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Haar Wavelet Method for Solving Nonlinear Integral Equations

by

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Committee Decision

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Dedication

To my parents Jameel and Basimah Hamdan, who always loved me unconditionally and whose good examples have taught me to work hard for the things that I aspire to achieve.

To my brothers and sisters who have been a constant source of support and encouragement during the challenges of graduate school and life.

To all people in my life that have touched my heart.

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Abstract

Haar Wavelet Method for Solving Nonlinear Integral Equations

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In this thesis, the solutions of one , two and three-dimensional nonlinear integral equations have been investigated and simulated using Matlab software. Indeed, the method of Haar wavelet is applied to approximate the solution of certain types of nonlinear integral equations. This method transforms the integral equation to a square system of nonlinear algebraic equations with the aid of collocation techiques. The present approach is based on the method introduced by Aziz, S. and Khan (2014) for solving two-dimensional nonlinear Fredholm and Volterra integral equations.

Using the special properties of Haar functions, Algorithms for one, two, and three-dimensional nonlinear integral equations are formed. Based on these algorithms no intermediate numerical integration is needed, which leads to high accuracy even in the case of a small number of collocation points.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Integral equations has many applications in physics, biology, and engineering [44]. Modeling a numerous phenomena and processes is based on them. Initial and boundary value problems of differential equations can be reformulated to have integral equations, where the solution of the integral equation satisfies the initial or boundary conditions automatically [34].

Many analytical and numerical methods are developed to solve integral equations. In [44], the author gathered many analytical methods to solve different types of integral equations like the Adomian decomposition method, Laplace transform method, successive approximations method, and series solution method . In [2], the discrete Galerkin method was employed to solve Fredholm integral equations numerically. In [3], Atkinson and Potra used Iterated projection methods for nonlinear integral equations.

In recent years, the wavelet methods have been employed widely to solve integral equations. Wavelets have been obtained by combination of pure and applied mathematics, they are functions that satisfy a certain mathematical conditions and used to represent functions [19]. It was first formulated in 1991 [1]. Then many types of wavelet functions are used for solving integral equations like Daubechies [41,45], lagendre [22,31,47], coifman [16], cohen [30], Hermite-type trigonometric [13], Walsh [36], and Albert [20] wavelets. Among them [31, 45, 47] are used for solving nonlinear integral equations. There are almost 50 papers used wavelets for integral equations [27].

The simplest and oldest wavelet among all wavelets is Haar wavelet. It has been known for hundred years by Alfred Haar and has been used in various Mathematical and physical fields [42,43]. Haar wavelet has been used to solve integral equations in 2004 by Ülo Lepik and Enn Tamme. At that time, linear Fredholm, Volterra, integro-differential and weakly singular integral equations are solved numerically using an efficient algorithm based on the Haar wavelet [28]. In 2005 another method based on Haar wavelet for solving linear Fredholm, Volterra integral equations of the second kind is produced [35]. In 2006, Ülo Lepik presented a method based on Haar wavelet applicable for nonlinear Volterra, integro-differential equations [26]. In 2007 and 2009 there are two independent numerical methods based on Haar wavelet approach were introduced for nonlinear Fredholm integral equations [8, 29].

In last four years, many authors used Haar wavelet to solve nonlinear integral equations. In 2013, Aziz and Siraj-ul-Islam [5] developed a new numerical method based on Haar wavelet to approximate the solution of nonlinear one-dimensional Fredholm and Volterra integral equations. In the same year, Siraj-ul-Islam, Aziz and Fayyz extended the method of [5] to the integro-differential equations [6]. In [4], the authors improved the methods in [5, 6] by providing Haar wavelet approximation of the kernel function. Later in 2014, Haar wavelet was employed to the numerical solution of non-linear two-dimensional Fredholm, Volterra and Volterra-Fredholm integral equations [7].

In the present work, the methodology presented in [7] is followed. In fact, the aim of this thesis is to solve the three-dimensional nonlinear Fredholm and Volterra integral equations of the second kind numerically using similar approach as in [7]. On the other hand, this method is applied to solve one-dimensional nonlinear integral equations.

The idea of the presented method is to approximate the Kernel function using Haar wavelet functions. No intermediate numerical integration for the Kernel function is needed. Instead, a square nonlinear system of algebraic equations is derived and solved using Newton-Raphson method.

In chapter two, basic definitions and classifications of integral equations, wavelet transform, Haar wavelet functions, Newton's method, and steepest descent method are discussed in some details.

Chapter three contains an efficient theorem for approximating functions of one variable by Haar wavelets. Also two algorithms for solving one-dimensional nonlinear Fredholm and Volterra integral equations.

In chapter four, a new proof is proposed for the theorem in [5] which used Haar functions to approximate functions of two variables. On the other hand, the numerical solution of two-dimenstional nonlinear Fredholm and Volterra integral equations are introduced with examples.

In Chapter five, a theorem for approximating functions of three variables is constructed and proved. In addition, Haar wavelet method is applied for solving the three-dimensional linear and nonlinear integral equations of the second kind.

Chapter 2

Preliminaries

2.1 Classification of Integral Equations

An integral equation is an equation in which the unknown function appears under one or more integral signs [21], it was used for the first time by the German Mathematician Paul Du Bois-Reymond in 1888 [15] . The most typical form of the integral equations is [44]:

$$h(x)U(x) = g(x)f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)U(t)dt \tag{2.1}$$

where $U(x)$ is an unknown function called the solution of the integral equation, $\alpha(x)$ and $\beta(x)$ are the limits of integration, λ is a nonzero constant parameter, $h(x)$,$f(x)$ and $K(x,t)$ are known functions. $K(x,t)$ is called the kernel or the nucleus of the integral equation. $K(x,t)$ is also called degenerate or separable if it can be written in the form:

$$K(x,t) = \sum_{n=1}^{k} A_n(x)B_n(t) \tag{2.2}$$

also the Kernel function should satisfy the regularity conditions for each $a \le x \le b$, and $c \le t \le d$ [21]:

- $\int_a^b \int_c^d |K(x,t)|^2 < \infty$

- $\int_a^b |K(x,t)|^2 dt < \infty$

- $\int_c^d |K(x,t)|^2 dx < \infty$

Such equations may be classified into two main types [21]:

1. Fredholm integral equation: the limits of integration in equation (1.1) are constant. If $h(x) = 0$, Fredholm integral equation is of the first kind. If $h(x) = 1$, Fredholm integral equation is of the second kind. The function $g(x)$ determines the homogeneity of equation (1.1).

2. Volterra integral equation: at least one of the integration limits is variable. Also, it can be characterized in a similar manner as in Fredholm integral equation.

If the integral equation has both derivatives and integrals of the unknown function, then it is called integro-differential equation [44]:

$$h(x)U^{(m)}(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)U(t)dt, \quad m = 1, 2, ..., n \tag{2.3}$$

where the power of U represents the m^{th} derivative.

The nonlinear integral equation has the following form [44]:

$$h(x)U(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)F(U(t))dt \tag{2.4}$$

where $F(U(t))$ is a nonlinear function of $U(t)$ such as $U^2(t)$, and $cos(U(t))$.

For the n-independent variables $x = (x_1, x_2, ..., x_n)$, the n-dimensional integral equation is:

$$h(x)U(x) = f(x) + \int_G K(x, s)U(s)ds \tag{2.5}$$

where $x, s \in \mathbb{R}^n, G \subseteq \mathbb{R}^n$.

2.2 The Need of Wavelets

The concept of Wavelets was derived in the early 1980's by combination of pure and applied Mathematics [19]. The Wavelets were used for the first time in geophysics to analyze data from the seismic surveys [9]. Afterwards, it was used in the fields of Mathematics, quantum physics, electrical engineering, and medical technology [19]. It is predicted to be used in diagnosing breast cancer, predicting the weather, DNA analysis, fingerprints verification, and many other applications [24].

The word "wavelet" comes from waving above and below the x-axis [43]. Wavelet analysis like Fourier uses basic functions to expand a given function [27]. Namely, it expands functions in term of wavelets which are functions obtained from two basic functions: the scaling function ϕ called "father wavelet", and the wavelet ψ called "mother wavelet" [9, 25]. For more details on Wavelet theory, see for example [24].

Fourier analysis is a good tool for studying a stationary data whose properties are invarient over time, but it is not so for data with transient events that can not be predicted from old data [24]. The Fourier transform of the function $s = s(t)$ is [27]:

$$S(w) = \int_{-\infty}^{\infty} s(t)e^{-iwt}dt \tag{2.6}$$

where w is the frequency, Fourier transform analyzing data over the whole domain, but it does not characterize the motion in time. Consider the two signal functions:

$$x_1(t) = sin3t + 0.8sin10t \qquad t \in [0, 10] \tag{2.7}$$

$$x_2(t) = \begin{cases} sin3t & \text{if t} \in [0, 5) \\ sin5t & \text{if t} \in [5, 10) \end{cases}$$

The Fourier transform of these signal functions are given in the following figure [27]

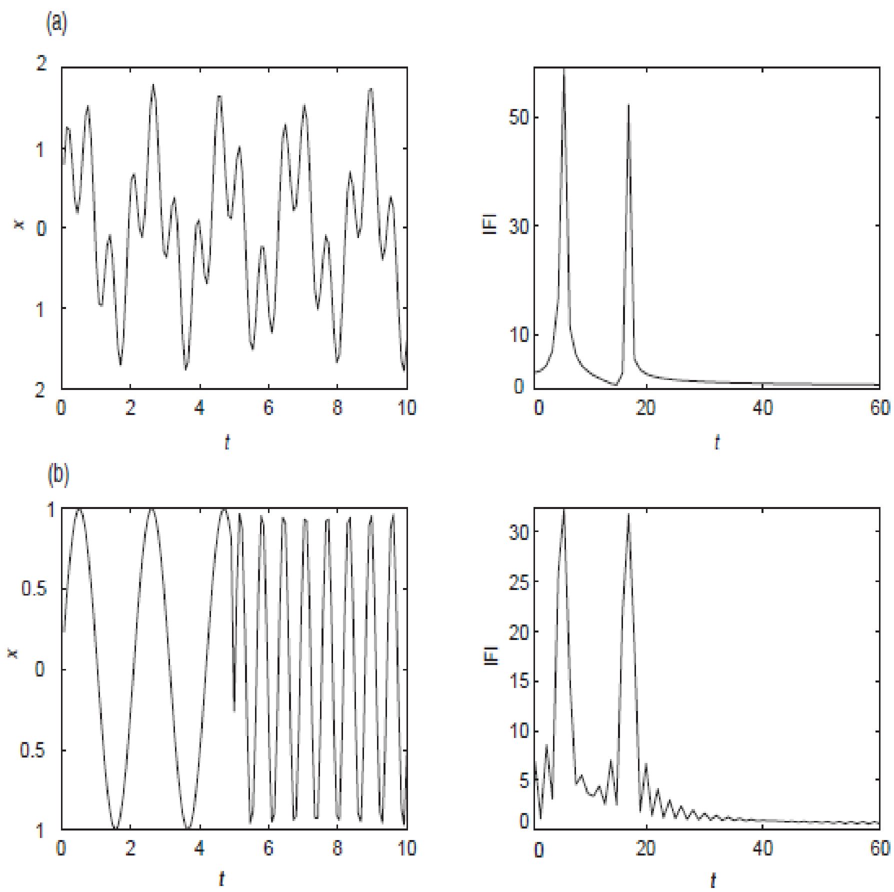

Fig. 2.1: Fourier transform of $x_1(t)$ and $x_2(t)$

Time history and Fourier diagram for (a) $x_1 = sin3t + 0.8sin10t$ for $t \in [0, 10]$ and (b) $x_2 = sin3t$ for t$\in$ $[0, 5]$; $x_2 = sin10t$ for $t \in [5, 10]$. Although the two signal functions are absolutely different, the Fourier transform of both of them is similar [27].

When Fourier is used to transform signals from time domain to frequency domain, time information is lost. However, In Wavelet transform, time information is not lost which is very useful in some context [27].

2.3 Wavelet Transform

Special conditions should be satisfied to obtain wavelet functions. they are localized waves functions defined in a finite domains used to represent functions. They vanish outside their finite domains intead of oscilating forever [24].

Definition 2.1 *[24, 25] A mother wavelet is a function ψ that satisfies the following conditions:*

$$\int_{-\infty}^{\infty} |\psi(t)| dt = 0 \tag{2.8}$$

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt < \infty \tag{2.9}$$

$$\int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty \tag{2.10}$$

where $\hat{\psi}(w)$ is the Fourier transform of $\psi(t)$, (2.5) is called the admissibility condition.

A wavelet $\psi_{a,b}(t)$ is defined as:

$$\psi_{ab}(t) = \frac{1}{\sqrt{|a|}} \psi(\frac{t-b}{a}) \quad a, b \in \mathbb{R}, a \neq 0. \tag{2.11}$$

where the parameters a is the scaling parameter decides the degree of compression, and b is the tranlation parameter decides the time location of wavelet. $\frac{1}{\sqrt{|a|}}$ is the normalizing factor.

Definition 2.2 *(The continuous wavelet transform) [14] Let $f(x)$ be a square integrable function, then the wavelet transform $W_\psi f$ of the function f with respect to the mother wavelet ψ is defined as:*

$$W_\psi f(a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} f(x)\bar{\psi}(\frac{x-b}{a})dx \tag{2.12}$$

where $\bar{\psi}$ is the conjugate of the function ψ, it is clear that the contiuous wavelet transform is a function in two variables.

If the continuous wavelet transform of the square integrable function f is given, then the inverse wavelet transform is given by [17]:

$$f(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{c_\psi a^2} W_f(a,b)\psi(\frac{x-b}{a})dadb \tag{2.13}$$

where

$$c_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|} dw \tag{2.14}$$

Example 2.1 *(The discrete Wavelet Transform) Choose $a = \{2^{-j}\}_{j\in\mathbb{Z}}$, $\{b = k2^{-j}\}_{j,k\in\mathbb{Z}}$, then the mother wavelet becomes [17]:*

$$\psi_{jk} = 2^{\frac{j}{2}}\psi(2^j x - k) \tag{2.15}$$

hence the discrete wavelet transform is :

$$c_{jk} = \int_{-\infty}^{\infty} f(x)\bar{\psi}_{jk}(x)dx \tag{2.16}$$

2.3.1 Wavelet Series

Suppose that $f(x) \in L^2(R)$, then it can be represented as a wavelet series as follows [14, 17]:

$$f(x) = \sum_j \sum_k < f, \psi_{jk} > \psi_{jk} \tag{2.17}$$

where $j, k \in \mathbb{Z}$, $< f, \psi_{jk} >= c_{jk}, \psi_{j,k} = 2^{\frac{j}{2}}\psi(2^j x - k)$. Note that equation (2.12) contains a double summation because wavelets have two parameters, which are the translation and scale parameters.

2.3.2 Haar Scaling Wavelets

Haar wavelet is the simplest and the oldest kind of all wavelets. The Haar scaling function ϕ is a box function whose figure is given as [9]:

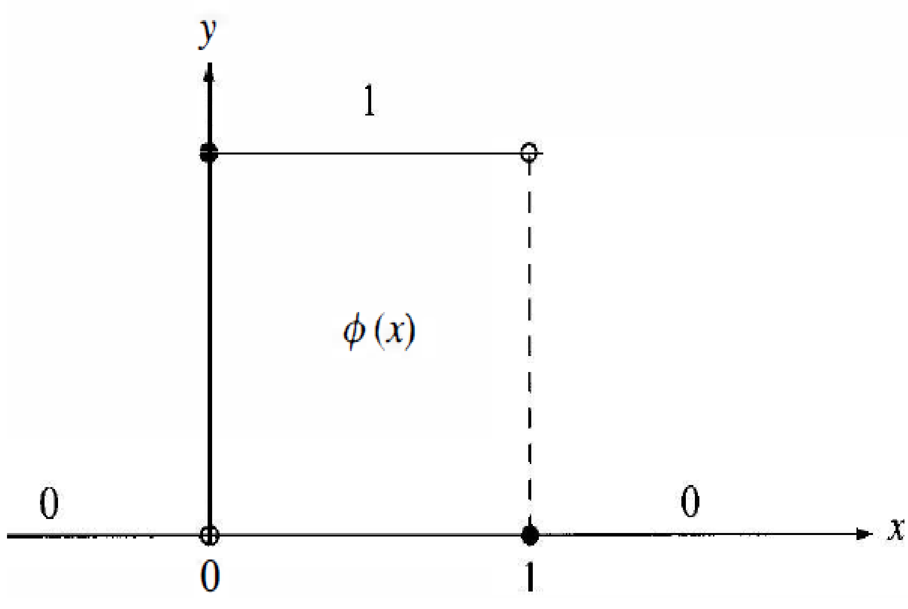

Fig. 2.2: Graph of the Haar scaling function

Definition 2.3 *Haar scaling function is defined as [9, 25]:*

$$\phi(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 0 & otherwise \end{cases}$$

Any signal is approximated using the dilations and translations of the previous basic function.

Example 2.2 *[9] The following two figures are the voltage from a faulty meter and it's approximation using the haar functions respectively.*

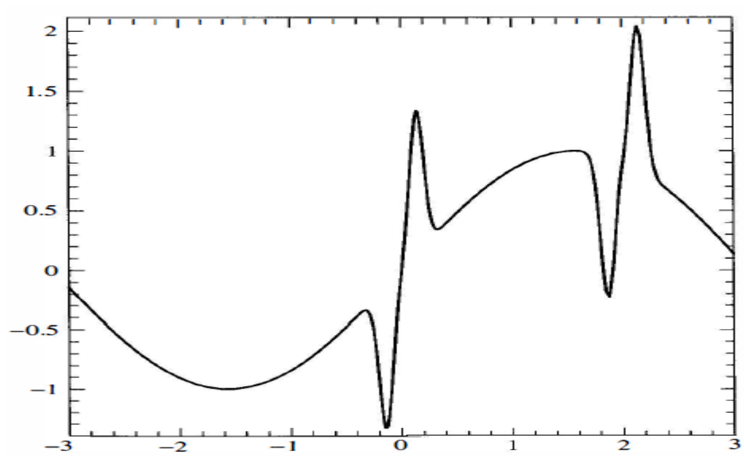

Fig. 2.3: Voltage from faulty meter

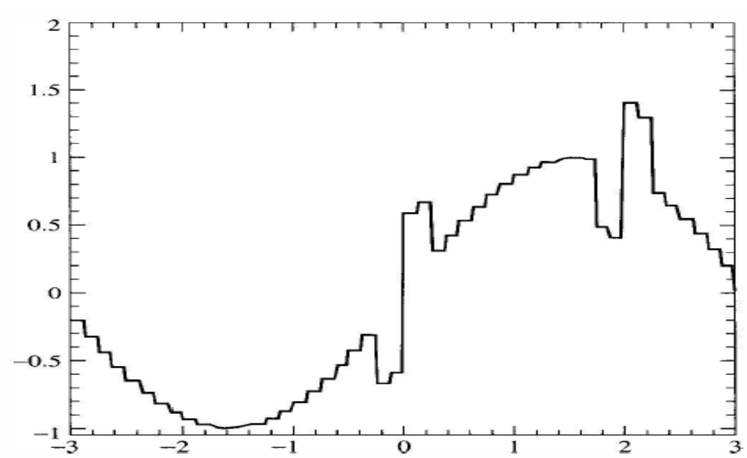

Fig. 2.4: Approximated voltage signal by Haar function.

Definition 2.4 *[9] let j be a nonnegative integer. The set V_j, is the space of piecewise constant functions of finite support with discontinuities contained in the set:$\{\frac{k}{2^j}\}_{k\in\mathbb{Z}}$, i.e the space of functions spanned by $\{\phi(2^j x - k)\}_{k\in\mathbb{Z}}$*

Remarks:

1. A function of finite support means that the function vanishes outside a finite interval.

2. If $f \in V_j$, then f can be expressed as a finite sum as follows:

$$f = \sum_k a_k \phi(2^j x - k), \qquad a_k \in \mathbb{R}. \tag{2.18}$$

3. $V_0 \subset V_1 \subset V_2 \subset ... \subset V_{j-1} \subset V_j \subset V_{j+1} \subset ...$, $\phi(4x)$ belongs to V_2 but does not belong to V_1 since it has a discontinuity point at $x = \frac{1}{4}$, so the containment is strict.

4. The reason of selecting $\phi(2^j x)$ instead of $\phi(ax)$ is if V_2 is defined as $\phi(3x - k)$, then V_1 is not subset of V_2

Theorem 2.1 *[9] The set of functions $\{2^{\frac{j}{2}}\phi(2^j x - k)\}$ form an orthonormal basis of V_j, $j \in \mathbb{Z}$.*

2.3.3 The Mother Haar Wavelet

To decompose V_1 as an orthogonal sum of V_0 and its complement, we need to characterize the orthogonal complement by a function ψ in which the following two conditions must be satisfied [9]:

1. $\psi = \sum_k a_k \phi(2x - k)$, where the sum is finite, $a_k \in \mathbb{R}$

2. ψ is orthogonal to any element in V_0 i.e $\int_{-\infty}^{\infty} \psi(x)\phi(x-k)dx = 0$, $\forall k \in \mathbb{Z}$

The simplest function ψ that satisfies the previous conditions is:

$$\psi(x) = \phi(2x) - \phi(2x - 1) \tag{2.19}$$

The graph of function $\psi(x)$ is given as:

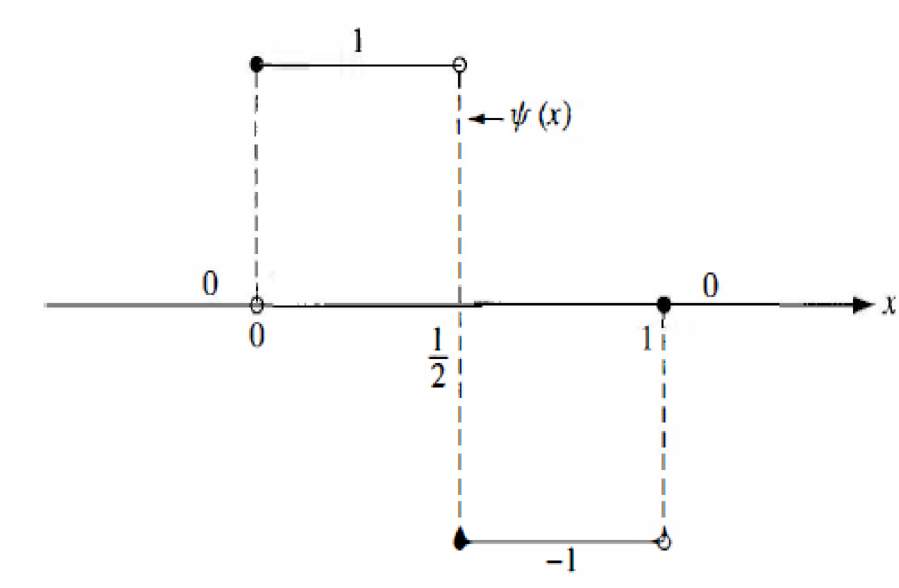

Fig. 2.5: The Haar wavelet $\psi(x)$.

Definition 2.5 *[33] The Haar wavelet function $\psi(x)$ is defined as:*

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.6 *[9] Let j be a nonnegative integer. W_j is the space of functions that spanned by the set $\{\psi(2^j x - k)\}_{k\in\mathbb{Z}}$, i.e if $f(x) \in W_j$, then f can be expressed as:*

$$f(x) = \sum_k a_k \psi(2^j x - k), \quad k \in \mathbb{Z}, a_k \in \mathbb{R} \tag{2.20}$$

Theorem 2.2 *[9] W_j is the orthogonal complement of V_j in V_{j+1}.*

Theorem 2.3 *[9] The space $L^2(\mathbb{R})$ can be decomposed as an infinite orthogonal direct sum i.e $L^2(\mathbb{R}) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus ....$*

Note that any $f(x) \in L^2(R)$ can be expressed as an infinite sum as follows:

$$f(x) = f_0 + \sum_{k=0}^{\infty} w_k \tag{2.21}$$

Where $f_0 \in V_0,\ w_k \in W_k$.

2.3.4 Haar Wavelet Family in [0, 1)

Let us restrict our work in the interval [0, 1). The Haar wavelet functions are [18, 28]:

$$h_1(x) = \begin{cases} 1 & \text{if x} \in [0,1) \\ 0 & otherwise \end{cases}$$

$$h_i(x) = \psi(2^j x - k) = \begin{cases} 1 & \text{if x} \in [\alpha_i, \beta_i) \\ -1 & \text{if x} \in [\beta_i, \gamma_i) \\ 0 & otherwise. \end{cases} \tag{2.22}$$

Where $\alpha_i = \frac{k}{m}, \beta_i = \frac{k+.5}{m}, \gamma_i = \frac{k+1}{m},\ m = 2^j, j = 0, 1, 2, ..., J,$ determine the level of wavelet, $k = 0, 1, 2, ..., m-1$, k is the translation parameter, J is the maximal level of resolution. The index i can be founded the relation $i = m + k + 1$, which means that the minimal level of i is 2, the maximal level of i is $l = 2^{J+1}$.

Theorem 2.4 *[43] The Haar wavelet family is an orthonormal basis for $L^2([0, 1))$ with the normalization factor 2^j.*

$$\int_0^1 h_i(x)h_r(x) = \frac{1}{2^j}\delta_{ir} \tag{2.23}$$

The set of the first eight Haar wavelet functions [35]

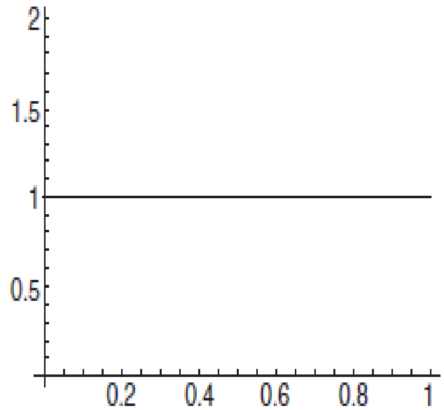

Figure 2.6:$h_1(x)$.

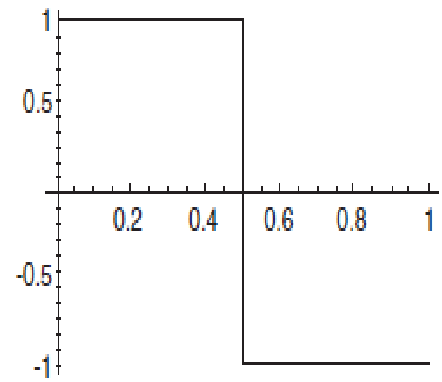

Figure 2.7: $h_2(x)$

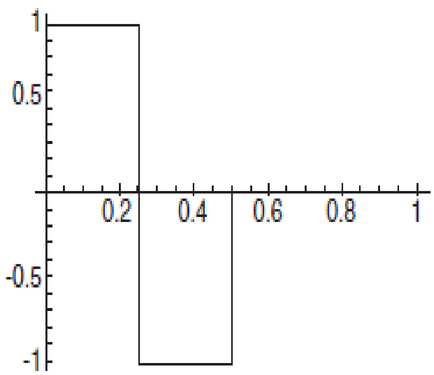

Figure 2.8: $h_3(x)$.

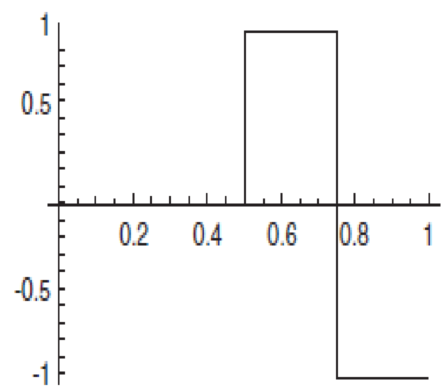

Figure 2.9: $h_4(x)$

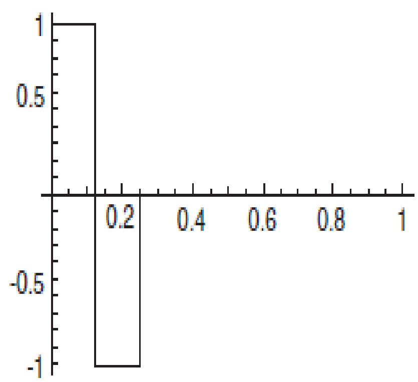

Figure 2.10: $h_5(x)$.

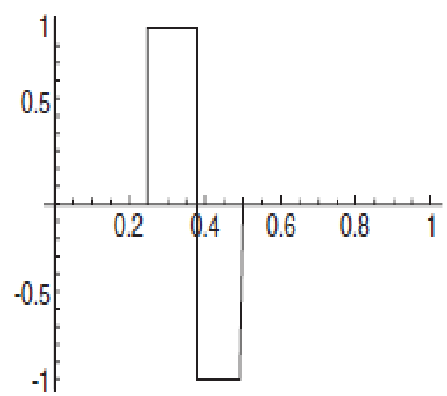

Figure 2.11: $h_6(x)$

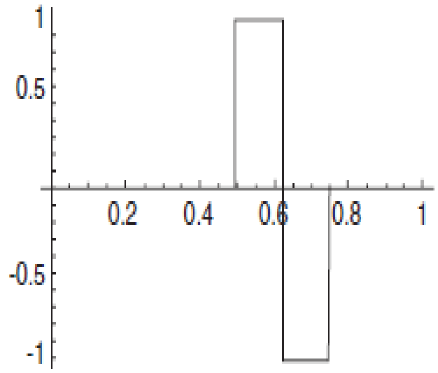

Figure 2.12: $h_7(x)$.

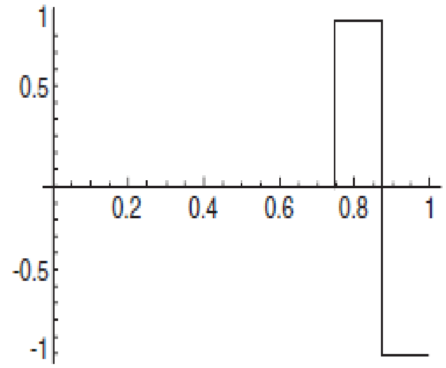

Figure 2.13: $h_8(x)$

2.3.5 Integration of Haar Functions

The method that we will use to solve the nonlinear integral equations needs to integrate Haar wavelet functions.

$$\int_0^1 h_i(t)dt = \begin{cases} 1 & \text{for i} = 1, \\ 0 & \text{for i} \geq 2. \end{cases} \tag{2.24}$$

Let $P_{i,1}$ be the integral of the i^{th} Haar wavelet from 0 to x [27]

$$P_{i,1} = \int_0^x h_i(t)dt = \begin{cases} x - \alpha_i & \text{if x} \in [\alpha_i, \beta_i) \\ \gamma_i - x & \text{if x} \in [\beta_i, \gamma_i) \\ 0 & otherwise. \end{cases} \tag{2.25}$$

and in general for $v = 1, 2, 3, ...$

$$P_{i,v+1} = \int_0^x P_{i,v}(t)dt \tag{2.26}$$

For evaluating $P_{i,2}, P_{i,3}, P_{i,4}$ see [27].

Now, divide the interval [0,1) into $2M$ subintervals of equal length where $M = 2^J$, then the grid points are given by:

$$x_p = \frac{p - .5}{2M}, \quad p = 1, 2, ..., 2M. \tag{2.27}$$

Set $H_1 = 1$, and suppose that H be the Haar coefficient $2M \times 2M$ matrix such that $h(i,p) = h_i(x_p)$, the operational matrix of integration P, which is also $2M \times 2M$ matrix is defined by the equation [18]:

$$(PH)_{il} = \int_0^{x_l} h_i(t)dt \tag{2.28}$$

Finally P is calculated using the following formula:

$$P = AH^{-1} \tag{2.29}$$

where $A = PH$, H^{-1} is the inverse of Haar coeficient matrix that can be founded using the formula [35]:

$$H^{-1} = \frac{1}{2M} H^T D^{-1}$$

where D is a $2M \times 2M$ diagnoal matrix such that:

$$D_{ii} = \int_0^1 h_i^2(x)dx, \quad i \in \{0, 1, 2, ..., 2M\}$$

Chen and Hsiao [12] provided a matrix equation for calculating the matrix P of order $2M$ which is:

$$P_{(2M)} = \frac{1}{4M} \begin{bmatrix} 4MP_M & -H_M \\ H_M^{-1} & O_M \end{bmatrix} \tag{2.30}$$

where O_M is a square zero matrix of size $M \times M$, P_1=0.5.

Example 2.3 *[42]* $P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$

2.4 Newton-Raphson Method

Newton-Raphson method is an iterative method for solving square system of algebraic equations.

Consider the system:

$$f_i(x_1, x_2, ..., x_n) = 0, \quad i = 1, 2, ..., n \tag{2.31}$$

This system can be written in matrix form as:

$$F(X) = O,$$

where $X = \begin{bmatrix} x_1 & x_2 & ... & x_n \end{bmatrix}^T$, $F = \begin{bmatrix} f_1 & f_2 & ... & f_n \end{bmatrix}^T$, O is the zero vector, Newton's method is based on constructing a sequence of vectors that converges to X such that $F(X) = 0$. Let $F(X)$ be a continuously differentiable function, then it can be approximated around the vector $X^{(0)}$ using Taylor expansion as follows [38]:

$$F(X^{(0)} + \Delta X) \approx F(X^{(0)}) + J(F(X^{(0)}))\Delta X, \tag{2.32}$$

where $J(F(X^{(0)})) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(X^{(0)}) & \frac{\partial f_1}{\partial x_2}(X^{(0)}) & ... & \frac{\partial f_1}{\partial x_n}(X^{(0)}) \\ \frac{\partial f_2}{\partial x_1}(X^{(0)}) & \frac{\partial f_2}{\partial x_2}(X^{(0)}) & ... & \frac{\partial f_2}{\partial x_n}(X^{(0)}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(X^{(0)}) & \frac{\partial f_n}{\partial x_2}(X^{(0)}) & ... & \frac{\partial f_n}{\partial x_n}(X^{(0)}) \end{bmatrix}$,

J is called the Jacobian. The vector $\hat{X}$ is founded such that $F(\hat{X}) = 0$.

The following equation is obtained:

$$F(X^{(0)}) + J(F(X^{(0)}))(\hat{X} - X^{(0)}) = 0$$

Given that the Jacobian matrix is invertible at X^0, the previous equation is solved for $\hat{X}$

$$\hat{X} = X^{(0)} - J^{-1}(F(X^{(0)}))F(X^{(0)})$$

In order to avoid finding inverse, it is reliable to solve the linear system

$$J(F(X^{(0)}))Y^{(0)} = -F(X^{(0)})$$

$J(F(X^{(0)})), F(X^{(0)})$ are knowns. Once unknown vector Y is evaluated, the initial guess is improved to $\hat{X} = X^{(0)} + Y^{(0)}$.

In general:

$$J(F(X^{(n-1)}))Y^{(n-1)} = -F(X^{(n-1)}) \tag{2.33}$$

Finally, $X^{(n)} = X^{(n-1)} + Y^{(n-1)}$ [46].

It is convenient to stop after a certain number of iterations, or when $|F(X^n)| < \varepsilon$.

Theorem 2.5 *[38] Let $F(X)$ be a continuously differentiable function in a neighberhood of $\hat{X}$, and the Jacobian matrix $J(F(\hat{X}))$ is invertable, then Newton's method converges to $\hat{X}$ given that $X^(0)$ is close enough to $\hat{X}$. Moreover the convergence rate is quadratic i.e for large enough k, there exists c such that:*

$$||X^{(k+1)} - \hat{X}||_2 < c||X^{(k)} - \hat{X}||_2^2 \tag{2.34}$$

In order to confirm that the initial guess $X^{(0)}$ is close enough such that the sequence of vector is coverging to $\hat{X}$, Steepest descent teqnique is helpful.

2.5 Steepest descent algorithm

The Steepest descend method is converging linearly to the solution, but it is applicable even for poor initial approximations [10]. To say that system (2.26) has a solution is equivalent to the problem : $g(X) = \sum_{i=1}^{n} f_i^2(X)$ has minimal value 0.

Definition 2.7 *[23] let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ the gradient of f is denoted by $\bigtriangledown f(X)$, and it is defined as:*

$$\bigtriangledown f(X) = \begin{bmatrix} \frac{\partial f}{\partial x_1}(X) & \frac{\partial f}{\partial x_2}(X) & ... & \frac{\partial f}{\partial x_n}(X) \end{bmatrix} \tag{2.35}$$

Definition 2.8 *[23] The directional derivative of f in the direction of the unit vector v is denoted by $D_v f(x)$, it is defined as:*

$$D_v f(x) = \lim_{h \to 0} \frac{f(X + hv) - f(X)}{h} = v. \bigtriangledown f(X) \tag{2.36}$$

The directional derivative of f in the direction of the unit vector v measures the change in f with respect to the change in variable in the direction of v. Its maximum and minimum values are attained when v is parallel to $\bigtriangledown f(X)$, and $v. \bigtriangledown f(X) = - \bigtriangledown f(X)$ respectively.

The goal is to minimize the value of the function g to reach zero value, so if we start with the initial vector $X^{(0)}$, it is reliable to move a way from $X^{(0)}$ in the opposite direction of $\bigtriangledown g$, so choose $X^{(1)}$ such that [10]:

$$X^{(1)} = X^{(0)} - \alpha \bigtriangledown g(X^{(0)}), \quad \alpha > 0. \tag{2.37}$$

The constant α is choosen in a way such that minimizes the one-variable function:

$$h(\alpha) = g(X^{(1)}) \tag{2.38}$$

Finding α requires the derivative of the function h, and then its critical points. in stead of that, three numbers $\alpha_1,\ \alpha_2,\ \alpha_3$ are choosen to be close to where the minimal value of $h(\alpha)$ occurs, then a quadretic polynomial P is constructed in the interval $[\alpha_1, \alpha_3]$ using Newton's forward divided-difference formula. Finally it is very easy to find the critical value of P, let $\hat{\alpha}$ be the value that minimizing the function P, then a new iterate is obtained as follows:

$$X^{(1)} = X^{(0)} - \hat{\alpha} g(X^{(0)}) \tag{2.39}$$

Since $g(X^{(0)})$ is available, α_1 is coosen to be zero, then α_3 is choosen such that $h(\alpha_3) < h(\alpha_1)$, and $\alpha_2 = \frac{\alpha_3}{2}$. For Steepest descend algorithm see [10]

Chapter 3

Haar Wavelet Method for Solving One-dimensional Nonlinear Integral Equations

This chapter focuses on the numerical solution of one-dimensional nonlinear integral equation of the second kind using Haar wavelet method. In section one, an important and efficient theorem for approximating functions of one variable by Haar wavelet functions is introduced. Section two, deals with solving Fredholm nonlinear integral equation. In section three, the numerical solution of nonlinear Volterra integral equation is investigated.

3.1 One-dimensional Haar Wavelet System

Let $f(x) \in L^2([0,1))$, then it can be expressed as an infinite sum as follows [7]:

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x) \tag{3.1}$$

where $\sum_{i=1}^{\infty} a_i h_i(x)$ converges to f uniformly [8], a_i are real constants such that

$$a_1 = \int_0^1 f(x)dx,$$

$$a_i = 2^j \int_0^1 f(x)h_i(x)dx,\ i \geq 2. \tag{3.2}$$

In matrix form:

$$f(x) = \hat{a}^T \hat{h}, \tag{3.3}$$

where $\hat{a} = (a_i)_{i\in\mathbb{N}^*}, \hat{h} = (h_i)_{i\in\mathbb{N}^*}$, $\hat{a}, \hat{h}$ are infinite column vectors.

For numerical reasons the infinite sum in equation (3.1) is truncated and $i = 1, 2, ..., l$ where l is a power of 2 that indicates the number of nodes, hence $f(x)$ can be written as

$$f_l(x) = \sum_{i=1}^{l} a_i h_i(x). \tag{3.4}$$

The nodes are defined as: [8]

$$x_p = \frac{2p-1}{2l}, \qquad p = 1, 2, ..., l. \tag{3.5}$$

If the nodes are substituted in equation (3.4), the following linear system of size $l \times l$ is obtained:

$$f(x_p) = \sum_{i=1}^{l} a_i h_i(x_p). \tag{3.6}$$

In matrix form equation (3.6) is:

$$F = AH \tag{3.7}$$

where $F = \begin{bmatrix} f_1 & f_2 & \cdots & f_l \end{bmatrix}$, $f_i = f(x_i)$, $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_l \end{bmatrix}$, $H = [h_{ij}],\ \ h_{ij} = h_i(x_j),\ i,\ j = 1, 2, ..., l.$

The following theorem is efficient to find these coefficients.

Theorem 3.1 *[5] If $f(x)$ is known at the nodes $x_p,\ p=1,2,...,l$, then the solution of system (3.7) is:*

$$a_1 = \frac{1}{l}\sum_{p=1}^{l} f(x_p) \tag{3.8}$$

$$a_i = \frac{1}{\rho_i}\left[\sum_{p=\alpha_i}^{\beta_i} f(x_p) - \sum_{p=\beta_i+1}^{\gamma_i} f(x_p)\right] \tag{3.9}$$

where $\tau_i = 2^{\lfloor log_2(i-1)\rfloor}$, $\sigma_i = (i-\tau_i)$, $\rho_i = \frac{l}{\tau_i}$, $\gamma_i = \rho_i\sigma_i$, $\beta_i = \rho_i(\sigma_i - 1) + \frac{\rho_i}{2}$, and $\alpha_i = \rho_i(\sigma_i - 1) + 1$.

Substitute equations (3.8) and (3.9) into equation 3.4 to get:

$$f(x) = \frac{1}{l}\sum_{p=1}^{l} f(x_p)h_1(x) + \sum_{i=2}^{l}\frac{1}{\rho_i}\left(\sum_{p=\alpha_i}^{\beta_i} f(x_p) - \sum_{p=\beta_i+1}^{\gamma_i} f(x_p)\right)h_i(x) \tag{3.10}$$

Example 3.1 *Consider the following function $f(x) = x - 1$, for $l = 2$, and equation (3.5) $x_1 = \frac{1}{4},\ x_2 = \frac{3}{4},\ \tau_2 = 1,\ \sigma_1 = 1,\ \rho_1 = 2,\ \gamma_2 = 2,\ \beta_2 = 1, \alpha_2 = 1$. the coefficients of $f(x)$ are: $a_1 = \frac{-1}{2},\ a_2 = \frac{-1}{4}$.*

Now, let $g(x, y)$ be a function of two-variables defined on $[0, 1) \times [0, 1)$, then $g(x, y)$ can be approximated using one-dimensional Haar wavelet functions as follows:

$$g(x, y) = \sum_{i=1}^{l} a_i(x)h_i(y) \tag{3.11}$$

From equation (3.5), $g(x, y)$ becomes:

$$g(x, y_p) = \sum_{i=1}^{l} a_i(x)h_i(y_p) \tag{3.12}$$

The variable coefficients $a_i(x)$, can be evaluated using the following corollary.

Corollary 3.1 *[7] The unknown coefficients $a_i(x)$ in equation (3.12) are given as follows:*

$$a_1(x) = \frac{1}{l}\sum_{j=1}^{l} g(x, y_p) \tag{3.13}$$

$$a_i(x) = \frac{1}{\rho_i}\left[\sum_{p=\alpha_i}^{\beta_i} g(x, y_p) - \sum_{p=\beta_i+1}^{\gamma_i} f(x, y_p)\right] \tag{3.14}$$

where τ_i, σ_i, ρ_i, γ_i, ρ_i, β_i, and α_i are defined as the same in theorem (3.1).

Example 3.2 *Consider the function $g(x,y) = e^{x+y}$. If $l = 2$, from equation (3.5), $y_1 = \frac{1}{4}$, $y_2 = \frac{3}{4}$, $\tau_2 = 1, \sigma_1 = 1$, $\rho_1 = 2$, $\gamma_2 = 2$, $\beta_2 = 1, \alpha_2 = 1$. using corollary (3.1) the coefficients of $g(x,y)$ are: $a_1(x) = \frac{e^{x+.25}+e^{x+.75}}{2}$, $a_2(x) = \frac{e^{x+.25}-e^{x+.75}}{2}$.*

3.2 Haar Wavelet Method for Nonlinear Fredholm Integral Equation

Consider the following one-dimensional nonlinear Fredholm integral equation of the second kind [44]:

$$U(x) = f(x) + \lambda \int_0^1 K(x,t)\Phi(U(t))dt \tag{3.15}$$

where $f(x)$ and $K(x,t)$ are known real valued functions defined respectively in [0,1], [0,1]×[0,1], Φ is a nonlinear function.

Theorem 3.2 *[48] Let the function $G(x,t,w) = K(x,t)\Phi(w)$ be a continuous function in the set, $[0,1] \times [0,1] \times \mathbb{R}$ and satisfies the Lipschitz condition with respect to w, i.e $|G(x,t,w_1) - G(x,t,w_2)| \le k|w_1 - w_2|$, $k > 0$, then equation (3.15) has unique solution in [0,1] if $|\lambda| < 1/k$.*

Now depending on Haar wavelets, the function $\lambda K(x,t)\phi(U(t))$ can be approximated as:

$$\lambda K(x,t)\phi(U(t)) \simeq \sum_{i=1}^{l} a_i(x)h_i(t). \tag{3.16}$$

Substituting equation (3.16) in equation (3.15) to get the following equation:

$$U(x) = f(x) + \sum_{i=1}^{l} a_i(x) \int_0^1 h_i(t)dt. \tag{3.17}$$

Using the properties of Haar wavelet functions under integral sign, equation (3.17) can be written as:

$$U(x) = f(x) + a_1(x). \tag{3.18}$$

Substituting the value of $a_1(x)$ given in corollary (3.1) implies that

$$U(x) = f(x) + \frac{\lambda}{l} \sum_{p=1}^{l} K(x,t)\Phi\big(U(t_p)\big) \tag{3.19}$$

Finally, applying the collocation points given in equation (3.5), the following system of nonlinear algebraic equations is produced:

$$U(x_r) = f(x_r) + \frac{\lambda}{l} \sum_{p=1}^{l} K(x_r,t_p)\Phi(U(t_p)), \tag{3.20}$$

where r,p = 1, 2, ..., l.

The matrix form of the system (3.20) is:

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_l \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_l \end{bmatrix} + \tfrac{\lambda}{l} \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1l} \\ K_{21} & K_{22} & \cdots & K_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ K_{l1} & K_{l2} & \cdots & K_{ll} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_l \end{bmatrix},$$

where $U_i = U(x_i),\ f_i = f(x_i),\ K_{ij} = K(x_i, t_j),\ \Phi_i = \Phi(U(t_i)),\ i, j = 1, 2, ..., l$

The previous nonlinear system can be solved for $U(x_r), r = 1, 2, ..., l$ using Newton's method. A maximum number of iterations 400, tolerance 10^{-5},and Initial guess $f(x_r), r = 1, 2, ..., l$ were used.

Remarks: To see the effectivity of the presented method, consider the following equation.

Example 3.3 *[29] Consider the following integral equation:*

$$U(x) = (\frac{1}{2} - ln2)x^2 + \sqrt{x} + \int_0^1 \frac{x^2t^2}{1 + (U^2(t))}dt, x \in [0, 1), \tag{3.21}$$

The exact solution is $U_{ex}(x) = \sqrt{x}$.

Using Haar wavelet approximation, the kernel function can be approximated as:

$$\frac{x^2t^2}{1 + (U^2(t))} \simeq \sum_{i=1}^{l} a_i(x)h_i(t). \tag{3.22}$$

Then from equation(3.20):

$$\begin{aligned} U(x) &= (\frac{1}{2} - ln2)x^2 + a_1(x) \\ &= (\frac{1}{2} - ln2)x^2 + \frac{1}{l}\sum_{p=1}^{l} \frac{x^2t_p^2}{1 + (U^2(t_p))} \end{aligned}$$

. For l=2, the matrix form is:

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0.487928 \\ 0.757380 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (\frac{1}{4})^2(\frac{1}{4})^2 & (\frac{1}{4})^2(\frac{3}{4})^2 \\ (\frac{3}{4})^2(\frac{1}{4})^2 & (\frac{3}{4})^2(\frac{3}{4})^2 \end{bmatrix} \begin{bmatrix} \frac{1}{1+U_1^2} \\ \frac{1}{1+U_2^2} \end{bmatrix}. \tag{3.23}$$

The solution of system (3.23) is $U(\frac{1}{4}) = 0.499574$ and $U(\frac{3}{4}) = 0.862193$.

Tab. 3.1: Error function for Example 3.3

J	l	$\|\|.\|\|_\infty$	$R_c(l)$
0	2	3.8×10^{-3}	-
1	4	1.3×10^{-3}	1.5475
2	8	3.8×10^{-4}	1.7744
3	16	1.03×10^{-4}	1.8834
4	32	2.6×10^{-5}	1.8961

The precision of the results is measured using the error function [27]:

$$||.||_\infty = Max_{1\leq i\leq l}|U_{ex}(x_i) - U(x_i)|$$

where $U_{ex}(x_i)$, and $U(x_i)$ are the excat and the approximate solutions of the integral equation at x_i respectively.

On the other hand, the experimental rate of convergence $R_c(l)$ is calculated [7].

$$R_c(l) = log_2(\frac{E_c(\frac{l}{2})}{E_c(l)}), \quad E_c(l) = ||.||_\infty.$$

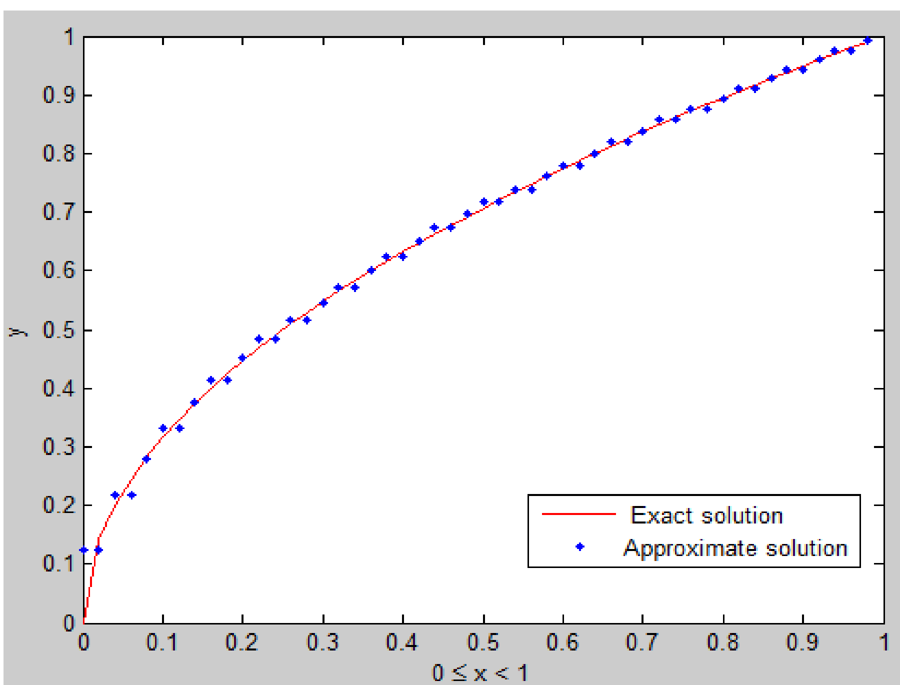

Fig. 3.1: Comparison plot of Exact and Approximate solutions at $l = 32$.

3.3 Haar Wavelet Method for Nonlinear Volterra Integral Equations

Consider the following one-dimensional nonlinear Volterra integral equation of the second kind [44]:

$$U(x) = f(x) + \lambda \int_0^x K(x,t)\Phi(U(t))dt \tag{3.24}$$

where $K(x,t)$ is defined on the set $D=\{(x,t), x\in [0,1],\ t\in [0,x)\}$, while the other variables are the same as equation (3.15).

Theorem 3.3 *[48] If $G(x,t,w) = K(x,t)\Phi(w)$ satisfies the Lipschitz condition with respect to w,*
i.e $|G(x,t,w_1) - G(x,t,w_1)| \leq k|w_2 - w_1|$ where $k > 0$, then equation (3.24) has unique solution in $[0,1]$ for any value of the parameter λ.

The function $\lambda K(x,t)\Phi(U(t))$ is approximated using one-dimensional Haar wavelet functions and and substituted in equation (3.24), to have:

$$U(x) = f(x) + \sum_{i=1}^{l} a_i(x)P_{i,1}(x). \tag{3.25}$$

Applying corollary (3.1), substituting the nodes defined in equation (3.5) to get:

$$\begin{aligned} U(x_r) &= f(x_r) + \frac{\lambda}{l}\sum_{p=1}^{l} K(x_r,t_p)\Phi(U(t_p))P_{1,1}(x_r) \\ &+ \sum_{i=2}^{l} \frac{\lambda}{\rho_i}\Bigg(\sum_{p=\alpha_i}^{\beta_i} K(x_r,t_p)\Phi(U(t_p)) - \sum_{p=\beta_i}^{\gamma_i} K(x_r,t_p)\Phi\big(U(t_p)\big)\Bigg)P_{i,1}(x_r), \end{aligned} \tag{3.26}$$

where $r = 1,2,...,l, p = 1,2,...,l, P_{i,1}(x) = \int_0^x h_i(t)dt$.

The previous system is a $l \times l$ nonlinear system of algebraic equations, which can be solved using Newton's method. Writing the previous equation in the matrix form:

$$U = F + \lambda T \Phi \tag{3.27}$$

where $U = \begin{bmatrix} U_1 & U_2 & \ldots & U_l \end{bmatrix}^T$, $F = \begin{bmatrix} f_1 & f_2 & \ldots & f_l \end{bmatrix}^T$.

When $l = 2,\ 4$, $T = \begin{bmatrix} \frac{K_{11}}{4} & 0 \\ \frac{K_{21}}{2} & \frac{K_{22}}{4} \end{bmatrix}$, $T = \begin{bmatrix} \frac{K_{11}}{8} & 0 & 0 & 0 \\ \frac{K_{21}}{4} & \frac{K_{22}}{8} & 0 & 0 \\ \frac{K_{31}}{4} & \frac{K_{32}}{4} & \frac{K_{33}}{8} & 0 \\ \frac{K_{41}}{4} & \frac{K_{42}}{4} & \frac{K_{43}}{4} & \frac{K_{44}}{8} \end{bmatrix}$, repectively

$U_r = U(x_r), f_r = f(x_r), K_{rj} = K(x_r, t_j),\ x_r = \frac{2r-1}{2l},\ i = 2, 3, ..., l,\ \ r, j = 1, 2, ..., l.$

Then $l \times l$ nonlinear system is solved for the unknowns $U(x_r),\ r = 1, 2, ..., l$ using Newton's method. Finally, the value of $U(x)$ at any point other than the nodes that defined in equation (3.4) is obtained using theorem (3.1).

Example 3.4 *[44] Consider the following integral equation:*

$$U(x) = cos(x) + \frac{cos(2x)}{8} - \frac{x^2}{4} - \frac{1}{8} + \int_0^x (x-t) U^2(t) dt, \tag{3.28}$$

The exact solution is $U_{ex}(x) = cos(x)$.

Using one$-$dimensional Haar wavelet approximation for the Kernel function

$$(x-t)U^2(t) \simeq \sum_{i=1}^{l} a_i(x) h_i(x). \tag{3.29}$$

Depending on equation (3.26), the following form is given:

$$\begin{aligned} U(x_r) &= cos(x_r) + \frac{cos(2x_r)}{8} - \frac{x_r^2}{4} - \frac{1}{8} + \frac{1}{l}\sum_{p=1}^{l}(x_r - t_p)U^2(t_p)p_{1,1}(x_r) \\ &+ \sum_{i=2}^{l}\frac{1}{\rho_i}\left(\sum_{p=\alpha_i}^{\beta_i}(x_r - t_p)U^2(t_p) - \sum_{p=\beta_i+1}^{\gamma_i}(x_r - t_p)U^2(t_p)\right) \end{aligned}$$

$r = 1, 2, ..., l, \;\; p = 1, 2, ..., l$.

For $l = 2$,

$$\begin{aligned} U_1 &= cos(\frac{1}{4}) + \frac{cos(\frac{2}{4})}{8} - \frac{1}{4^3} - \frac{1}{8} + \frac{1}{2}\frac{1}{4}(\frac{1}{4} - \frac{3}{4})U_2^2 - \frac{1}{2}\frac{1}{4}(\frac{1}{4} - \frac{3}{4})U_2^2, \\ U_2 &= cos(\frac{3}{4}) + \frac{cos(\frac{6}{4})}{8} - \frac{3}{4^3} - \frac{1}{8} + \frac{1}{2}\frac{3}{4}(\frac{3}{4} - \frac{1}{4})U_1^2 + \frac{1}{2}\frac{1}{4}(\frac{3}{4} - \frac{1}{4})U_1^2. \end{aligned}$$

In matrix form:

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + (\tfrac{1}{2}\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix})\begin{bmatrix} U_1^2 \\ U_2^2 \end{bmatrix}$$

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & 0 \end{bmatrix}\begin{bmatrix} U_1^2 \\ U_2^2 \end{bmatrix}$$

which implies that: $U_{(\frac{1}{4})} = f(\frac{1}{4}) = 0.937985$, $U_{(\frac{3}{4})} = f(\frac{3}{4}) + \frac{1}{4}U^2(\frac{1}{4}) = 0.694860$, the method is very more efficient when l is changed to be larger.

Tab. 3.2: Error function for Example 3.4

J	l	$\|\|.\|\|_\infty$	$R_c(l)$
0	2	3.7×10^{-2}	-
1	4	1×10^{-2}	1.8876
2	8	2.7×10^{-3}	1.8890
3	16	7.1×10^{-4}	1.9271
4	32	1.8×10^{-4}	1.9798
5	64	4.5×10^{-5}	2

The following figure shows the numerical and the exact solution at $l = 64$

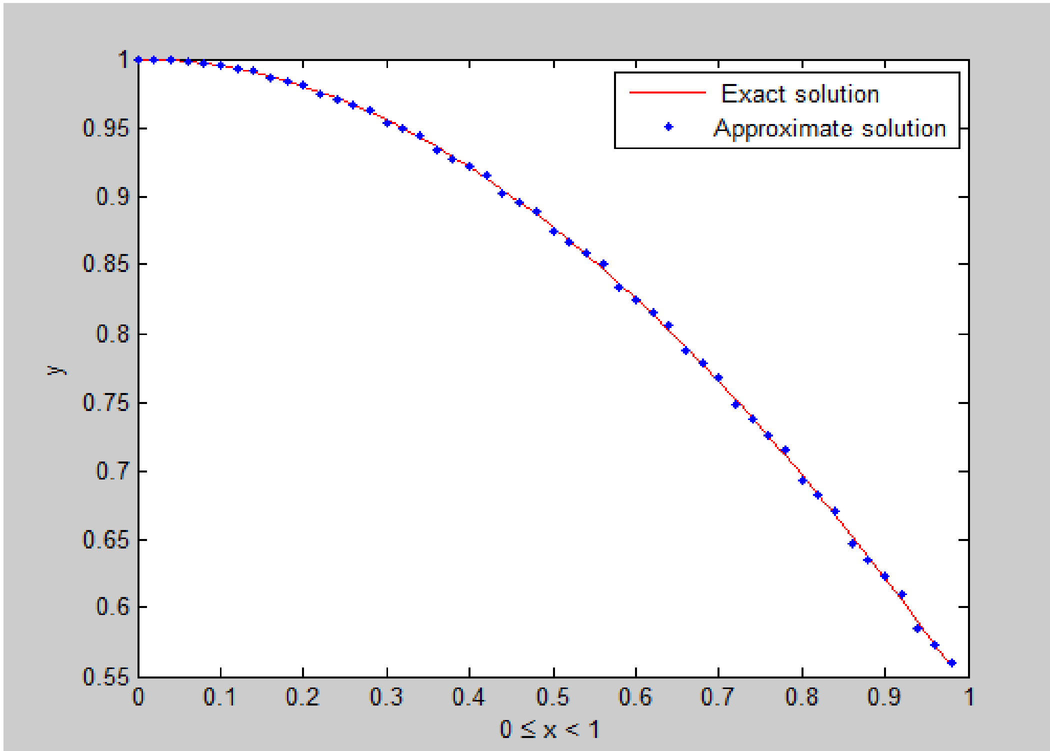

Fig. 3.2: Exact and Approximate solutions at $l = 64$.

Haar functions are defined in the interval [0,1]. If equation (3.15) was defined on the interval [a,b], then the following change of variable should be done [27]: $\hat{x} = \frac{x-a}{b-a}$, $\hat{t} = \frac{t-a}{b-a}$.

Chapter 4

Haar Wavelet Method for Solving Two-dimensional Nonlinear Integral Equations

In this chapter, Haar wavelet method was introduced for two-dimensional nonlinear integral equations of certain types. In first section, a theorem for approximating functions of two variables depending on Haar wavelets is presented. In the second and third sections, Fredholm and Volterra nonlinear integral equations are solved numerically using Matlab.

4.1 Two-dimensional Haar Wavelet System

Consider the following collocation points [7]:

$$x_m = \frac{2m-1}{2l_1} \tag{4.1}$$

$$y_m = \frac{2n-1}{2l_2} \tag{4.2}$$

where $m = 1, 2, ..., l_1$, $n = 1, 2, ..., l_2$, $l_1 = 2^{J_1+1}$, $l_2 = 2^{J_2+1}$.

Any continuous real valued function of two variables defined on the set $[0, 1) \times [0, 1)$

can be written as an infinite sum of Haar wavelet functions as follows:

$$f(x,y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}b_{ij}h_i(x)h_j(y), \tag{4.3}$$

where the coefficients $b_{ij},\ \ i,j\in\mathbb{N}$ are real constants can be calculated by,

$$\begin{aligned}
b_{11} &= \int_0^1\int_0^1 f(x,y)h_1(x)h_1(y)dxdy,\\
b_{i1} &= 2^{j_1}\int_0^1\int_0^1 f(x,y)h_i(x)h_1(y)dxdy,\\
b_{1j} &= 2^{j_2}\int_0^1\int_0^1 f(x,y)h_1(x)h_j(y)dxdy,\\
b_{ij} &= 2^{j_1+j_2}\int_0^1\int_0^1 f(x,y)h_i(x)h_j(y)dxdy,
\end{aligned} \tag{4.4}$$

where $\forall i\geq 2,\ k_1=0,1,\ 2,\ 2^{j_1}-1,\ j_1=0,1,2,..,J_1,\ i=2^{j_1}+k_1+1,\ l_1=2^{J_1+1}$, $\forall j\geq 2,\ k_2=0,1,\ 2,\ 2^{j_2}-1,\ j_2=0,1,2,..,J_2,\ j=2^{j_2}+k_2+1,\ l_2=2^{J_2+1}$.

The series in equation (4.3) is uniformly convergent to $f(x,y)$ [37]. The function $f(x,y)$ can be expressed by a finite sum as follows :

$$f(x,y)\simeq\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}b_{ij}h_i(x)h_j(y). \tag{4.5}$$

Also, equation (4.5) can be expressed in matrix form as follows:

$$f(x,y)\simeq\hat{h}(x)B\hat{h}(y) \tag{4.6}$$

Where $\hat{h}(x)=\begin{bmatrix}h_1(x) & h_2(x) & \dots & h_{l_1}(x)\end{bmatrix}$, $\hat{h}(y)=\begin{bmatrix}h_1(y) & h_2(y) & ... & h_{l_2}(y)\end{bmatrix}^T$,
$B=[b_{ij}],\ i=1,2,...,l_1, j=1,2,3,...,l_2$..

Substituting the collocation points defined in equations (4.1, 4.2), the previous equation will be:

$$f(x_m,y_n)=\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}b_{ij}h_i(x_m)h_j(y_n) \tag{4.7}$$

which gives a square linear system of size $l_1 l_2$ in which b_{ij} are the unknowns.

To find the unknowns b_{ij} the following theorem is needed.

Theorem 4.1 *[5] Let $F(x,y)$ be a real valued function of two-variables defined on the set [0,1)×[0,1). If the value of $F(x,y)$ is Known on the set of collocation points (x_p, y_q) where $p = 1,2,...,l_1,\ p = 1,2,...,l_2,$ then $F(x,y)$ can be approximated using Haar wavelet approximation as:*

$$F(x,y) = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} b_{ij} h_i(x) h_j(y)$$

with the real constant coefficients $b_{ij},\ \ i = 1,2,...,l_1,\ \ j = 1,2,...,l_2.$

$$\begin{aligned}
b_{11} &= \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2} F(x_p, y_q)\\
b_{i1} &= \frac{1}{\rho_{1i} l_2}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2} F(x_p, y_q) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2} F(x_p, y_q)\Bigg)\\
b_{1j} &= \frac{1}{l_1 \rho_{2j}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q) - \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p, y_q)\Bigg)\\
b_{ij} &= \frac{1}{\rho_{1i}\rho_{2j}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q) - \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p, y_q)\\
&\quad - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p, y_q)\Bigg)
\end{aligned}$$

$i = 2,3,...,l_1, j = 2,3,...,l_2.$

$\tau_{1i} = 2^{\lfloor log_2(i-1)\rfloor},\ \tau_{2j} = 2^{\lfloor log_2(j-1)\rfloor},$

$\sigma_{1i} = i - \tau_{1i},\ \sigma_{2j} = j - \tau_{2j},$

$\rho_{1i} = \frac{l_1}{\tau_{1i}}.\ \rho_{2j} = \frac{l_2}{\tau_{2j}},\ \gamma_{1i} = \rho_{1i}\sigma_{1i},\ \gamma_{2j} = \rho_{2j}\sigma_{2j},$

$\beta_{1i} = \rho_{1i}(\sigma_{1i} - 1) + \frac{\rho_{1i}}{2},\ \ \beta_{2j} = \rho_{2j}(\sigma_{2j} - 1) + \frac{\rho_{2j}}{2},$

$\alpha_{1i} = \rho_{1i}(\sigma_{1i} - 1) + 1,$ and $\alpha_{2j} = \rho_{2j}(\sigma_{2j} - 1) + 1.$

Proof: Fix y, and using one-dimensional Haar wavelet functions, the function $F(x,y)$ can be written as:

$$\begin{aligned} F(x,y) &= \sum_{i=1}^{l_1} a_i(y)h_i(x) \\ &= a_1(y)h_1(x) + \sum_{i=2}^{l_1} a_i(y)h_i(x). \end{aligned} \tag{4.8}$$

From corollary (3.1) and equation (4.8), we have:

$$F(x,y) = \frac{1}{l_1}\sum_{p=1}^{l_1} F(x_p,y)h_1(x) + \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\left(\sum_{p=\alpha_{1i}}^{\beta_{1i}} F(x_p,y) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} F(x_p,y)\right)h_i(x). \tag{4.9}$$

Also, for each collocation point x_p , the function $F(x_p, y)$ can be approximated using one-dimensional Haar wavelet functions as follows:

$$F(x_p,y) = \sum_{j=1}^{l_2} a_{pj}h_j(y). \tag{4.10}$$

If equation (4.10) is substituted in (4.9), then the following form is obtained:

$$F(x,y) = \frac{1}{l_1}\sum_{p=1}^{l_1}\sum_{j=1}^{l_2} a_{pj}h_j(y)h_1(x) + \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\left(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{j=1}^{l_2} a_{pj}h_j(y) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{j=1}^{l_2} a_{pj}h_j(y)\right)h_i(x) \tag{4.11}$$

Define I_1, I_2, and I_3 such that:

$$\begin{aligned} I_1 &= \frac{1}{l_1}\sum_{p=1}^{l_1}\sum_{j=1}^{l_2} a_{pj}h_j(y)h_1(x) \\ I_2 &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{j=1}^{l_2} a_{pj}h_j(y)h_i(x) \\ I_3 &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{j=1}^{l_2} a_{pj}h_j(y)h_i(x) \end{aligned}$$

$$\begin{aligned}
I_1 &= \frac{1}{l_1}\sum_{p=1}^{l_1}\Big(\sum_{j=1}^{l_2} a_{pj}h_j(y)\Big)h_1(x) \\
&= \frac{1}{l_1}\sum_{p=1}^{l_1}\Big(a_{p_1}h_1(y)+\sum_{j=2}^{l_2} a_{pj}h_j(y)\Big)h_1(x) \\
&= \frac{1}{l_1}\sum_{p=1}^{l_1}\frac{1}{l_2}\sum_{q=1}^{l_2}F(x_p,y_q)h_1(y)h_1(x)+\frac{1}{l_1}\sum_{p=1}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{2j}}\Big(\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q) \\
&- \sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Big)h_j(y)h_1(x) \\
&= \frac{1}{l_1l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}F(x_p,y_q)h_1(y)h_1(x)+\frac{1}{l_1}\sum_{p=1}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{2j}}\Big(\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q) \\
&- \sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Big)h_j(y)h_1(x) \\
&= \frac{1}{l_1l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}F(x_p,y_q)h_1(y)h_1(x)+\sum_{j=2}^{l_2}\frac{1}{l_1\rho_{2j}}\Big(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_2}F(x_p,y_q) \\
&- \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Big)h_j(y)h_1(x) \qquad (4.12)
\end{aligned}$$

Assume that:

$$b_{11} = \frac{1}{l_1l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}F(x_p,y_q) \qquad (4.13)$$

$$b_{1j} = \frac{1}{l_1\rho_{2j}}\Big(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q)-\sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Big) \qquad (4.14)$$

Substituting equations (4.13), and (4.14) in equation (4.12) implies that:

$$I_1 = b_{11}h_1(x)h_1(y)+\sum_{j=2}^{l_2}b_{1j}h_1(x)h_j(y) \qquad (4.15)$$

$$
\begin{aligned}
I_2 &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{j=1}^{l_2}a_{pj}h_j(y)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}a_{p1}h_1(y)h_i(x)+\sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{j=2}^{l_2}a_{pj}h_j(y)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\frac{1}{l_2}\sum_{q=1}^{l_2}F(x_p,y_q)h_1(y)h_i(x)+\sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{j=2}^{l_2}\frac{1}{\rho_{2j}}\Bigg(\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q)\\
&- \sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Bigg)h_j(y)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_1}F(x_p,y_q)h_1(y)h_i(x)+\sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q)\\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Bigg)h_j(y)h_i(x) \qquad (4.16)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{j=1}^{l_2}a_{pj}h_j(y)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\left(a_{p1}h_1(y)+\sum_{j=2}^{l_2}a_{pj}h_j(y)\right)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}a_{p1}h_1(y)h_i(x)+\sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{j=2}^{l_2}a_{pj}h_j(y)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\frac{1}{l_2}\sum_{q=1}^{l_2}F(x_p,y_q)h_1(y)h_i(x)+\sum_{i=2}^{l_1}\frac{1}{\rho_{1i}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{j=2}^{l_2}\frac{1}{\rho_{2j}}\Bigg(\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q)\\
&- \sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Bigg)h_j(y)h_i(x)\\
&= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2}F(x_p,y_q)h_1(y)h_i(x)+\sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\Bigg(\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}F(x_p,y_q)\\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p,y_q)\Bigg)h_i(x)h_j(y) \qquad (4.17)
\end{aligned}
$$

Add equations (4.15) with (4.16) and then subtract equation (4.17) will produce the following:

$$
\begin{aligned}
F(x,y) &= b_{11}h_1(x)h_1(y) + \sum_{j=2}^{l_2} b_{1j}h_1(x)h_j(y) \\
&+ \sum_{i=2}^{l_1} \frac{1}{\rho_{1i}l_2}\left(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} F(x_p,y_q) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} F(x_p,y_q) \right) h_i(x)h_1(y) \\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2} \frac{1}{\rho_{1i}\rho_{2j}} \left(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p,y_q) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p,y_q) \right. \\
&- \left. \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p,y_q) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p,y_q) \right) \qquad (4.18)
\end{aligned}
$$

Assume that:

$$
b_{i1} = \frac{1}{\rho_{1i}l_2}\left(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} F(x_p,y_q) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} F(x_p,y_q) \right) \qquad (4.19)
$$

$$
\begin{aligned}
b_{ij} &= \frac{1}{\rho_{1i}\rho_{2j}} \left(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p,y_q) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p,y_q) \right. \\
&- \left. \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p,y_q) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p,y_q) \right) \qquad (4.20)
\end{aligned}
$$

Substituting equations (4.19) and (4.20) into equation (4.18) completes the proof. □

Corollary 4.1 *[7] Let $F(x,y,s,t)$ be a real valued function of four variables defined on the set [0,1)×[0,1)×[0,1)×[0,1), then $F(x,y,s,t)$ can be approximated using two-dimensional Haar wavelet approximation as:*

$$
F(x,y,s,t) = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2} b_{ij}(x,y)h_i(s)h_j(t) \qquad (4.21)
$$

where:

$$b_{11} = \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2} F(x,y,s_p,t_q) \tag{4.22}$$

$$b_{i1} = \frac{1}{\rho_{1i} l_2}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2} F(x,y,s_p,t_q) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2} F(x,y,s_p,t_q)\Bigg) \tag{4.23}$$

$$b_{1j} = \frac{1}{l_1 \rho_{2j}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x,y,s_p,t_q) - \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x,y,s_p,t_q)\Bigg) \tag{4.24}$$

$$\begin{aligned} b_{ij} = {} & \frac{1}{\rho_{1i}\rho_{2j}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x,y,s_p,t_q) - \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x,y,s_p,t_q) \\ & - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x,y,s_p,t_q) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x,y,s_p,t_q)\Bigg) \end{aligned} \tag{4.25}$$

where $\alpha_{1i},\ \alpha_{2j},\ \beta_{1i}, \beta_{2j},\ \rho_{1i},\ \rho_{2j},\ \gamma_{1i},\ \gamma_{2j},\ \sigma_{1i},\ \sigma_{2j}$. and $\tau_{1i},\ \tau_{2j}$ are defined as the same in theorem (4.1).

4.2 Two-dimensional Nonlinear Fredholm Integral Equations

Consider the following nonlinear Fredholm integral equation of the second kind [11]:

$$U(x,y) = f(x,y) + \lambda \int_0^1 \int_0^1 K(x,y,s,t,U(s,t))dsdt, \tag{4.26}$$

where (x,y)∈[0,1)×[0,1), $f(x,y)$, and $K(x,y,s,t,U(s,t))$ are a continuous real valued functions defined respectively on [0,1)×[0,1), [0,1)×[0,1)×[0,1)×[0,1)×ℝ , $U(x,y)$ is the unknown function called the solution of two-dimensional integral equation,

Theorem 4.2 *[32] If the function $K(x,y,s,t,U(s,t))$ is a Lipschitz function of the form $|K(x,y,s,t,U_2) - K(x,y,s,t,U_1)| \leq L|U_2 - U_1|$, and $|\lambda| < \frac{1}{L}$ then equation (4.26) has unique solution.*

Let's approximate the kernel function using two-dimensional Haar wavelet approximation

$$K(x, y, s, t, U(s, t)) \simeq \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} b_{ij}(x, y) h_i(s) h_j(t) \tag{4.27}$$

Substituting the approximation of the function K in equation (4.27), and using the fact that: $\int_0^1 \int_0^1 h_i(s) h_j(t) ds dt = 1$, for $i, j = 1$ and zero otherwise. Equation (4.26) can be written as:

$$U(x, y) = f(x, y) + b_{11}(x, y) \tag{4.28}$$

The value of b_{11} which obtained in equation (4.22), and the collocation points (x_m, y_n), $m = 1, 2, ..., l_1,\ n = 1, 2, ..., l_2$ are substituted,

$$U(x_m, y_n) = f(x_m, y_n) + \frac{1}{l_1 l_2} \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} K(x_m, y_n, s_p, t_q, U(s_p, t_q)) \tag{4.29}$$

which is a nonlinear system of size $l_1 l_1$ with unknowns $U(x_m, y_n)$. Assume that the function K can be represented as:

$$K(x, y, s, t, U(s, t)) = G(x, y, s, t)\Phi(U(s, t))$$

where Φ is a nonlinear function, then the matrix form of the system (4.29) is:

$$U = F + M\Phi \tag{4.30}$$

such that:

$$\begin{aligned} U &= \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1l_2} & U_{21}, U_{22} & \cdots & U_{2l_2} & \cdots & U_{l_1 l_2} \end{bmatrix}^T \\ \Phi &= \begin{bmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1l_2} & \Phi_{21}, \Phi_{22} & \cdots & \Phi_{2l_2} & \cdots & \Phi_{l_1 l_2} \end{bmatrix}^T \\ M &= \frac{1}{l_1 l_2} \begin{bmatrix} G_{ijpq} \end{bmatrix} \end{aligned}$$

$$F = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1l_2} & f_{21} & f_{22} & \cdots & f_{l_1l_2} \end{bmatrix}^T$$

The solution $U(x,y)$ of the nonlinear $l_1l_2 \times l_1l_2$ system of algebraic equations at collocation points (x_m, y_n) can be derived using Newton's method with tolerance 10^{-5} and maximum number of iterations 400. Finally the solution at any point differ from the collocation points is computed using theorem (4.1).

Example 4.1 *[11] consider the following two-dimensional nonlinear Fredholm integral equation of the second kind:*

$$U(x,y) = \frac{1}{(x+y+1)^2} - \frac{x}{6(1+y)} + \int_0^1 \int_0^1 \frac{x}{1+y}(1+s+t)U^2(s,t)dsdt, \quad (x,y) \in [0,1]\times[0,1] \tag{4.31}$$

The exact solution is $U_{ex}(x,y) = \frac{1}{(x+y+1)^2}$.

From equation (4.29) we have:

$$U(x_m,y_n) = \frac{1}{(x_m+y_n+1)^2} - \frac{x_m}{6(1+y_n)} + \frac{1}{l_1l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2} \frac{x_m}{1+y_n}(1+s_p+t_q)U^2(s_p,t_q), \tag{4.32}$$

where $m,p = 1,2,..,l_1, \quad n,q = 1,2,...,l_2$. Let $l_1 = l_2 = 2$, then the Matrix form of the system is:

$$\begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} f_{11} \\ f_{12} \\ f_{21} \\ f_{22} \end{bmatrix} + \frac{1}{l_1 l2} \begin{bmatrix} G_{1111} & G_{1112} & G_{1121} & G_{1122} \\ G_{1211} & G_{1212} & G_{1221} & G_{1222} \\ G_{2111} & G_{2112} & G_{2121} & G_{2122} \\ G_{2211} & G_{2212} & G_{2221} & G_{2222} \end{bmatrix} \begin{bmatrix} U_{11}^2 \\ U_{12}^2 \\ U_{21}^2 \\ U_{22}^2 \end{bmatrix}$$

$$\begin{bmatrix} U_{11} \\ U_{12} \\ U_{21} \\ U_{22} \end{bmatrix} = \begin{bmatrix} 0.41111 \\ 0.22619 \\ 0.15 \\ 0.08857 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 0.3 & 0.4 & 0.4 & 0.5 \\ 0.21428 & 0.28571 & 0.28571 & 0.35714 \\ 0.9 & 1.2 & 1.2 & 1.5 \\ 0.64285 & 0.85714 & 0.85714 & 1.07142 \end{bmatrix} \begin{bmatrix} U_{11}^2 \\ U_{12}^2 \\ U_{21}^2 \\ U_{22}^2 \end{bmatrix}$$

The solution of the previous system for $U(x_m, y_n)$, $m = 1, 2,\ n = 1, 2$ by Matlab is:

$$\begin{bmatrix} U11 & U12 & U21 & U22 \end{bmatrix} = \begin{bmatrix} 0.44022 & 0.24699 & 0.23735 & 0.15096 \end{bmatrix}$$

The following table shows that when $l_1 l_2$ are increasing, the error of approximation decreased.

Tab. 4.1: Error function for Example 4.1

l_1	l_2	$\|\|.\|\|_\infty$	R_c
2	2	1.3×10^{-2}	-
4	4	4.6×10^{-3}	1.4988
8	8	1.4×10^{-3}	1.7162

The following figure shows the approximate solution at $l_1 = l_2 = 8$.

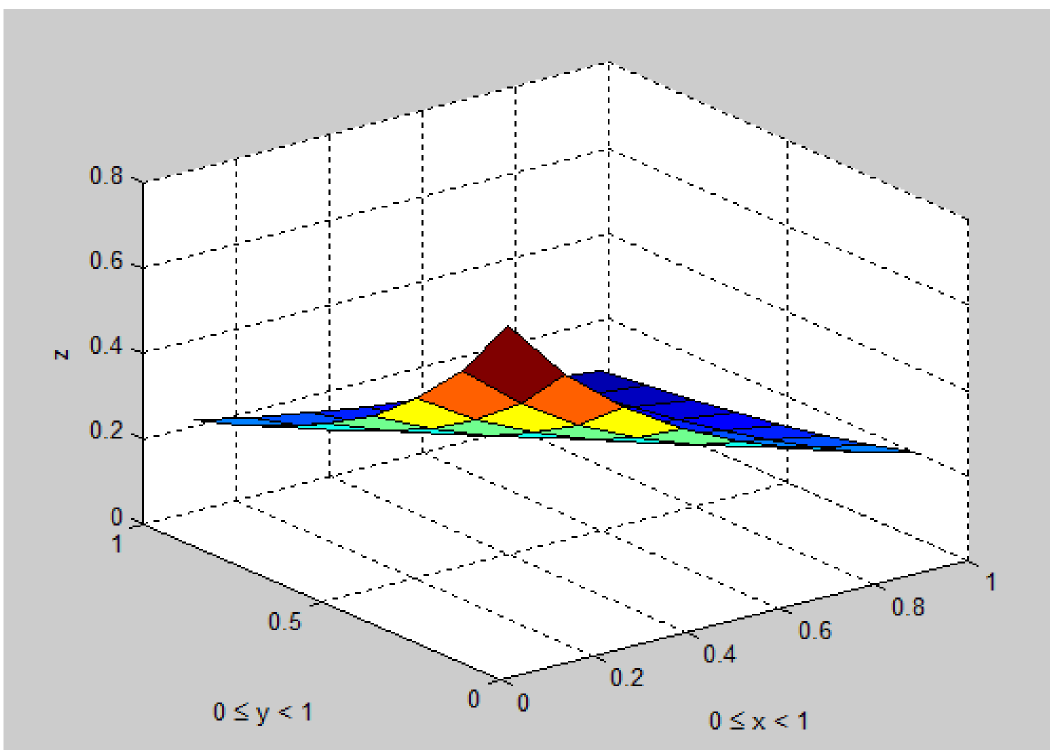

Fig. 4.1: Approximate solution of Example 4.1, for $l_1 = l_2 = 8$.

4.3 Two-dimensional Nonlinear Volterra Integral Equations

In this section, the numerical solution of two-dimensional nonlinear Volterra integral equation was proposed using Haar wavelet method. A numerical example is discussed to demonstrate the accuracy of the approximate solution.

Consider the following nonlinear Volterra integral equation of the second kind

$$U(x,y) = f(x,y) + \int_0^y \int_0^x K(x,y,s,t,U(s,t))dsdt, \quad (x,y) \in [0,1) \times [0,1), \tag{4.33}$$

where f and K are a continuous real valued functions in [0,1]×[0,1], $D := \{(x,y,s,t,U) : 0 \le s \le x \le 1, \quad 0 \le t \le y \le 1, \quad U \in \mathbb{R}\}$ respectively. If the function $K(x,y,s,t,u(s,t))$ satisfies Lipschitz condition of the form: $|K(x,y,s,t,u_1) - K(x,y,s,t,u_2)| \le L|u_1 - u_2|$, then equation (4.33) has unique solution [39].

The function $K(x,y,s,t,U(s,t))$ is approximated using two-dimensional Haar wavelet functions, then it's approximation is substituted in equation (4.33).

$$U(x,y) = f(x,y) + \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} b_{ij}(x,y) P_{i,1}(x) P_{j,1}(y), \tag{4.34}$$

where $P_{i,1} = \int_0^x h_i(s)ds$, and $P_{j,1} = \int_0^y h_j(t)dt$. The values of the coefficients b_{ij} given in corollary (4.1), and the collocation points in equations (4.1), and (4.2) are substituted:

$$\begin{aligned} U(x_m,y_n) &= f(x_m,y_n) + \frac{P_{1,1}(x_m)P_{1,1}(y_n)}{l_1 l_2} \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} K\big(x_m,y_n,s_p,t_q,U(s_p,t_q)\big) \\ &+ \sum_{i=2}^{l_1} \frac{P_{i,1}(x_m)P_{1,1}(y_n)}{\rho_1 l_2} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} K\big(x_m,y_n,s_p,t_q,U(s_p,t_q)\big) \\ &- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} K\big(x_m,y_n,s_p,t_q,U(s_p,t_q)\big) \Bigg) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=2}^{l_2} \frac{P_{1,1}(x_m)P_{j,1}(y_n)}{l_1\rho_{2j}} \Bigg(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} K\big(x_m, y_n, s_p, t_q, U(s_p, t_q)\big) \\
&- \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} K\big(x_m, y_n, s_p, t_q, U(s_p, t_q)\big) \Bigg) \\
&+ \sum_{i=2}^{l_1} \sum_{j=2}^{l_2} \frac{P_{i,1}(x_m)P_{j,1}(y_n)}{\rho_{1i}\rho_{2j}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} K\big(x_m, y_n, s_p, t_q, U(s_p, t_q)\big) \\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} K\big(x_m, y_n, s_p, t_q, U(s_p, t_q)\big) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} K\big(x_m, y_n, s_p, t_q, U(s_p, t_q)\big) \\
&+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} K\big(x_m, y_n, s_p, t_q, U(s_p, t_q)\big) \Bigg). \qquad (4.35)
\end{aligned}
$$

where $m = 1, 2, ..., l_1, \quad n = 1, 2, ..., l_2$.

If the function K can be written as $K(x, y, s, t, U(s, t)) = G(x, y, s, t)\Phi\big(U(s, t)\big)$, then the previous system can be written in a matrix form as:

$$U = F + T\Phi \qquad (4.36)$$

where:

$$
\begin{aligned}
U &= \begin{bmatrix} U_{11} & \cdots & U_{1l_2} & U_{21} & \cdots & U_{2l_2} & U_{31} & \cdots & U_{l_1l_2} \end{bmatrix}^T, \\
F &= \begin{bmatrix} f_{11} & \cdots & f_{1l_2} & f_{21} & \cdots & f_{2l_2} & \cdots & f_{31} & \cdots & f_{l_1l_2} \end{bmatrix}^T, \\
\Phi &= \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1l_2} & \Phi_{21} & \cdots & \Phi_{2l_2} & \cdots & \Phi_{31} & \cdots & \Phi_{l_1l_2} \end{bmatrix}^T.
\end{aligned}
$$

If $l_1 = l_2 = 2, T = \begin{bmatrix} \frac{G_{1111}}{16} & 0 & 0 & 0 \\ \frac{G_{1211}}{8} & \frac{G_{1212}}{16} & 0 & 0 \\ \frac{G_{2111}}{8} & 0 & \frac{G_{2121}}{16} & 0 \\ \frac{G_{2211}}{4} & \frac{G_{2212}}{8} & \frac{G_{2221}}{8} & \frac{G_{2222}}{16} \end{bmatrix}$.

For $l_1 = 2,\ l_2 = 4$, the following form of the matrix T is obtained:

$$T = \begin{bmatrix} \frac{G_{1111}}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{G_{1211}}{16} & \frac{G_{1212}}{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{G_{1311}}{16} & \frac{G_{1312}}{16} & \frac{G_{1313}}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{G_{1411}}{16} & \frac{G_{1412}}{16} & \frac{G_{1413}}{16} & \frac{G_{1414}}{32} & 0 & 0 & 0 & 0 \\ \frac{G_{2111}}{16} & 0 & 0 & 0 & \frac{G_{2121}}{32} & 0 & 0 & 0 \\ \frac{G_{2211}}{8} & \frac{G_{2212}}{16} & 0 & 0 & \frac{G_{2221}}{16} & \frac{G_{2222}}{32} & 0 & 0 \\ \frac{G_{2311}}{8} & \frac{G_{2312}}{8} & \frac{G_{2313}}{16} & 0 & \frac{G_{2321}}{16} & \frac{G_{2322}}{16} & \frac{G_{2313}}{32} & 0 \\ \frac{G_{2411}}{8} & \frac{G_{2412}}{8} & \frac{G_{2413}}{8} & \frac{G_{2414}}{16} & \frac{G_{2421}}{16} & \frac{G_{2422}}{16} & \frac{G_{2424}}{16} & \frac{G_{2424}}{32} \end{bmatrix},$$

for $l_1 = 4,\ l_2 = 2$, T has the following form:

$$T = \begin{bmatrix} \frac{G_{1111}}{32} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{G_{1211}}{16} & \frac{G_{1212}}{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2111}{16} & 0 & \frac{G_{2121}}{32} & 0 & 0 & 0 & 0 & 0 \\ \frac{G_{2211}}{8} & \frac{G_{2212}}{16} & \frac{G_{2221}}{16} & \frac{G_{2222}}{32} & 0 & 0 & 0 & 0 \\ \frac{G_{3111}}{16} & 0 & \frac{G_{3121}}{16} & 0 & \frac{G_{3131}}{32} & 0 & 0 & 0 \\ \frac{G_{3211}}{8} & \frac{G_{3212}}{16} & \frac{G_{3221}}{8} & \frac{G_{3222}}{16} & \frac{G_{3231}}{16} & \frac{G_{3232}}{32} & 0 & 0 \\ \frac{G_{4111}}{16} & 0 & \frac{G_{4121}}{16} & 0 & \frac{G_{4131}}{16} & 0 & \frac{G_{4141}}{32} & 0 \\ \frac{G_{4211}}{8} & \frac{G_{4212}}{16} & \frac{G_{4221}}{8} & \frac{G_{4222}}{16} & \frac{G_{4231}}{8} & \frac{G_{4232}}{16} & \frac{G_{4241}}{16} & \frac{G_{4241}}{32} \end{bmatrix}.$$

In order to show the reliability of this method, consider following equation.

Example 4.2 *[40] Consider the integral equation:*

$$U(x, y) = x^2 e^y + \frac{x^7}{14} - \frac{x^7 e^{2y}}{14} - \frac{x^5 y}{5} + \int_0^y \int_0^x (s^2 + e^{-2t}) U^2(s, t) ds dt, \qquad x, y \in [0, 1], \quad (4.37)$$

The exact solution $U_{ex}(x, y) = x^2 e^y$.

Expand the function $(s^2+e^{-2t})U^2(s,t)$ by two-dimensional Haar wavelet functions and from corollary (4.1) the following for is implied:

$$(s^2+e^{-2t})U^2(s,t)=\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}b_{ij}h_i(s)h_j(t) \tag{4.38}$$

On the other hand, equations (4.35, 4.1) and (4.2) give the following equation:

$$\begin{aligned}
U(x_m,y_n) &= x^2e^{y_n}+\frac{x_m^7}{14}-\frac{x_m^7e^{2y_n}}{14}-\frac{x_m^5y_n}{5}+\frac{P_{1,1}(x_m)P_{1,1}(y_n)}{l_1l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\\
&+ \sum_{i=1}^{l_1}\frac{P_{i,1}(x_m)P_{1,1}(y_n)}{\rho_{1i}l_2}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\\
&- \sum_{p=\beta_{1i}}^{\gamma_{1i}}\sum_{q=1}^{l_2}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\Bigg)\\
&+ \sum_{j=2}^{l_2}\frac{P_{1,1}(x_m)P_{j,1}(y_n)}{l_1\rho_{2j}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\\
&- \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\Bigg)\\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{P_{i,1}(x_m)P_{j,1}(y_n)}{\rho_{1i}\rho_{2j}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{\alpha_{2j}}^{\beta_{2j}}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{\beta_{2j}+1}^{\gamma_{2j}}(s_p^2+e^{-2t_q})U^2(s_p,t_q)-\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{\alpha_{2j}}^{\beta_{2j}}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\\
&+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{\beta_{2j}+1}^{\gamma_{2j}}(s_p^2+e^{-2t_q})U^2(s_p,t_q)\Bigg)
\end{aligned} \tag{4.39}$$

The previous system of nonlinear algebraic equations can be solved using Newoton's method. For simplicity , choose $l_1=l_2=2$, then the matix form system (4.39) is:

$$\begin{bmatrix}U_{11}\\U_{12}\\U_{21}\\U_{22}\end{bmatrix}=\begin{bmatrix}0.08019\\0.132158\\0.704213\\1.122020\end{bmatrix}+\begin{bmatrix}0.041814&0&0&0\\0.083629&0.017852&0&0\\0.083629&0&0.073064&0\\0.167258&0.03570&0.146129&0.049102\end{bmatrix}\begin{bmatrix}U_{11}^2\\U_{12}^2\\U_{21}^2\\U_{22}^2\end{bmatrix}$$

Solving the previous nonlinear system using Matlab program yields:

$$\begin{bmatrix} U_{11} & U_{12} & U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} 0.080471 & 0.133008 & 0.745346 & 1.286137 \end{bmatrix}$$

For larger values of $l_1,\ l_2$, the error will decrease, as the following table shows.

Tab. 4.2: Error function for Example 4.2

l_1	l_2	$\|\|.\|\|_\infty$	R_c
2	2	9.5×10^{-2}	-
4	4	6.4×10^{-2}	0.57
8	8	2.5×10^{-2}	1.3651

The following figure shows the approximate solution at $l_1 = l_2 = 8$

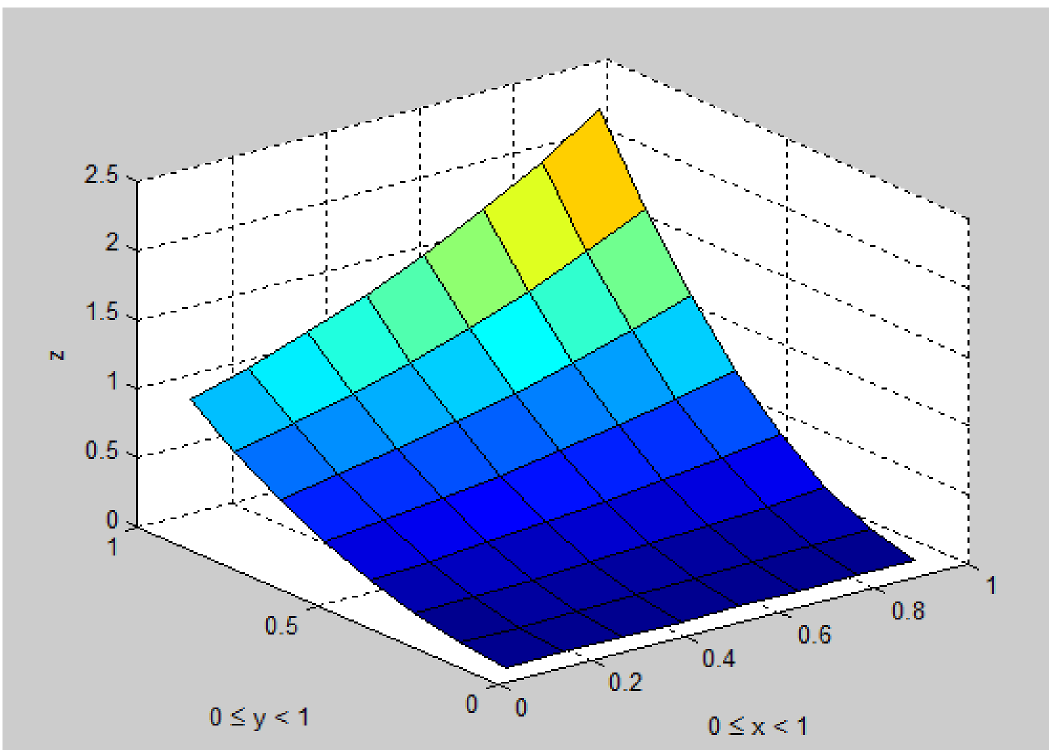

Fig. 4.2: Approximate solution of Example 4.2 at $l_1 = l_2 = 8$.

Chapter 5

Haar Wavelet Method for Solving Three-dimensional Integral Equations

In this chapter, three-dimensional functions were approximated using Haar wavelet functions. Moreover, some results concerning Haar wavelet method for the numerical solution of three-dimensional linear, and nonlinear Fredholm, and Volterra integral equations are given.

5.1 Three-dimensional Haar Wavelet System

For Haar wavelet approximations, the following collocation points are considered:

$$x_p = \frac{2p-1}{2l_1}, \ p = 1, 2, ..., l_1, \tag{5.1}$$

$$y_q = \frac{2q-1}{2l_2}, \ q = 1, 2, ..., l_2, \tag{5.2}$$

$$z_r = \frac{2r-1}{2l_3}, \ r = 1, 2, ..., l_3. \tag{5.3}$$

Let $f(x, y, z)$ be a real valued function of three variables x, y, z defined in the domain $D := \{(x, y, z) \in [0, 1) \times [0, 1) \times [0, 1)\}$, then f can be written as an infinite sum of

Haar wavelet functions as follows:

$$f(x,y,z)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}c_{ijk}h_i(x)h_j(y)h_k(z), \tag{5.4}$$

where c_{ijk} are real constants can be founded using the orhogonality proberty of Haar wavelet functions. In order to approximate three-dimensional functions, equation (5.4) is truncated to get:

$$f(x,y,z)\approx\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\sum_{k=1}^{l_3}c_{ijk}h_i(x)h_j(y)h_k(z), \tag{5.5}$$

such that:

$$\forall i\geq 2,\ k_1=0,1,...,\ 2^{j_1}-1,\ i=2^{j_1}+k_1+1,\ j_1=0,1,2,...,J_1,\ l_1=2^{J_1+1}$$

$$\forall j\geq 2, k_2=0,1,...,\ 2^{j_2}-1,\ j=2^{j_2}+k_2+1,\ j_2=0,1,...,J_2,\ l_2=2^{J_2+1}$$

$$\forall k\geq 2, k_3=0,1,...,\ 2^{j_3}-1,\ j=2^{j_3}+k_3+1,\ j_3=0,1,...,J_3,\ l_3=2^{J_3+1}$$

If collocation points in equations (5.1-5.3) are substituted in equaion(5.5), the following $l_1l_2l_3\times l_1l_2l_3$ linear system is obtained:

$$f(x_p,y_q,z_r)\approx\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\sum_{k=1}^{l_3}c_{ijk}h_i(x_p)h_j(y_q)h_k(z_r). \tag{5.6}$$

To calculate the unknowns c_{ijk}, the following theorem is needed.

Theorem 5.1 *Let $F(x,y,z)$ is a real valued function of three-variables defined on the set [0,1)×[0,1)×[0,1). If the value of $F(x,y,z)$ is Known on the set of collocation points (x_p,y_q,z_r) where $p=1,2,...,l_1$, $p=1,2,...,l_2$, $r=1,2,...,l_3$, then $F(x,y,z)$ can be approximated using Haar wavelet approximation as follows:*

$$F(x,y,z)\approx\sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\sum_{k=1}^{l_3}c_{ij,k}h_i(x)h_j(y)h_k(z).$$

With the real constant coefficients c_{ijk}, $i = 1, 2, ..., l_1$, $j = 1, 2, ..., l_2$, $k = 1, 2, ..., l_3$, *such that:*

$$c_{111} = \frac{1}{l_1 l_2 l_3} \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r),$$

$$c_{i11} = \frac{1}{\rho_{1i} l_2 l_3} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{k=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) \Bigg),$$

$$i = 2, 3, ..., l_1,$$

$$c_{1j1} = \frac{1}{l_1 \rho_{2j} l_3} \Bigg(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) \Bigg),$$

$$j = 2, 3, ..., l_2,$$

$$c_{11k} = \frac{1}{l_1 l_2 \rho_{3k}} \Bigg(\sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Bigg),$$

$$k = 2, 3, ..., l_3,$$

$$c_{ij1} = \frac{1}{\rho_{1i} \rho_{2j} l_3} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r)$$

$$- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} F(x_P, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} F(x_P, y_q, z_r) \Bigg),$$

$$i = 2, 3, ..., l_1, \ \ j = 2, 3, ..., l_2,$$

$$c_{i1k} = \frac{1}{\rho_{1i} l_2 \rho_{3k}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)$$

$$- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Bigg),$$

$$i = 2, 3, ..., l_1, \ \ k = 2, 3, ..., l_3,$$

$$c_{1jk} = \frac{1}{l_1\rho_{2j}\rho_{3k}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)$$
$$-\sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_P, y_q, z_r) + \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_P, y_q, z_r)\Bigg),$$
$$j = 2, 3, ..., l_2,\;\; k = 2, 3, ..., l_3,$$

$$c_{ijk} = \frac{1}{\rho_{1i}\rho_{2j}\rho_{3k}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)$$
$$-\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_P, y_q, z_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r)$$
$$+\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_P, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_P, y_q, z_r)$$
$$+\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)\Bigg),$$
$$i = 2, 3, ..., l_1,\;\; j = 2, 3, ..., l_2,\;\; k = 2, 3, ..., l_3.$$

Proof: Fix z and approximate the function $F(x, y, z)$ using two-dimensional Haar wavelet functions as follows:

$$F(x, y, z) = b_{11}(z)h_1(x)h_1(y) + \sum_{j=2}^{l_2} b_{1j}(z)h_1(x)h_j(y) + \sum_{i=2}^{l_1} b_{i1}(z)h_i(x)h_1(y) + \sum_{i=2}^{l_1}\sum_{j=2}^{l_2} b_{ij}(z)h_i(x)h_j(y). \tag{5.7}$$

In order to make the calculations as simple as possible, equation (5.7) is partitioned to four parts.

$$I_1 = b_{11}(z)h_1(x)h_1(y) = \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2} F(x_p, y_q, z)h_1(x)h_1(y). \tag{5.8}$$

One-dimensional Haar functions are used to approximate the functions of one-variable $F(x_p, y_q, z),\ p = 1, 2, ..., l_1,\ q = 1, 2, ..., l_2$ as follows:

$$\begin{aligned} I_1 &= \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{k=1}^{l_3} a_{kpq} h_k(z) h_1(x) h_1(y) \\ &= \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2} a_{1pq} h_1(z) h_1(x) h_1(y) + \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{k=2}^{l_3} a_{kpq} h_k(z) h_1(x) h_1(y) \end{aligned}$$

The values of a_{kpq} are substituted using one-dimensional Haar wavelet functions, and then the summations are reordered:

$$\begin{aligned} I_1 &= \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\frac{1}{l_3}\sum_{r=1}^{l_3} F(x_p, y_q, z_r) h_1(z) h_1(x) h_1(y) + \frac{1}{l_1 l_2}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{k=2}^{l_3}\frac{1}{\rho_{3k}}\Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) \\ &- \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)\Bigg) h_k(z) h_1(x) h_1(y) \\ &= \frac{1}{l_1 l_2 l_3}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=1}^{l_3} F(x_p, y_q, z_r) h_1(x) h_1(y) h_1(z) + \sum_{k=2}^{l_3}\frac{1}{l_1 l_2 \rho_{3k}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) \\ &- \sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)\Bigg) h_1(x) h_1(y) h_k(z) \end{aligned} \tag{5.9}$$

Define $c_{111},\ c_{11k},\ k = 2, 3, ..., l_3$ as follows:

$$c_{111} = \frac{1}{l_1 l_2 l_3}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=1}^{l_3} F(x_p, y_q, z_r) \tag{5.10}$$

$$\begin{aligned} c_{11k} &= \frac{1}{l_1 l_2 \rho_{3k}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) \\ &- \sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)\Bigg) h_1(x) h_1(y) h_k(z) \end{aligned} \tag{5.11}$$

Substituting equations (5.10, 5.11) into equation (5.9) to get:

$$I_1 = c_{111} h_1(x) h_1(y) h_1(z) + \sum_{k=2}^{l_3} c_{11k} h_1(x) h_1(y) h_k(z). \tag{5.12}$$

Let $I_2 = \sum_{j=2}^{l_2} b_{1j}(z)h_1(x)h_j(y)$, then from (4.14) I_2 becomes:

$$
\begin{aligned}
I_2 &= \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \Big(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q, z) - \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p, y_q, z) \Big) h_1(x)h_j(y) \\
&= \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q, z)h_1(x)h_j(y) - \sum_{j=2}^{l_2} \frac{1}{l_1\rho_2} \sum_{p=1}^{l_1} \sum_{q=\beta_2+1}^{\gamma_2} F(x_p, y_q, z)h_1(x)h_j(y) \\
&= \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \Big(a_{1pq}h_1(z) + \sum_{k=2}^{l_3} a_{kpq}h_k(z) \Big) h_1(x)h_j(y) \\
&- \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \Big(a_{1pq}h_1(z) + \sum_{k=2}^{l_3} a_{kpq}h_k(z) \Big) h_1(x)h_j(y) \\
&= \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \frac{1}{l_3} \sum_{r=1}^{l_3} F(x_p, y_q, z_r)h_1(z)h_1(x)h_j(y) \\
&+ \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{k=2}^{l_3} \frac{1}{\rho_{3k}} \Big(\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Big) h_k(z)h_1(x)h_j(y) \\
&- \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \frac{1}{l_3} \sum_{r=1}^{l_3} F(x_p, y_q, z_r)h_1(z)h_1(x)h_j(y) \\
&- \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}} \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{k=2}^{l_3} \frac{1}{\rho_{3k}} \Big(\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Big) h_k(z)h_1(x)h_j(y).
\end{aligned}
$$

Rearrange the above summation to reach the following form for I_2:

$$
\begin{aligned}
I_2 &= \sum_{j=2}^{l_2} \frac{1}{l_1\rho_{2j}l_3} \Big(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) \Big) h_1(x)h_j(y)h_1(z) \\
&+ \sum_{j=2}^{l_2} \sum_{k=2}^{l_3} \frac{1}{l_1\rho_{2j}\rho_{3k}} \Big(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \\
&- \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) + \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Big) h_1(x)h_j(y)h_k(z).
\end{aligned} \tag{5.13}
$$

Define $c_{1j1},\ c_{1jk},\ j = 2, 3, ..., l_2,\ k = 2, 3, ..., l_3$ as follows:

$$\begin{aligned} c_{1j1} &= \frac{1}{l_1\rho_{2j}l_3}\Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=1}^{l_3}F(x_p, y_q, z_r) - \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}F(x_p, y_q, z_r)\Bigg) \quad (5.14)\\ c_{1jk} &= \frac{1}{l_1\rho_{2j}\rho_{3k}}\Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p, y_q, z_r) - \sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}F(x_p, y_q, z_r)\\ &- \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p, y_q, z_r) + \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}F(x_p, y_q, z_r)\Bigg) \quad (5.15)\end{aligned}$$

Substitute (5.14) and (5.15) into equation (5.13) to get:

$$I_2 = \sum_{j=2}^{l_2} c_{1j1}h_1(x)h_j(y)h_1(z) + \sum_{j=2}^{l_2}\sum_{k=2}^{l_3} c_{1jk}h_1(x)h_j(y)h_k(z). \quad (5.16)$$

Now, let I_3 be as follows:

$$I_3 = \sum_{i=2}^{l_1} b_{i1}(z)h_i(x)h_1(y).$$

Substituting the value of the coefficient $b_{i1},\ i = 2, 3, ..., l_1$ in equation (4.19),

$$\begin{aligned} I_3 &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}F(x_p, y_q, z) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_1}F(x_p, y_q, z)\Bigg)h_i(x)h_1(y)\\ &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}F(x_p, y_q, z)h_i(x)h_1(y) - \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2}F(x_p, y_q, z)h_i(x)h_1(y)\end{aligned}$$

Substituting the approximation of the one-variable function $F(x_p, y_q, z)$ as follows:

$$\begin{aligned} I_3 &= \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}\Bigg(a_{1pq}h_1(z) + \sum_{k=2}^{l_3}a_{kpq}h_k(z)\Bigg)h_i(x)h_1(y)\\ &- \sum_{i=2}^{l_1}\frac{1}{\rho_{1i}l_2}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2}\Bigg(a_{1pq}h_1(z) + \sum_{k=2}^{l_3}a_{kpq}h_k(z)\Bigg)h_i(x)h_1(y).\end{aligned}$$

Substituting the value of the constant coefficients $a_{kpq},\ k = 1, 2, ..., l_3$, using equations (3.8, 3.9) gives:

$$
\begin{aligned}
I_3 &= \sum_{i=2}^{l_1} \frac{1}{\rho_{1i} l_2} \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \frac{1}{l_3} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) h_1(z) h_i(x) h_1(y) \\
&+ \sum_{i=2}^{l_1} \frac{1}{\rho_{1i} l_2} \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{k=2}^{l_3} \frac{1}{\rho_{3k}} \Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Bigg) h_k(z) h_i(x) h_1(y) \\
&- \sum_{i=2}^{l_1} \frac{1}{\rho_{1i} l_2} \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \frac{1}{l_3} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) h_1(z) h_i(x) h_1(y) \\
&- \sum_{i=2}^{l_1} \frac{1}{\rho_{1i} l_2} \sum_{p=\beta_{1i}+1}^{\gamma_1} \sum_{q=1}^{l_2} \sum_{k=2}^{l_3} \frac{1}{\rho_{3k}} \Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Bigg) h_k(z) h_i(x) h_1(y).
\end{aligned}
$$

Collecting the similar terms, and reordering the summations in order to have the following form:

$$
\begin{aligned}
I_3 &= \sum_{i=2}^{l_1} \frac{1}{\rho_{1i} l_2 l_3} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) \Bigg) h_i(x) h_1(y) h_1(z) \\
&+ \sum_{i=2}^{l_1} \sum_{k=2}^{l_3} \frac{1}{\rho_{1i} l_2 \rho_{3k}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Bigg) h_i(x) h_1(y) h_k(z).
\end{aligned} \tag{5.17}
$$

Defining $c_{i11},\ c_{i1k},\ i = 2, 3, ..., l_1,\ k = 2, 3, ..., l_3$ as follows:

$$
c_{i11} = \frac{1}{\rho_{1i} l_2 l_3} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) \Bigg), \tag{5.18}
$$

$$
\begin{aligned}
c_{i1k} &= \frac{1}{\rho_{1i} l_2 \rho_{3k}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r), \\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \Bigg).
\end{aligned} \tag{5.19}
$$

Substituting equations (5.18, 5.19) in equation 5.17 to have:

$$I_3 = \sum_{i=2}^{l_1} c_{i11} h_i(x) h_1(y) h_1(z) + \sum_{i=2}^{l_1} \sum_{k=2}^{l_3} c_{i1k} h_i(x) h_1(y) h_k(z) \tag{5.20}$$

Let I_4 be defined as follows:

$$I_4 = \sum_{i=1}^{l_1} \sum_{j=2}^{l_2} b_{ij}(z) h_i(x) h_j(y).$$

Substituting the value of the variable coefficients $b_{ij}(z), i = 2, 3, ..., l_1,\ j = 2, 3, ..., l_2$, using equation (4.20)

$$\begin{aligned} I_4 &= \sum_{i=2}^{l_1} \sum_{j=2}^{l_2} \frac{1}{\rho_{1i}\rho_{2j}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q, z) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p, y_q, z) \\ &- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} F(x_p, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} F(x_p, y_q, z) \Bigg) h_i(x) h_j(y). \end{aligned}$$

Estimating the value of the one-variable function $F(x_p, y_q, z)$, using one-dimensional Haar wavelet functions,

$$\begin{aligned} I_4 &= \sum_{i=2}^{l_1} \sum_{j=2}^{l_2} \frac{1}{\rho_{1i}\rho_{2i}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} (a_{1pq} h_1(z) + \sum_{k=2}^{l_3} a_{kpq} h_k(z)) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \big(a_{1pq} h_1(z) \\ &+ \sum_{k=2}^{l_3} a_{kpq} h_k(z)\big) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} (a_{1pq} h_1(z) + \sum_{k=2}^{l_3} a_{kpq} h_k(z)) \\ &+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \big(a_{1pq} h_1(z) + \sum_{k=2}^{l_3} a_{kpq} h_k(z)\big) \Bigg). \end{aligned}$$

Substituting the value of the constant coefficient $a_{kpq}, k = 1, 2, ..., l_3$ using equations (3.8, 3.9)

$$I_4 = \sum_{i=2}^{l_1} \sum_{j=2}^{l_2} \frac{1}{\rho_{1i}\rho_{2j}} \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \frac{1}{l_3} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) h_1(z) h_i(x) h_j(y)$$

$$
\begin{aligned}
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_3}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{k=2}^{l_3}\frac{1}{\rho_{3k}}\Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p,y_q,z_r)\\
&- \sum_{r=\beta_{3k}}^{\gamma_{3k}}F(x_p,y_q,z_r)\Bigg)h_k(z)h_i(x)h_j(y)\\
&- \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\frac{1}{l_3}\sum_{r=1}^{l_3}F(x_p,y_q,z_r)h_1(z)h_i(x)h_j(y)\\
&- \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{k=2}^{l_3}\frac{1}{\rho_{3k}}\Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p,y_q,z_r)\\
&- \sum_{r=\beta_{3k}+1}^{\gamma_{3k}}F(x_p,y_q,z_r)\Bigg)h_k(z)h_i(x)h_j(y)\\
&- \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\frac{1}{l_3}\sum_{r=1}^{l_3}F(x_p,y_q,z_r)h_1(z)h_i(x)h_j(y)\\
&- \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{k=2}^{l_3}\frac{1}{\rho_{3k}}\Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p,y_q,z_r)\\
&- \sum_{r=\beta_{3k}+1}^{\gamma_{3k}}F(x_p,y_q,z_r)\Bigg)h_k(z)h_i(x)h_j(y)\\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\frac{1}{l_3}\sum_{r=1}^{l_3}F(x_p,y_q,z_r)h_1(z)h_i(x)h_j(y)\\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}}\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{k=2}^{l_3}\frac{1}{\rho_{3k}}\Bigg(\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p,y_q,z_r)\\
&- \sum_{r=\beta_{3k}+1}^{\gamma_{3k}}F(x_p,y_q,z_r)\Bigg)h_k(z)h_i(x)h_j(y),
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{1}{\rho_{1i}\rho_{2j}l_3}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=1}^{l_3}F(x_p,y_q,z_r)-\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=1}^{l_3}F(x_p,y_q,z_r)\\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=1}^{l_3}F(x_P,y_q,z_r)+\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2}}\sum_{r=1}^{l_3}F(x_P,y_q,z_r)\Bigg)h_i(x)h_j(y)h_1(z)\\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\sum_{k=2}^{l_3}\frac{1}{\rho_{1i}\rho_{2j}\rho_{3j}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p,y_q,z_r)-\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}F(x_p,y_q,z_r)\\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_P,y_q,z_r)-\sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}F(x_p,y_q,z_r)
\end{aligned}
$$

$$+ \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_P, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_P, y_q, z_r)$$

$$+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)\Bigg) h_i(x) h_j(y) h_k(z), \tag{5.21}$$

Defining $c_{ij1}, c_{ijk}, i = 2, 3, ..., l_1,\ j = 2, 3, ..., l_2,\ k = 2, 3, ..., l_3$ as follows:

$$\begin{aligned} c_{ij1} &= \frac{1}{\rho_{1i}\rho_{2j} l_3}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) \\ &- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} F(x_p, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_2} \sum_{r=1}^{l_3} F(x_p, y_q, z_r)\Bigg) \end{aligned} \tag{5.22}$$

$$\begin{aligned} c_{ijk} &= \frac{1}{\rho_{1i}\rho_{2j}\rho_{3j}}\Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \\ &- \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) \\ &+ \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r) \\ &+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{\alpha_{3k}}^{\beta_{3k}} F(x_p, y_q, z_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} F(x_p, y_q, z_r)\Bigg) \end{aligned} \tag{5.23}$$

Substituting equations (5.22, 5.23) into equation (5.21) to obtain:

$$I_4 = \sum_{i=2}^{l_1} \sum_{j=2}^{l_2} c_{ij1} h_i(x) h_j(y) h_1(z) + \sum_{i=2}^{l_1} \sum_{j=2}^{l_2} \sum_{k=2}^{l_3} c_{ijk} h_i(x) h_j(y) h_k(z) \tag{5.24}$$

where:

$$\begin{aligned} \tau_{1i} &= 2^{\lfloor log_2(i-1) \rfloor},\ \tau_{2j} = 2^{\lfloor log_2(j-1) \rfloor},\ \tau_{3k} = 2^{\lfloor log_2(k-1) \rfloor}, \\ \sigma_{1i} &= i - \tau_{1i},\ \sigma_{2j} = j - \tau_{2j},\ \sigma_{3k} = j - \tau_{3k}, \end{aligned}$$

$$\begin{aligned}
\rho_{1i} &= \frac{l_1}{\tau_{1i}},\ \rho_{2j} = \frac{l_2}{\tau_{2j}},\ \rho_{3k} = \frac{l_2}{\tau_{3k}}, \\
\gamma_{1i} &= \rho_{1i}\sigma_{1i},\ \gamma_{2j} = \rho_{2j}\sigma_{2j},\ \gamma_{3k} = \rho_{3k}\sigma_{3k}, \\
\beta_{1i} &= \rho_{1i}(\sigma_{1i}-1) + \frac{\rho_{1i}}{2},\ \beta_{2j} = \rho_{2j}(\sigma_{2j}-1) + \frac{\rho_{2j}}{2},\ \beta_{3k} = \rho_{3k}(\sigma_{3k}-1) + \frac{\rho_{3k}}{2}, \\
\alpha_{1i} &= \rho_{1i}(\sigma_{1i}-1) + 1,\ \alpha_{2j} = \rho_{2j}(\sigma_{2j}-1) + 1,\ \alpha_{3k} = \rho_{3k}(\sigma_{3k}-1) + 1.
\end{aligned}$$

Adding Equations (5.12), (5.16), (5.20), and (5.24) completes the proof. □

Example 5.1 *Consider the function $f(x, y, z) = xy^2 + z^3$, if $l_1 = l_2 = l_3 = 2$.*
from equations (5.1-5.3) the collocation points are: $s_1 = t_1 = w_1 = \frac{1}{4}$, $s_2 = t_2 = w_2 = \frac{3}{4}$,
for $i, j, k = 2$ the following holds , $\tau_{12} = \tau_{22} = \tau_{32} = 1$, $\sigma_{12} = \sigma_{22} = \sigma_{32} = 1, \rho_{12} = \rho_{22} = \rho_{32} = 2$, $\gamma_{12} = \gamma_{22} = \gamma_{32} = 2$, $\beta_{12} = \beta_{22} = \beta_{32} = 1$, $\alpha_{12} = \alpha_{22} = \alpha_{32} = 1$.
From the theorem (5.1) the coefficients of the function $F(x, y, z)$ are:
$c_{111} = \frac{3}{8}$, $c_{112} = \frac{-13}{64}$, $c_{121} = \frac{-1}{8}$, $c_{211} = \frac{-5}{64}$, $c_{221} = \frac{1}{16}$, $c_{122} = c_{212} = c_{222} = 0$.

For three-dimensional integral equations, Haar wavelet method is based on approximating the Kernel function which is a six-variable function, the following corollary is helpful for that purpose.

Corollary 5.1 *Let $K(x, y, z, s, t, w)$ The real valued function of six-variables defined on the set $D := [0, 1)^6$. Then $K(x, y, z, s, t, w)$ can be approximated using three-dimensional Haar wavelet approximation as:*

$$F(x, y, z, s, t, w) = \sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\sum_{k=1}^{l_3} c_{ijk}(x, y, z)h_i(s)h_j(t)h_k(w), \tag{5.25}$$

with the variable coefficients c_{ijk}, $i = 1, 2, ..., l_1$, $j = 1, 2, ..., l_2$, $k = 1, 2, ..., l_3$, such that:

$$c_{111} = \frac{1}{l_1 l_2 l_3}\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=1}^{l_3} K(x, y, z, s_p, t_q, w_r),$$

$$c_{i11} = \frac{1}{\rho_{1i} l_2 l_3} \Bigg(\sum_{p=\alpha_1}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{k=1}^{l_3} K(x,y,z,s_p,t_q,w_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} K(x,y,z,s_p,t_q,w_r) \Bigg),$$

$$i = 2, 3, ..., l_1,$$

$$c_{1j1} = \frac{1}{l_1 \rho_{2j} l_3} \Bigg(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} K(x,y,z,s_p,t_q,w_r) - \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} K(x,y,z,s_p,t_q,w_r), \Bigg)$$

$$j = 2, 3, ..., l_2,$$

$$c_{11k} = \frac{1}{l_1 l_2 \rho_{3k}} \Bigg(\sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_p,t_q,w_r) - \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_p,t_q,w_r) \Bigg),$$

$$k = 2, 3, ..., l_3,$$

$$c_{ij1} = \frac{1}{\rho_{1i} \rho_{2j} l_3} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} K(x,y,z,s_p,t_q,w_r) - \sum_{p=\alpha_1}^{\beta_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} K(x,y,z,s_p,t_q,w_r),$$
$$- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=1}^{l_3} K(x,y,z,s_P,t_q,w_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=1}^{l_3} K(x,y,z,s_P,t_q,w_r) \Bigg),$$

$$i = 2, 3, ..., l_1, \;\; j = 2, 3, ..., l_2,$$

$$c_{i1k} = \frac{1}{\rho_1 l_2 \rho_{3k}} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_p,t_q,w_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_p,t_q,w_r)$$
$$- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_P,t_q,w_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=1}^{l_2} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_P,t_q,w_r) \Bigg),$$

$$i = 2, 3, ..., l_1, \;\; k = 2, 3, ..., l_3,$$

$$c_{1jk} = \frac{1}{l_1 \rho_{2j} \rho_{3k}} \Bigg(\sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\alpha_{3k}}^{\beta_3} K(x,y,z,s_p,t_q,w_r) - \sum_{p=1}^{l_1} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_p,t_q,w_r)$$
$$- \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_P,t_q,w_r) + \sum_{p=1}^{l_1} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_P,t_q,w_r) \Bigg),$$

$$j = 2, 3, ..., l_2, \;\; k = 2, 3, ..., l_3,$$

$$c_{ijk} = \frac{1}{\rho_{1i}\rho_{2j}\rho_{3k}}\Big(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_p,t_q,w_r) - \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_p,t_q,w_r)$$
$$- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_P,t_q,w_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_p,t_q,w_r)$$
$$+ \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_P,t_q,w_r) + \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_P,t_q,w_r)$$
$$+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{\alpha_{3k}}^{\beta_{3k}} K(x,y,z,s_p,t_q,w_r) - \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K(x,y,z,s_p,t_q,w_r)\Big),$$
$$i = 2,3,...,l_1,\ j = 2,3,...,l_2,\ k = 2,3,...,l_3.$$

Example 5.2 *Consider the function* $F(x,y,z,s,t,w) = s^2cos(xy) + (wt)sin(z)$, *assume that* $l_1 = l_2 = l_3 = 2$.
From equations (5.1-5.3), we have: $s_1 = t_1 = w_1 = \frac{1}{4},\ s_2 = t_2 = w_2 = \frac{3}{4}$, *for* $i,j,k = 2$ *the following holds* , $\tau_{12} = \tau_{22} = \tau_{32} = 1,\ \sigma_{12} = \sigma_{22} = \sigma_{32} = 1, \rho_{12} = \rho_{22} = \rho_{32} = 2,\ \gamma_{12} = \gamma_{22} = \gamma_{32} = 2,\ \beta_{12} = \beta_{22} = \beta_{32} = 1,\ \alpha_{12} = \alpha_{22} = \alpha_{32} = 1.$
From the previous corollary the coefficients of the function $F(x,y,z,s,t,w)$ *are:*
$c_{111} = \frac{5cos(xy)+4sin(z)}{16},\ c_{211} = \frac{-cos(xy)}{4},\ c_{121} = \frac{-sin(z)}{8},\ c_{112} = \frac{-sin(z)}{8},\ c_{221} = c_{212} = c_{222} = 0,\ c_{122} = \frac{sin(z)}{16}.$

5.2 Three-dimensional Fredholm Integral Equation

The aim of this section is to derive a numerical solution of three-dimensional non-linear integral equation using Haar Wavelet method, and then some illustrative examples will be solved using the presented method.

Consider the following three-dimensional non-linear Fredholm integral equation

$$U(x,y,z) = f(x,y,z) + \int_0^1\int_0^1\int_0^1 K(x,y,z,s,t,w,U(s,t,w))dsdtdw, \qquad (5.26)$$

where (x,y,z)∈[0,1]×[0,1]×[0,1], $f(x,y,z), K(x,y,z,s,t,w)$ are continuous functions. If the Kernel function is non-linear in U, then equation (5.26) is called non-linear, otherwise it is called linear.

From corollary (5.1), the function $K(x, y, z, s, t, w, U(s, t, w))$ can be approximated as

$$K(x, y, z, s, t, w, U(s, t, w)) \approx \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} \sum_{k=1}^{l_3} c_{ijk}(x, y, z) h_i(s) h_j(t) h_k(w). \tag{5.27}$$

Note that:

$$\int_0^1 \int_0^1 \int_0^1 h_i(s) h_j(t) h_k(w) ds dt dw = \begin{cases} 1 & \text{if} \quad \text{i = j = k = 1} \\ 0 & \text{if} \quad \text{i > 1 , or j > 1, or k>1.} \end{cases}$$

Substituting equation (5.26) in equation (5.26):

$$U(x, y, z) = f(x, y, z) + c_{111}(x, y, z). \tag{5.28}$$

Substituting the collocation points defined in equations (5.1-5.3), and the value of the variable coefficient $c_{111}(x, y, z)$ which is given in corollary (5.1) the following equation is derived:

$$U(x_m, y_n, z_v) = f(x_m, y_n, z_v) + \frac{1}{l_1 l_2 l_3} \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big), \tag{5.29}$$

where $m,\ p = 1, 2, ..., l_1,\ n,\ q = 1, 2, ..., l_2,\ v,\ r = 1, 2, ..., l_3,$

equation (5.29) gives an $l_1 l_2 l_3 \times l_1 l_2 l_3$ system of non-linear algebraic equations can be solved using Newoton's method. Assuming that the function K can be decompsed to the form:

$$K(x, y, z, s, t, w, U) = G(x, y, z, s, t, w) \Phi(U(s, t, w)) \tag{5.30}$$

then the system (5.29) can be represented in matrix form as follows:

$$U = F + \frac{1}{l_1 l_2 l_3} M\Phi \tag{5.31}$$

For $l_1 = l_2 = l_3 = 2,\ M = \begin{bmatrix} G^{111}_{111} & G^{111}_{112} & G^{111}_{121} & G^{111}_{122} & G^{111}_{211} & G^{111}_{212} & G^{111}_{221} & G^{111}_{222} \\ G^{112}_{111} & G^{112}_{112} & G^{112}_{121} & G^{112}_{122} & G^{112}_{211} & G^{112}_{212} & G^{112}_{221} & G^{112}_{222} \\ G^{121}_{111} & G^{121}_{112} & G^{121}_{121} & G^{121}_{122} & G^{121}_{211} & G^{121}_{212} & G^{121}_{221} & G^{121}_{222} \\ G^{122}_{111} & G^{122}_{112} & G^{122}_{121} & G^{122}_{122} & G^{122}_{211} & G^{122}_{212} & G^{122}_{221} & G^{122}_{222} \\ G^{211}_{111} & G^{211}_{112} & G^{211}_{121} & G^{211}_{122} & G^{211}_{211} & G^{211}_{212} & G^{211}_{221} & G^{211}_{222} \\ G^{212}_{111} & G^{212}_{112} & G^{212}_{121} & G^{212}_{122} & G^{212}_{211} & G^{212}_{212} & G^{212}_{221} & G^{212}_{222} \\ G^{221}_{111} & G^{221}_{112} & G^{221}_{121} & G^{221}_{122} & G^{221}_{211} & G^{221}_{212} & G^{221}_{221} & G^{221}_{222} \\ G^{222}_{111} & G^{222}_{112} & G^{222}_{121} & G^{222}_{122} & G^{222}_{211} & G^{222}_{212} & G^{222}_{221} & G^{222}_{222} \end{bmatrix}$

$U = \begin{bmatrix} U_{111} & U_{112} & U_{121} & U_{122} & U_{211} & U_{212} & U_{221} & U_{222} \end{bmatrix}^T,\ U_{mnv} = U(x_m, y_n, z_v)$

$F = \begin{bmatrix} f_{111} & f_{112} & f_{121} & f_{122} & f_{211} & f_{212} & f_{221} & f_{222} \end{bmatrix}^T,\ f_{mnv} = f(x_m, y_n, z_v),$

$m, n, v = 1,\ 2.$ The solution at any point other than (x_m, y_n, z_v) can be approximated using theorem (5.1).

Example 5.3 *Consider the following integral equation:*

$$U(x, y, z) = \frac{13x + 17yz}{24} + \int_0^1 \int_0^1 \int_0^1 (xs + yzt^2) U(s, t, w) ds dt dw, \quad (x, y, z) \in [0, 1][0, 1][0, 1]. \tag{5.32}$$

The exact solution is $U_{ex}(x, y, z) = x + yz$.

$K\big(x, y, z, s, t, w, U(s, t, w)\big) = (xs + yzt^2)U(s, t, w)$, then from equation (5.29) we have:

$$U(x_m, y_n, z_v) = f(x_m, y_n, z_v) + \frac{1}{l_1 l_2 l_3} \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \sum_{r=1}^{l_3} (x_m s_p + y_n z_v t_q^2) U(s_p, t_q, w_r). \tag{5.33}$$

For $l_1 = l_2 = l_3 = 2$, we have:

$$U(x_m, y_n, z_v) = \frac{13x_m + 14y_n z_v}{24} + \frac{1}{8}\sum_{p=1}^{2}\sum_{q=1}^{2}\sum_{r=1}^{2}(x_m s_p + y_n z_v t_q^2)U(s_p, t_q, w_r),$$

the previous form can be expressed as a matrix form as $U = F + \frac{1}{8}MU$, where:

$$F = \begin{bmatrix} 0.179687 & 0.268229 & 0.268229 & 0.533854 & 0.450520 & 0.539062 & 0.539062 & 0.804687 \end{bmatrix}^T$$

$$M = \begin{bmatrix} 0.066406 & 0.066406 & 0.097656 & 0.097656 & 0.191406 & 0.191406 & 0.222656 & 0.222656 \\ 0.074219 & 0.0742187 & 0.167967 & 0.167969 & 0.199219 & 0.199219 & 0.292969 & 0.292969 \\ 0.074219 & 0.0742187 & 0.167967 & 0.167969 & 0.199219 & 0.199219 & 0.292969 & 0.292969 \\ 0.097656 & 0.097656 & 0.378906 & 0.378906 & 0.222656 & 0.222656 & 0.503906 & 0.503906 \\ 0.191406 & 0.191406 & 0.222656 & 0.222656 & 0.566406 & 0.566406 & 0.597656 & 0.597656 \\ 0.199219 & 0.199219 & 0.292969 & 0.292969 & 0.574219 & 0.574219 & 0.667969 & 0.667969 \\ 0.199219 & 0.199219 & 0.292969 & 0.292969 & 0.574219 & 0.574219 & 0.667969 & 0.667969 \\ 0.222656 & 0.222656 & 0.503906 & 0.503906 & 0.597656 & 0.597656 & 0.878906 & 0.878906 \end{bmatrix}$$

$IU = F + \frac{1}{8}M$

$(I - \frac{1}{8}M)U = F$, which implies that $U = (I - \frac{1}{8}M)^{-1}F$,

$$U = \begin{bmatrix} 0.301071 & 0.421609 & 0.421609 & 0.783223 & 0.782674 & 0.903212 & 0.903212 & 1.264827 \end{bmatrix}^T$$

The Matlab program is used for larger l_1, l_2, l_3. The following table shows the absolute maximum error at various values of l_1, l_2, l_3.

Tab. 5.1: Error function for Example 5.3

l_1	l_2	l_3	$\|\|.\|\|_\infty$	R_c
2	2	2	4.8×10^{-2}	-
4	4	4	1.5×10^{-2}	1.7807
8	8	8	4.3×10^{-3}	1.8026

5.3 Three-dimensional Volterra Integral Equations

Consider the following integral equation:

$$U(x,y,z) = f(x,y,z) + \int_0^z \int_0^y \int_0^x K(x,y,z,s,t,w,U(s,t,w))dsdtdw \ (x,y,z) \in [0,1]^3. \tag{5.34}$$

Then, using the Haar wavelet approximation which described previously, equation ((5.34)) becomes:

$$U(x,y,z) = f(x,y,z) + \sum_{i=1}^{l_1}\sum_{j=1}^{l_2}\sum_{k=1}^{l_3} c_{ijk} P_{i,1}(x)P_{j,1}(y)P_{k,1}(z) \tag{5.35}$$

Also, from equations (5.1-5.3) and the expressions of the variable coefficients c_{ijk} which are given in corollary (5.1), equation (5.35) can be expressed as follows:

$$\begin{aligned}
U(x_m,y_n,z_v) &= f(x_m,y_n,z_v) \\
&+ \frac{P_{1,1}(x_m)P_{1,1}(y_n)P_{1,1}(z_v)}{l_1 l_2 l_3} \sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=1}^{l_3} K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big) \\
&+ \sum_{k=2}^{l_3} \frac{P_{1,1}(x_m)P_{1,1}(y_n)P_{k,1}(z_v)}{l_1 l_2 \rho_{3k}} \Bigg(\sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=\alpha_{3k}}^{\beta_{3k}} K(x_m,y_n,z_v,s_p,t_q,U(s_p,t_q,w_r)) \\
&- \sum_{p=1}^{l_1}\sum_{q=1}^{l_2}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K\big(x_m,y_n,z_v,s_p,t_q,U(s_p,t_q,w_r)\big)\Bigg) \\
&+ \sum_{j=2}^{l_2} \frac{P_{1,1}(x_m)P_{j,1}(y_n)P_{1,1}(z_v)}{l_1 \rho_{2j} l_3} \Bigg(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=1}^{l_3} K(x_m,y_n,z_v,s_p,t_q,U(s_p,t_q,w_r)) \\
&- \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=1}^{l_3} K\big(x_m,y_n,z_v,s_p,t_q,U(s_p,t_q,w_r)\big)\Bigg) \\
&+ \sum_{i=2}^{l_1} \frac{P_{i,1}(x_m)P_{1,1}(y_n)P_{1,1}(z_v)}{\rho_{1i} l_2 l_3} \Bigg(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}\sum_{r=1}^{l_3} K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big) \\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2}\sum_{r=1}^{l_3} K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=2}^{l_2}\sum_{k=2}^{l_3}\frac{P_{1,1}(x_m)P_{j,1}(y_n)P_{k,1}(z_v)}{l_1\rho_{2j}\rho_{3k}}\Big(\sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&- \sum_{p=1}^{l_1}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&- \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&+ \sum_{p=1}^{l_1}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\Big)\\
&+ \sum_{i=2}^{l_1}\sum_{k=2}^{l_2}\frac{P_{i,1}(x_m)P_{1,1}(y_n)P_{k,1}(z_v)}{\rho_{1i}l_2\rho_{3k}}\Big(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}\sum_{r=\alpha_{3k}}^{\beta_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=1}^{l_2}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2}\sum_{r=\alpha_{3k}}^{\beta_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=1}^{l_2}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\Big)\\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\frac{P_{i,1}(x_m)P_{j,1}(y_n)P_{1,1}(z_v)}{\rho_{1i}\rho_{2j}l_3}(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=1}^{l_3}K(x_m,y_,,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r))\\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=1}^{l_3}K(x_m,y_,,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r))\\
&- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=1}^{l_3}K(x_m,y_,,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r))\\
&+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}}\sum_{q=\beta_{2j}+1}^{\gamma_{2j}}\sum_{r=1}^{l_3}K(x_m,y_,,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)))\\
&+ \sum_{i=2}^{l_1}\sum_{j=2}^{l_2}\sum_{k=2}^{l_3}\frac{P_{i,1}(x_m)P_{j,1}(y_n)P_{k,1}(z_v)}{\rho_{1i}\rho_{2j}\rho_{3k}}\Big(\sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\alpha_{3k}}^{\beta_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)\\
&- \sum_{p=\alpha_{1i}}^{\beta_{1i}}\sum_{q=\alpha_{2j}}^{\beta_{2j}}\sum_{r=\beta_{3k}+1}^{\gamma_{3k}}K\big(x_m,y_n,z_v,s_p,t_q,w_r,U(s_p,t_q,w_r)\big)
\end{aligned}
$$

$$- \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big)$$

$$- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big)$$

$$+ \sum_{p=\alpha_{1i}}^{\beta_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big)$$

$$+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\alpha_{2j}}^{\beta_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big)$$

$$+ \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\alpha_{3k}}^{\beta_{3k}} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big)$$

$$- \sum_{p=\beta_{1i}+1}^{\gamma_{1i}} \sum_{q=\beta_{2j}+1}^{\gamma_{2j}} \sum_{r=\beta_{3k}+1}^{\gamma_{3k}} K\big(x_m, y_n, z_v, s_p, t_q, w_r, U(s_p, t_q, w_r)\big)\Bigg), \tag{5.36}$$

which is an $l_1 l_2 l_3 \times l_1 l_2 l_3$ non-linear system if the Kernel function is nonlinear in U or linear system if not. If the function K can be written as:

$$K(x, y, z, s, t, w, U(s, t, w)) = G(x, y, z, s, t, w)\Phi(U(s, t, w)), \tag{5.37}$$

then the matix form of the system of equation (5.36) is:

$$U = F + T\Phi \tag{5.38}$$

For $l_1 = l_2 = l_3 = 2$,

$U = \begin{bmatrix} U_{111} & U_{112} & U_{121} & U_{122} & U_{211} & U_{212} & U_{221} & U_{222} \end{bmatrix}^T, \; U_{mnv} = U(x_m, y_n, z_v),$

$F = \begin{bmatrix} f_{111} & f_{112} & f_{121} & f_{122} & f_{211} & f_{212} & f_{221} & f_{222} \end{bmatrix}^T, \; f_{mnv} = f(x_m, y_n, z_v),$

$\Phi = \begin{bmatrix} \Phi_{111} & \Phi_{112} & \Phi_{121} & \Phi_{122} & \Phi_{211} & \Phi_{212} & \Phi_{221} & \Phi_{222} \end{bmatrix}^T, \; \Phi_{mnv} = \Phi\big(U(x_m, y_n, z_v)\big),$

$$m,n,v=1,\ 2.\ T=\begin{bmatrix} \frac{G^{111}_{111}}{64} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{G^{112}_{111}}{32} & \frac{G^{112}_{112}}{64} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{G^{121}_{111}}{32} & 0 & \frac{G^{121}_{121}}{64} & 0 & 0 & 0 & 0 & 0 \\ \frac{G^{122}_{111}}{16} & \frac{G^{122}_{112}}{32} & \frac{G^{122}_{121}}{32} & \frac{G^{122}_{122}}{64} & 0 & 0 & 0 & 0 \\ \frac{G^{211}_{111}}{32} & 0 & 0 & 0 & \frac{G^{211}_{211}}{64} & 0 & 0 & 0 \\ \frac{G^{212}_{111}}{16} & \frac{G^{212}_{112}}{32} & 0 & 0 & \frac{G^{212}_{211}}{32} & \frac{G^{212}_{212}}{64} & 0 & 0 \\ \frac{G^{221}_{111}}{16} & 0 & \frac{G^{221}_{121}}{32} & 0 & \frac{G^{221}_{211}}{32} & 0 & \frac{G^{221}_{221}}{64} & 0 \\ \frac{G^{222}_{111}}{8} & \frac{G^{222}_{112}}{16} & \frac{G^{222}_{121}}{16} & \frac{G^{222}_{122}}{32} & \frac{G^{222}_{211}}{16} & \frac{G^{222}_{212}}{32} & \frac{G^{222}_{221}}{32} & \frac{G^{222}_{222}}{64} \end{bmatrix}$$

Example 5.4 *[49] Consider the integral equation:*

$$U(x,y,z)=xcosz-\frac{x^3y^3}{9}sinz+\int_0^z\int_0^y\int_0^x st^2U^2(s,t,w)dsdtdw,\ (x,y,z)\in[0,1]\times[0,1]\times[0,1] \tag{5.39}$$

The exact solution is $U_{ex}=xcosz$.

Solution: Approximating the function $st^2U^2(s,t,w)$ using three-dimensional Haar functions using corollary (5.1) and choosing $l_1=l_2=l_3=2$ implies that:

$$\begin{aligned} U(x_m,y_n,z_v) &= x_m cos(z_v)\frac{x_m^3y_n^3}{9}sin(z_v) \\ &+ \frac{P_{1,1}(x_m)P_{1,1}(y_n)P_{1,1}(z_v)}{8}\sum_{p=1}^{2}\sum_{q=1}^{2}\sum_{r=1}^{2}s_pt_q^2U^2(s_p,t_q,w_r) \\ &- \frac{P_{1,1}(x_m)P_{1,1}(y_n)P_{2,1}(z_v)}{8}\Bigg(\sum_{p=1}^{2}\sum_{q=1}^{2}s_pt_q^2U^2(s_p,t_q,w_1) \\ &- \sum_{p=1}^{2}\sum_{q=1}^{2}s_pt_q^2U^2(s_p,t_q,w_2)\Bigg) \\ &- \frac{P_{1,1}(x_m)P_{2,1}(y_n)P_{1,1}(z_v)}{8}\Bigg(\sum_{p=1}^{2}\sum_{r=1}^{2}s_pt_1^2U^2(s_p,t_1,w_r) \\ &- \sum_{p=1}^{2}\sum_{r=1}^{2}s_pt_2^2U^2(s_p,t_2,w_r)\Bigg) \end{aligned}$$

$$
\begin{aligned}
&- \frac{P_{2,1}(x_m)P_{1,1}(y_n)P_{1,1}(z_v)}{8}\Bigg(\sum_{q=1}^{2}\sum_{r=1}^{2} s_1 t_q^2 U^2(s_1,t_q,w_r) \\
&- \sum_{q=1}^{2}\sum_{r=1}^{2} s_2 t_q^2 U^2(s_2,t_q,w_r)\Bigg) \\
&+ \frac{P_{1,1}(x_m)P_{2,1}(y_n)P_{2,1}(z_v)}{8}\Bigg(\sum_{p=1}^{2} s_p t_1^2 U^2(s_p,t_1,w_1) - \sum_{p=1}^{2} s_p t_1^2 U^2(s_p,t_1,w_2) \\
&- \sum_{p=1}^{2} s_p t_2^2 U^2(s_p,t_2,w_1) + \sum_{p=1}^{2} s_p t_2^2 U^2(s_p,t_2,w_2)\Bigg) + \frac{P_{2,1}(x_m)P_{1,1}(y_n)P_{2,1}(z_v)}{8} \\
&\Bigg(\sum_{q=1}^{2} s_1 t_q^2 U^2(s_1,t_q,w_1) - \sum_{q=1}^{2} s_1 t_q^2 U^2(s_1,t_q,w_2) - \sum_{q=1}^{2} s_2 t_q^2 U^2(s_2,t_q,w_1) \\
&+ \sum_{q=1}^{2} s_2 t_q^2 U^2(s_2,t_q,w_2)\Bigg) + \frac{P_{2,1}(x_m)P_{2,1}(y_n)P_{1,1}(z_v)}{8}\Bigg(\sum_{r=1}^{2} s_1 t_1^2 U(s_1,t_1,w_r) \\
&- \sum_{r=1}^{2} s_1 t_2^2 U(s_1,t_2,w_r) - \sum_{r=1}^{2} s_2 t_1^2 U(s_2,t_1,w_r) + \sum_{r=1}^{2} s_2 t_2^2 U(s_2,t_2,w_r)\Bigg) \\
&+ \frac{P_{2,1}(x_m)P_{2,1}(y_n)P_{2,1}(z_v)}{8}\Bigg(s_1 t_1^2 U(s_1,t_1,w_1) - s_1 t_1^2 U(s_1,t_1,w_2) \\
&- s_2 t_1^2 U(s_2,t_1,w_1) - s_2 t_1^2 U(s_2,t_1,w_1) + s_1 t_2^2 U(s_1,t_2,w_2) \\
&+ s_2 t_1^2 U(s_2,t_1,w_2) + s_2 t_2^2 U(s_2,t_2,w_1) - s_2 t_2^2 U(s_2,t_2,w_2)\Bigg),
\end{aligned}
\tag{5.40}
$$

where $m, n, v = 1, 2$. The previous 8×8 non-linear system is solved using Newoton's method. It can be written in a matrix form as follows:

$$
U = F + T\Phi, \tag{5.41}
$$

where: $U = \begin{bmatrix} U_{111} & U_{112} & U_{121} & U_{122} & U_{211} & U_{212} & U_{221} & U_{222} \end{bmatrix}^T$,

$F = \begin{bmatrix} 0.242221 & 0.182904 & 0.242047 & 0.182423 & 0.726503 & 0.548267 & 0.721792 & 0.535287 \end{bmatrix}^T$,

$$T = \begin{bmatrix} 0.000244 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.000488 & 0.000244 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.000488 & 0 & 0.002197 & 0 & 0 & 0 & 0 & 0 \\ 0.000977 & 0.000488 & 0.004394 & 0.002197 & 0 & 0 & 0 & 0 \\ 0.000488 & 0 & 0 & 0 & 0.000732 & 0 & 0 & 0 \\ 0.000977 & 0.000488 & 0 & 0 & 0.001465 & 0.000732 & 0 & 0 \\ 0.000977 & 0 & 0.004394 & 0 & 0.001465 & 0 & 0.006592 & 0 \\ 0.001953 & 0.000977 & 0.00879 & 0.004394 & 0.002930 & 0.001465 & 0.013184 & 0.006592 \end{bmatrix},$$

$\Phi = \begin{bmatrix} U_{111}^2 & U_{112}^2 & U_{121}^2 & U_{122}^2 & U_{211}^2 & U_{212}^2 & U_{221}^2 & U_{222}^2 \end{bmatrix}^T$, solving the non-linear system implies that:

$$U = \begin{bmatrix} 0.242281 & 0.183067 & 0.242698 & 0.184220 & 0.727154 & 0.550061 & 0.728965 & 0.555087 \end{bmatrix}^T$$

The Matlab program is used for larger l_1, l_2, l_3. The following table shows the absolute maximum error for several values of l_1, l_2, l_3

Tab. 5.2: Error function for Example 5.4

l_1	l_2	l_3	$\|\|.\|\|_\infty$	R_c
2	2	2	6.3×10^{-3}	-
4	4	4	3×10^{-3}	1.0704
8	8	8	1×10^{-3}	1.5850

Conclusion

In this thesis, the method of Haar wavelet functions is employed to the numerical solutions of one, two and three-dimensional nonlinear integral equations of the second kind.

Some algorithms are produced theoretically alongside the numerical solutions. These algorithms approximate the Kernel function using Haar wavelets. There is no need to solve any linear system for evaluating Haar coefficients. After that, the collocation points are applied for the variables of the unknown function. The Numerical solutions of such equations were obtained using Matlab software, and compared with the exact solutions. High accuracy is noticed even in the case of small number of collocation points.

As a future work, one can employ this method to the first kind of Volterra and Fredhilm integral equations, and to systems of integral equations.

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ملخص

طريقة مويجة هار لحل المعادلات التكاملية غير الخطية

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في هذه الأطروحة، الحلول العددية للمعادلات التكاملية احادية ثنائية وثلاثية الابعاد قد درست وتم محاكاتها باستخدام برنامج الماتلاب. في الواقع طريقة مويجات هار سوف تطبق لتقريب حلول أنواع محددة من المعادلات التكاملية غير الخطية . هذه الطريقة تحول المعادلة التكاملية الى نظام مربع من المعادلات الجبرية غير الخطية بالاستفادة من نقاط التجميع. الطريقة المستخدمة مرتكزة على طريقة عزيز و خان (٢٠١٤) لحل معادلات فولتيرا وفردهلوم التكاملية غير الخطية ثنائية الابعاد .

باستخدام الخصائص المميزة لاقترانات هار، خوارزميات لحل المعادلات التكاملية احادية وثنائية وثلاثية الأبعاد تم تكوينها . اعتمادا على هذه الخوارزميات لا حاجة لتكامل عددي وسيط مما يؤدي الى دقة عالية حتى في حالة استخدام عدد قليل من النقاط التجميعية.