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Semi-Analytical Solution of Volterra Integral Equation by DTM and RDTM

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Dedication

To my parents,

To my husband,

To my dear children,

To all my family.

Acknowledgment

First of all, a great and continuous thank to Allah who gave me the effort and the ability to overcome all difficulties that I faced during the last three years in my study.

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I will never forget my dear husband, who without his support, I could not manage to achieve my goal.

I must acknowledge as well the many friends, colleagues, students, teachers who assisted, advised, and supported my research over the years.

Abstract

The differential transform method (DTM) and the reduced differential transform method (RDTM) are semi-analytical numerical methods which can be applied to various kinds of ordinary and partial differential equations besides several types of integral equations.

In this work, the theory of DTM and RDTM of one and two-dimensional Volterra Integral Equation (VIE) have been introduced. The solution of three- dimensional VIE has been investigated. In addition, we have proved some additional theorems related to this type of integral equations. Furthermore, the theory of three dimensional DTM has been successfully extended to three dimensional RDTM. Finally, a comparison between the two methods has been carried and to show the advantage of RDTM over DTM in solving three dimensional VIE.

حل شبه تحليلي لمعادلة فولتيرا التكاملية باسلوب التحويل التفاضلي واسلوب التحويل التفاضلي المختزل

ملخص

أسلوبي التحويل التفاضلي و أسلوب التحويل التفاضلي المختزل هي عمليات شبه تحليلية عددية يمكن تطبيقها على أنواع مختلفة من المعادلات التفاضلية العادية و الجزئية و كذلك المعادلات التكاملية .

في هذا البحث، تم تقديم الجانب النظري لأسلوبي التحويل التفاضلي و التحويل التفاضلي المختزل لمعادلات فولتيرا ذات البعد الواحد و البعدين، ثم دراسة حل معادلة فولتيرا ذات البعد الثلاثي بشكل مستفيض و إضافة نظريات و أمثلة متعلقة بهذا النوع من المعادلات. علاوة على ذلك، استطعنا توسيع الجانب النظري لأسلوب التحويل التفاضلي ذو البعد الثلاثي إلى أسلوب التحويل التفاضلي المختزل ذو البعد الثلاثي. و أخيراً، تم إجراء مقارنة بين الاسلوبين وأظهرت النتائج أفضلية أسلوب التحويل التفاضلي المختزل على التحويل التفاضلي في حل معادلة فولتيرا التكاملية ذات البعد الثالث.

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Introduction

Integral equations are equations in which the function to be determined appears under one or more integral signs. The concept of integral equations interferes with many issues in the field of pure and applied mathematics. Many initial and boundary value problems can be reformulated to the shape of integral equations. Further, in physics and engineering fields, integral equations have wide applications, for example, potential theory, scattering in quantum mechanics, and mathematical physics modeling [1, 2, 3]. The most frequently used integral equations are of two major types, namely Volterra (an Italian mathematician Vito volterra,1860-1920) and Fredholm (Swedish mathematician Ivar Fredholm, 1866-1927) integral equations [2]. Volterra Integral Equation (VIE) has been used to model several kinds of problem. For example, they are considered to be stochastic integrals of a time-dependent kernel with respect to a standard Brownian motion [4]. In addition, they have many applications in demography and actuarial science [5].

In the literature, numerical and analytical methods have been developed in the last and the present centuries to solve VIE. One of these methods is the differential transform method (DTM). Recently DTM has been modified to the reduced differential transform method (RDTM). However, the differential transform method is considered to be one of the useful techniques in solving VIE, despite the significant time needed to calculate the higher derivatives [1].

Since the beginning of 1986, Zhou was the first who initiated and introduced the idea of the DTM [8]. He showed the efficiency of the method by a successful application to the electrical circuit's analysis problem. In the last and in the present decades, researchers have applied the method and its variants to various linear and nonlinear ordinary and partial differential equations [7-24]. In 2007 Z. M. Odibat [25] was the first who successfully applied the differential transform method (DTM) to find exact solutions to linear and nonlinear VEI of dimension one, he has managed to express the solution of such problems in Taylor series expansion form. The method could overcome the computational difficulties of other methods and all calculations can be simply manipulated. He tested many examples by applying DTM and the results illustrated the reliability and the performance of the DTM. In 2009 A. Tari, M.Y. Rahimi and S. Shahmorad developed the two dimensional DTM for solving two dimensional linear and nonlinear VIE [26]. They gave examples to demonstrate the accuracy of the method. Jang [27], introduced some properties concerning the two dimensional VIE . In 2009, M. Mohseni Moghadam and H.A. Saeedi [28], used DTM to solved Volterra integro-differential equations. In 2012, M. Bakhshi, M. Asghari-larimi, Mohammad Asghari-Larimi [29] have successfully tested the three -dimensional DTM on a nonlinear three-dimensional VIE. In 2013, Reza Abazari and AdemKilicman [30], developed the two dimensional RDTM for solving two dimensional VIE, they also made a comparison between the two solutions getting from RDTM and DTM. They concluded that RDTM is more accurate, it has faster convergence rate, and has less complicated computational process.

In this thesis, a numerical solution of three-dimensional VIE is proposed. We solve such equation using RDTM, then, a comparison between this method and DTM is done.

In chapter one, basic definitions of integral equations are presented. Chapter two contains a complete proof of some fundamental theorems concerning the solution of one and two dimensional VIE using DTM. Also, it includes some illustrative examples.

Chapter three contains some basic concepts on the solution of three dimensional VIE using DTM and some numerical examples on the presented theory.
In chapter four, the technique of RDTM in solving two dimensional VIE is introduced. For that, some basic definitions and properties of RDTM are given. Then, we apply the method on some examples and clarify the rapid convergence of this method by some figures.
In chapter five, we present results relating to the solution of three dimensional VIE via RDTM. Also, we apply the produced theory on some examples which are solved earlier using DTM. Finally, we compare the results obtained from the two methods by showing the relative and absolute error at several points in the domain of the integral equation.

Chapter One

Preliminaries

In this chapter, the definitions of VIE in one, two and three variables are given in section one. In section two, we introduce the differential transform method.

1.1 Volterra Integral Equation

The integral equation, which is first introduced in 1888 by Paul Du Bois-Reymond [31], is an equation in which the unknown function appears under one or more integral sings. A standard form of the integral equation in $u(x)$ is given by

$$\phi(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt$$

where $k(x,t)$ is called the Kernel of the integral equation, $\alpha(x)$ and $\beta(x)$ are limits of integration, λ is nonzero constant, $f(x)$ is any given function. The function $u(x)$ that will be determined appears inside the integral sing and outside the integral sing, where the limits of integration $\alpha(x)$ and $\beta(x)$ may be variables, constant or mixed.

If the limit of integration are fixed, then the integral equation is called Fredholm integral equation given by the form:

$$\phi(x)u(x) = f(x) + \lambda \int_{a}^{b} k(x,t)u(t)dt$$

If one of the limits is variable, then the integral equation is called Volterra integral equation which is given by the form:

$$\phi(x)u(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt$$

If the function $\phi(x) = 1$, then we have the second kind of VIE

$$u(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt$$

While $\phi(x) = 0$ leads to the first kind of VIE

$$f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt = 0$$

VIE can be classified according to linearity and homogeneity concept. If the integral equation contains nonlinear function of $u(x)$, such as $e^{u(x)}$, $\sin u(x)$, $\cosh u(x)$, $\ln(u(x))$ or $(u(x))^n$ where $n \neq 1$, the integral equation is called nonlinear. The integral equation is called linear if $n = 1$.

VIE of the second kind are classified as homogeneous or inhomogeneous, if the function $f(x)$ is identically zero, then VIE of second kind is called homogeneous. Otherwise it is called inhomogeneous.

Theorem 1.1[33]: For each $f(x) \in C[a,b]$, the Volterra integral equation of the second kind with continuous kernel $k(x,t)$ has a unique solution $u(x) \in C[a,b]$.

A function which can be represented by a convergent power series is called an ***analytic function***. In others words, functions that are infinitely differentiable. Moreover, a function is analytic if and only if its Taylor series about a point x_0 converges to that function in some neighborhood for every x in its domain.

In the next two definitions, we present the two and three dimensional VIE of the second kind.

Definition 1.2[34]: The two dimensional VIE is defined by:

$$u(x,t) = f(x,t) + \int_{t_0}^{t}\int_{x_0}^{x} K(x,t,y,z)u(y,z)dydz \tag{1.1}$$

Where the kernel K has the following degenerate form

$$K(x,t,y,z) = \sum_{i=0}^{n} M_i(x,t)N_i(y,z) \tag{1.2}$$

Definition 1.3[29]: The three dimensional VIEs is defined by

$$u(x,y,t) = f(x,y,t) + \int_{t_0}^{t}\int_{y_0}^{y}\int_{x_0}^{x} K(x,y,t,\omega,z,\tau)u(\omega,z,\tau)d\omega dz d\tau \tag{1.3}$$

The kernel K is degenerate of the form

$$K(x,y,t,\omega,z,\tau) = \sum_{i=0}^{n} M_i(x,y,t)N_i(\omega,z,\tau) \tag{1.4}$$

The functions M_i and N_i appear in equations (1.2) and (1.4) are assumed to be analytic for $i = 1,2,\dots,n$.

Finally, we close this section by the so called Leibniz formula, such formula plays a main role in solving integral equations.

Leibniz Rule [35]: let $R = \{(x,t) : a < x < b, c < t < d\}$ be a region in the xt - plane. Let $f(x,t)$ be a continuous function defined on R, such that $f_x(x,t)$ is continuous on R. Moreover, assume that $\int_{A(x)}^{B(x)} |f(x,t)|dt < \infty$, for each t in (c,d) for each continuous functions $A(x)$ and $B(x)$. Suppose also there is a piecewise continuous function $g(t)$ such that for all (x,t) in R, $|f_x(x,t)| < g(t)$ and $\int_{A(x)}^{B(x)} g(t)dt < \infty$. Then

$$\frac{d}{dx}\int_{A(x)}^{B(x)} f(x,t)dt = \int_{A(x)}^{B(x)} f_x(x,t)dt + f(x,B(x))B'(x) - f(x,A(x),)A'(x) \tag{1.5}$$

For special case, the partial derivatives of a function of two variables under double integral signs are listed below:

$$\frac{\partial^m}{\partial x^m}\int_0^t\int_0^x u(y,z)dydz = \int_0^t \frac{\partial^{m-1}}{\partial x^{m-1}}u(x,z)dz$$, and

$$\frac{\partial^n}{\partial t^n}\int_0^t\int_0^x u(y,z)dydz = \int_0^x \frac{\partial^{n-1}}{\partial t^{n-1}}u(y,t)dy$$

Leibniz Formula [36]: Let $u(x)$ and $v(x)$ be functions having derivatives up to order n, then the n^{th} derivatives of their product gives by:

$$\frac{d^n}{dx^n}(u(x)v(x)) = \sum_{m=0}^{n}\binom{n}{m}\frac{d^{n-m}}{dx^{n-m}}u(x)\frac{d^m}{dx^m}v(x) \tag{1.6}$$

1.2 The Differential Transform Method

In this section, we give the definitions of the differential transform and the differential inverse transform of functions in one variables.

Definition 1.4[25]: The differential transform of the k^{th} derivative of function $f(x)$ is defined by:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{1.7}$$

Definition 1.5 [25]: The differential inverse transform of $F(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k \,. \tag{1.8}$$

From equations (1.7) and (1.8), we get

$$f(x) = \sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{1.9}$$

This means that the concept of differential transform is derived from the Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, the relative derivatives are calculated by an iterative way which is described by the transformed equations of the original function.

In real applications, the function $f(x)$ is expressed by a finite series and equation (1.8) can be written as

$$f(x) = \sum_{k=0}^{n} F(k)(x-x_0)^k \tag{1.10}$$

The DTM determines the coefficients of the Taylor series of a function by solving recursive equations from the given differential equation.

Assuming the one dimensional VIE, there are a host of available solution techniques, one of the most important classical techniques are the method of successive approximations and the method of successive substitutions [37]. In addition, the series method and the direct computational method are also suitable for some problems [38]. The recently developed methods, namely the Adomain decomposition method (ADM) and the modified decomposition method are gaining popularity among scientists and engineers for solving highly nonlinear integral equations and singular integral equations [37,38], encountered by Abel [33]. The aim of the next chapter is to present the technique of DTM for solving one and two dimensional VIE.

Chapter Two

Solving One and Two- Dimensional Volterra Integral Equation by DTM

This chapter consists of two sections. In section one, we give a complete proof of some fundamental theorems on 1-dimentional VIE using the DTM. In section two, we discuss the solvability of 2-dimentioal VIE using DTM.

2.1 Solving One Dimensional Volterra Integral Equation by DTM

In this section, we introduce fundamental theorems on the one dimensional VIE and present their proofs in more details. Moreover, we apply the method on various kinds of differential equations appear in [8-9, 25].
The following theorems give the differential transformation for some functions that have the first kind of VIE shape. For convenience, we will use the notations $U(k)$,$G(k)$ and $F(k)$ to indicate the differential transform of the functions $u(x)$, $g(x)$ and $f(x)$ respectively.

The proof of the following theorem can be deduced directly from equations (1.7-8) and Leibniz formula so we skip the proof.

Theorem 2.1[10, 25]: If $f(x)=\int_0^x u(t)dt$, then $F(k)=\dfrac{U(k-1)}{k}$ and $F(0)=0$

Theorem 2.2[10, 25]: If $f(x)=\int_0^x g(t)u(t)dt$, then $F(k)=\sum_{l=0}^{k-1} G(l)\dfrac{U(k-1-l)}{k}$,

where $F(0)=0$.

Proof: Using Leibniz formula, the k^{th} derivative of the given VIE is

$$\left[\frac{d^k f(x)}{dx^k}\right]=\frac{d^{k-1}(g(x)u(x))}{dx^{k-1}}$$

$$=\sum_{l=0}^{k-1}\binom{k-1}{l}\frac{d^l g(x)}{dx^l}\frac{d^{k-1-l}u(x)}{dx^{k-1-l}}$$

Therefore,

$$\left[\frac{d^k f(x)}{dx^k}\right]_{x=0}=\sum_{l=0}^{k-1}\binom{k-1}{l}l!G(l)(k-1-l)!U(k-1-l)$$

$$=(k-1)!\sum_{l=0}^{k-1}G(l)U(k-1-l)$$

And equation (1.7) completes the proof of the theorem.

Theorem 2.3 [10, 25]: If $f(x)=g(x)\int_0^x u(t)dt$ then $F(k)=\sum_{l=0}^{k-1}G(l)\frac{U(k-l-1)}{k-l}$, where $F(0)=0$.

Proof: Assume that $k\geq 1$ and let $v(x)=\int_0^x u(t)dt$. Thus, f can be written as $f(x)=g(x)v(x)$. From equation (1.6), the k^{th} derivative is

$$\left[\frac{d^k f(x)}{dx^k}\right]=\frac{d^k(g(x)v(x))}{dx^k}=\sum_{l=0}^{k}\binom{k}{l}\frac{d^l g(x)d^{k-l}v(x)}{dx^l dx^{k-l}}.$$

But since for $k=l$, $\left[\frac{d^{k-l}v(x)}{dx^{k-l}}\right]_{\substack{x=x_0\\t=t_0}}=0$ the upper limit sum in the above equation can be reduced to $k-1$. Therefore,

$$\left[\frac{d^k f(x)}{dx^k}\right]_{x=0}=\sum_{l=0}^{k-1}\binom{k}{l}l!G(l)(k-l)!V(k-l)=k!\sum_{l=0}^{k-1}G(l)V(k-l)$$

Taking in account that $v(x)=\int_0^x u(t)dt$, the result can be drawn directly from theorem (2.1), and this finished the proof of the theorem.

Now, we give some examples to show how the above theorems are applicable.

Example 2.1: Consider the linear VIE.

$$u(x)=(1+x)+\lambda\int_0^x (x-t)u(t)dt \quad , \; 0<x<1$$

According to theorems (2.2-3) and from the table in appendix A.1, we have the following recurrence relation:

$$U(k)=\delta(k-1)+\lambda\sum_{l=0}^{k-1}\delta(l-1)\frac{U(k-l-1)}{k-1}-\lambda\sum_{l=0}^{k-1}\delta(l-1)\frac{U(k-l-1)}{k},k\geq 1$$

where $U(0)=u(0)=1.$ Then

$$U(0)=U(1)=1,\; U(2)=\frac{\lambda}{2!},\; U(3)=\frac{\lambda}{3!},\; U(4)=\frac{\lambda^2}{4!},\; U(5)=\frac{\lambda^2}{5!},....$$

From the above sequence we conclude the following general term

$$U(k)=\begin{cases}\frac{\lambda^{k/2}}{k!}, & k \text{ is even}\\[2ex] \frac{\lambda^{(k-1)/2}}{k!}, & k \text{ is odd}\end{cases}.$$

From equation (1.8), the solution of the integral equation is

$$u(x) = 1 + x + \frac{\lambda}{2!}x^2 + \frac{\lambda}{3!}x^3 + \frac{\lambda^2}{4!}x^4 + \frac{\lambda^2}{5!}x^5 + \dots\dots\dots$$

As a special case when $\lambda = 1$, the exact solution of the integral equation is $u(x) = e^x$.

Example 2.2: consider the nonlinear Volterra integral equation

$$u(x) = \sin x + \frac{1}{8}\sin(2x) - \frac{x}{4} + \frac{1}{2}\int_0^x u^2(t)dt$$

From theorem (2.2) and the table in appendix A.1, we have the following recurrence relation

$$U(k) = \frac{1}{k!}\sin\frac{(\pi k)}{2} + \frac{2^{k-3}}{k!}\sin\frac{(\pi k)}{2} - \frac{\delta(k-1)}{4} + \frac{1}{2}\sum_{l=0}^{k-1} U(l)\frac{U(k-l-1)}{k}.$$

Direct substitutions give the following sequence:

$$U(0) = 0, U(1) = 1,\ U(2) = 0,\ U(3) = -\frac{1}{3!},\ U(4) = 0,\ U(5) = \frac{1}{5!}, \dots$$

Hence $U(2k) = 0$, and $U(2k+1) = \dfrac{(-1)^k}{(2k+1)!}$.

From equation (1.7), the solution of the integral equation is given by

$u(x) = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots = \sin x$ (Which is the exact solution)

2.2 Solving Two Dimensional Volterra Integral Equation by DTM

This section contains definitions and fundamental theorems on VIE of dimension two. Further, examples using the method of two dimensional DTM has been introduced.

Consider a function of two variables $w(x,t)$ and suppose that it can be represented by a product of two functions each of which is of one variable, i.e. $w(x,t) = f(x)g(t)$. So, on the light of one-dimensional differential transform properties, the function $w(x,t)$ can be represented by [27]:

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j}^{\infty} G(j)t^j = \sum_{i=0}^{\infty} W(i,j)x^i t^j \tag{2.1}$$

Where $W(i,j) = F(i)G(j)$ is the spectrum of $w(x,t)$.

The basic definitions and operations for two-dimensional transform will be listed below:

Definition 2.1[26]: If $w(x,t)$ is analytic and continuously differentiable in the domain of interest, then the differential transform function of $w(x,t)$ is define by

$$W(m,n)=\frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n}w(x,t)\right]_{x=x_0,t=t_0} \tag{2.2}$$

Definition 2.2 [26]: The differential inverse transform of $W(m,n)$ is defined by

$$w(x,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}W(m,n)(x-x_0)^m\ (t-t_0)^n. \tag{2.3}$$

Combining (2.1) and (2.2) gives

$$w(x,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n}w(x,t)\right]_{x=x_0,t=t_0} \tag{2.4}$$

In real application, the function $w(x,t)$ in (2.4) can be approximated around $(0,0)$

$$w_{M,N}(x,t)=\sum_{m=0}^{M}\sum_{n=0}^{N}W(m,n)x^m t^n+R_{M,N}(x,t). \tag{2.5}$$

Using the assumption $w(x,t)=f(x)g(t)$ and expressing $f(x)$ and $g(t)$ as power series around (x_0,t_0), we get

$$w(x,t)=\sum_{i=0}^{\infty}F(i)(x-x_0)^i\sum_{j=0}^{\infty}G(j)(t-t_0)^j$$

$$\frac{\partial w(x,t)}{\partial x}=i\sum_{i=1}^{\infty}F(i)(x-x_0)^{i-1}\sum_{j=0}^{\infty}G(j)(t-t_0)^j$$

$$\frac{\partial w(x,t)}{\partial t}=\sum_{i=0}^{\infty}F(i)(x-x_0)^i\,(j)\sum_{j=1}^{\infty}G(j)(t-t_0)^{j-1}$$

$$\frac{\partial^2 w(x,t)}{\partial x\partial t}=i\sum_{i=1}^{\infty}F(i)(x-x_0)^{i-1}(j)\sum_{j=1}^{\infty}G(j)(t-t_0)^{j-1}$$

So, inductively we conclude the following formula

$$\frac{\partial^{m+n}w(x,t)}{\partial x^m\partial t^n}=((i)(i-1)...(i-m+1)\sum_{i=m}^{\infty}F(i)(x-x_0)^{i-m})((j)(j-1)...(j-n+1)\sum_{j=n}^{\infty}G(j)(t-t_0)^{j-n}) \tag{2.6}$$

Set $i=m$ and $j=n$ to reduce (2.6) to the following formula

$$\frac{\partial^{m+n}w(x,t)}{\partial x^m\partial t^n}=(m!)F(m)(n!)G(n)\cdot$$

Therefore, $\frac{1}{m!n!}[\frac{\partial^{m+n}w(x,t)}{\partial x^m\partial t^n}]=F(m)G(n)$

Finally, assume $W(m,n)=F(m)G(n)$ to be the transformed form of $w(x,y)$. Then

$$W(m,n)=\frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m \partial t^n}w(x,t)\right]_{\substack{x=x_0\\t=t_0}}.$$

From the above analysis we conclude that the concept of the two-dimensional differential transform method is derived from the two- dimensional Taylor series expansion.

Below, we give theorems which are useful in solving some kinds of integral equations. To each theorem a complete proof will be given. We suppose also that $W(m,n)$, $V(m,n)$ and $U(m,n)$ are the differential transformations of the functions $w(x,y)$, $v(x,y)$ and $u(x,y)$ respectively.

Theorem2.4 [26, 27]: If $w(x,t)=\int_0^t\int_0^x u(z,\tau)dzd\tau$, then

$$W(m,n)=\begin{cases}\dfrac{1}{nm}U(m-1,n-1) & n,m=1,2,...\\ 0 & Otherwise\end{cases}$$

Proof*:* From equation (2.4), we have

$$W(0,0)=\frac{1}{0!0!}\left[\frac{\partial^{0+0}}{\partial x^0\partial t^0}W(x,t)\right]_{x=0,t=0}=\left[w(x,t)\right]_{x=0,t=0}=\int_0^0\int_0^0 u(z,\tau)dzd\tau=0$$

The m^{th} partial derivative with respect to x is $\dfrac{\partial^m w(x,t)}{\partial x^m}=\int_0^t\dfrac{\partial^{m-1}}{\partial t^{m-1}}(u(x,\tau))d\tau$,

and the n^{th} partial derivative with respect to t is $\dfrac{\partial^n w(x,t)}{\partial t^n}=\int_0^x\dfrac{\partial^{n-1}}{\partial t^{n-1}}(u(z,t))dz$.

Therefore, $\left[\dfrac{\partial^n w(x,t)}{\partial t^n}\right]_{x=0,t=0}=0$, and $\left[\dfrac{\partial^m w(x,t)}{\partial x^m}\right]_{x=0,t=0}=0$. This proves the first assertion of the theorem. For the next part of the theorem, differentiate $w(x,t)$ with respect to x and then with respect to t, with the Leibniz formula and appendix A.2, we have

$(m+1)(n+1)W(m+1,n+1)=U(m,n)$, $m,n=0,1,2,\ldots.$

Replacing $m+1$ by m and $n+1$ by n in the last equation will finish the proof.

Theorem 2.5[26, 27]: If $w(x,t)=\int_0^t\int_0^x u(z,\tau)v(z,\tau)dzd\tau$ then

$$W(m,n)=\begin{cases}\dfrac{1}{nm}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}U(k,l)V(m-k-1,n-l-1), & n,m=1,2,...\\ 0 & Otherwise\end{cases}$$

Proof: Assume $m,n \geq 1$. According to the fundamental operation of two dimensional DTM listed in Appendix A.2 and from Leibniz formula, we get

$$\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right] = \frac{\partial^{m+n-2}(u(x,t)v(x,t))}{\partial x^{m-1}\partial t^{n-1}}$$

$$= \frac{\partial^{m-1}}{\partial x^{m-1}}\left[\frac{\partial^{n-1} u(x,t)v(x,t)}{\partial t^{n-1}}\right]$$

$$= \frac{\partial^{m-1}}{\partial x^{m-1}}\left[\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{\partial^l u(x,t)}{\partial t^l}\frac{\partial^{n-1-l} v(x,t)}{\partial t^{n-1-l}}\right]$$

$$= \sum_{l=0}^{n-1}\binom{n-1}{l}\left[\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\frac{\partial^l u(x,t)}{\partial t^l}\frac{\partial^{n-1-l} v(x,t)}{\partial t^{n-1-l}}\right)\right]$$

$$= \sum_{l=0}^{n-1}\binom{n-1}{l}\left[\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{\partial^{k+l} u(x,t)}{\partial x^k \partial t^l}\frac{\partial^{m+n-l-k-2} v(x,t)}{\partial x^{m-k-1}\partial t^{n-l-1}}\right]$$

$$= \sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\binom{n-1}{l}\binom{m-1}{k}\left(\frac{\partial^{k+l} u(x,t)}{\partial x^k \partial t^l}\right)\left(\frac{\partial^{m+n-l-k-2} v(x,t)}{\partial x^{m-k-1}\partial t^{n-l-1}}\right)$$

Therefore,

$$\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right]_{\substack{x=0\\t=0}} = \sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\binom{n-1}{l}\binom{m-1}{k}\left[k!l!U(k,l)\right]\left[(m-k-1)!(n-l-1)!V(m-k-1,n-l-1)\right]$$

$$= (m-1)!(n-1)!\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}U(k,l)V(m-k-1,n-l-1)$$

Equation (2.2) implies the result and the proof is completed.

Theorem 2.6 [26, 27]: If $w(x,t) = h(x,t)\int_0^t\int_0^x u(z,\tau)\,dz\,d\tau$, then

$$W(m,n) = \begin{cases} \displaystyle\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\frac{H(k,l)U(m-k-1,n-l-1)}{(m-k)(n-l)}, & n,m = 1,2,\ldots \\ 0 & Otherwise \end{cases}$$

Proof: Assume $m,n \geq 1$ and for convenience, we set $v(x,t) = \int_0^t\int_0^x u(z,\tau)\,dz\,d\tau$

Then, $w(x,t) = h(x,t)v(x,t)$ and the $m+n$ partial derivatives with respect to x and t is

$$\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right] = \frac{\partial^{m+n} h(x,t)v(x,t)}{\partial x^m \partial t^n}$$

$$= \frac{\partial^m}{\partial x^m}\left[\frac{\partial^n (h(x,t)v(x,t))}{\partial t^n}\right]$$

$$= \frac{\partial^m}{\partial x^m}\left[\sum_{l=0}^{n}\binom{n}{l}\frac{\partial^l h(x,t)}{\partial t^l}\frac{\partial^{n-l} v(x,t)}{\partial t^{n-l}}\right]$$

$$= \sum_{l=0}^{n}\binom{n}{l}\left[\frac{\partial^m}{\partial x^m}\left(\frac{\partial^l h(x,t)}{\partial t^l}\frac{\partial^{n-l} v(x,t)}{\partial t^{n-l}}\right)\right]$$

$$= \sum_{l=0}^{n}\binom{n}{l}\left[\sum_{k=0}^{m}\binom{m}{k}\frac{\partial^{k+l} h(x,t)}{\partial x^k \partial t^l}\frac{\partial^{m+n-l-k} v(x,t)}{\partial x^{n-k-1}\partial t^{n-l-1}}\right]$$

$$= \sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\left(\frac{\partial^{k+l} h(x,t)}{\partial x^k \partial t^l}\right)\left(\frac{\partial^{m+n-l-k} v(x,t)}{\partial x^{m-k}\partial t^{n-l}}\right)$$

Since $\left[\frac{\partial^{m+n-l-k} v(x,t)}{\partial x^{m-k}\partial t^{n-l}}\right]_{\substack{x=x_0\\ t=t_0}} = 0$ when $m=k, n=l$, the upper limits in the last equation can be replaced by $m-1$ and $n-1$, so it becomes

$$\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right]_{\substack{x=x_0\\ t=t_0}} = \sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\binom{n}{l}\binom{m}{k}\big[k!l!H(k,l)\big]\big[(m-k)!(n-l)!V(m-k,n-l)\big]$$

On the other hand, theorem (2.4) implies $V(m,n) = \frac{1}{mn}U(m-1,n-1)$ and the proof is completed.

Theorem 2.7 [26,27]: If $w(x,t) = \int_0^t\int_0^y \frac{u(z,\tau)}{v(z,\tau)}dz d\tau$, then

$$U(m,n) = \begin{cases} \sum_{k=0}^{m}\sum_{l=0}^{n} V(k,l)(m-k+1)(n-l+1)W(m-k+1,n-l+1), & n,m = 1,2,\ldots \\ 0, & Otherwise \end{cases}$$

Proof: The second partial derivative of $w(x,t)$ with respect to x and t is

$$\frac{\partial^2 w(x,t)}{\partial t \partial x} = \frac{u(x,t)}{v(x,t)}.$$

Let $\tilde{w}(x,t) = \frac{\partial^2 w(x,t)}{\partial t\partial x}$ then we have $v(x,t)\tilde{w}(x,t) = u(x,t)$.

From the table in appendix A.2, we have $U(m,n) = \sum_{k=0}^{m}\sum_{l=0}^{n} V(k,l)\tilde{W}(m-k,n-l)$, where $\tilde{W}(m,n)$ the differential transform of is $\tilde{w}(x,t)$.

Thus, $\tilde{W}(m,n) = (m+1)(n+1)W(m+1,n+1)$ and the proof is finished.

Theorem 2.8 [26, 27] If $w(x,t)=1/v(x,t)\int_0^t\int_0^x u(z,\tau)dzd\tau$,then

$$U(m,n)=\begin{cases}(m+1)(n+1)\sum_{k=0}^{m+1}\sum_{l=0}^{n+1}V(k,l)(W(m-k+1,n-l+1), & n,m=1,2,...\\ 0, & Otherwise\end{cases}$$

Proof: Suppose that $f(x,t)=v(x,t)w(x,t)$ and let $F(m,n)$ be the differential transform of the function $f(x,t)$. From appendix A.2, we have

$$F(m,n)=\sum_{k=0}^{m}\sum_{l=0}^{n}V(k,l)W(m-k,n-l).$$

Also, since $\frac{\partial^2 f(x,t)}{\partial t\partial x}=u(x,t)$, $(m+1)(n+1)F(m+1,n+1)=U(m,n)$ for $m,n=0,1,...$

Therefore, $F(m+1,n+1)=\sum_{k=0}^{m+1}\sum_{l=0}^{n+1}V(k,l)W(m-k+1,n-l+1)=\dfrac{U(m,n)}{(m+1)(n+1)}$

The proof of the theorem is completed.

Theorem 2.9 [39]: If $w(x,t)=\int_0^t\int_0^x\prod_{i=1}^{k}u_i(z,\tau)dzd\tau$ then,

$W(m,0)=W(0,n)=0$ for $m,n=0,1,...$, and

$$W(m,n)=\frac{1}{mn}\sum_{r_{k-1}=0}^{m-1}\sum_{s_{k-1}=0}^{n-1}\sum_{r_{k-2}=0}^{r_{k-1}}\sum_{s_{k-2}=0}^{s_{k-1}}....\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}[U_1(r_1,s_1)U_2(r_2-r_1,s_2-s_1)...$$
$$U_{k-1}(r_{k-1}-r_{k-2},s_{k-1}-s_{k-2})U_k(m-1-r_{k-1},n-1-s_{k-1})]$$

Proof: First set $v(x,t)=\prod_{i=1}^{k}u_i(x,t)$, then $w(x,t)=\int_0^t\int_0^x v(z,\tau)dzd\tau$. Apply Theorem (2.4) to get $W(m,n,)=\frac{1}{mn}V(m-1,n-1)$. From appendix A.2 we get

$$V(m,n)=\sum_{r_{k-1}=0}^{m}\sum_{s_{k-1}=0}^{n}\sum_{r_{k-2}=0}^{r_{k-1}}\sum_{s_{k-2}=0}^{s_{k-1}}....\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}[U_1(r_1,s_1)U_2(r_2-r_1,s_2-s_1)...$$
$$U_{k-1}(r_{k-1}-r_{k-2},s_{k-1}-s_{k-2})U_k(m-r_{k-1},n-s_{k-1})]$$

Hence,

$$W(m,n)=\frac{1}{mn}\sum_{r_{k-1}=0}^{m-1}\sum_{s_{k-1}=0}^{n-1}\sum_{r_{k-2}=0}^{r_{k-1}}\sum_{s_{k-2}=0}^{s_{k-1}}....\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}[U_1(r_1,s_1)U_2(r_2-r_1,s_2-s_1)...$$
$$U_{k-1}(r_{k-1}-r_{k-2},s_{k-1}-s_{k-2})U_k(m-1-r_{k-1},n-1-s_{k-1})]$$

The following example is mentioned in [24], here we give the solution in details.

Example 2.3[24]: Consider the following Volterra integral equation

$$u(x,t)=xe^{-t}+t-\frac{1}{3}x^4-xt+\frac{1}{3}x^4e^{-t}-\frac{1}{2}x^2t^2-\frac{1}{4}x^3t^2-xt^2e^t+xte^t+\int_0^t\int_0^x(xz+te^{\tau})u(z,\tau)dzd\tau$$

According to theorems (2.5-6) and the transformations given in appendix A.2, we have the following recurrence relation:

$$U(m,n)=\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\delta_{r,1}\delta_{s,0}\frac{1}{(m-r)(n-s)}\left[\sum_{l=0}^{n-s-1}\sum_{k=0}^{m-r-1}\delta_{k,1}\delta_{l,0}U(m-r-k-1,n-s-l-1)\right]$$

$$+\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\delta_{r,0}\delta_{s,1}\frac{1}{(m-r)(n-s)}\sum_{l=0}^{n-s-1}\sum_{k=0}^{m-r-1}\delta_{k,0}\frac{1}{l!}U(m-r-k-1,n-s-l-1)+\delta_{m,1}\frac{(-1)^n}{n!}$$

$$+\delta_{m,0}\delta_{n,1}-\frac{1}{3}\delta_{m,4}\delta_{n,0}-\delta_{m,1}\delta_{n,1}+\frac{1}{3}\delta_{m,4}\frac{(-1)^n}{n!}-\frac{1}{2}\delta_{m,2}\delta_{n,2}-\frac{1}{4}\delta_{m,3}\delta_{n,2}$$

$$+\sum_{r=0}^{m}\sum_{s=0}^{n}\delta_{r,1}\delta_{s,1}\delta_{m-r,0}\frac{1}{(n-s)!}-\sum_{r=0}^{m}\sum_{s=0}^{n}\delta_{r,1}\delta_{s,2}\delta_{m-r,0}\frac{1}{(n-s)!}$$

Where $U(0,0)=0$, $U(m,0)=F(m,0)$ and $U(0,n)=F(0,n)$, $m,n=1,2,\ldots$.
The computations from the previous equation can be summarized in the following matrix:

$$\begin{bmatrix} U(0,0) & U(1,0) & U(2,0) & U(3,0) & U(4,0) & \cdots \\ U(0,1) & U(1,1) & U(2,1) & U(3,1) & U(4,1) & \cdots \\ U(0,2) & U(1,2) & U(2,2) & U(3,2) & U(4,2) & \cdots \\ U(0,3) & U(1,3) & U(2,3) & U(3,3) & U(4,3) & \cdots \\ U(0,4) & U(1,4) & U(2,4) & U(3,4) & U(4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2!} & 0 & 0 & 0 & \cdots \\ 0 & \frac{-1}{3!} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{4!} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Substituting this relation in equation (2.3), we conclude that the solution is of the form

$$u(x,t)=t+x-xt+\frac{1}{2!}xt^2-\frac{1}{3!}xt^3+\frac{1}{4!}xt^4-\ldots$$

$$=\quad t+x(1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4-\ldots.)$$

$$=(x+t)e^{-t}$$

Which is the exact solution.

The example below has been solved in [40, 41] in different methods, we solve it using DTM.

Example 2.4: Consider the following Volterra integral equation

$$u(x,t)=e^{x+3t}+e^{3x+t}-e^{3(x+t)}+4\int_0^t\int_0^x e^{x+t}u^2(z,\tau)dzd\tau \text{ where } x,t\in[0,1]$$

According to theorems (2.5) and (2.6) and the operation transformation given in appendix A.2, we obtain

$U(0,0)=F(0,0)=1, U(m,0)=F(m,0)=\frac{1}{m!}$, $U(0,n)=F(0,n)=\frac{1}{n!}$ and we have for $r\neq m, s\neq n$ the following recursive relation:

$$U(m,n)=\frac{(3^n+3^m-3^{m+n})}{m!n!}+4\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\frac{1}{r!s!(m-r)(n-s)}\sum_{l=0}^{n-s-1}\sum_{k=0}^{m-r-1}U(k,l)U(m-r-k-1,n-s-l-1)$$

.

By solving the recursive equation for $m,n=0,1,2,3,4$ the results can be listed as follows

$$\begin{bmatrix} U(0,0) & U(1,0) & U(2,0) & U(3,0) & U(4,0)\ldots \\ U(0,1) & U(1,1) & U(2,1) & U(3,1) & U(4,1)\ldots \\ U(0,2) & U(1,2) & U(2,2) & U(3,2) & U(4,2)\ldots \\ U(0,3) & U(1,3) & U(2,3) & U(3,3) & U(4,3)\ldots \\ U(0,4) & U(1,4) & U(2,4) & U(3,4) & U(4,4)\ldots \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!}\ldots \\ 1 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!}\ldots \\ \frac{1}{2!} & \frac{1}{2!} & \frac{1}{2!2!} & \frac{1}{3!2!} & \frac{1}{4!2!}\ldots \\ \frac{1}{3!} & \frac{1}{3!} & \frac{1}{2!3!} & \frac{1}{3!3!} & \frac{1}{4!3!}\ldots \\ \frac{1}{4!} & \frac{1}{4!} & \frac{1}{2!4!} & \frac{1}{3!4!} & \frac{1}{4!4!}\ldots \end{bmatrix}$$

Therefore, from equation (2.3), we get

$$\begin{aligned} u(x,t) &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots \\ &+t+xt+\frac{x^2t}{2!}+\frac{x^3t}{3!}+\frac{x^4t}{4!}+\ldots \\ &+\frac{t^2}{2!}+\frac{xt^2}{2!}+\frac{x^2t^2}{2!2!}+\frac{x^3t^2}{3!2!}+\frac{x^4t^2}{4!2!}+\ldots \\ &+\frac{t^3}{3!}+\frac{xt^3}{3!}+\frac{x^2t^3}{2!3!}+\frac{x^3t^3}{3!3!}+\frac{x^4t^3}{4!3!}+\ldots \\ &+\frac{t^4}{4!}+\frac{xt^4}{4!}+\frac{x^2t^4}{2!4!}+\frac{x^3t^4}{3!4!}+\frac{x^4t^4}{4!4!}+\ldots \end{aligned}$$

Thus,

$$u(x,t)=1+(x+t)+\frac{1}{2!}(x+t)^2+\frac{1}{3!}(x+t)^3+\frac{1}{4!}(x+t)^4+\ldots=e^{x+t}.$$

Chapter Three

Solving Three- Dimensional Volterra Integral Equation by DTM

This chapter contains basic concepts on VIE of dimension three and some numerical examples on the presented theory.

Consider a function of three variable $w(x,y,t)$ and suppose it can be represented as a product of three functions namely, $w(x,y,t)=F(x)G(y)H(t)$ i.e. the function $w(x,y,t)$ is separable. The differential transformation of $w(x,y,t)$ is

$$w(x,y,t)=\sum_{i=0}^{\infty}F(i)x^i\sum_{j=0}^{\infty}G(j)\sum_{k=0}^{\infty}H(k)t^k=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W(i,j,k)x^iy^jt^k\ . \tag{3.1}$$

Where $W(i,j,k)=F(i)G(j)H(k)$ is called the spectrum of $w(x,y,t)$.

Definition3.1 [29]: Let $w(x,y,t)$ be analytic and defined on a domain $D\subset R^3$ and let $(x_0,y_0,t_0)\in D$ be a fixed point. The three dimensional differential transform of $w(x,y,t)$ is defined

$$W(m,n,l)=\frac{1}{m!n!l!}\left[\frac{\partial^{m+n+l}}{\partial x^m\partial y^n\partial t^l}W(x,y,t)\right]_{x=x_0,y=y_0,t=t_0} \tag{3.2}$$

Definition3.2 [29]: The differential inverse transform of $W(m,n,l)$ is defined by

$$w(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}W(m,n,l)(x-x_0)^m(y-y_0)^n(t-t_0)^l \tag{3.3}$$

Combining equations (3.2) and (3.3) we get

$$w(x,y,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{1}{m!n!l!}\left[\frac{\partial^{m+n+l}}{\partial x^m\partial y^n\partial t^l}w(x,y,t)\right]_{x=x_0,y=y_0,t=t_0} \tag{3.4}$$

In fact, the function $w(x,y,t)$ can be represented by a finite series around $(0,0,0)$

$$w_{M,N,L}(x,y,t)=\sum_{m=0}^{M}\sum_{n=0}^{N}W(m,n,L)x^my^nt^l+R_{M,N,L}(x,y,t) \tag{3.5}$$

Below, we give theorems which are useful in solving some kinds of VIEs in three dimensions using DTM. From now, denote the functions $W(m,n,l)$, $V(m,n,l)$, $H(m,n,l)$ and $U(m,n)$ to be the differential transformations of the functions $w(x,y,t)$, $v(x,y,t)$ $h(x,y,t)$ and $u(x,y,t)$ respectively.

Theorem3.1 [29]*:* If $w(x,y,t)=\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$ then

$$W(m,n,l)=\begin{cases}\dfrac{1}{mnl}U(m-1,n-1,l-1), & n,m,l=1,2,...\\ 0 & Otherwise\end{cases}$$

Proof*:* From definition (3.1), we have

$$W(0,0,0)=\frac{1}{0!0!0!}\left[\frac{\partial^{0+0+0}}{\partial x^0\partial y^0\partial t^0}W(x,y,t)\right]_{\substack{x=0\\y=0\\t=0}}=\left[w(x,y,t)\right]_{\substack{x=0\\y=0\\t=0}}=\int_0^0\int_0^0\int_0^0(u(z,\omega,\tau))dzd\omega d\tau=0$$

The partial derivative of $w(x,y,t)$ with respect to x,y and t is $u(x,y,t)=\dfrac{\partial^3 w(x,y,t)}{\partial x\partial y\partial t}$.

Also, DTM operations imply that

$$(m+1)!(n+1)!(l+1)!W(m+1,n+1,l+1)=m!n!l!U(m,n,l)$$

Finally, replacing $(m+1)$ by m, $(n+1)$ by n and $(l+1)$ by l yield the results and the proof is completed.

Theorem 3.2[29]: If $w(x,y,t)=\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)v(z,\omega,\tau)dzd\omega d\tau$, then

$$W(m,n,l)=\begin{cases}\dfrac{1}{mnl}\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\sum_{p=0}^{m-1}U(p,s,k)V(m-1-p,n-1-s,l-1-k), & n,m=1,2,...\\ 0, & Otherwise\end{cases}$$

Proof: The m^{th} partial derivative with respect to x is

$$\frac{\partial^m w(x,y,t)}{\partial x^m}=\int_0^t\int_0^y\frac{\partial^{m-1}}{\partial x^{m-1}}\ (u(x,z,\tau)v(x,z,\tau))dzd\tau$$

Thus, $\left[\dfrac{\partial^m w(x,y,t)}{\partial x^m}\right]_{x=0,y=0,t=0}=0$.

Similarly we have $\left[\dfrac{\partial^n w(x,y,t)}{\partial y^n}\right]_{x=0,y=0,t=0}=0$, $\left[\dfrac{\partial^l w(x,y,t)}{\partial t^l}\right]_{x=0,y=0,t=0}=0$.

Hence $W(m,0,0)=0, W(0,n,0)=W(0,0,l)=0$, $m,n,l=0,1,...$

Now, for $m\ge1, n\ge1, l\ge1$, by Leibniz formula

$$\left[\frac{\partial^{m+n+l}w(x,y,t)}{\partial x^m\partial y^n\partial t^l}\right]=\frac{\partial^{m+n+l-3}(u(x,y,t)v(x,y,t))}{\partial x^{m-1}\partial y^{n-1}\partial t^{l-1}}$$

$$=\frac{\partial^{m-1}}{\partial x^{m-1}}\frac{\partial^{n-1}}{\partial y^{n-1}}\left[\frac{\partial^{l-1}u(x,y,t)v(x,y,t)}{\partial t^{l-1}}\right]$$

$$=\frac{\partial^{m-1}}{\partial x^{m-1}}\frac{\partial^{n-1}}{\partial y^{n-1}}\left[\sum_{k=0}^{l-1}\binom{l-1}{k}\frac{\partial^{k}u(x,y,t)}{\partial t^{k}}\frac{\partial^{l-1-k}v(x,y,t)}{\partial t^{l-1-k}}\right]$$

$$=\frac{\partial^{m-1}}{\partial x^{m-1}}\sum_{k=0}^{l-1}\binom{l-1}{k}\left[\frac{\partial^{n-1}}{\partial y^{n-1}}\left(\frac{\partial^{k}u(x,y,t)}{\partial t^{k}}\frac{\partial^{l-1-k}v(x,y,t)}{\partial t^{l-1-k}}\right)\right]$$

$$=\frac{\partial^{m-1}}{\partial x^{m}}\sum_{k=0}^{l-1}\binom{l-1}{k}\left[\sum_{s=0}^{n-1}\binom{n-1}{s}\frac{\partial^{k+s}u(x,y,t)}{\partial y^{s}\partial t^{k}}\frac{\partial^{l+n-k-s-2}v(x,y,t)}{\partial y^{n-s-1}\partial t^{l-k-1}}\right]$$

$$=\sum_{k=0}^{l-1}\binom{l-1}{k}\sum_{s=0}^{n-1}\binom{n-1}{s}\left[\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\frac{\partial^{k+s}u(x,y,t)}{\partial y^{s}\partial t^{k}}\frac{\partial^{l+n-k-s-2}v(x,y,t)}{\partial y^{n-s-1}\partial t^{l-k-1}}\right)\right]$$

$$=\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\binom{l-1}{k}\binom{n-1}{s}\left[\sum_{p=0}^{m-1}\binom{m-1}{p}\frac{\partial^{k+s+p}u(x,y,t)}{\partial x^{p}\partial y^{s}\partial t^{k}}\frac{\partial^{l+n+m-k-s-p-3}v(x,y,t)}{\partial x^{m-p-1}\partial y^{n-s-1}\partial t^{l-k-1}}\right]$$

$$=\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\sum_{p=0}^{m-1}\binom{l-1}{k}\binom{n-1}{s}\binom{m-1}{p}\left[\frac{\partial^{k+s+p}u(x,y,t)}{\partial x^{p}\partial y^{s}\partial t^{k}}\frac{\partial^{l+n+m-k-s-p-3}v(x,y,t)}{\partial x^{m-p-1}\partial y^{n-s-1}\partial t^{l-k-1}}\right]$$

$$=\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\sum_{p=0}^{m-1}\binom{l-1}{k}\binom{n-1}{s}\binom{m-1}{p}\left[\begin{array}{l}p!s!k!U(p,s,k)\bullet(m-p-1)!(n-s-1)!(l-k-1)!\\ \times V(m-p-1,n-s-1,l-k-1)\end{array}\right]$$

$$=(l-1)!(n-1)!(m-1)!\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\sum_{p=0}^{m-1}U(p,s,k)V(m-p-1,n-s-1,l-k-1)$$

Therefore, the $m+n+l$ partial derivative with respect to x,y and t at $(0,0,0)$

$$\left[\frac{\partial^{m+n+l}w(x,y,t)}{\partial x^m\partial y^n\partial t^l}\right]_{\substack{x=0\\y=0\\t=0}}=(l-1)!(n-1)!(m-1)!\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\sum_{p=0}^{m-1}U(p,s,k)V(m-p-1,n-s-1,l-k-1).$$

This ends the proof of the theorem.

Theorem 3.3[29]: If $w(x,y,t)=\int_0^t\int_0^y\int_0^x\left[\prod_{q=1}^{n}u_q(z,\omega,\tau)\right]dz\,d\omega\,d\tau$, then

$$W(m,n,l)=\frac{1}{mnl}\sum_{r_{k-1}=0}^{m-1}\sum_{r_{k-2}=0}^{r_{n-1}}\ldots\sum_{r_1=}^{r_2}\sum_{s_{k-1}=0}^{n-1}\sum_{s_{k-2}=0}^{s_{n-1}}\ldots\sum_{s_1=0}^{s_2}\sum_{p_{k-1}=0}^{l-1}\sum_{p_{k-2}=0}^{p_{k-1}}\ldots\sum_{p_1=0}^{p_2}[U_1(r_1,s_1,t_1)$$
$$\times U_2(r_2-r_1,s_2-s_1,p_2-p_1)\times\ldots\times U_n(m-1-r_{k-1},n-1-s_{k-1},l-1-p_{k-1})]$$

Proof: Set $v(x,y,t)=\prod_{i=1}^{k}u_i(x,y,t)$, then $w(x,y,t)=\int_0^t\int_0^y\int_0^x v(z,\omega,\tau)dz\,d\omega\,d\tau$

From Theorem (3.1), $W(m,n,l)=\frac{1}{mnl}V(m-1,n-1,l-1)$. Also, from Appendix A.3 we have the following formula

$$V(m,n,l)=\sum_{r_{k-1}=0}^{m}\sum_{s_{k-1}=0}^{n}\sum_{p_{k-1}=0}^{l}\sum_{r_{k-2}=0}^{r_{k-1}}.\sum_{s_{k-2}}^{s_{k-1}}\sum_{p_{k-2}}^{p_{k-1}}....\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}\sum_{p_1=0}^{p_2}[U_1(r_1,s_1,p_1)U_2(r_2-r_1,s_2-s_1,p_2-p_1)...$$
$$\times\ U_{k-1}(r_{k-1}-r_{k-2},s_{k-1}-s_{k-2},p_{k-1}-p_{k-2})U_k(m-r_{k-1},n-s_{k-1},l-p_{k-1})].$$

Hence

$$W(m,n,l)=\frac{1}{mnl}\sum_{r_{k-1}=0}^{m-1}\sum_{s_{k-1}=0}^{n-1}\sum_{p_{k-1}=0}^{l-1}\sum_{r_{k-2}=0}^{r_{k-1}}.\sum_{s_{k-2}}^{s_{k-1}}\sum_{p_{k-2}}^{p_{k-1}}....\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}\sum_{p_1=0}^{p_2}[U_1(r_1,s_1,p_1)U_2(r_2-r_1,s_2-s_1,p_2-p_1)...$$
$$\times\ U_{k-1}(r_{k-1}-r_{k-2},s_{k-1}-s_{k-2},p_{k-1}-p_{k-2})U_k(m-1-r_{k-1},n-1-s_{k-1},l-1-p_{k-1})]$$

The proof is completed.

Theorem *3.4[28]*: If $w(x,y,t)=h(x,y,t)\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$ then

$$W(m,n,l)=\sum_{k=1}^{l}\sum_{s=1}^{n}\sum_{p=1}^{m}\frac{1}{(l-k)(n-s)(m-p)}H(p,s,k)U(m-1-p,n-1-s,l-1-k) \tag{3.9}$$

Proof*:* Let $v(x,y,t)=\int_0^t\int_0^y\int_0^x u(\omega,z,\tau)d\omega dzd\tau$, therefore the left hand side of the integral equation takes the form $w(x,y,t)=h(x,y,t)v(x,y,t)$. So, for $m,n,l\ge 1$
We have

$$\left[\frac{\partial^{m+n+l}w(x,y,t)}{\partial x^m\partial y^n\partial t^l}\right]_{\substack{x=0\\y=0\\t=0}}=\frac{\partial^{m+n+l}}{\partial x^m\partial y^n\partial t^l}(h(x,y,t)v(x,y,t))=\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}\left[\frac{\partial^l}{\partial t^l}(h(x,y,t)v(x,y,t))\right]$$

From Leibniz formula we have

$$\left[\frac{\partial^{m+n+l}w(x,y,t)}{\partial x^m\partial y^n\partial t^l}\right]=\frac{\partial^m}{\partial x^m}\frac{\partial^n}{\partial y^n}\left[\sum_{k=0}^{l}\binom{l}{k}\frac{\partial^k h(x,y,t)}{\partial t^k}\frac{\partial^{l-k}v(x,y,t)}{\partial t^{l-k}}\right]$$
$$=\frac{\partial^m}{\partial x^m}\sum_{k=0}^{l}\binom{l}{k}\left[\frac{\partial^n}{\partial y^n}\left(\frac{\partial^k h(x,y,t)}{\partial t^k}\frac{\partial^{l-k}v(x,y,t)}{\partial t^{l-k}}\right)\right]$$
$$=\frac{\partial^m}{\partial x^m}\sum_{k=0}^{l}\binom{l}{k}\left[\sum_{s=0}^{n}\binom{n}{s}\frac{\partial^{k+s}h(x,y,t)}{\partial y^s\partial t^k}\frac{\partial^{l+n-k-s}v(x,y,t)}{\partial y^{n-s}\partial t^{l-k}}\right]$$
$$=\sum_{k=0}^{l}\binom{l}{k}\sum_{s=0}^{n}\binom{n}{s}\left[\frac{\partial^m}{\partial x^m}\left(\frac{\partial^{k+s}h(x,y,t)}{\partial y^s\partial t^k}\frac{\partial^{l+n-k-s}v(x,y,t)}{\partial y^{n-s}\partial t^{l-k}}\right)\right]$$

The last formula can be simplified as

$$\frac{\partial^{m+n+l} w(x,y,t)}{\partial x^m \partial y^n \partial t^l} = \sum_{k=0}^{l}\sum_{s=0}^{n}\sum_{p=0}^{m}\binom{l}{k}\binom{n}{s}\binom{m}{p}\left[\frac{\partial^{k+s+p} h(x,y,t)}{\partial x^p \partial y^s \partial t^k}\frac{\partial^{l+n+m-k-s-p} v(x,y,t)}{\partial x^{m-p}\partial y^{n-s}\partial t^{l-k}}\right]$$

$$= \sum_{k=0}^{l}\sum_{s=0}^{n}\sum_{p=0}^{m}\binom{l}{k}\binom{n}{s}\binom{m}{p}\left[p!s!k!H(p,s,k)(m-p)!(n-s)!(l-k)!V(m-p,n-s,l-k)\right]$$

But since for $m=p, n=s, l=k$, $\left[\frac{\partial^{m+n+l-p-s-k} v(x,y,t)}{\partial x^{m-p}\partial y^{n-s}\partial t^{l-k}}\right]_{\substack{x=0\\y=0\\t=0}} = 0$, the upper indices in the above sums reduce to $m-1, n-1$ and $l-1$ and hence $v(x,y,t) = \int_0^t\int_0^y\int_0^x u(\omega,z,\tau)\,d\omega dz d\tau$

Therefore, $V(m,n,l) = \frac{1}{mnl}U(m-1,n-1,l-1)$ and

$$W(m,n,l) = \frac{1}{mnl}\sum_{k=0}^{l-1}\sum_{s=0}^{n-1}\sum_{p=0}^{m-1} H(p,s,k)\frac{U(m-p-1,n-s-1,l-k-1)}{(m-p)(n-s)(l-k)}$$

The proof is finished.

The two theorems listed below are a natural extension to theorems 2.7and 2.8 which are mentioned in the previous chapter.

Theorem *3.5*: If $w(x,y,t) = \int_0^t\int_0^y\int_0^x \frac{u(z,\omega,\tau)}{v(z,\omega,\tau)}dz d\omega d\tau$ then $W(m.0,0) = W(0,n,0) = W(0,0,l) = 0$, $m,n,l = 0,1,...$ and

$$U(m,n,l) = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=0}^{l} V(r,s,p)(m-r+1)(n-s+1)(l-p+1)W(m-r+1,n-s+1,l-p+1)$$

Proof: Let $\frac{\partial^3 w(x,y,t)}{\partial x\partial y\partial t} = \frac{u(x,y,t)}{v(x,y,t)}$, also let $\tilde{w}(x,y,t) = \frac{\partial^3 w(x,y,t)}{\partial x\partial y\partial t}$. Then we have $v(x,y,t)\tilde{w}(x,y,t) = u(x,y,t)$

From appendix A.3 we get $U(m,n,l) = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=0}^{l} V(r,s,p)\tilde{W}(m-r,n-s,l-p)$ where $\tilde{W}(m,n,l)$ is the differential transform of $\tilde{w}(x,y,t)$.

Thus, $\widetilde{W}(m,n,l)=(m+1)(n+1)(l+1)W(m+1,n+1,l+1)$ and

$$U(m,n,l)=\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=0}^{l}V(r,s,p)(m-r+1)(n-s+1)(l-p+1)W(m-r+1,n-s+1,l-p+1)$$

The proof of the theorem is completed.

Theorem *3.6*: If $w(x,y,t)=\dfrac{1}{v(x,y,t)}\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$.

Then $W(m,0,0)=W(0,n,0)=W(0,0,l)=0$, $m,n,l=0,1,\ldots$ and

$$U(m,n,l)=(m+1)(n+1)(l+1)\sum_{r=0}^{m+1}\sum_{s=0}^{n+1}\sum_{p=0}^{l+1}V(r,s,p)W(m-r+1,n-s+1,l-p+1)$$

Proof: Let $f(x,y,t)=v(x,y,t)w(x,y,t)$. Then its differential transform is

$$F(m,n,l)=\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=0}^{l}V(r,s,p)W(m-r,n-s,l-p)$$

On the other hand, $\dfrac{\partial^3 f(x,y,t)}{\partial x\partial y\partial t}=u(x,y,t)$ so we have

$(m+1)(n+1)(l+1)F(m+1,n+1,l+1)=U(m,n,l)$, $m,n,l=0,1,\ldots.$.

The last equation leads to the end of the proof.

Now we discus some examples [29] using DTM. We give the solutions in more details.

Example 3.1: Consider the following linear Volterra Integral Equation

$$u(x,y,t)=g(x,y,t)-\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$$

Such that $0\le x,y,t\le 1$ and $g(x,y,t)=\dfrac{x^2yz+xy^2z+xyz^2}{2}+x+y+t$.

Let $f(x,y,t)=\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$, then we have $u(x,y,t)=g(x,y,t)-f(x,y,t)$

Now from appendix A.3, we get $U(m,n,l)=G(m,n,l)-F(m,n,l)$, where

$$G(m,n,l)=\frac{1}{2}(\delta(m-2)\delta(n-1)\delta(l-1)+\delta(m-1)\delta(n-2)\delta(l-1)+\delta(m-1)\delta(n-1)\delta(l-2))$$
$$+\delta(m-1)\delta(n)\delta(l)+\delta(m)\delta(n-1)\delta(l)+\delta(m)\delta(n)\delta(l-1),$$

$$F(m,n,l)=\frac{1}{mnl}U(m-1,n-1,l-1)$$

The solutions of the above equations can be organized as follows:

$$\begin{bmatrix} U(0,0,0) & U(0,01) & U(0,0,2) & U(0,0,3) & .. \\ U(0,1,0) & U(0,1,1) & U(0,1,2) & U(0,1,3) & \cdots \\ U(0,2,0) & U(0,2,1) & U(0,2,2) & U(0,2,3) & \cdots \\ U(1,0,0) & U(1,0,1) & U(1,0,2) & U(1,0,3) & \cdots \\ U(1,2,0) & U(1,1,1) & U(1,1,2) & U(1,2,3) & \cdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots. \\ 0 & 0 & 0 & 0 & \cdots. \\ 1 & 0 & 0 & 0 & \cdots. \\ 0 & 0 & 0 & 0 & \cdots. \end{bmatrix}$$

Now substituting the result in equation (3.3) to get

$$u(x,y,t) = x + y + t$$

Example 3.2: Consider the linear VIE

$$u(x,y,t) = g(x,y,t) - 24x^2 y \int_0^t \int_0^y \int_0^x u(z,\omega,\tau)\,dz\,d\omega\,d\tau$$

Such that $0 \le x, y, t \le 1$ and $g(x,y,t) = 4x^5y^3t + 4x^3y^3t^3 + 3x^4y^3t^2 + x^2y + yt^2 + xyt$

Set $f(x,y,t) = -24x^2 y \int_0^t \int_0^y \int_0^x u(z,\omega,\tau)\,dz\,d\omega\,d\tau$, then we have

$u(x,y,t) = g(x,y,t) + f(x,y,t)$ now using the properties in table in appendix A.1 , we get

$U(m,n,l) = G(m,n,l) + F(m,n,l)$

Where

$$\begin{aligned} G(m,n,l) = {} & 4\delta(m-5)\delta(n-3)\delta(l-1) + 4\delta(m-3)\delta(n-3)\delta(l-3) \\ & + 3\delta(m-4)\delta(n-3)\delta(l-2) + \delta(m-2)\delta(n-1)\delta(l-0) \\ & + \delta(m-0)\delta((n-1)\delta(l-2) + \delta(m-1)\delta(n-1)\delta(l-1) \end{aligned}$$

and we also have from theorem (3.4)

$$\begin{aligned} F(m,n,l) = {} & -24\sum_{k=0}^{m-1}\sum_{h=0}^{n-1}\sum_{r=0}^{l-1} \frac{1}{(m-k)(n-h)(l-r)} \delta(m-2)\delta(n-1)\delta(l-0) \\ & \times U(m-k-1, n-h-1, l-r-1) \end{aligned}$$

Solving the above equation recursively, the results can be listed as follows:

$$\begin{bmatrix} U(0,0,0) & U(0,0,1) & U(0,0,2) & U(0,0,3) & \ldots \\ U(0,1,0) & U(0,1,1) & U(0,1,2) & U(0,1,3) & \ldots \\ U(1,0,0) & U(1,0,1) & U(1,0,2) & U(1,0,3) & \ldots \\ U(1,1,0) & U(1,1,1) & U(1,1,2) & U(1,1,3) & \ldots \\ U(2,0,0) & U(2,0,1) & U(2,0,2) & U(2,0,3) & \ldots \\ U(2,1,0) & U(2,1,1) & U(2,1,2) & U(2,1,3) & \ldots \\ U(3,0,0) & U(3,0,1) & U(3,0,2) & U(3,0,3) & \ldots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & \ldots \end{bmatrix}$$

Now substituting this relation in equation (3.3), we get the exact solution

$$u(x,y,t) = x^2 y + yt^2 + xyt$$

Chapter Four

Solving Two - Dimensional Volterra Integral Equation by RDTM

In chapter two, we introduced the differential transform method and showed how this method could solve the two dimensional VIEs. The purpose of the current chapter is to solve the two dimensional Volterra Integral equation using the reduced differential transform method (RDTM). For that, we give basic definitions and some properties of RDTM, then we apply the method on some examples and finally we present some figures to show the rapid convergence of the method.

4.1 Two Dimensional RTDM

Consider a function of two variable $w(x,t)$ and suppose it can be represented as a product of two functions as follows $w(x,t) = F(x)G(t)$. Then the function $w(x,t)$ can be represented in the following form [42]:

$$w(x,t) = \sum_{i=0}^{\infty} F(i)x^i \sum_{j=0}^{\infty} G(j)t^j = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} W(i)x^i t^j \ . \tag{4.1}$$

Where $W(i) = F(i)G(j)$ is called the spectrum of $w(x,t)$

Definition4.1 [30]: If $w(x,t)$ is analytic and continuous in the domain of interest, then

$$W_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=t_0} \tag{4.2}$$

is called the reduced transformed function of $w(x,t)$.

Definition4.2 [30]: The differential inverse reduced transform of $W_k(x)$ is defined as

$$w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^k \ . \tag{4.3}$$

Combining equations (4.2) and (4. 3) imply

$$w(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} w(x,t)\right]_{t=t_0} (t-t_0)^k \tag{4.4}$$

In fact, the function $w(x,t)$ in equation (4.4) can be written as

$$w_n(x,t) = \sum_{k=0}^{n} W_k(x)t^k + R_n(x,t) \tag{4.5}$$

Equation (4.5) implies that $R_n(x,t) = \sum_{k=n+1}^{\infty} W_k(x)t^k$ is negligibly small.

Throughout this chapter, we assume that $W_k(x)$, $U_k(x)$ and $V_k(x)$ are the reduced differential transform method of the functions, $w(x,t)$, $u(x,t)$ and $v(x,t)$ respectively.

In the following theorems we consider various shapes of two dimensional integral equations and we give a full proof for the presented results.

Theorem 4.1[30]: If $w(x,t)=\int_0^t\int_0^x u(y,z)dydz$ then

$$W_k(x)=\frac{1}{k}\int_0^x U_{k-1}(y)dy \tag{4.6}$$

Proof: The k^{th} derivative with respect to t is

$$\frac{\partial^k}{\partial t^k}w(x,t)=\frac{\partial^k}{\partial t^k}(\int_0^t\int_0^x u(y,z)dydz)=\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}u(y,t)dy$$

Now, equation (4.2) gives $k!W_k(x)=\int_0^x (k-1)!U_{k-1}(y)dy$. Therefore,

$W_k(x)=\frac{1}{k}\int_0^x U_{k-1}(y)dy$ and the proof is completed.

Theorem 4.2[30]: If $w(x,t)=\int_0^t\int_0^x u(y,z)v(y,z)dydz$ then

$$W_k(x)=\frac{1}{k}\int_0^x\sum_{r=0}^{k-1}U_r(y)V_{k-r-1}(y)dy \tag{4.7}$$

Proof: From Appendix B.1 and Leibnitz formula, we have

$$\begin{aligned}\frac{\partial^k}{\partial t^k}w(x,t)&=\frac{\partial^k}{\partial t^k}(\int_0^t\int_0^x u(y,z)v(y,z)dydz\\&=\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}(u(y,t)v(y,t))dy\end{aligned} \tag{4.8}$$

From equations (1.6) and (4.8), we obtain

$$\frac{\partial^k}{\partial t^k}w(x,t) \quad = \quad \int_0^x\{\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{\partial^r}{\partial t^r}u(y,t)\frac{\partial^{k-r-1}}{\partial t^{k-r-1}}v(y,t)\}dy$$

Therefore,

$$\begin{aligned}\left[\frac{\partial^k}{\partial t^k}w(x,t)\right]_{t=0}&=\int_0^x\sum_{r=0}^{k-1}\binom{k-1}{r}r!(k-r-1)!U_r(y)V_{k-r-1}(y)dy\\&=(k-1)!\int_0^x\sum_{r=0}^{k-1}U_r(y)V_{k-r-1}(y)dy\end{aligned}$$

Finally, equation (4.2) implies

$$W_k(x)=\frac{1}{k}\int_0^x\sum_{r=0}^{k-1}U_r(y)V_{k-r-1}(y)dy \quad \text{for each k=1,2,....}$$

Theorem 4.3[30]: If $w(x,t)=h(x,t)\int_0^t\int_0^x u(y,z)dydz$ then

$$W_k(x)=\sum_{r=0}^{k-1}\frac{1}{k-r}H_r(x)\int_0^x U_{k-r-1}(y)dy \tag{4.9}$$

Proof: The k^{th} derivative of $w(x,t)$ with respect to t is

$$\frac{\partial^k}{\partial t^k}w(x,t)=\frac{\partial^k}{\partial t^k}(h(x,t)\int_0^t\int_0^x u(y,z)dydz$$

$$=\sum_{r=0}^{k-1}\binom{k}{r}\frac{\partial^r}{\partial t^r}h(x,t)\int_0^x\frac{\partial^{k-r-1}}{\partial t^{k-r-1}}u(y,t)dy$$

So ,

$$\left[\frac{\partial^k}{\partial t^k}w(x,t)\right]_{t=0}=\sum_{r=0}^{k-1}\binom{k}{r}r!(k-r-1)!H_r(y)\int_0^x U_{k-r-1}(y)dy$$

$$=\sum_{r=0}^{k-1}\frac{k!}{k-r}H_r(x)\int_0^x U_{k-r-1}(y)dy$$

Thus,

$$W_k(x)=\sum_{r=0}^{k-1}\frac{1}{k-r}H_r(x)\int_0^x U_{k-r-1}(y)dy$$

This ends the proof.

Theorem 4.4[30]: If $w(x,t)=\int_0^t\int_0^x\frac{u(y,z)}{v(y,z)}dydz$ then

$$U_k(x)=\sum_{r=0}^{k}(r+1)\frac{\partial W_{r+1}(x)}{\partial x}.V_{k-r}(y) \tag{4.10}$$

Proof: The partial derivative of $w(x,t)$ with respect to t and x is

$$\frac{\partial^2 w(x,t)}{\partial x\partial t}=\frac{u(x,t)}{v(x,t)}$$

Hence,

$$u(x,t)=\frac{\partial^2 w(x,t)}{\partial x\partial t}v(x,t)$$

So, the k^{th} derivative of $u(x,t)$ with respect to t is

$$\frac{\partial^k}{\partial t^k}u(x,t)=\frac{\partial^k}{\partial t^k}\left[\frac{\partial^2 w(x,t)}{\partial x\partial t}v(x,t)\right]$$

$$=\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^{r+2}w(x,t)}{\partial x\partial t^{r+1}}\frac{\partial^{k-r}v(x,t)}{\partial t^{k-r}}$$

Therefore, equation (4.2) implies that

$$k!U_k(x)=\sum_{r=0}^{k}\binom{k}{r}(r+1)!(k-r)!\frac{\partial\ W_{r+1}(x)}{\partial x}V_{k-r}(x)$$

Then $U_k(x)=\sum_{r=0}^{k}(r+1)\frac{\partial W_{r+1}(x)}{\partial x}.V_{k-r}(x)$

This completes of the proof.

Theorem4.5 [30]: If $w(x,t)=(1/v(x,t))\int_0^t\int_0^y u(y,z)dydz$ then

$$k\sum_{r=0}^{k}W_r(x)V_{k-r}(x)=\int_0^x U_{k-1}(y)dy \tag{4.11}$$

Proof: Rewrite the equation as $w(x,t)v(x,t)=\int_0^t\int_0^x u(y,z)dudz$. The k^{th} derivative with respect to t is

$$\frac{\partial^k}{\partial t^k}(w(x,t)v(x,t))=\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}u(y,t)dy\ .$$

Using equation (1.6) and Leibnitz formula

$$\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r}w(x,t)\frac{\partial^{k-r}}{\partial t^{k-r}}v(x,t)=\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}u(y,t)dy$$

Then from equation (4.2), we get

$$k!\sum_{r=0}^{k}W_r(x)V_{k-r}(x)=(k-1)!\int_0^x U_{k-1}(y)dy$$

then $k\sum_{r=0}^{k} W_r(x)V_{k-r}(x) = \int_0^x U_{k-1}(y)dy$

The proof is completed.

4.2 Example

Now we solve an example which is given in [40] and [41] by RDTM.

Example 4.1: Consider the following VIE

$u(x,t) = e^{x+3t} + e^{3x+t} - e^{3(x+t)} + 4\int_0^t\int_0^x e^{x+t}u^2(z,\tau)dzd\tau$ where $x,t \in [0,1]$

$U_k(x) = F_k(x) + G_k(x)$ where $f(x,t) = e^{x+3t} + e^{3x+t} - e^{3(x+t)}$

From table appendix B.1 we have $F_k(x) = \frac{3^k e^x}{k!} + \frac{e^{3x}}{k!} - \frac{3^k e^{3x}}{k!}$

Therefore, using theorems (4.2) and (4.3) we get

$$U_k(x) = F_k(x) + 4\sum_{r=0}^{k} \frac{e^x}{r!}.\frac{1}{k-r}\int_0^x \sum_{l=0}^{k-r-1} U_l(z)U_{k-r-l-1}(z)dz$$

Where $U_0(x) = F_0(x) = e^x$

Now we find,

$$U_1(x) = 3e^x + e^{3x} - 3e^{3x} + 4\sum_{r=0}^{1} \frac{e^x}{r!}.\frac{1}{1-r}\int_0^x \sum_{l=0}^{1-r-1} U_l(z)U_{k-r-l-1}(z)dz$$

$$= 3e^x - 2e^{3x} + 4(e^x\int_0^x U_0(z)U_0(z)dz)$$

$$= 3e^x - 2e^{3x} + 4(e^x\int_0^x e^z.e^z dz)$$

$$= 3e^x - 2e^{3x} + 4e^x(\frac{e^{2x}-1}{2}) = e^x$$

Similarly, we find $U_2(x)$ and $U_3(x)$ as follows:

$$U_2(x)=\frac{1}{2!}(3^2e^x+e^{3x}-3^2e^{3x})+4\sum_{r=0}^{2}\frac{e^x}{r!}\frac{1}{2-r}\int_0^x\sum_{l=0}^{2-r-1}U_l(z)U_{2-r-l-1}(z)dz$$

$$=\frac{1}{2}(9e^x-8e^{3x})+4[(e^x\frac{1}{2}\int_0^x\sum_{l=0}^{1}U_l(z)U_{2-l-1}(z)dz))+(e^x\int_0^x\sum_{l=0}^{0}U_l(z)U_{1-l-1}(z)dz)]$$

$$=\frac{1}{2}(9e^x-8e^{3x})+4[(\frac{e^x}{2}\int_0^x(U_0(z)U_1(z)+U_1(z)U_0(z))dz)+e^x\int_0^xU_0(z)U_0(z)dz$$

$$=\frac{1}{2}(9e^x-8e^{3x})+4[(\frac{e^x}{2}\int_0^x2e^{2z}dz)+e^x\int_0^xe^{2t}dt]$$

$$=\frac{1}{2}(9e^x-8e^{3x})+4e^{3x}-4e^x=\frac{1}{2!}(e^x)$$

$$U_3(x)=\frac{1}{3!}(3^3e^x+e^{3x}-3^3e^{3x})+4\sum_{r=0}^{3}\frac{e^x}{r!}\cdot\frac{1}{3-r}\int_0^x\sum_{l=0}^{3-r-1}U_l(z)U_{3-r-l-1}(z)dz$$

$$=\frac{1}{3!}(27e^x-26e^{3x})+4[(e^x.\frac{1}{3}\int_0^x\sum_{l=0}^{2}U_l(z)U_{3-l-1}(z)dz))+(e^x.\frac{1}{2}\int_0^x\sum_{l=0}^{1}U_l(z)U_{1-l}(z)dz)$$

$$+(\frac{e^x}{2!}\int_0^x\sum_{l=0}^{0}U_l(z)U_0(z)dz)]$$

$$=\frac{1}{3!}(27e^x-26e^{3x})+4[(\frac{e^x}{3}\int_0^x(U_0(z)U_2(z)+U_1(z)U_1(z)+U_2(z)U_0(z))dz)$$

$$+(\frac{e^x}{2}\int_0^x(U_0(z)U_1(z)+U_1(z)U_0(z))dz)+(\frac{e^x}{2}\int_0^xU_0(z)U_0(z)dz)]$$

$$=\frac{1}{3!}(27e^x-26e^{3x})+4[(\frac{e^x}{3}\int_0^x2e^{2z}dz)+(e^x\int_0^xe^{2z}dz)+(\frac{e^x}{2}\int_0^xe^{2z}dz)]=\frac{1}{3!}(e^x)$$

Therefore, we conclude that the general formula $U_k(x)=\frac{e^x}{k!}$ and from equation (4.3) we get

$$u(x,t)=U_0(x)t^0+U_1(x)t^1+U_2(x)t^2+U_3(x)t^3+U_4(x)t^4+.......$$

$$=e^x+e^xt+\frac{e^x}{2}t^2+\frac{e^x}{3!}t^3+\frac{e^x}{4!}t^4+......$$

$$=e^x(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+......)$$

Therefore, $u(x,t)=e^x.e^t=e^{x+t}$ which is the exact solution

The following figures illustrate the approach of RDTM solution of $u(x,t)$ at different values of t . Also, the figures show the rapid convergence of this method

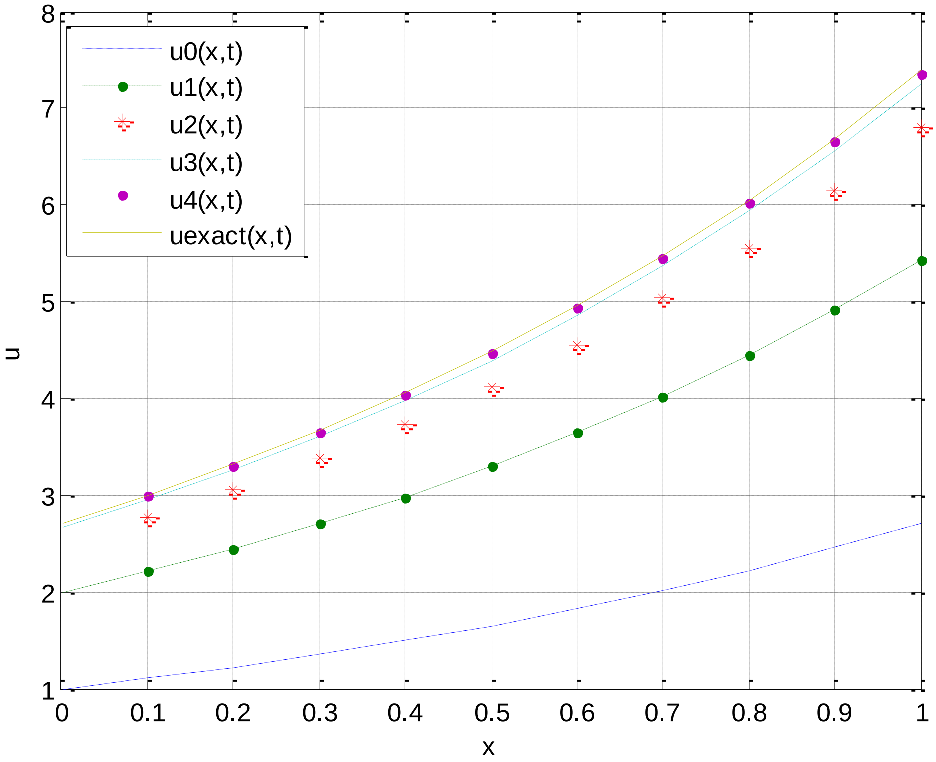

Figure 4.1: The numerical result of RDTM at t=1, by comparing the exact solution by the first five iteration.

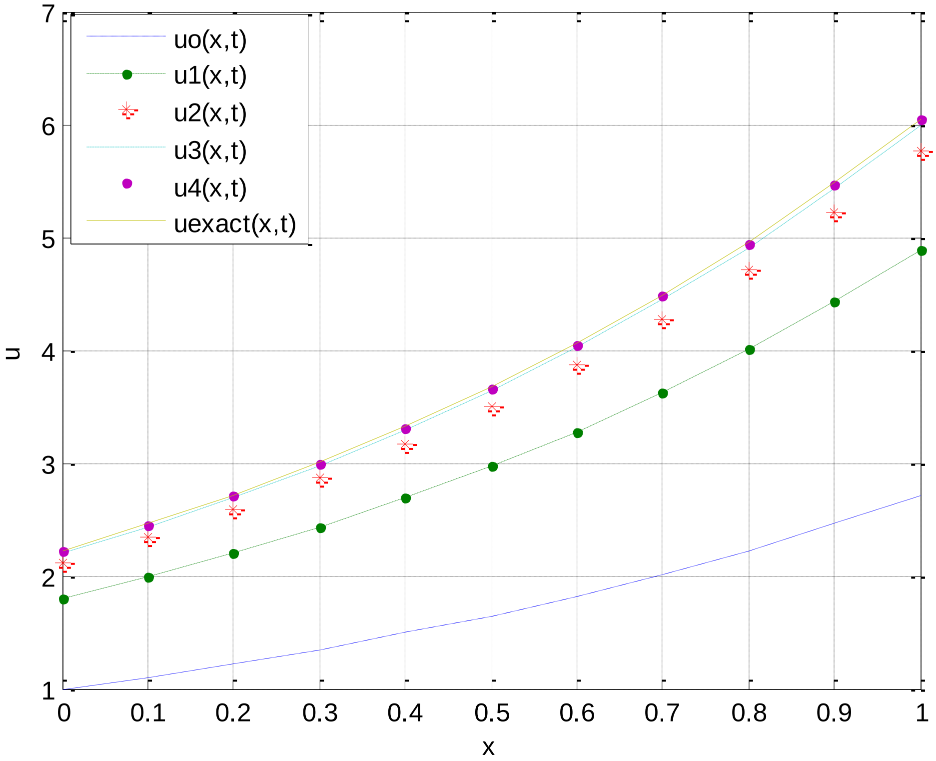

Figure 4.2: The numerical result of RDTM at t=0.8, by comparing the exact solution by the first five iteration.

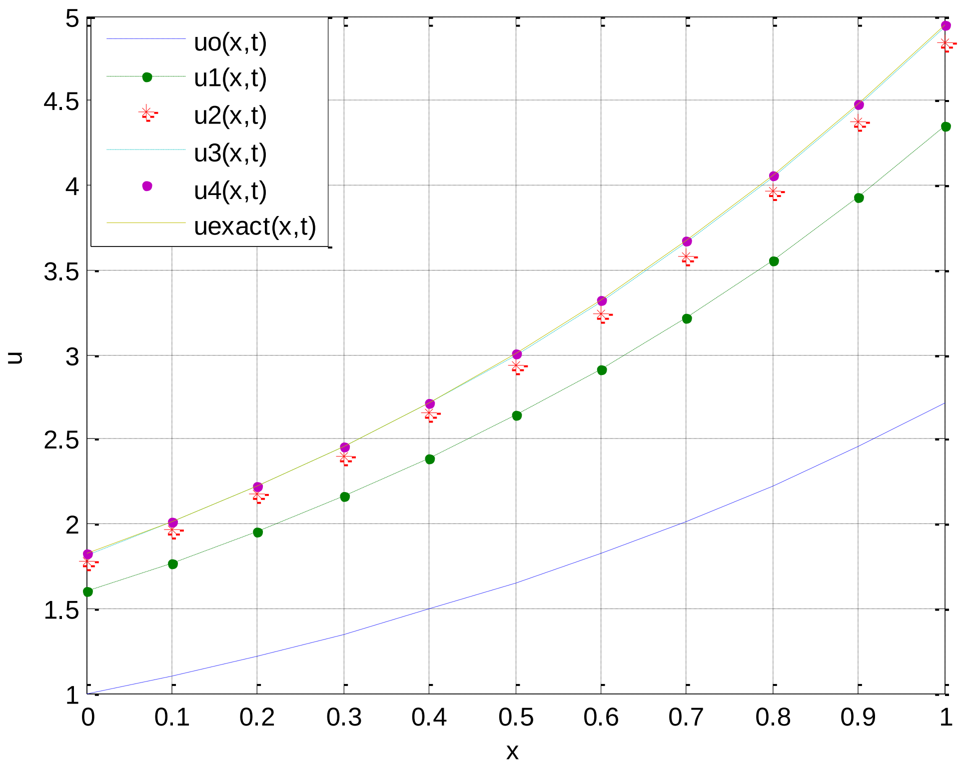

Figure 4.3: The numerical result of RDTM at t=0.6, by comparing the exact solution by the first five iteration.

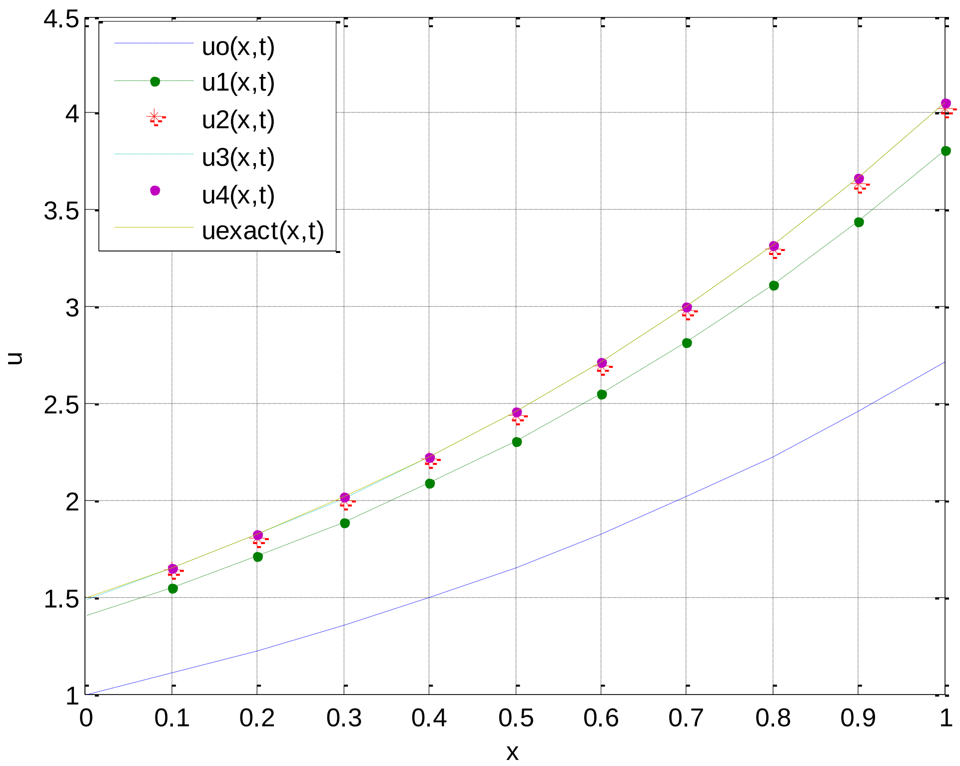

Figure 4.4: The numerical result of RDTM at t=0.4, by comparing the exact solution by the first five iteration.

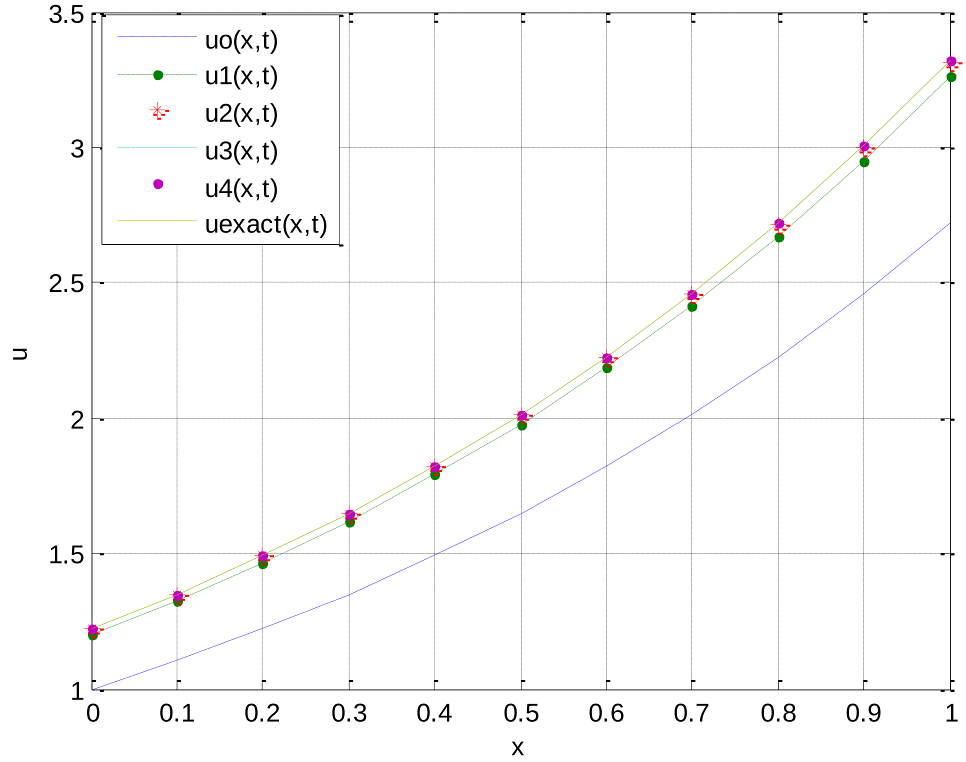

Figure 4.5: The numerical result of RDTM at t=0.2, by comparing the exact solution by the first five iteration.

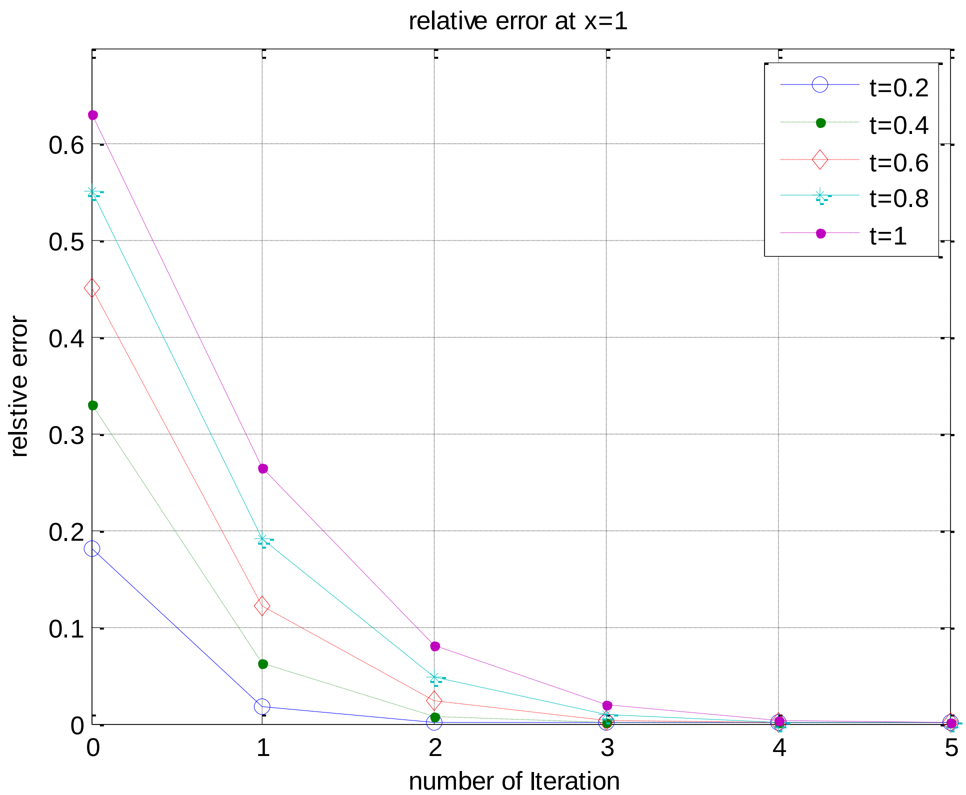

Figure 4.6: The relative error at $x=1$ within five iteration at five value of t.

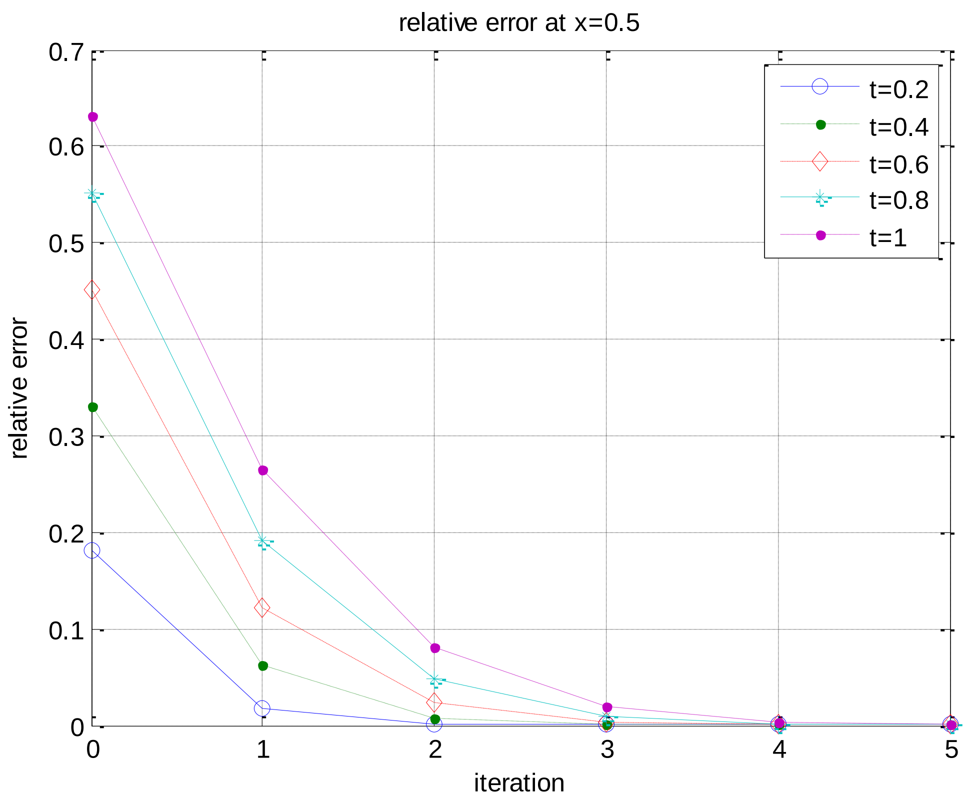

Figure 4.7: The relative error at $x = 0.5$ also at five iteration at five value of t

Chapter Five

Solving Three- Dimensional Volterra Integral Equation by RDTM

In this chapter, we establish some theorems on three dimensional Volterra integral equation which are similarly to that in chapter 3 and give in details the proof of such theorems. Also, we apply the produced theory on some examples which are solved earlier using DTM method. Finally, we compare the results obtained from the two method.

5.1 Three Dimensional RDTM

Let $w(x,y,t)$ be a function of three variables, and assume it can be represented as product of two functions as follows $w(x,y,t)=F(x,y)G(t)$. Then the function $w(x,y,t)$ can be written as [24]:

$$w(x,y,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}F(i,j)x^i y^j\sum_{k=0}^{\infty}G(k)t^k=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}W(i,j)x^i y^j t^k \ . \tag{5.1}$$

The function $W(i,j)=F(i,j)G(k)$ is called the spectrum of $w(x,y,t)$.

Definition 5.1[43]: If $w(x,y,t)$ is analytic and continuous in the domain of interest, then its reduced transformed function is

$$W_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}w(x,y,t)\right]_{t=t_0} \tag{5.2}$$

Definition 5.1[43]: The differential inverse reduced transform of $W_k(x,y)$ is defined by

$$w(x,y,t)=\sum_{k=0}^{\infty}W_k(x,y)(t-t_0)^k. \tag{5.3}$$

Combining equations (5.2) and (5.3), gives

$$w(x,y,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}w(x,y,t)\right]_{t=t_0}(t-t_0)^k \tag{5.4}$$

In fact, the function $w(x,y,t)$ in equation (5.4) can be written as

$$w_n(x,y,t)=\sum_{k=0}^{n}W_k(x,y)t^k+R_n(x,y,t) \tag{5.5}$$

The tail function $R_n(x,y,t)=\sum_{k=n+1}^{\infty}W_k(x,y)t^k$ in equation (5.5) is negligibly small.

Suppose that $W_k(x,y)$, $U_k(x,y)$ and $V_k(x,y)$ are the reduce differential transform functions of $w(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$ respectively.

Now, we introduce some theorems for various shapes of VIE that will useful in solving three dimension Volterra integral equation.

Theorem 5.1: If $w(x,y,t)=\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$ then

$$W_k(x,y)=\frac{1}{k}\int_0^y\int_0^x U_{k-1}(z,\omega)dzd\omega \tag{5.6}$$

Proof: The k^{th} partial derivatives with respect to t have the form

$$\frac{\partial^k}{\partial t^k}w(x,y,t)=\frac{\partial^k}{\partial t^k}(\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau).$$

Leibnitz formula simplifies the last equation to $\frac{\partial^k}{\partial t^k}w(x,y,t)=\int_0^y\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}u(z,\tau,t)dzd\tau$

Now, equation (5.2) gives $\frac{k!}{(k-1)!}W_k(x,y)=\int_0^y\int_0^x U_{k-1}(z,\tau)dzd\tau$ as desired.

Theorem 5.2: If $w(x,y,t)=\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)v(z,\omega,\tau)dzd\omega d\tau$ then,

$$W_k(x,y)=\frac{1}{k}\int_0^y\int_0^x\sum_{r=0}^{k-1}U_r(z,\omega)V_{k-r-1}(z,\omega)dzd\omega \tag{5.7}$$

Proof: From appendix B.2 and Leibnitz formula , we have

$$\frac{\partial^k}{\partial t^k}w(x,y,t)=\frac{\partial^k}{\partial t^k}(\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)v(z,\omega,\tau)dzd\omega d\tau)$$

$$=\int_0^y\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}(u(z,\omega,t)v(z,\omega,t)dzd\omega$$

Equation (1.6) leads to

$$\frac{\partial^k}{\partial t^k}w(x,y,t)=\int_0^y\int_0^x\sum_{r=0}^{k-1}\binom{k-1}{r}\frac{\partial^{k-r}}{\partial t^{k-r}}u(z,\omega,t)\frac{\partial^{k-r-1}}{\partial t^{k-r-1}}v(z,\omega,t)dzd\omega$$

While equation (5.2) provides the following equation

$$\left[\frac{\partial^k}{\partial t^k} w(x,y,t)\right]_{t=0} = \int_0^y\int_0^x \sum_{r=0}^{k-1}\binom{k-1}{r}.r!(k-r-1)!.U_r(z,\omega).V_{k-r-1}(z,\omega)dzd\omega$$

$$= (k-1)!\int_0^y\int_0^x \sum_{r=0}^{k-1} U_r(z,\omega).V_{k-r-1}(z,\omega)dzd\omega$$

From equation (5.2) we get the result and the proof is completed.

Theorem5.3: If $w(x,y,t) = h(x,y,t)\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau)$, then

$$W_k(x,y) = \sum_{r=0}^{k-1}\frac{1}{k-r}H_r(x,y)\int_0^y\int_0^x U_{k-r-1}(z,\omega)dzd\omega \tag{5.8}$$

Proof: The k^{th} partial derivative with respect to t is

$$\frac{\partial^k}{\partial t^k} w(x,y,t) = \frac{\partial^k}{\partial t^k}(h(x,y,t)\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau)$$

$$= \sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r}h(x,y,t)\int_0^y\int_0^x \frac{\partial^{k-r-1}}{\partial t^{k-r-1}}u(z,\omega,t)dzd\omega$$

Therefore, equation (5.2) implies that

$$\left[\frac{\partial^k}{\partial t^k} w(x,y,t)\right]_{t=0} = k!W(x,y) = \sum_{r=0}^{k}\binom{k}{r}.r!(k-r-1)!H_r(x,y)\int_0^y\int_0^x U_{k-r-1}(z,\omega)dzd\omega$$

$$= \sum_{r=0}^{k}\frac{k!}{k-r}H_r(x,y)\int_0^y\int_0^x U_{k-r-1}(z,\omega)dzd\omega$$

But since for $k = r, [\frac{\partial^{k-r}}{\partial t^{k-r}}(\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau)]_{t_0=0} = 0$

and then using (5.2) for k=1,2,…

$$W_k(x,y) = \sum_{r=0}^{k-1}\frac{1}{k-r}H_r(x,y)\int_0^y\int_0^x U_{k-r-1}(z,\omega)dzd\omega$$

The proof is completed

Theorem5.4: If $w(x,y,t) = \int_0^t\int_0^y\int_0^x \frac{u(z,\omega,\tau)}{v(z,\omega,\tau)}dzd\omega d\tau$**.** Then

$$U_k(x,y)=\sum_{r=0}^{k}(r+1)\frac{\partial^2 W_{r+1}(x,y)}{\partial x\partial y}.V_{k-r}(x,y) \tag{5.9}$$

Proof: The partial derivatives of $w(x,y,t)$ with respect to x , y , and t is

$$\frac{\partial^3 w(x,y,t)}{\partial x\partial y\partial t}=\frac{u(x,y,t)}{v(x,y,t)}$$

Hence,

$$u(x,y,t)=\frac{\partial^3 w(x,y,t)}{\partial x\partial y\partial t}v(x,y,t) \tag{5.10}$$

Then, The k^{th} partial derivative of equation (5.10) with respect to t is

$$\frac{\partial^k}{\partial t^k}u(x,y,t)=\frac{\partial^k}{\partial t^k}\left[\frac{\partial^3 w(x,y,t)}{\partial x\partial y\partial t}.v(x,y,t)\right]$$

$$=\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^{r+3} w(x,y,t)}{\partial x\partial y\partial t^{r+1}}\frac{\partial^{k-r} v(x,y,t)}{\partial t^{k-r}}$$

Therefore

$$k!U_k(x,y)=\sum_{r=0}^{k}\binom{k}{r}(r+1)!(k-r)!\frac{\partial^2 W_{r+1}(x,y)}{\partial x\partial y}V_{k-r}(x,y)$$

So,

$$U_k(x,y)=\sum_{r=0}^{k}(r+1)\frac{\partial^2 W_{r+1}(x,y)}{\partial x\partial y}.V_{k-r(x,y)}$$

This completes the proof.

Theorem 5.5: If $w(x,y,t)=(1/v(x,y,t))\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$, then

$$k\sum_{r=0}^{k}W_r(x,y)V_{k-r}(x,y)=\int_0^y\int_0^x U_{k-r}(z,\omega)dzd\omega \tag{5.11}$$

Proof: Rewrite the integral equation as

$$w(x,y,t)v(x,y,t)=\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau \tag{5.12}$$

The k^{th} partial derivative of equation (5.12) with respect to t is

$$\frac{\partial^k}{\partial t^k}(w(x,y,t)v(x,y,t))=\int_0^y\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}u(z,\omega,t)dzd\omega$$

Now, using equation (1.6), we get

$$\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r}w(x,y,t)\frac{\partial^{k-r}}{\partial t^{k-r}}v(x,y,t)=\int_0^y\int_0^x\frac{\partial^{k-1}}{\partial t^{k-1}}u(z,\omega,t)dzd\omega$$

Then from equation (5.2) we get

$$k!\sum_{r=0}^{k}W_r(x,y)V_{k-r}(x,y)=(k-1)!\int_0^y\int_0^x U_{k-r}(z,\omega)dzd\omega$$

$$k\sum_{r=0}^{k}W_r(x,y)V_{k-r}(x,y)=\int_0^y\int_0^x U_{k-r}(z,\omega)dzd\omega$$

Which is the end of the proof

5.2 Examples

In this section, we apply RDTM on the examples which are given and solved in [29] by DTM.

Example 5.1: Consider the linear VIE.

$$u(x,y,t)=g(x,y,t)-\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$$ where $(x,y,t)\in[0,1]\times[0,1]\times[0,1]$ and

$$g(x,y,t)\frac{x^2yt+xy^2t+xyt^2}{2}+x+y+t$$

Its exact solution is $u(x,y,t)=x+y+t$

It easy to see that $g(x,y,0)=x+y$ and therefore RDTM version is $G_0(x,y)=x+y$ where $U_0(x,y)=G_0(x,y)$

Theorem 5.1 implies for $k=1,2,\ldots$ $U_k(x,y)=G_k(x,y)-\frac{1}{k}\int_0^y\int_0^x U_{k-1}(z,\omega)dzd\omega$

where $G_k(x,y)=\frac{x^2y}{2}\delta(k-1)+\frac{xy^2}{2}\delta(k-1)+\frac{xy}{2}\delta(k-2)+x\delta(k)+y\delta(k)+\delta(k-1)$,

thus $G_1(x,y)=\frac{x^2y}{2}+\frac{xy^2}{2}+1$, $G_2(x,y)=\frac{xy}{2}$ and $G_k(x,y)=0$ when $k=3,4,\ldots..$

Now,

$$U_1(x,y)=G_1(x,y)-\int_0^y\int_0^x U_0(z,\omega)dzd\omega$$
$$=\frac{x^2y}{2}+\frac{xy^2}{2}+1-\int_0^y\int_0^x (z+\omega)dzd\omega$$
$$=\frac{x^2y}{2}+\frac{xy^2}{2}+1-(\frac{x^2y}{2}+\frac{xy^2}{2})=1$$

$$U_2(x,y)=G_2(x,y)-\frac{1}{2}\int_0^y\int_0^x U_1(z,\omega)dzd\omega$$
$$=\frac{xy}{2}-\frac{1}{2}\int_0^y\int_0^x (1)dzd\omega$$
$$=\frac{xy}{2}-\frac{xy}{2}=0$$

and

$U_3(x,y)=G_3(x,y)-\frac{1}{3}\int_0^y\int_0^x U_2(z,\omega)dzd\omega=0,$ since $G_3(x,y)=0$, and we obtain that $G_k(x,y)=0$, when $k=2,3,4,......$

Using equation (5.3) the solution can expressed as

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k$$
$$=U_0(x,y)t^0+U_1(x,y)t^1+U_2(x,y)t^2+.......$$
$$=x+y+t$$

Example 5.2: Consider the linear VIE

$u(x,y,t)=g(x,y,t)-24x^2y\int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$ where $(x,y,t)\in[0,1]\times[0,1]\times[0,1]$ and

$g(x,y,t)=4x^5y^3t+4x^3y^3t^3+3x^4y^3t^2+x^2y+yt^2+xyt$

The exact solution is $u(x,y,t)=x^2y+yt^2+xyt$

It easy to see that $g(x,y,0)=x^2y$ and therefore RDTM version is $G_0(x,y)=x+y$ where $U_0(x,y)=G_0(x,y)$

Apply theorem (5.2) for $k=1,2,\ldots$ to get

$U_k(x,y)=G_k(x,y)-(24x^2y)\cdot\frac{1}{k}\int_0^y\int_0^x U_{k-1}(z,\omega)dzd\omega$. Where

$G_k(x,y)=4x^5y^3\delta(k-1)+4x^3y^3\delta(k-3)+3x^4y^3\delta(k-2)+x^2y\delta(k)+y\delta(k-2)+xy\delta(k-1)$

$G_1(x,y)=4x^5y^3+xy$, $G_2(x,y)=3x^4y^3+y$, $G_3(x,y)=4x^3y^3$ and $G_k(x,y)=0$, $k\geq 4$.

So, according to the above computations we have

$$U_1(x,y) = G_1(x,y) - (24x^2 y)\int_0^y\int_0^x U_0(z,\omega)dzd\omega$$

$$= 4x^5y^3 + xy - 24x^2 y\int_0^y\int_0^x (z^2\omega)dzd\omega$$

$$= 4x^5y^3 + xy - 24x^2 y(\frac{x^3y^2}{6})$$

$$= 4x^5y^3 + xy - 4x^5y^3$$

$$= xy$$

$$U_2(x,y) = G_2(x,y) - (24x^2 y)\frac{1}{2}\int_0^y\int_0^x U_1(z,\omega)dzd\omega$$

$$= 3x^4y^3 + y - 12x^2 y\int_0^y\int_0^x (z^2\omega)dzd\omega$$

$$= 3x^4y^3 + y - 12x^2 y(\frac{x^2y^3}{4})$$

$$= 3x^4y^3 + y - 3x^4y^3$$

$$= y$$

$$U_3(x,y) = G_3(x,y) - (24x^2 y)\frac{1}{3}\int_0^y\int_0^x U_2(z,\omega)dzd\omega$$

$$= 4x^3y^3 + -8x^2 y\int_0^y\int_0^x (\omega)dzd\omega$$

$$= 4x^3y^3 + -8x^2 y(\frac{xy^2}{2})$$

$$= 4x^3y^3 - 4x^3y^3$$

$$= 0$$

On the other hand, since $G_k(x,y) = 0$ $k \geq 4$, $U_k(x,y) = 0$ when $k \geq 3$

Using equation (5.3) ,the solution can be expressed as

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k$$

$$= U_0(x,y)t^0 + U_1(x,y)t^1 + U_2(x,y)t^2 + U_3(x,y)t^3 + ...$$

$$= x^2y + xyt + yt^2$$

Example 5.3: Consider the linear VIE

$u(x,y,t) = g(x,y,t) + \int_0^t\int_0^y\int_0^x u(z,\omega,\tau)dzd\omega d\tau$ where $(x,y,t) \in [0,1]\times[0,1]\times[0,1]$ and

$$g(x,y,t) = e^{x+y} + e^{x+t} + e^{y+t} - e^x - e^y - e^t + 1$$

Its exact solution is $u(x,y,t)=e^{x+y+t}$

It easy to see that $g(x,y,0)=e^{x+y}$ and therefore RDTM version is $G_0(x,y)=e^{x+y}$
where $U_0(x,y)=G_0(x,y)$
Now, by using theorem (5.1),for $k=1,2,\ldots\ldots,$ we get

$$U_k(x,y)=G_k(x,y)+\frac{1}{k}\int_0^y\int_0^x U_{k-1}(z,\omega)dzd\omega$$ where $G_k(x,y)=\frac{e^x}{k!}+\frac{e^y}{k!}-\frac{1}{k!}$

Now,

$$U_1(x,y)=G_1(x,y)+\int_0^y\int_0^x U_0(z,\omega)dzd\omega$$

$$=e^x+e^y-1+\int\int e^{z+\omega}dzd\omega$$

$$=e^x+e^y-1+(e^{x+y}-e^y-e^x+1)$$

$$=e^{x+y}$$

$$U_2(x,y)=G_2(x,y)+\frac{1}{2}\int_0^y\int_0^x U_1(z,\omega)dzd\omega$$

$$=\frac{e^x}{2}+\frac{e^y}{2}-\frac{1}{2}+\frac{1}{2}\int_0^y\int_0^x e^{z+\omega}dzd\omega$$

$$=\frac{e^x}{2}+\frac{e^y}{2}-\frac{1}{2}+\frac{1}{2}(e^{x+y}-e^y-e^x+1)$$

$$=\frac{e^{x+y}}{2}$$

$$U_3(x,y)=G_3(x,y)+\frac{1}{3}\int_0^y\int_0^x U_2(z,\omega)dzd\omega$$

$$=\frac{e^x}{3!}+\frac{e^y}{3!}-\frac{1}{3!}+\frac{1}{3}\int_0^y\int_0^x\frac{1}{2}(e^{z+\omega})dzd\omega$$

$$=\frac{e^x}{3!}+\frac{e^y}{3!}-\frac{1}{3!}+\frac{1}{3!}(e^{x+y}-e^y-e^x+1)$$

$$=\frac{e^{x+y}}{3!}$$

Therefore, we conclude that the general formula $U_k(x,y)=\frac{e^{x+y}}{k!}$

and from equation (5.3) we get

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^k$$

$$=U_0(x,y)t^0+U_1(x,y)t^1+U_2(x,y)t^2+U_3(x,y)t^3+.......$$

$$=e^{x+y}+t(e^{x+y})+\frac{t^2}{2}(e^{x+y})+\frac{t^3}{3!}(e^{x+y})+.........$$

$$=\quad e^{x+y}(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+.......)=e^{x+y}.e^t$$

$$=\quad e^{x+y+t}.$$

The figures below illustrate the approach of RDTM solution of $u(x,y,t)$,at different values of t , also show the rapid convergence of this method, where the approximated value converge up to the exact solution.

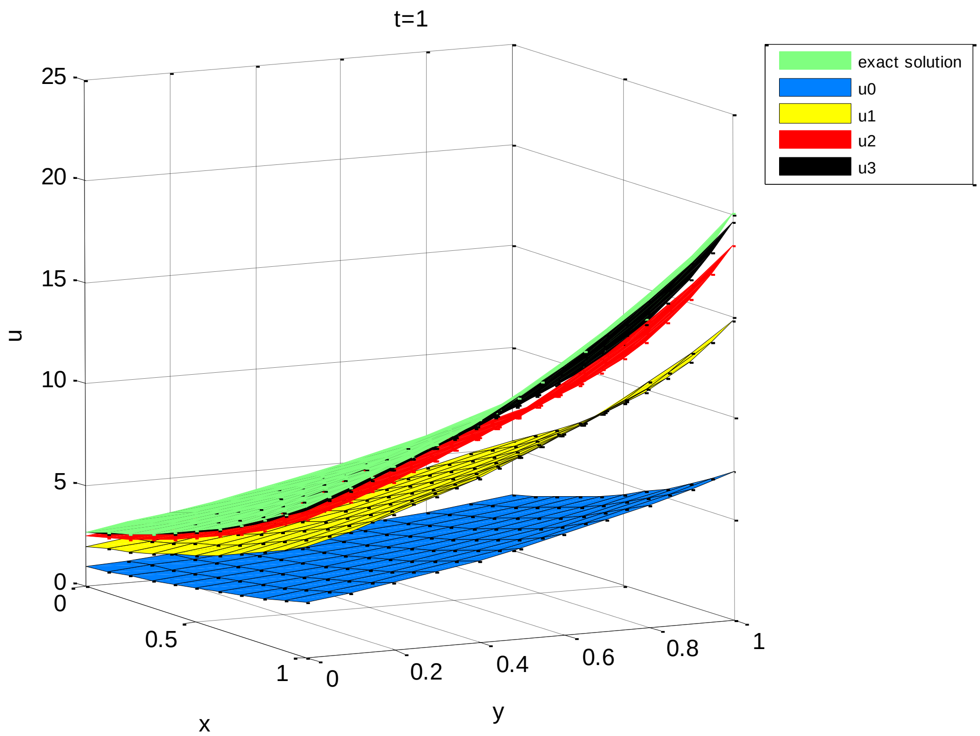

Figure 5.1: The numerical result of RDTM at t=1, by comparing the exact solution by the first four iteration for example 5.3.

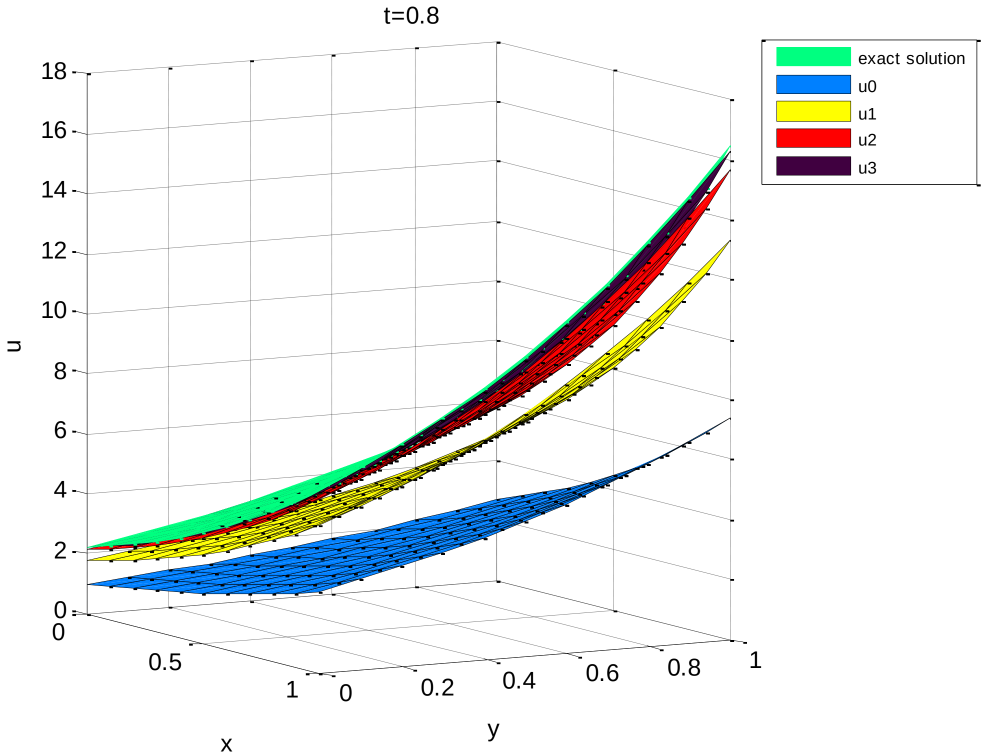

Figure 5.2: The numerical result of RDTM at t=0.8, by comparing the exact solution by the first four iteration for example 5.3.

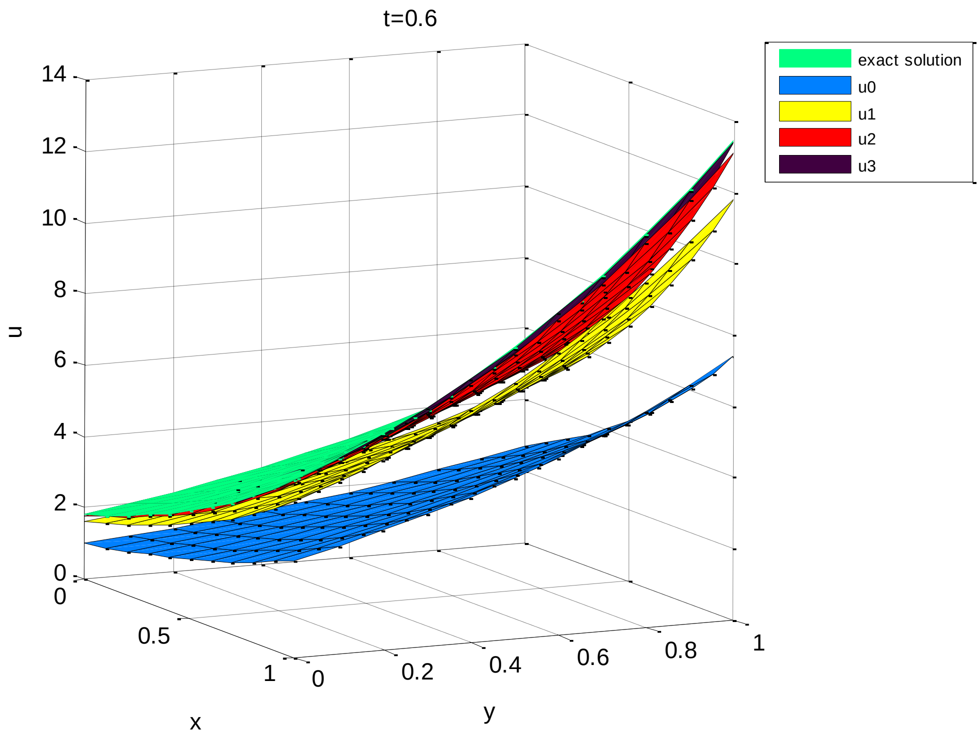

Figure 5.3: The numerical result of RDTM at t=0.6, by comparing the exact solution by the first four iteration for example 5.3.

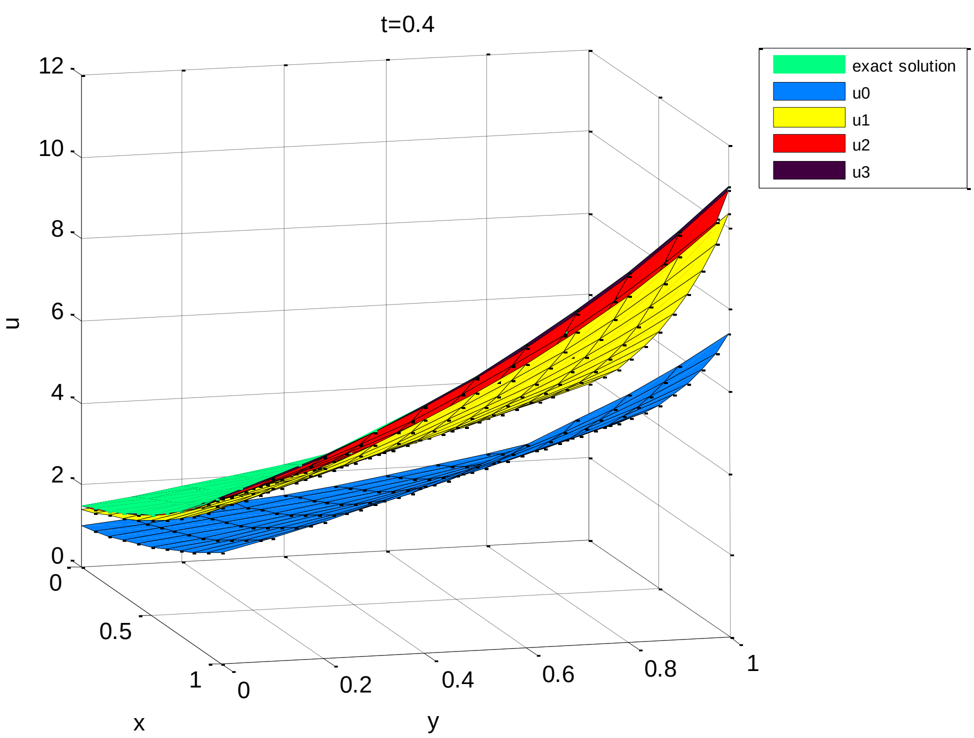

Figure 5.4: The numerical result of RDTM at t=0.4, by comparing the exact solution by the first four iteration for example 5.3.

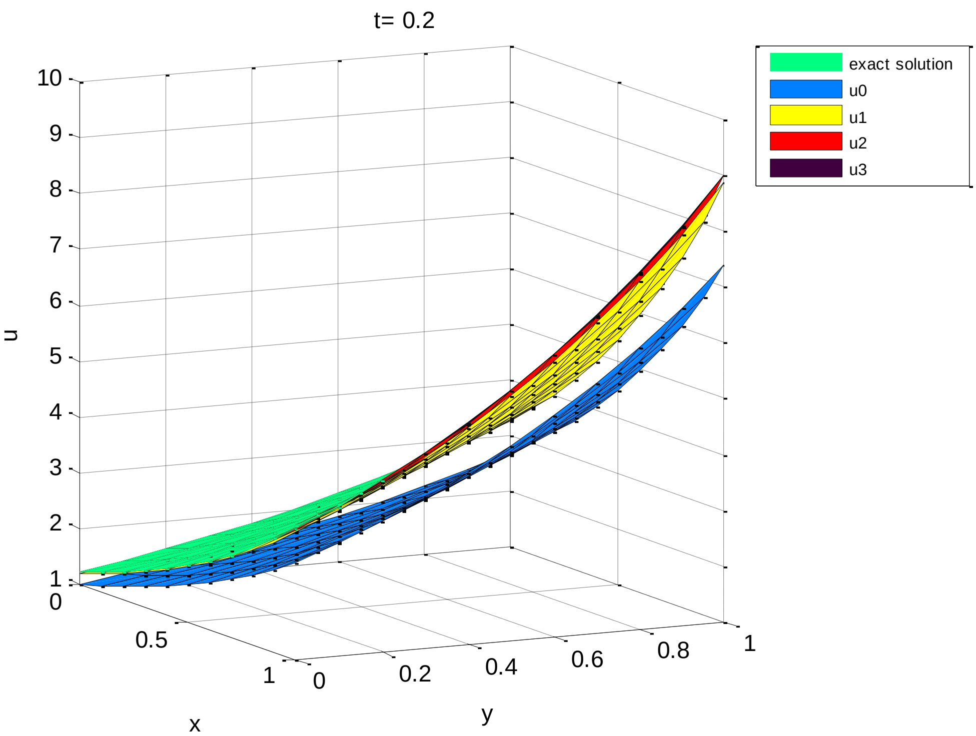

Figure 5.5: The numerical result of RDTM at t=0.2, by comparing the exact solution by the first four iteration for example 5.3.

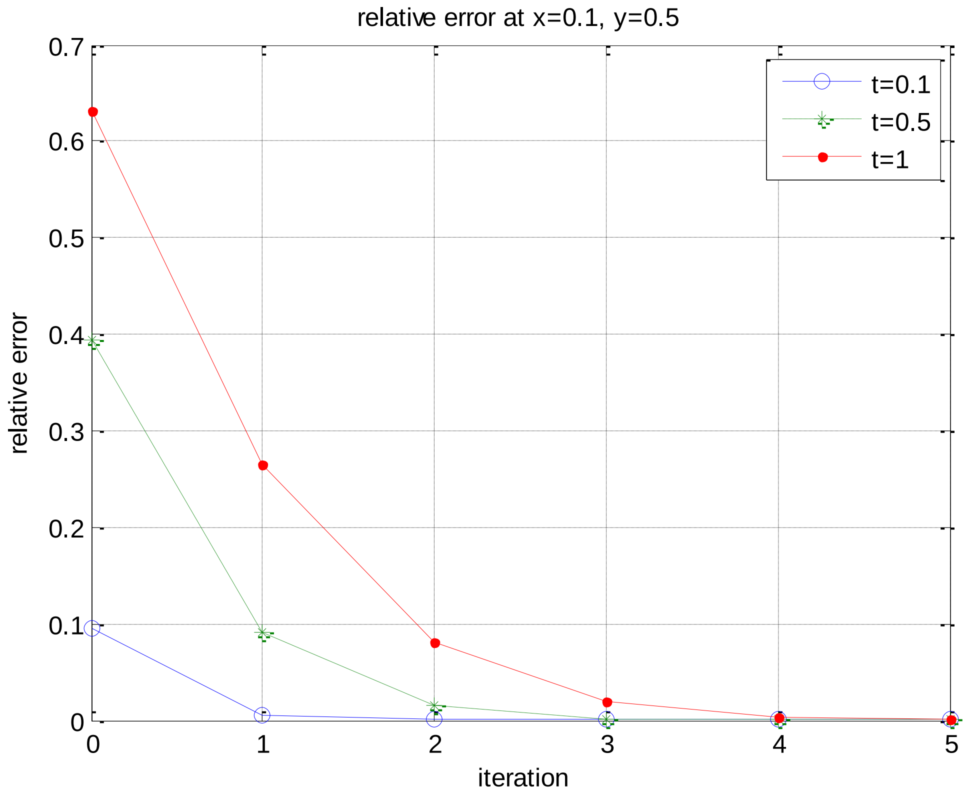

Figure 5.6: The relative error within five iteration at $x = 0.1, y = 0.5$ for example 5.3

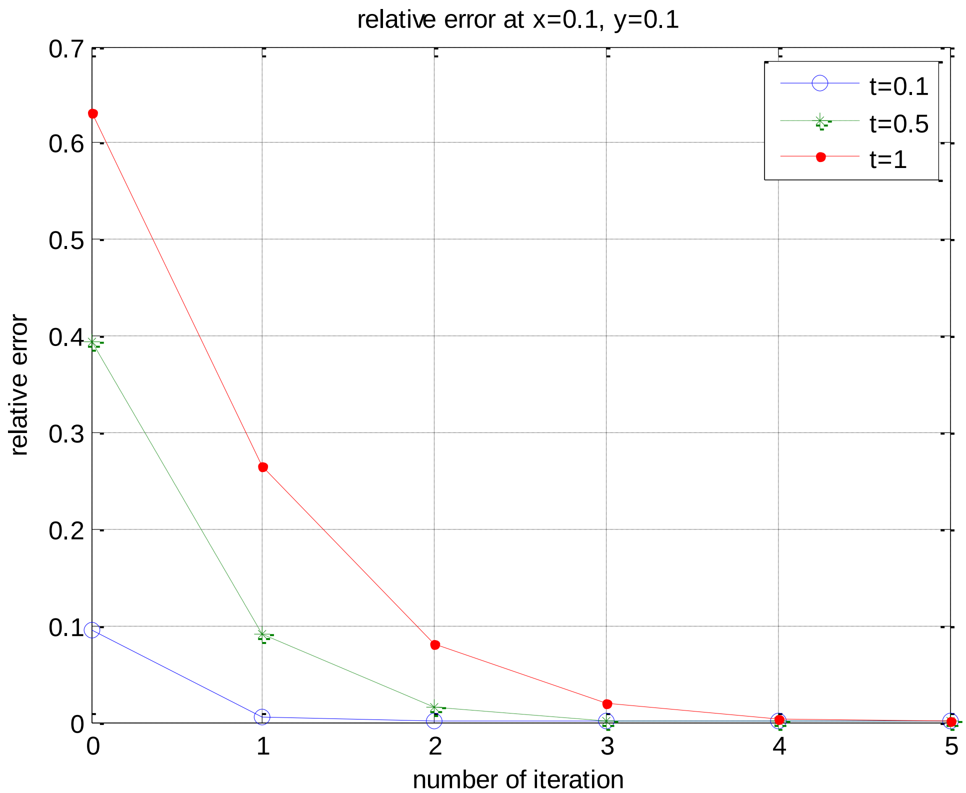

Figure 5.7: The relative error within five iteration at $x = 0.1, y = 0.1$ for example 5.3

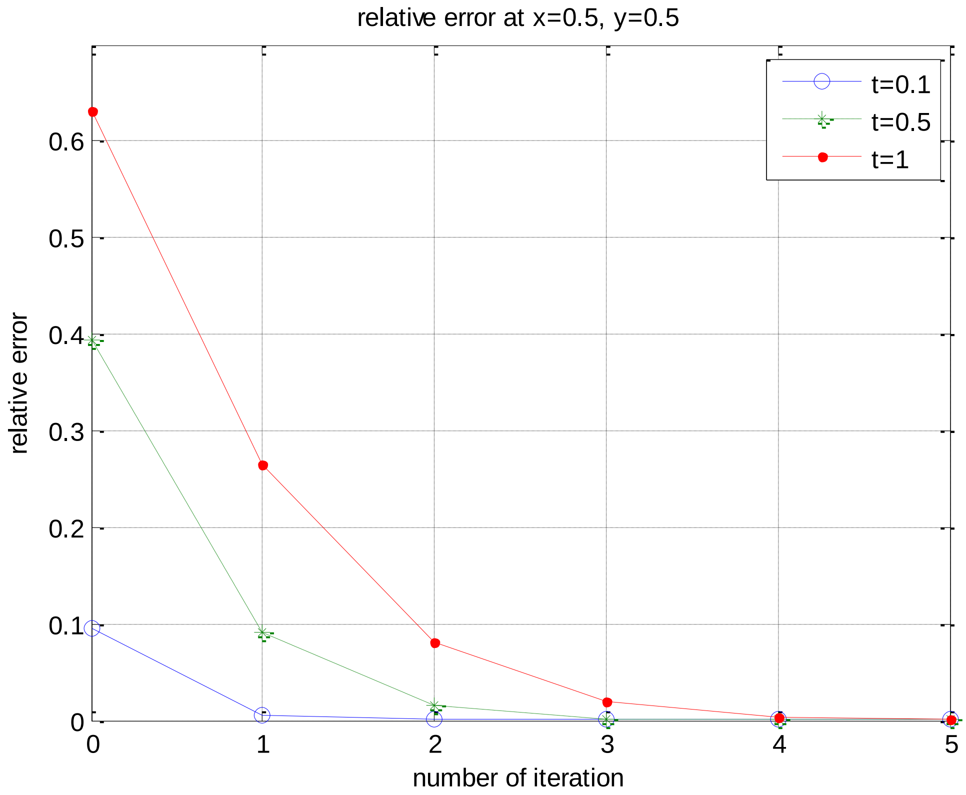

Figure 5.8: The relative error within five iteration at $x = 0.5, y = 0.5$ for example 5.3

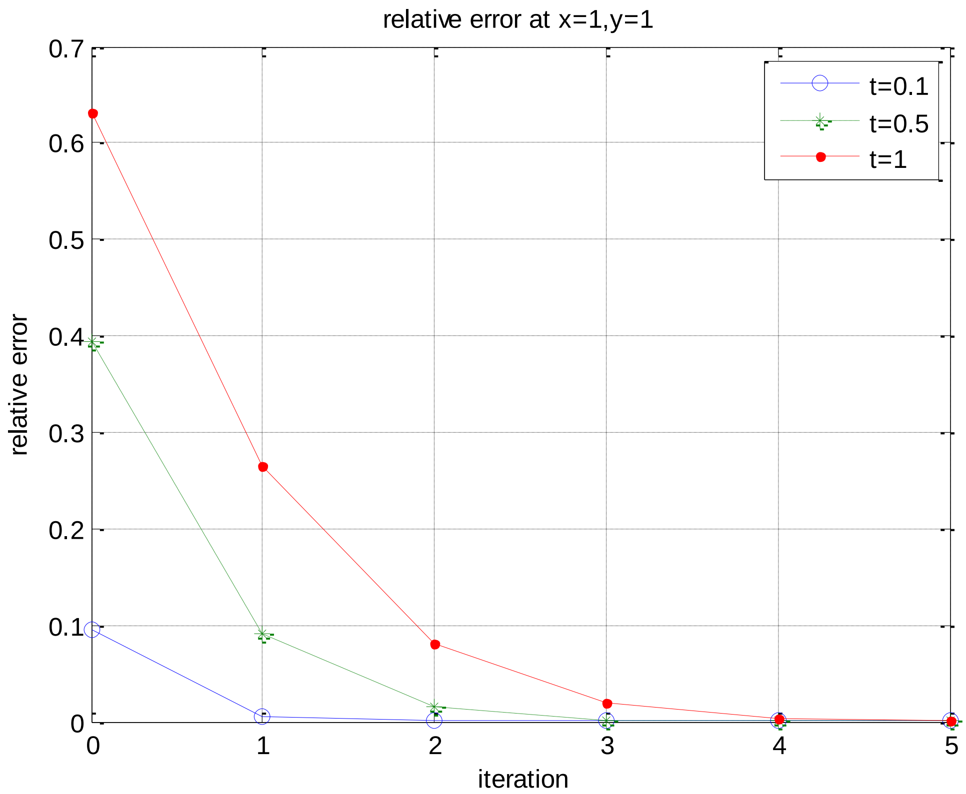

Figure 5.9: The relative error within five iteration at $x = 1, y = 1$ for example 5.3

According to these figure the result indicated that using small t lead to error approach to zero (see Appendix C). In addition, if we look at these figure we conclude that the figure is identical, although we use different value of x and y , which means that if we want get a small error we choose a small value to the reduced variable.

5.3 Comparison

In this section, we will compare between the methods DTM and RDTM using the examples introduced in the previous section to show the accuracy of these two methods. For example 5.3, the exact solution is $u(x,y,t)=e^{x+y+t}$. The absolute error of the iterative solutions $u_{2,2,2}(x,y,t)$, $u_{3,3,3}(x,y,t)$ obtained by DTM [28] and $u_2(x,y,t)$, $u_3(x,y,t)$ obtained by RDTM at some test point (x,y,t) are calculated in the following tables.

Table 5.1. Absolute error using DTM [28]

x	y	t	Exact solution	$\lvert u_{exact}-u_{2,2,2}(x,y,t)\rvert$	$\lvert u_{exact}-u_{3,3,3}(x,y,t)\rvert$
1	1	1	20.085536923	11.585536	7.085536
0.1	0.5	1	4.9530324243	1.07303242	3.9036575×10^{-1}
0.5	0.5	0.5	4.4816890703	8.5668907×10^{-1}	2.941890×10^{-1}
0.1	0.1	0.1	1.3498588076	1.6261825×10^{-3}	1.557798×10^{-5}
0.01	0.1	0.1	1.2336780600	4.8175870×10^{-4}	9.492003×10^{-6}
0.01	0.01	0.1	1.1274968516	1.5113801×10^{-4}	4.338226×10^{-6}
0.01	0.01	0.01	1.0304545340	1.5113801×10^{-6}	1.280568×10^{-9}
0.001	0.01	0.01	1.0212220516	4.3803265×10^{-7}	8.421920×10^{-10}
0.001	0.001	0.01	1.0120722889	1.7775346×10^{-7}	4.165485×10^{-10}
0.001	0.001	0.001	1.0030045045	1.5043042×10^{-9}	3.637978×10^{-12}

Table5.2. Absolute error using RDTM

x	y	t	Exact solution	$\lvert u_{exact}-u_2(x,y,t)\rvert$	$\lvert u_{exact}-u_3(x,y,t)\rvert$
1	1	1	20.0855369231	1.61289667586104	3.81387326×10^{-1}
0.1	0.5	1	4.95303242439	$3.977354234\times 10^{-1}$	9.40489566×10^{-2}
0.5	0.5	0.5	4.48168907033	$6.448109909\times 10^{-2}$	7.85022766×10^{-3}
0.1	0.1	0.1	1.3498588076	2.08759809×10^{-4}	5.19268266×10^{-6}
0.01	0.1	0.1	1.2336780600	1.90792099×10^{-4}	4.7457546×10^{-6}
0.01	0.01	0.1	1.1274968516	1.74370849×10^{-4}	4.3372931×10^{-6}
0.01	0.01	0.01	1.0304545340	1.7045949×10^{-7}	4.2594×10^{-10}
0.001	0.01	0.01	1.0212220516	1.6893224×10^{-7}	4.2212×10^{-10}
0.001	0.001	0.01	1.0120722889	1.6741867×10^{-7}	4.1834×10^{-10}
0.001	0.001	0.001	1.0030045045	1.6704×10^{-10}	4×10^{-15}

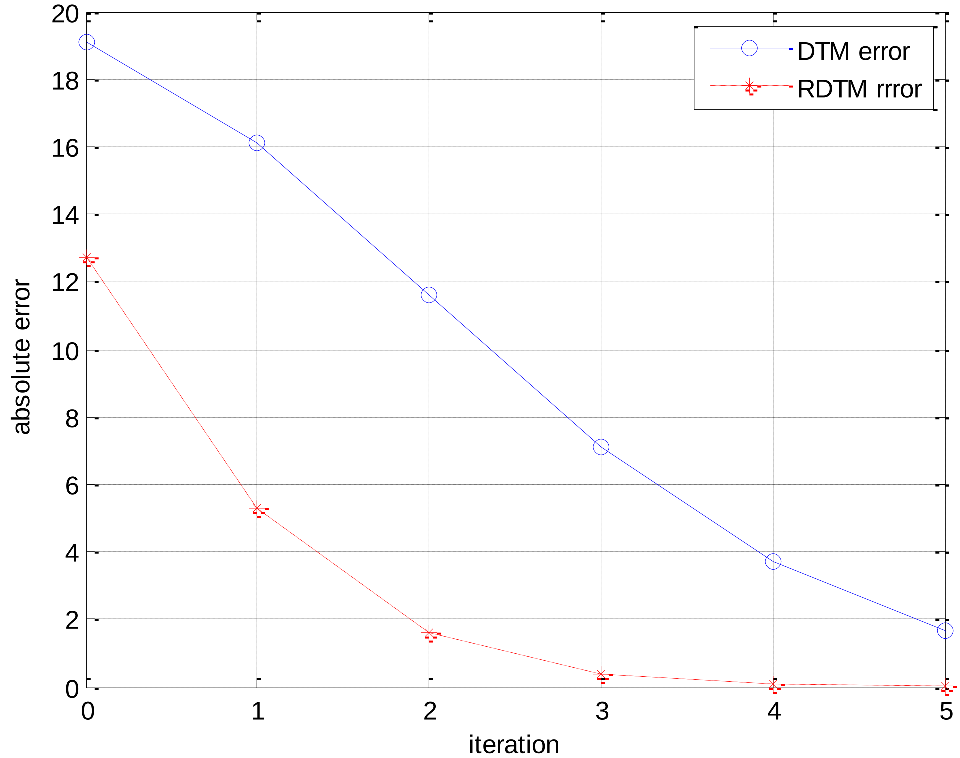

Figure 5.10 : The absolute error for five iteration at $x=1, y=1, t=1$ obtaining from DTM, and RDTM for example 5.3.

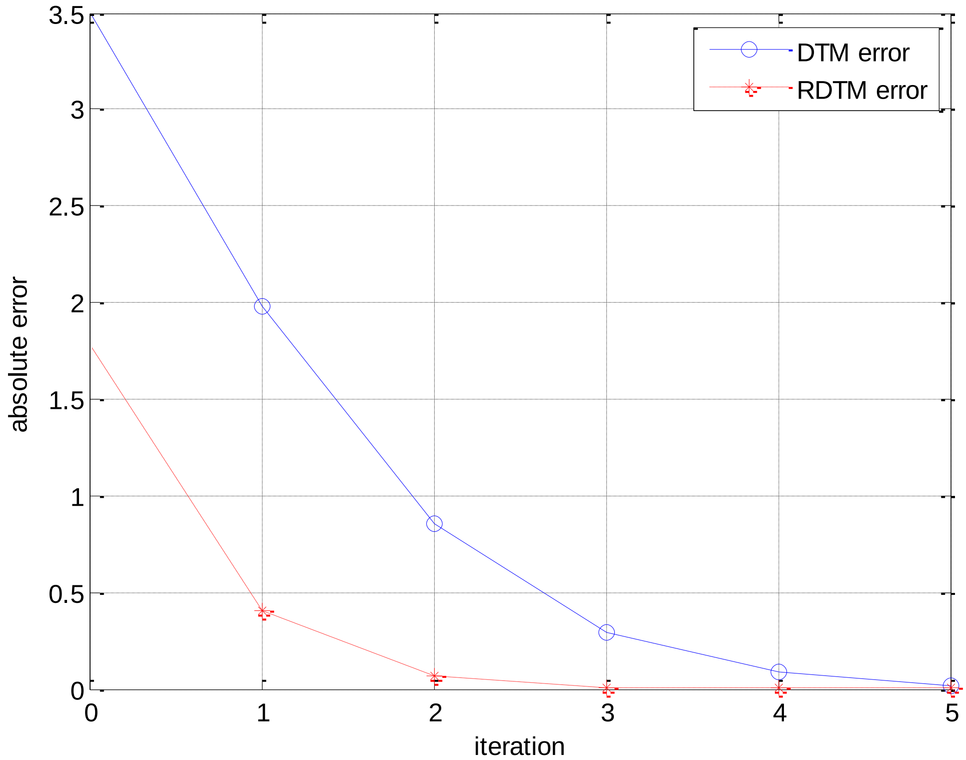

Figure 5.11:The absolute error within five iteration at $x=0.5, y=0.5, t=0.5$ obtaining from DTM, and RDTM for example 5.3.

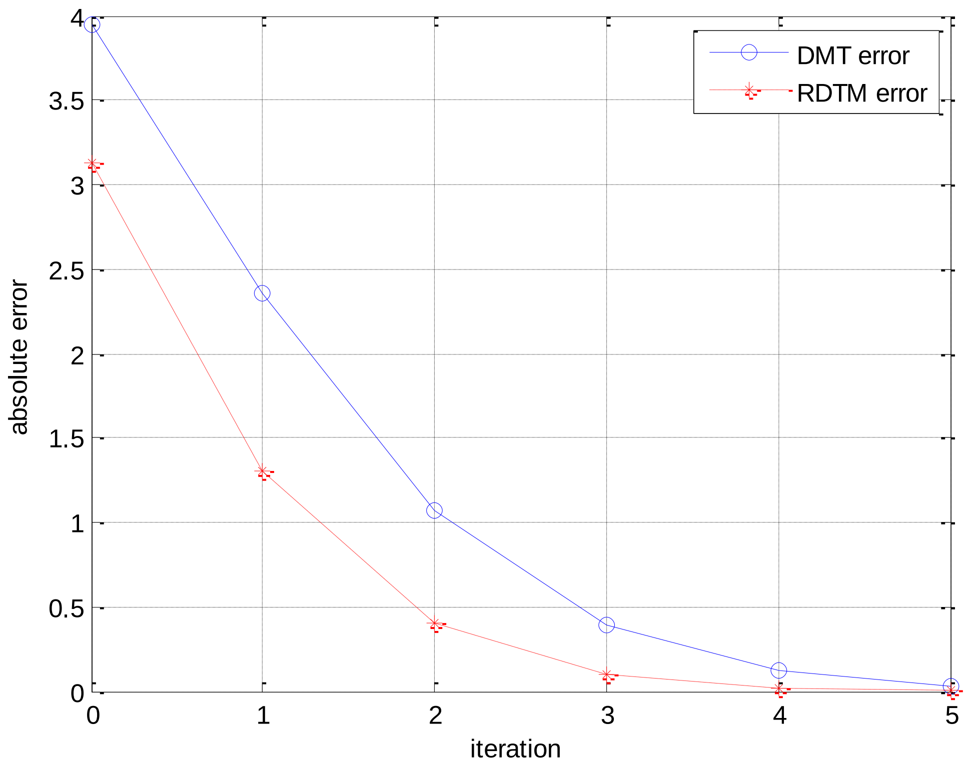

Figure 5.12 :The absolute error within five iteration at $x=0.1, y=0.5, t=1$ obtaining from DTM, and RDTM for example 5.3.

5.4 Discussion and Conclusion

We successfully extended the RDTM method to solve three dimensional VIEs. For that purpose, we produced theoretical approach to the extended method, also a numerical analysis has been made in order to check the effectiveness of the method.
Regarding the examples that have been discussed, we can conclude the followings:

1) RDTM minimizes the number of iterations to reach the exact solution.
2) The approximated solution using RDTM approaches rabidly to the exact solution, see figures (5.1- 5.5).
3) The value of the error decreases whenever the iteration increases, see figures (5.6-5.9).

In this thesis, the three dimensional linear and nonlinear integral equations are solved by using RDTM. However, this method works when the kernel is analytic and separable. It is worth noting that RDTM does not require complex computational work like DTM. It can be easily implemented, its convergence is rapid and its approximation is accurate. In general, it can be concluded that RDTM is a powerful tool for solving many linear and nonlinear three dimensional integral equations

Appendix A
Differential Transform Tables

A.1 : DTM Operation in One Dimension

Suppose that $U(k)$, $V(k)$ and $F(k)$ are the differential transformations of the functions $u(x)$ $v(x)$ and $f(x)$, respectively. Using one dimensional DTM, the fundamental operations [44-64] are listed in following table

Original function	Transformed function
$f(x)=u(x)\pm v(x)$	$F(k)=U(k)\pm V(k)$
$f(x)=\alpha u(x)$	$F(k)=\alpha U(k)$
$f(x)=u(x)v(x)$	$F(k)=\sum_{l=0}^{k} U(l)V(k-1)$
$f(x)=\dfrac{du(x)}{dx}$	$F(k)=(k+1)U(k+1)$
$f(x)=\dfrac{d^m u(x)}{dx^m}$	$F(k)=(k+1)........(k+m)U(k+m)$
$f(x)=x^m$	$F(k)=\delta(k-m)$
$f(x)=e^{\alpha x}$	$F(k)=\dfrac{\alpha^k}{k!}$
$f(x)=\sin(\omega x+\alpha)$	$F(k)=\dfrac{\omega^k}{k!}\sin(\dfrac{\pi k}{2}+\alpha)$
$f(x)=\cos(\omega x+\alpha)$	$F(k)=\dfrac{\omega^k}{k!}\cos(\dfrac{\pi k}{2}+\alpha)$

A.2 : DTM Operation in Two Dimension

Suppose that $W(m,n)$, $V(m,n)$ and $U(m,n)$ are the differential transformations of the functions $w(x,y)$, $v(x,y)$ and $u(x,y)$, respectively. The fundamental mathematical differential transform [47,48] listed in next table.

Original function	Transformed function
$w(x,y)=u(x,t)\pm v(x,t)$	$W(m,n)=U(m,n)\pm V(m,n)$
$w(x,t)=cu(x,t)$	$W(m,n)=cU(m,n)$
$w(x,t)=\dfrac{\partial}{\partial x}u(x,t)$	$W(m,n)=(m+1)U(m+1,n)$
$w(x,t)=\dfrac{\partial}{\partial t}u(x,t)$	$W(m,n)=(n+1)U(m,n+1)$
$w(x,t)=\dfrac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$	$W(m,n)=\dfrac{(m+r)!(n+s)!}{m!n!}U(m+r,n+s)$
$w(x,t)=u(x,t)v(x,t)$	$W(m,n)=\sum_{r=0}^{m}\sum_{s=0}^{n}U(r,s)V(m-r,n-s)$
$w(x,t)=x^{\alpha}t^{\beta}$	$W(m,n)=\delta(m-\alpha,n-\beta)\begin{cases}1, m=\alpha, n=\beta\\ 0, ow\end{cases}$
$w(x,t)=x^k e^{\alpha t}$	$W(m,n)=\dfrac{\alpha^n}{n!}\delta_{m,k}$
$w(x,t)=e^{\alpha x+\beta t}$	$W(m,n)=\dfrac{\alpha^m\beta^n}{m!n!}$
$w(x,t)=x^k\sin(\alpha t+\beta)$	$W(m,n)=\dfrac{\alpha^n}{n!}\delta_{m,k}\sin(\dfrac{n\pi}{2}+\beta)$
$w(x,t)=x^k\cos(\alpha t+\beta)$	$W(m,n)=\dfrac{\alpha^n}{n!}\delta_{m,k}\cos(\dfrac{n\pi}{2}+\beta)$
$w(x,t)=\sin(\alpha x+\beta t)$	$W(m,n)=\dfrac{\alpha^m\beta^n}{m!n!}\sin(\dfrac{(m+n)\pi}{2})$
$w(x,t)=\cos(\alpha x+\beta t)$	$W(m,n)=\dfrac{\alpha^m\beta^n}{m!n!}\cos(\dfrac{(m+n)\pi}{2})$
$w(x,t)=\prod_{i=1}^{k}u_i(x,t)$	$W(m,n)=\sum_{r_{k-1}=0}^{m}\sum_{s_{k-1}=0}^{n}\sum_{r_{k-2}=0}^{r_{k-1}}\sum_{s_{k-2}=0}^{s_{k-1}}\ldots\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}[U_1(r_1,s_1)U_2(r_2-r_1,s_2-s_1)\ldots U_{k-1}(r_{k-1}-r_{k-2},s_{k-1}-s_{k-2})U_k(m-r_{k-1},n-s_{k-1})]$

A.3 : DTM Operation in Three Dimension

Suppose that $W(m,n,l), V(m,n,l)$ and $U(m,n,l)$ are the differential transformations of the functions $w(x,y,t), v(x,y,t)$ and $u(x,y,t)$, respectively. The fundamental mathematical differential transform [59,50] listed in next table.

Original function	Transformed function
$w(x,y,t)=u(x,y,t)\pm v(x,y,t)$	$W(m,n,l)=U(m,n,l)\pm V(m,n,l)$
$w(x,y,t)=cu(x,y,t)$	$W(m,n,l)=cU(m,n,l)$
$\frac{\partial}{\partial x}u(x,y,t)$	$(m+1)U(m+1,n,l)$
$\frac{\partial}{\partial y}u(x,y,t)$	$(n+1)U(m,n+1,l)$
$\frac{\partial}{\partial t}u(x,y,t)$	$(l+1)U(m,n,l+1)$
$\frac{\partial^{r+s+p}}{\partial x^r \partial y^s \partial t^p}u(x,y,t)$	$\frac{(m+r)!(n+s)!(l+p)!}{r!s!p!}U(m+r,n+s,l+p)$
$w(x,y,t)=u(x,y,t)v(x,y,t)$	$W(m,n,l)=\sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=0}^{l}U(r,s,p)V(m-r,n-s,l-p)$
$w(x,y,t)=\prod_{i=1}^{k}u_i(x,y,t)$	$W(m,n,l)=\sum_{r_{k-1}=0}^{m}\sum_{s_{k-1}=0}^{n}\sum_{p_{k-1}=0}^{l}\sum_{r_{k-2}=0}^{r_{k-1}}.\sum_{s_{k-2}}^{s_{k-1}}\sum_{p_{k-2}}^{p_{k-1}}\ldots\sum_{r_1=0}^{r_2}\sum_{s_1=0}^{s_2}\sum_{p_1=0}^{p_2}[U_1(r_1,s_1,p_1)$ $\times\ U_2(r_2-r_1,s_2-s_1,p_2-p_1)..\ \times U_k(m-r_{k-1},n-s_{k-1},l-p_{k-1})]$
$w(x,y,t)=x^p y^q t^r$	$W(m,n,l)=\delta(m-p)\delta(n-q)\delta(l-r)$
$w(x,y,t)=\sin(ax+by+cz)$	$W(m,n,l)=\frac{a^m b^n c^l}{m!n!l!}\sin(\frac{m+n+l}{2}\pi)$
$w(x,y,t)=\cos(ax+by+cz)$	$W(m,n,l)=\frac{a^m b^n c^l}{m!n!l!}\cos(\frac{m+n+l}{2}\pi)$
$w(x,y,t)=e^{ax+by+cz}$	$W(m,n,l)=\frac{a^m b^n c^l}{m!n!l!}$

Appendix B
Reduce Differential Transform tables

B.1 : RDTM Operation in Two Dimension

Suppose that $W_k(x), U_k(x)$ and $V_k(x)$ are the reduce differential transform method of the function $w(x,t), u(x,t)$ and $v(x,t)$ respectively. The fundamental mathematical differential transform [51-53] listed in next table.

Original function	Reduce transformed function
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \alpha u(x,t) \pm \beta v(x,t)$	$W_k(x) = \alpha U_k(x) \pm \beta V_k(x)$
$\frac{\partial}{\partial t} u(x,t)$	$(k+1)U_{k+1}(x)$
$\frac{\partial^{r+s}}{\partial x^r \partial t^s} u(x,t)$	$\frac{(k+s)!}{k!} \frac{\partial^r}{\partial x^r} U_{k+s}(x)$
$x^m t^n$	$x^m \delta(k-n)$
$e^{\alpha t}$	$\frac{\alpha^k}{k!}$
$\sin(\alpha x + \beta t)$	$\frac{\beta^k}{k!} \sin(\frac{\pi k}{2!} + \alpha x)$
$\cos(\alpha x + \beta t)$	$\frac{\beta^k}{k!} \cos(\frac{\pi k}{2!} + \alpha x)$

B.2 : RDTM Operation in Three Dimension

Suppose that $W_k(x,y)$, $U_k(x,y)$ and $V_k(x,y)$ are the reduce differential transform method of the function $w(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$ respectively. The fundamental mathematical differential transform [54] listed in next table.

Original function	Reduce transformed function
$w(x,y,t)=u(x,y,t)v(x,y,t)$	$W_k(x,y)=\sum_{r=0}^{k}U_r(x,y)V_{k-r}(x,y)$
$w(x,y,t)=\alpha u(x,y,t)\pm\beta v(x,y,t)$	$W_k(x,y)=\alpha U_k(x,y)\pm\beta V_k(x,y)$
$\frac{\partial^N}{\partial t^N}u(x,y,t)$	$\frac{(k+N)!}{k!}U_{k+N}(x,y)$
$\frac{\partial^{m+n+s}}{\partial x^m\partial y^n\partial t^s}u(x,y,t)$	$\frac{(k+s)!}{k!}\frac{\partial^{m+n}}{\partial x^m\partial y^n}U_{k+s}(x,y)$
$x^m y^n t^q$	$\begin{cases}x^m y^n, k=q\\ 0, ow\end{cases}$
$e^{\alpha t}$	$\frac{\alpha^k}{k!}$
$\sin(\alpha x+\beta y+\omega t)$	$\frac{\omega^k}{k!}\sin(\frac{\pi k}{2!}+\alpha x+\beta y)$
$\cos(\alpha x+\beta y+\omega t)$	$\frac{\omega^k}{k!}\cos(\frac{\pi k}{2!}+\alpha x+\beta y)$

Appendix C
Relative Error for Example 5.2.3 Using RDTM

x	Y	T	Iteration 0	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5
0.1	0.1	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
	0.5	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
	1	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
0.5	0.1	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
	0.5	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
	1	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
1	0.1	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
	0.5	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$
	1	0.1	$9.516258*10^{-2}$	$4.67884*10^{-3}$	$1.54653*10^{-4}$	$3.84683*10^{-6}$	$7.66780*10^{-8}$	$1.27490*10^{-9}$
		0.5	$3.934693*10^{-1}$	$9.02040*10^{-2}$	$1.43876*10^{-2}$	$1.75162*10^{-3}$	$1.72115*10^{-4}$	$1.41649*10^{-5}$
		1	$6.321205*10^{-1}$	$2.64241*10^{-1}$	$8.03013*10^{-2}$	$1.89881*10^{-2}$	$3.65984*10^{-3}$	$5.94184*10^{-4}$

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