\end{array} \right] \\[4ex]
\displaystyle\sum_{l=m}^{k-1} (\alpha^{k-1}_{h=l+2}) \frac{b_l}{\delta_l-\delta_m}
\left[\begin{array}{c}
\displaystyle\sum_{j=1}^{m} (\alpha^{l-1}_{i=j}) n_j(0) + \\[2ex]
\displaystyle\sum_{z=1}^{m-2} (\alpha^{m-1}_{w=z+2}) \frac{b_z}{\delta_z-\delta_m}
\left[\begin{array}{c}
\displaystyle\sum_{g=1}^{z} (\alpha^{z-1}_{f=g}) n_g(0) + \\[2ex]
\ldots +
\end{array} \right] \\[4ex]
\displaystyle\sum_{p=m}^{l-2} (\alpha^{l-1}_{p=q+2}) \frac{b_p}{\delta_p-\delta_m}
\left[\begin{array}{c}
\displaystyle\sum_{R=1}^{p} (\alpha^{p-1}_{s=R}) n_R(0) + \\[2ex]
...+
\end{array} \right]
\end{array} \right]
\end{array} \right]
$$

□

Example

Consider the following system

$$\begin{bmatrix} \dot{n}_1(t) \\ \dot{n}_2(t) \\ \dot{n}_3(t) \\ \dot{n}_4(t) \\ \dot{n}_5(t) \\ \dot{n}_6(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & -4 & 0 & 0 & 0 & \dots \\ 0 & 2 & 3 & -5 & 0 & 0 & \dots \\ 0 & 0 & 2 & 3 & -6 & 0 & \dots \\ 0 & 0 & 0 & 3 & 3 & -7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \\ n_5(t) \\ n_6(t) \\ \vdots \end{bmatrix}$$

where:

$n_1(0) = 4, n_2(0) = 2, n_3(0) = 2, n_4(0) = \frac{5}{3}, n_5(0) = \frac{1}{3}, n_6(0) = \frac{3}{5}$.

First we solve this system using the regular technique, then we use theorem (4.1.2) to solve this system.

Solution:

$\dot{n}_1(t) = -2n_1(t) \Longrightarrow \dot{n}_1(t) + 2n_1(t) = 0$

$e^{2t}\dot{n}_1(t) + 2e^{2t}n_1(t) = 0 \Longrightarrow \frac{d}{dt}\left(e^{2t}n_1(t)\right) = 0$

$e^{2t}n_1(t) = n_1(0) \Longrightarrow n_1(t) = 4e^{-2t}$

$\dot{n}_2(t) = n_1(t) - 3n_2(t) \Longrightarrow e^{3t}\dot{n}_2(t) + 3e^{3t}n_2(t) = e^{3t}n_1(t)$

$\frac{d}{dt}\left(e^{3t}n_2(t)\right) = e^{3t}\left(e^{-2t}n_1(0)\right) \Longrightarrow e^{3t}n_2(t) - n_2(0) = \int_0^t e^{(3-2)\tau}d\tau$

$e^{3t}n_2(t) = 4\left[\frac{e^1-1}{1}\right] + n_2(0) \Longrightarrow n_2(t) = 4e^{-2t} - 2e^{-3t}$

$$\dot{n}_3(t) = 2n_1(t) + 2n_2(t) - 4n_3(t)$$

$$e^{4t}\dot{n}_3(t) + 4e^{4t}n_3(t) = 2e^{4t}n_1(t) + 2e^{4t}n_2(t)$$

$$\frac{d}{dt}\left(e^{4t}n_3(t)\right) = 2e^{4t}(4e^{-2t} + 4e^{-2t} - 2e^{-3t})$$

$$e^{4t}n_3(t) = 2e^{4t}\int_0^t (8e^{-2t} - 2e^{-3t})d\tau$$

$$n_3(t) = 8e^{-2t} - 4e^{-3t} - 2e^{-4t}$$

$$\dot{n}_4(t) = 2n_2(t) + 3n_3(t) - 5n_4(t)$$

$$\frac{d}{dt}\left(e^{5t}n_4(t)\right) = e^{5t}\left[2(4e^{-2t} - 2e^{-3t}) + 3(8e^{-2t} - 4e^{-3t} - 2e^{-4t})\right]$$

$$e^{-5t}n_4(t) = 8\left[\frac{e^{5-2t}-1}{3}\right] - 4\left[\frac{e^{5-3t}-1}{2}\right] + 24\left[\frac{e^{5-2t}-1}{3}\right] - 12\left[\frac{e^{5-3t}-1}{2}\right] - 6(e^{5-4t} - 1) + \frac{10}{3}$$

$$n_4(t) = \frac{32}{3}e^{-2t} - 8e^{-3t} - 6e^{-4t} + 5e^{-5t}$$

$$\dot{n}_5(t) = 2n_3(t) + 3n_4(t) - 6n_5(t)$$

$$\frac{d}{dt}\left(e^{6t}n_5(t)\right) = e^{6t}\left[2(8e^{-2t} - 4e^{-3t} - 2e^{-4t}) + 3(\frac{32}{3}e^{-2t} - 8e^{-3t} - 6e^{-4t} + 5e^{-5t})\right]$$

$$e^{6t}n_5(t) = 12e^{4t} - \frac{32}{3}e^{3t} - 11e^{2t} + 15e^{t} - 5$$

$$n_5(t) = 12e^{-2t} - \frac{32}{3}e^{-3t} - 11e^{-4t} + 15e^{-5t} - 5e^{-6t}$$

$$\dot{n}_6(t) = 3n_4(t) + 3n_5(t) - 7n_6(t)$$

$$\frac{d}{dt}\left(e^{7t}n_6(t)\right) = e^{7t}\left[\begin{array}{c} 3(\frac{32}{3}e^{-2t} - 8e^{-3t} - 6e^{-4t} + 5e^{-5t}) \\ +3(12e^{-2t} - \frac{32}{3}e^{-3t} - 11e^{-4t} + 15e^{-5t} - 5e^{-6t} \end{array}\right]$$

$$\Rightarrow n_6(t) = \frac{68}{5}e^{-2t} - 14e^{-3t} - 17e^{-4t} + 30e^{-5t} - 15e^{-6t} + 3e^{-7t}.$$

The solution by the theorem

$$\begin{bmatrix} \dot{n}_1(t) \\ \dot{n}_2(t) \\ \dot{n}_3(t) \\ \dot{n}_4(t) \\ \dot{n}_5(t) \\ \dot{n}_6(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & -3 & 0 & 0 & 0 & 0 & \dots \\ 2 & 2 & -4 & 0 & 0 & 0 & \dots \\ 0 & 2 & 3 & -5 & 0 & 0 & \dots \\ 0 & 0 & 2 & 3 & -6 & 0 & \dots \\ 0 & 0 & 0 & 3 & 3 & -7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \\ n_5(t) \\ n_6(t) \\ \vdots \end{bmatrix}$$

Solution:

$$e^{-\delta_1 t}\left[\begin{array}{c} \frac{a_1a_2a_3a_4a_5n_1(0)}{(\delta_2-\delta_1)(\delta_3-\delta_1)(\delta_4-\delta_1)(\delta_5-\delta_1)(\delta_6-\delta_1)} + \frac{a_3a_4a_5b_1n_1(0)}{(\delta_3-\delta_1)(\delta_4-\delta_1)(\delta_5-\delta_1)(\delta_6-\delta_1)} \\ +\frac{a_4a_5b_2a_1n_1(0)}{(\delta_4-\delta_1)(\delta_5-\delta_1)(\delta_6-\delta_1)(\delta_2-\delta_1)} + \frac{a_5b_3a_1a_2n_1(0)}{(\delta_5-\delta_1)(\delta_6-\delta_1)(\delta_3-\delta_1)(\delta_2-\delta_1)} \\ +\frac{a_5b_3b_1n_1(0)}{(\delta_5-\delta_1)(\delta_3-\delta_1)(\delta_6-\delta_1)} + \frac{b_4a_1a_2a_3n_1(0)}{(\delta_3-\delta_1)(\delta_2-\delta_1)(\delta_6-\delta_1)} \\ +\frac{b_4a_3b_1n_1(0)}{(\delta_4-\delta_1)(\delta_3-\delta_1)(\delta_6-\delta_1)} + \frac{b_4b_2a_1n_1(0)}{(\delta_4-\delta_1)(\delta_2-\delta_1)(\delta_6-\delta_1)} \end{array}\right]$$

$$+e^{-\delta_2 t}\left[\begin{array}{c} \frac{a_1a_2a_3a_4a_5n_1(0)}{(\delta_1-\delta_2)(\delta_3-\delta_2)(\delta_4-\delta_2)(\delta_5-\delta_2)(\delta_6-\delta_2)} + \frac{a_2a_3a_4a_5n_1(0)}{(\delta_3-\delta_2)(\delta_4-\delta_2)(\delta_5-\delta_2)(\delta_6-\delta_2)} \\ +\frac{a_4a_5b_2a_1n_1(0)}{(\delta_4-\delta_2)(\delta_5-\delta_2)(\delta_6-\delta_2)(\delta_1-\delta_2)} + \frac{a_4a_5b_2n_2(0)}{(\delta_4-\delta_2)(\delta_5-\delta_2)(\delta_6-\delta_2)} \\ +\frac{a_5b_3a_1a_2n_1(0)}{(\delta_5-\delta_2)(\delta_3-\delta_2)(\delta_6-\delta_2)(\delta_1-\delta_2)} + \frac{a_5b_3a_2n_2(0)}{(\delta_5-\delta_2)(\delta_3-\delta_2)(\delta_6-\delta_2)} \\ +\frac{b_4a_1a_2a_3n_1(0)}{(\delta_4-\delta_2)(\delta_3-\delta_2)(\delta_6-\delta_2)(\delta_1-\delta_2)} + \frac{b_4a_2a_3n_2(0)}{(\delta_4-\delta_2)(\delta_3-\delta_2)(\delta_6-\delta_2)} \\ +\frac{b_4b_2n_2(0)}{(\delta_4-\delta_2)(\delta_6-\delta_2)} + \frac{b_4b_2a_1n_1(0)}{(\delta_4-\delta_2)(\delta_6-\delta_2)(\delta_1-\delta_2)} \end{array}\right]$$

$$+e^{-\delta_3 t}\left[\begin{array}{c} \frac{a_1a_2a_3a_4a_5n_1(0)}{(\delta_1-\delta_3)(\delta_2-\delta_3)(\delta_4-\delta_3)(\delta_5-\delta_3)(\delta_6-\delta_3)} + \frac{a_2a_3a_4a_5n_2(0)}{(\delta_2-\delta_3)(\delta_4-\delta_3)(\delta_5-\delta_3)(\delta_6-\delta_3)} \\ +\frac{a_3a_4a_5n_3(0)}{(\delta_4-\delta_3)(\delta_5-\delta_3)(\delta_6-\delta_3)} + \frac{a_3a_4a_5b_1n_1(0)}{(\delta_6-\delta_3)(\delta_4-\delta_3)(\delta_5-\delta_3)(\delta_1-\delta_3)} \\ +\frac{a_5b_3a_1a_2n_1(0)}{(\delta_5-\delta_3)(\delta_6-\delta_3)(\delta_1-\delta_3)(\delta_2-\delta_3)} + \frac{a_5b_3a_2n_2(0)}{(\delta_5-\delta_3)(\delta_6-\delta_3)(\delta_2-\delta_3)} \\ +\frac{a_5b_3n_3(0)}{(\delta_5-\delta_3)(\delta_6-\delta_3)} + \frac{a_5b_3b_1n_1(0)}{(\delta_5-\delta_3)(\delta_6-\delta_3)(\delta_1-\delta_3)} \\ +\frac{b_4a_1a_2a_3n_1(0)}{(\delta_4-\delta_3)(\delta_1-\delta_3)(\delta_2-\delta_3)(\delta_6-\delta_3)} + \frac{b_4a_2a_3n_2(0)}{(\delta_4-\delta_3)(\delta_2-\delta_3)(\delta_6-\delta_3)} \\ +\frac{b_4a_3n_3(0)}{(\delta_4-\delta_3)(\delta_6-\delta_3)} + \frac{b_4a_3b_1n_1(0)}{(\delta_4-\delta_3)(\delta_6-\delta_3)(\delta_1-\delta_3)} \end{array}\right]$$

$$+e^{-\delta 4t}\left[\begin{array}{c}\frac{a_1a_2a_3a_4a_5n_1(0)}{(\delta_1-\delta_4)(\delta_2-\delta_4)(\delta_3-\delta_4)(\delta_5-\delta_4)(\delta_6-\delta_4)}+\frac{a_2a_3a_4a_5n_2(0)}{(\delta_2-\delta_4)(\delta_3-\delta_4)(\delta_5-\delta_4)(\delta_6-\delta_4)}\\ +\frac{a_3a_4a_5n_3(0)}{(\delta_3-\delta_4)(\delta_5-\delta_4)(\delta_6-\delta_4)}+\frac{a_4a_5n_4(0)}{(\delta_5-\delta_4)(\delta_6-\delta_4)}\\ +\frac{a_3a_4a_5b_1n_1(0)}{(\delta_3-\delta_4)(\delta_5-\delta_4)(\delta_6-\delta_4)(\delta_1-\delta_4)}+\frac{a_4a_5b_2a_1n_1(0)}{(\delta_5-\delta_4)(\delta_6-\delta_4)(\delta_2-\delta_4)(\delta_1-\delta_4)}\\ +\frac{a_4a_5b_2n_2(0)}{(\delta_5-\delta_4)(\delta_6-\delta_4)(\delta_2-\delta_4)(\delta_1-\delta_4)}+\frac{b_4a_1a_2a_3n_1(0)}{(\delta_6-\delta_4)(\delta_1-\delta_4)(\delta_2-\delta_4)(\delta_3-\delta_4)}\\ +\frac{b_4a_2a_3n_2(0)}{(\delta_6-\delta_4)(\delta_2-\delta_4)(\delta_3-\delta_4)}+\frac{b_4a_3n_3(0)}{(\delta_6-\delta_4)(\delta_3-\delta_4)}+\frac{b_4n_4(0)}{(\delta_6-\delta_4)}\\ +\frac{b_4a_3b_1n_1(0)}{(\delta_6-\delta_4)(\delta_3-\delta_4)(\delta_1-\delta_4)}+\frac{b_4b_2a_1n_1(0)}{(\delta_6-\delta_4)(\delta_2-\delta_4)(\delta_1-\delta_4)}+\frac{b_4b_2n_2(0)}{(\delta_6-\delta_4)(\delta_2-\delta_4)}\end{array}\right]$$

$$+e^{-\delta 5t}\left[\begin{array}{c}\frac{a_1a_2a_3a_4a_5n_1(0)}{(\delta_1-\delta_5)(\delta_2-\delta_5)(\delta_3-\delta_5)(\delta_4-\delta_5)(\delta_6-\delta_5)}+\frac{a_2a_3a_4a_5n_2(0)}{(\delta_2-\delta_5)(\delta_3-\delta_5)(\delta_4-\delta_5)(\delta_6-\delta_5)}\\ +\frac{a_3a_4a_5n_3(0)}{(\delta_3-\delta_5)(\delta_4-\delta_5)(\delta_6-\delta_5)}+\frac{a_4a_5n_4(0)}{(\delta_4-\delta_5)(\delta_6-\delta_5)}+\frac{a_5n_5(0)}{(\delta_6-\delta_5)}\\ +\frac{a_3a_4a_5b_1n_1(0)}{(\delta_3-\delta_5)(\delta_4-\delta_5)(\delta_6-\delta_5)(\delta_1-\delta_5)}+\frac{a_4a_5b_2a_1n_1(0)}{(\delta_4-\delta_5)(\delta_6-\delta_5)(\delta_2-\delta_5)(\delta_1-\delta_5)}\\ +\frac{a_4a_5b_2n_2(0)}{(\delta_4-\delta_5)(\delta_6-\delta_5)(\delta_2-\delta_5)}+\frac{a_5b_3a_1a_2n_1(0)}{(\delta_6-\delta_5)(\delta_3-\delta_5)(\delta_1-\delta_5)(\delta_2-\delta_5)}\\ +\frac{a_5b_3a_2n_2(0)}{(\delta_6-\delta_5)(\delta_3-\delta_5)(\delta_2-\delta_5)}+\frac{a_5b_3n_3(0)}{(\delta_3-\delta_5)(\delta_6-\delta_5)}+\frac{a_5b_3b_1n_1(0)}{(\delta_6-\delta_5)(\delta_3-\delta_5)(\delta_1-\delta_5)}\end{array}\right]$$

$$+e^{-\delta 6t}\left[\begin{array}{c}\frac{a_1a_2a_3a_4a_5n_1(0)}{(\delta_1-\delta_6)(\delta_2-\delta_6)(\delta_3-\delta_6)(\delta_4-\delta_6)(\delta_5-\delta_6)}+\frac{a_2a_3a_4a_5n_2(0)}{(\delta_2-\delta_6)(\delta_3-\delta_6)(\delta_4-\delta_6)(\delta_5-\delta_6)}\\ +\frac{a_3a_4a_5n_3(0)}{(\delta_3-\delta_6)(\delta_4-\delta_6)(\delta_5-\delta_6)}+\frac{a_4a_5n_4(0)}{(\delta_4-\delta_6)(\delta_5-\delta_6)}+\frac{a_5n_5(0)}{(\delta_5-\delta_6)}+n_6(0)\\ +\frac{a_3a_4a_5b_1n_1(0)}{(\delta_3-\delta_6)(\delta_4-\delta_6)(\delta_5-\delta_6)(\delta_1-\delta_6)}+\frac{a_4a_5b_2n_2(0)}{(\delta_4-\delta_6)(\delta_5-\delta_6)(\delta_1-\delta_6)}\\ +\frac{a_4a_5b_2a_1n_1(0)}{(\delta_4-\delta_6)(\delta_5-\delta_6)(\delta_2-\delta_6)(\delta_1-\delta_6)}+\frac{a_5b_3a_1a_2n_1(0)}{(\delta_5-\delta_6)(\delta_3-\delta_6)(\delta_1-\delta_6)(\delta_2-\delta_6)}\\ +\frac{a_5b_3a_2n_2(0)}{(\delta_5-\delta_6)(\delta_3-\delta_6)(\delta_2-\delta_6)}+\frac{a_5b_3n_3(0)}{(\delta_5-\delta_6)(\delta_3-\delta_6)}+\frac{a_5b_3b_1n_1(0)}{(\delta_5-\delta_6)(\delta_3-\delta_6)(\delta_1-\delta_6)}\\ +\frac{b_4a_1a_2a_3n_1(0)}{(\delta_4-\delta_6)(\delta_1-\delta_6)(\delta_2-\delta_6)(\delta_3-\delta_6)}+\frac{b_4a_2a_3n_2(0)}{(\delta_4-\delta_6)(\delta_2-\delta_6)(\delta_3-\delta_6)}+\frac{b_4a_3n_3(0)}{(\delta_4-\delta_6)(\delta_3-\delta_6)}\\ +\frac{b_4n_4(0)}{(\delta_4-\delta_6)}+\frac{b_4a_3b_1n_1(0)}{(\delta_4-\delta_6)(\delta_3-\delta_6)(\delta_1-\delta_6)}+\frac{b_4b_2a_1n_1(0)}{(\delta_4-\delta_6)(\delta_2-\delta_6)(\delta_1-\delta_6)}+\frac{b_4b_2n_2(0)}{(\delta_4-\delta_6)(\delta_2-\delta_6)}\end{array}\right]$$

$$=e^{-2t}\left[\begin{array}{c}\frac{9}{5}+\frac{9}{5}+\frac{6}{5}\\ +\frac{6}{5}+\frac{6}{5}+\frac{12}{5}\\ +\frac{12}{5}+\frac{8}{5}\end{array}\right]+e^{-3t}\left[\begin{array}{c}-9+\frac{9}{2}-3\\ +\frac{3}{2}-4+2\\ -3+\frac{9}{2}+\frac{3}{2}-3\end{array}\right]$$

$$+e^{-4t}\begin{bmatrix} 18-18+9 \\ -18+4-4 \\ 2-4+12 \\ -12+6-12 \end{bmatrix} + e^{-5t}\begin{bmatrix} -18+27-27 \\ +\frac{15}{2}+36+6 \\ -9-6+9 \\ -9+\frac{5}{2}+12 \\ +2-3 \end{bmatrix}$$

$$+e^{-6t}\begin{bmatrix} 9-18+27 \\ -15+1-27 \\ -6+12-2 \\ +4-6+6 \end{bmatrix} + e^{-7t}\begin{bmatrix} \frac{-9}{5}+\frac{9}{2}-9 \\ \frac{15}{2}+1+\frac{3}{5}+\frac{32}{5} \\ +\frac{18}{5}+\frac{9}{5}+\frac{4}{5} \\ -2+4-\frac{16}{5} \\ \frac{3}{5}-\frac{3}{2}+3 \\ \frac{-5}{2}-\frac{12}{5}-\frac{3}{5}-\frac{3}{2} \end{bmatrix}$$

$$\Rightarrow n_6(t) = \frac{68}{5}e^{-2t} - 14e^{-3t} - 17e^{-4t} + 30e^{-5t} - 15e^{-6t} + 3e^{-7t}.$$

4.1.1 The Mathlab Code

```
function [s]=test(k)

disp('Enter sigma')

for i=1:k

sigma(i)=input('')

end

disp('Enter a')

for i=1:k-1

a(i)=input('')

end

disp('Enter b')

for i=1:k-2

b(i)=input('')

end

disp('Enter n')

for i=1:k

n(i)=input('')

end

OUT=[];

for m=1:2%k

OUT=[OUT Funcx(m,k,n,a,sigma,k)+Funcy(m,k,b,a,sigma,k,n) +Funcz(m,k,b,a,sigma,k,n)];

end
```

OUT

```
function sum1 = Funcx(m,y,n,a,sigma,k)
sum1=0;
for j=1:m
product=1;
for i=j:y-1;
if i >= y
break;
end
if i==m
product=product*a(i)/(sigma(k)-sigma(m));
else
product=product*a(i)/(sigma(i)-sigma(m));
end
end
sum1=sum1+product*n(j)
% u=sum1
end
end
function sum1=Funcy(m,y,b,a,sigma,k,n)
u=0;
sum1=0;
if y < 3
```

```
sum1=0;
else
for j=1:m-2
sum2=0;
product=1;
for i=j+2:k-1;
if i >= y
break;
end
if i==m
product=product*a(i)/(sigma(k)-sigma(m));
else
product=product*a(i)/(sigma(i)-sigma(m));
end
end
%xx=product
if j==m
sum2=sum2+product*b(j)/(sigma(k)-sigma(m));
else
sum2=sum2+product*b(j)/(sigma(j)-sigma(m));
end
sum2 = sum2*(Func1(m,j,n,a,sigma,k)+Func2(m,j,b,a,sigma,k,n));
u(j)=sum2;
```

```
end

sum1=sum(u)

% sum1=sum1+sum2

end

end

function sum1=Funcz(m,y,b,a,sigma,k,n)

sum1=0;

if y < 2 + m

sum1=0;

else

for j=m:k-2

sum2=0;

product=1;

for i=j+2:k-1;

if i >= k

break;

end

if i==m

product=product*a(i)/(sigma(k)-sigma(m));

else

product=product*a(i)/(sigma(i)-sigma(m));

end

end
```

```
if j==m
sum2=sum2+product*b(j)/(sigma(k)-sigma(m));
else
sum2=sum2+product*b(j)/(sigma(j)-sigma(m));
end
sum2 = sum2*(Funcx(m,j,n,a,sigma,k)+Func2(m,m,b,a,sigma,k,n)+Func3(m,j,b,a,sigma,k,n));
u(j)=sum2;
end
sum1=sum(u)
%sum1=sum1+sum2
end
end
function sum1 = Func1(m,y,n,a,sigma,k)
sum1=0;
for j=1:y
product=1;
for i=j:y-1;
if i >= y
break;
end
if i==m
product=product*a(i)/(sigma(k)-sigma(m));
else
```

```
product=product*a(i)/(sigma(i)-sigma(m));
end
end
sum1=sum1+product*n(j)
% u=sum1
end
end
function sum1=Func2(m,y,b,a,sigma,k,n)
sum1=0;
if y < 3
sum1=0;
else
for j=1:y-2
sum2=0;
product=1;
for i=j+2:y-1;
if i >= y
break;
end
if i==m
product=product*a(i)/(sigma(k)-sigma(m));
else
product=product*a(i)/(sigma(i)-sigma(m));
```

```
end
end
%xx=product
if j==m
sum2=sum2+product*b(j)/(sigma(k)-sigma(m));
else
sum2=sum2+product*b(j)/(sigma(j)-sigma(m));
end
sum2 = sum2*(Func1(m,j,n,a,sigma,k)+Func2(m,j,b,a,sigma,k,n));
u(j)=sum2;
end
sum1=sum(u)
% sum1=sum1+sum2
end
end
function sum1=Func3(m,y,b,a,sigma,k,n)
sum1=0;
if y < 2+m
sum1=0;
else
for j=m:y-2
sum2=0;
product=1;
```

```
for i=j+2:y-1;
if i >= y
break;
end
if i==m
product=product*a(i)/(sigma(k)-sigma(m));
else
product=product*a(i)/(sigma(i)-sigma(m));
end
end
if j==m
sum2=sum2+product*b(j)/(sigma(k)-sigma(m));
else
sum2=sum2+product*b(j)/(sigma(j)-sigma(m));
end
sum2 = sum2*(Funcx(m,j,n,a,sigma,k)+Func2(m,m,b,a,sigma,k,n)+Func3(m,j,b,a,sigma,k,n));
u(j)=sum2;
sum1=sum(u)
end
%sum1=sum1+sum2
end
end
```

4.2 The Explicit Solution for Countable ODEs for Constant Lower Triangular Matrix

In this section, we find the solution for system (3.1) when

A is an infinite lower Triangular matrix given by:

$$A = \begin{bmatrix} -\sigma_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ a_1 & -\sigma_2 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ b_1 & a_2 & -\sigma_3 & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ c_1 & b_2 & a_3 & -\sigma_4 & \ddots & \vdots & \vdots & \vdots & \cdots \\ d_1 & c_2 & \ddots & \ddots & \ddots & 0 & 0 & 0 & \cdots \\ e_1 & d_2 & \ddots & b_k & a_k & -\sigma_k & \ddots & 0 & \cdots \\ \vdots & e_2 & \ddots & c_{k+1} & b_{k+1} & a_{k+1} & -\sigma_{k+1} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \end{bmatrix}$$

The coefficients $\sigma_j, a_i, b_g, c_m, e_x, d_l...$ are constants, and $\sigma_i \neq \sigma_j$, if $i \neq j$.

Theorem 4.2.1 *The solution of this system is given by :*

$$
n_k(t) =
\sum_{m=1}^{k} \frac{e^{-\delta_m t}}{\delta_k-\delta_m} \left[\begin{matrix}
\sum_{j=1}^{m}(\alpha_{i=j}^{k-1}) n_j(0)+ \\
\sum_{y=1}^{m-2}(\alpha_{x=y+2}^{k-1})\frac{b_y}{\delta_y-\delta_m} \left[\begin{matrix} \sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \sum_{z=1}^{y-2}(\alpha_{w=z+2}^{y-1})\frac{b_z}{\delta_z-\delta_m}\left[\begin{matrix}\sum_{g=1}^{z}(\alpha_{f=g}^{z-1})n_g(0)+\\ \ldots+\end{matrix}\right]+ \\ \sum_{h=1}^{y-3}(\alpha_{f=h+2}^{y-1})\frac{c_h}{\delta_h-\delta_m}\left[\begin{matrix}\sum_{g=1}^{h}(\alpha_{f=g}^{h-1})n_g(0)+\\ \ldots+\end{matrix}\right]+ \\ \ldots+ \end{matrix}\right] \\
+ \\
\sum_{y=1}^{m-3}(\alpha_{x=y+3}^{k-1})\frac{c_y}{\delta_y-\delta_m} \left[\begin{matrix} \sum_{e=1}^{y}(\alpha_{d=e}^{y-1})n_e(0)+ \\ \sum_{z=1}^{y-2}(\alpha_{w=z+2}^{y-1})\frac{b_z}{\delta_z-\delta_m}\left[\begin{matrix}\sum_{g=1}^{z}(\alpha_{f=g}^{z-1})n_g(0)+\\ \ldots+\end{matrix}\right]+ \\ \sum_{h=1}^{y-3}(\alpha_{f=h+2}^{y-1})\frac{c_h}{\delta_h-\delta_m}\left[\begin{matrix}\sum_{g=1}^{h}(\alpha_{f=g}^{h-1})n_g(0)+\\ \ldots+\end{matrix}\right]+ \\ \ldots+ \end{matrix}\right] \\
+\ldots+ \\
\sum_{l=m}^{k-2}(\alpha_{h=l+2}^{k-1})\frac{b_l}{\delta_l-\delta_m}\left[\begin{matrix} \sum_{j=1}^{m}(\alpha_{i=j}^{l-1})n_j(0)+ \\ \sum_{z=1}^{m-2}(\alpha_{w=z+2}^{m-1})\frac{b_z}{\delta_z-\delta_m}\left[\begin{matrix}\sum_{g=1}^{z}(\alpha^{z-1}_{f=g})n_g(0)+\\ \ldots+\end{matrix}\right]+ \\ \sum_{p=m}^{l-2}(\alpha_{p=q+2}^{l-1})\frac{b_p}{\delta_p-\delta_m}\left[\begin{matrix}\sum_{R=1}^{p}(\alpha_{s=R}^{p-1})n_R(0)+\\ \sum_{z=1}^{p-2}(\alpha_{w=z+2}^{p-1})\frac{b_z}{\delta_z-\delta_m}\left[\begin{matrix}\sum_{g=1}^{z}(\alpha_{f=g}^{z-1})n_g(0)+\\ \ldots+\end{matrix}\right]\\ + \end{matrix}\right]\end{matrix}\right] \\
\ldots)
\end{matrix}\right]
$$

The proof of this theorem is the same of proof of Th4.1.2

Example:

Consider the following system

$$\begin{bmatrix} \dot{n}_1(t) \\ \dot{n}_2(t) \\ \dot{n}_3(t) \\ \dot{n}_4(t) \\ \dot{n}_5(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & ... \\ 3 & -4 & 0 & 0 & 0 & ... \\ 6 & 4 & -3 & 0 & 0 & ... \\ 18 & 6 & 3 & -2 & 0 & ... \\ 12 & 12 & 12 & 4 & -1 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \\ n_5(t) \\ \vdots \end{bmatrix}$$

$n_1(0) = \frac{1}{3}, n_2(0) = 2, n_3(0) = 3, n_4(0) = 4, n_5(0) = 5.$

First, we solve this system using the regular technique, then we use theorem (4.2.1).

To solve this system solution:

$\dot{n}_1(t) = -5n_1(t) \Longrightarrow \dot{n}_1(t) + 5n_1(t) = 0$

$e^{5t}\dot{n}_1(t) + 5e^{5t}n_1(t) = 0 \Longrightarrow \frac{d}{dt}\left(e^{5t}n_1(t)\right) = 0$

$e^{5t}n_1(t) = n_1(0) \Longrightarrow n_1(t) = \frac{1}{3}e^{-5t}$

$\dot{n}_2(t) = 3n_1(t) - 4n_2(t) \Longrightarrow e^{4t}\dot{n}_2(t) + 4e^{4t}n_2(t) = e^{4t}n_1(t)$

$\frac{d}{dt}\left(e^{4t}n_2(t)\right) = e^{4t}\left(e^{-5t}n_1(0)\right) \Longrightarrow e^{4t}n_2(t) - n_2(0) = \int_0^t e^{(4-5)\tau}d\tau$

$e^{4t}n_2(t) = \left[\frac{e^{1}-1}{1}\right] + n_2(0) \Longrightarrow n_2(t) = -e^{-5t} + e^{-4t} + 2e^{-4t}$

$then\ n_2(t) = -e^{-5t} + 3e^{-4t}$

$$\dot{n}_3(t) = 6n_1(t) + 4n_2(t) - 3n_3(t)$$

$$e^{3t}\dot{n}_3(t) + 3e^{3t}n_3(t) = 6e^{3t}n_1(t) + 4e^{3t}n_2(t)$$

$$\frac{d}{dt}\left(e^{3t}n_3(t)\right) = 6e^{3t}\left[\frac{1}{3}e^{-5t}\right] + 4e^{3t}\left[-e^{-5t} + 3e^{-4t}\right]$$

$$e^{3t}n_3(t) = \int_0^t (2e^{-2t} - 4e^{-2t} + 3e^{-t})d\tau + n_3(0)$$

$$e^{3t}n_3(t) = 2\left(\frac{e^{-2t}-1}{-2}\right) + 12\left(\frac{e^{-t}-1}{-1}\right) + 3$$

$$n_3(t) = e^{-5t} - 12e^{-4t} + 14e^{-3t}$$

$$\dot{n}_4(t) = 18n_1(t) + 6n_2(t) + 3n_3(t) - 2n_4(t)$$

$$\frac{d}{dt}\left(e^{2t}n_4(t)\right) = e^{2t}\left[18(\tfrac{1}{3}e^{-5t}) + 6(-e^{-5t} + 3e^{-4t}) + 3(e^{-5t} - 12e^{-4t} + 14e^{-3t})\right]$$

$$e^{-2t}n_4(t) = 3\left[\frac{e^{2-5t}-1}{-3}\right] - 18\left[\frac{e^{2-4t}-1}{-2}\right] + 42\left[\frac{e^{2-3t}-1}{-1}\right] + n_4(0)$$

$$n_4(t) = -e^{-5t} + 9e^{-4t} - 42e^{-3t} + 38e^{-2t}$$

$$\dot{n}_5(t) = -n_5(t) + 4n_4(t) + 12n_3(t) + 12n_2(t) + 12n_1(t)$$

$$\frac{d}{dt}\left(e^{t}n_5(t)\right) = e^{t}\left[\begin{array}{c} 4(-e^{-5t} + 9e^{-4t} - 42e^{-3t} + 38e^{-2t}) \\ +12(e^{-5t} - 12e^{-4t} + 14e^{-3t}) + 12(-e^{-5t} + 3e^{-4t}) + 12(\tfrac{1}{3}e^{-5t}) \end{array}\right]$$

$$e^{t}n_5(t) = -72(\tfrac{e^{-3t}-1}{-3}) + 152(\tfrac{e^{-t}-1}{-1}) + 5$$

$$\textit{then } n_5(t) = 24e^{-4t} - 152e^{-2t} + 133e^{-t}.$$

Now, the solution of the example by the theorem:

$$\begin{bmatrix} \dot{n}_1(t) \\ \dot{n}_2(t) \\ \dot{n}_3(t) \\ \dot{n}_4(t) \\ \dot{n}_5(t) \\ \vdots \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 & 0 & ... \\ 3 & -4 & 0 & 0 & 0 & ... \\ 6 & 4 & -3 & 0 & 0 & ... \\ 18 & 6 & 3 & -2 & 0 & ... \\ 12 & 12 & 12 & 4 & -1 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \times \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ n_4(t) \\ n_5(t) \\ \vdots \end{bmatrix}$$

$n_1(0) = \frac{1}{3}, n_2(0) = 2, n_3(0) = 3, n_4(0) = 4, n_5(0) = 5.$

Solution:

$$e^{-\delta_1 t} \begin{bmatrix} \frac{a_1a_2a_3a_4n_1(0)}{(\delta_2-\delta_1)(\delta_3-\delta_1)(\delta_4-\delta_1)(\delta_5-\delta_1)} + \frac{a_3a_4b_1n_1(0)}{(\delta_3-\delta_1)(\delta_4-\delta_1)(\delta_5-\delta_1)} \\ +\frac{a_4b_2a_1n_1(0)}{(\delta_4-\delta_1)(\delta_5-\delta_1)(\delta_2-\delta_1)} + \frac{b_3a_1a_2n_1(0)}{(\delta_3-\delta_1)(\delta_5-\delta_1)(\delta_2-\delta_1)} \\ +\frac{b_3b_1n_1(0)}{(\delta_3-\delta_1)(\delta_5-\delta_1)} + \frac{a_4c_1n_1(0)}{(\delta_4-\delta_1)(\delta_5-\delta_1)} \\ +\frac{c_2a_1n_1(0)}{(\delta_2-\delta_1)(\delta_5-\delta_1)} + \frac{d_1n_1(0)}{(\delta_5-\delta_1)} \end{bmatrix}$$

$$+e^{-\delta_2 t} \begin{bmatrix} \frac{a_1a_2a_3a_4n_1(0)}{(\delta_1-\delta_2)(\delta_3-\delta_2)(\delta_4-\delta_2)(\delta_5-\delta_2)} + \frac{a_2a_3a_4n_1(0)}{(\delta_3-\delta_2)(\delta_4-\delta_2)(\delta_5-\delta_2)} \\ +\frac{a_4b_2a_1n_1(0)}{(\delta_4-\delta_2)(\delta_5-\delta_2)(\delta_1-\delta_2)} + \frac{a_4b_2n_2(0)}{(\delta_4-\delta_2)(\delta_5-\delta_2)} \\ +\frac{b_3a_1a_2n_1(0)}{(\delta_3-\delta_2)(\delta_1-\delta_2)(\delta_5-\delta_2)} + \frac{b_3a_2n_2(0)}{(\delta_3-\delta_2)(\delta_5-\delta_2)} \\ +\frac{c_2a_1n_1(0)}{(\delta_5-\delta_2)(\delta_1-\delta_2)} + \frac{c_2n_2(0)}{(\delta_5-\delta_2)} \end{bmatrix}$$

$$+e^{-\delta_3 t} \begin{bmatrix} \frac{a_1a_2a_3a_4n_1(0)}{(\delta_1-\delta_3)(\delta_2-\delta_3)(\delta_4-\delta_3)(\delta_5-\delta_3)} + \frac{a_2a_3a_4n_2(0)}{(\delta_2-\delta_3)(\delta_4-\delta_3)(\delta_5-\delta_3)} \\ +\frac{a_3a_4n_3(0)}{(\delta_4-\delta_3)(\delta_5-\delta_3)} + \frac{a_3a_4b_1n_1(0)}{(\delta_4-\delta_3)(\delta_5-\delta_3)(\delta_1-\delta_3)} \\ +\frac{b_3a_1a_2n_1(0)}{(\delta_5-\delta_3)(\delta_1-\delta_3)(\delta_2-\delta_3)} + \frac{b_3a_2n_2(0)}{(\delta_5-\delta_3)(\delta_2-\delta_3)} \\ +\frac{b_3n_3(0)}{(\delta_5-\delta_3)(\delta_6-\delta_3)} + \frac{b_3b_1n_1(0)}{(\delta_5-\delta_3)(\delta_6-\delta_3)(\delta_1-\delta_3)} \end{bmatrix}$$

$$+e^{-\delta 4t} \begin{bmatrix} \frac{a_1a_2a_3a_4n_1(0)}{(\delta_1-\delta_4)(\delta_2-\delta_4)(\delta_3-\delta_4)(\delta_5-\delta_4)} + \frac{a_2a_3a_4n_2(0)}{(\delta_2-\delta_4)(\delta_3-\delta_4)(\delta_5-\delta_4)} \\ +\frac{a_3a_4n_3(0)}{(\delta_3-\delta_4)(\delta_5-\delta_4)} + \frac{a_4n_4(0)}{(\delta_5-\delta_4)} + \frac{a_3a_4b_1n_1(0)}{(\delta_3-\delta_4)(\delta_5-\delta_4)(\delta_1-\delta_4)} \\ +\frac{a_4b_2a_1n_1(0)}{(\delta_5-\delta_4)(\delta_2-\delta_4)(\delta_1-\delta_4)} + \frac{a_4b_2n_2(0)}{(\delta_5-\delta_4)(\delta_2-\delta_4)} + \frac{a_4c_1n_1(0)}{(\delta_5-\delta_4)(\delta_1-\delta_4)} \end{bmatrix}$$

$$+e^{-\delta 5t}\begin{bmatrix} \frac{a_1a_2a_3a_4n_1(0)}{(\delta_1-\delta_5)(\delta_2-\delta_5)(\delta_3-\delta_5)(\delta_4-\delta_5)} + \frac{a_2a_3a_4n_2(0)}{(\delta_2-\delta_5)(\delta_3-\delta_5)(\delta_4-\delta_5)} \\ +\frac{a_3a_4n_3(0)}{(\delta_3-\delta_5)(\delta_4-\delta_5)} + \frac{a_4n_4(0)}{(\delta_4-\delta_5)} + n_5(0) \\ +\frac{a_3a_4b_1n_1(0)}{(\delta_3-\delta_5)(\delta_4-\delta_5)(\delta_1-\delta_5)} + \frac{a_4b_2a_1n_1(0)}{(\delta_4-\delta_5)(\delta_2-\delta_5)(\delta_1-\delta_5)} \\ +\frac{a_4b_2n_2(0)}{(\delta_4-\delta_5)(\delta_2-\delta_5)} + \frac{b_3a_1a_2n_1(0)}{(\delta_3-\delta_5)(\delta_1-\delta_5)(\delta_2-\delta_5)} \\ +\frac{b_3a_2n_2(0)}{(\delta_3-\delta_5)(\delta_2-\delta_5)} + \frac{b_3n_3(0)}{(\delta_3-\delta_5)} + \frac{b_3b_1n_1(0)}{(\delta_3-\delta_5)(\delta_1-\delta_5)} \\ \frac{a_4c_1n_1(0)}{(\delta_4-\delta_5)(\delta_1-\delta_5)} + \frac{c_2a_1n_1(0)}{(\delta_2-\delta_5)(\delta_1-\delta_5)} + \frac{c_2n_2(0)}{(\delta_2-\delta_5)} + \frac{d_1n_1}{(\delta_1-\delta_5)} \end{bmatrix}$$

$$= e^{-5t}\begin{bmatrix} 2+-1 \\ -2-6 \\ +3+2 \\ +3-1 \end{bmatrix} + e^{-4t}\begin{bmatrix} -8-16 \\ +4+8 \\ +16+32 \\ -4-8 \end{bmatrix}$$

$$e^{-3t}\begin{bmatrix} 12+48 \\ +18+6 \\ -12-48 \\ -18-6 \end{bmatrix} + e^{-2t}\begin{bmatrix} -8-48 \\ -48-8 \\ -4-4 \\ -24-8 \end{bmatrix}$$

$$e^{-t}\begin{bmatrix} 2+16+18 \\ +16+5+3 \\ +2+16+2 \\ +16+8+3 \\ +6+1+8+1 \end{bmatrix}$$

$then\ n_5(t) = 24e^{-4t} - 152e^{-2t} + 133e^{-t}.$

Conclusions

In this thesis, explicit solutions to an infinite countable system of Ordinary Linear Differential Equations with constant coefficients are found. For this purpose, several theorems are provided and proved. A very important lemma, which can be used in a very wide range of Mathematical fields, is used to prove this theorems. an explicit solution to an infinite countable system of linear ODE for diagonal and bi-diagonal is provided. Also, a solution of the infinite countable system of linear ODE is given when the coefficient matrix is tri-diagonal and lower diagonal. Mathlab code are given for the bi-diagonal and tri-diagonal cases.

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Abstract

الحل الصريح لحل نظام لانهائي من المعدلات التفاضليه العاديه الخطيه ذات المعاملات الثابته لمصفوفه ثلاثيه الاقطار

حنين وليد كامل زغلول

في هذه الأطروحة،قمنا باشتقاق الحل الصريح لنظام لانهائي من المعادلات التفاضليه العاديه الخطيه ذات المعاملات الثابته. طبقنا بعض التقنيات الرياضيه وقد وجدنا الحل لنظام عند مصفوفه معملات تلاتثه الاقطار ومصفوفه معاملات قطريه . بالاضافه لبرنامج الماثلاب.