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Faculty of Graduate Studies

Local Fractional Fourier Series Method for Solving Local Fractional Fredholm Integral Equation

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This Thesis was Submitted
in Partial Fulfillment of the Requirements for the Degree of
Master of Science
in
Applied Mathematics

January, 2016

Committee Decision

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Dedication

To my husband Rami Sweis, my children Hussam and Hamza for their assistance, tolerance and endless support.

To my father Dr. Khalid Sweis and my mother, there is no doubt in my mind that without there continued support I could not have completed my study.

To my uncle Hussam Sweis and his wife, a great thank to them for their support and encouragement .

To my brothers Ahmad and Abd-Al Rahman and my sister Noora.

To my honoured teachers and professors.

Acknowledgements

First of all, the great continuous thank to Allah who gave me the ability and patience to handle the years I spent during my study.

I would like to express my deepest appreciation to my supervisors Dr. Abdelhalim Ziqan and Dr. Iyad Suwan for the generosity, guidance and helpful suggestions throughout the preparation of this work.

My sincere thanks to the examination committee for their careful reading to my thesis and valuable feedback.

I will never forget my family, my parents and my husband for their infinite support, also my children Hussam and Hamza. Finally, my great gratitude to Mathematics Department at The Arab American University.

List of Symbols

${}_{x_0}D_x^{\alpha}$	:	Local fractional derivative operator
d^{α}	:	Local fractional differential operator
${}_aI_b^{(\alpha)}$	:	Local fractional integral operator
Γ	:	Gamma function
$E_{\alpha}\left(x\right)^{\alpha}$	:	Mittag-Leffler function
C_{α}	:	Local fractional continuous

Abstract

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In this thesis, the solution of the local fractional Fredholm integral equation is studied using the local fractional Fourier series. For that purpose, we present some definitions and theorems on local fractional calculus. Also, the one and two-dimensional local fractional series are studied extensively and some examples as well, which are solved in detail. Finally, we solved some examples on the two types of local fractional Fredholm integral equations through integrating the one and two-dimensional local fractional Fourier series in the integral equations.

ملخص

طريقة متسلسلة فورير الجزئية المحلية لحل معادلة فردهولم التكاملية الجزئية المحلية

هند خالد محمد صويص

في هذه الأطروحة ، تم دراسة معادلة فريدهولم التكاملية الجزئية المحلية باستخدام متسلسلات فورير الجزئية المحلية ولهذا الغرض، قدمنا بعض التعاريف والنظريات حول حساب التفاضل والتكامل الجزئي المحلي. أيضا، تم دراسة المتسلسلات الجزئية المحلية بشكل موسع بالاضافة الى حل أمثلة بصورة مفصلة. واخيرا، حللنا أمثلة على النوع الاول والثاني من معادلة فريدهولم التكاملية الجزئية المحلية من خلال دمج المتسلسلات ذات البعد الواحد و ذات البعدين في المعادلة التكاملية.

Chapter 1

Introduction

Local fractional calculus or fractal calculus is a branch of mathematical analysis which deals with derivatives and integrals of arbitrary orders.The theory of local fractional calculus is important to solve the fractal and continuously non-differential functions [1–4]. It can be used to solve many problems in many disciplines, such as physics, biology, engineering, chemistry, economics, control theory, image processing, fractal elasticity, fractal release equation, geophysics and relaxation [3, 5]. Also, the local fractional calculus plays a vital role in solving many problems in applied mathematics. For example, fractal wave equation [6, 7] ,fractal signal [5, 8], fractional differential equations, which was applied to describe the dynamical systems in physics and engineering [9], local fractional laplace transform, the Hölder inequality in fractal space, the local fractional wavelet transform, local fractional Fourier transform [10], the Yang Fourier transform, the discrete Fourier transform and the fractional complex transform [11–15]. Moreover, the local fractional ordinary and partial differential equations which can be solved by different methods, such as the fractional variational iteration method [16, 17].

The theory of local fractional integrals and derivatives deals with fractal functions, it has many definitions and theories like the modified fractal derivative, Riemann-Liouville derivative, P-G local fractional derivative and Jumarie fractional integral [4, 18].

The local fractional Fourier analysis, which is based on local fractional calculus is a mathematical method used to transform a periodic function in physics, engineering, optics, signal processing and quantum mechanics [1].

In applied mathematics, the local fractional Fourier series can be used to solve some nonlinear equations within local fractional differential or fractional integral operator [1,19], partial differential equations, integral - differential equations, ordinary differential equations [1,19], local fractional wave equation in fractal vibrating string [11], fractal nonhomogeneous heat equation arising in fractal heat flow with local fractional derivative [20], fractal signals [1, 21]and local fractional heat conduction equation [22].

The local fractional integral equations are one of the applications on local fractional calculus. There are several types of local fractional integral equations like the local fractional Fredholm integral equation, the local fractional volterra integral equation, the local fractional nonlinear integral equation, the local fractional singular integral equation and the local fractional integro-differential equation.

In this thesis, we investigate the local fractional Fredholm integral equation. Such a type of equations has many applications in physical processes; such as acoustics, chemistry, biology ,electromagnetism, heat conduction, viscoelastic materials, signal processing, which appears in solving local fractional partial and ordinary differential equations that is used in physics , mechanics and other natural science [13, 14, 23, 24]. Besides, some boundary value problems can be converted into local fractional fredholm integral equation [12]. In literature, different methods have been used to solve this type of integral equations, such as Newman series [25] and the local fractional Adomain's Decomposition Method (ADM) [26].

The local fractional Fredholm integral equation has the following standard form

$$h(x)u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t)u(t)(dt)^\alpha. \tag{1.1}$$

The function $K(x,t)$ is called the kernel of the local fractional integral equation. $f(x)$ is a local fractional continuous function and λ^α is a parameter. The limits of integration a and b are constants, and the unknown function $u(x)$ appears linearly under the integral sign [18].

If $h(x) = 0$, then the equation is called the local fractional Fredholm integral equation of the first kind and has the form:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t)u(t)\,(dt)^\alpha. \tag{1.2}$$

If $h(x) = 1$, then the equation becomes

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t)u(t)(dt)^\alpha, \tag{1.3}$$

and it is called the local fractional Fredholm integral equation of the second kind.

The aim of this thesis is to display the local fractional Fourier series of one dimension and gives new examples; then to derive a formula of the local fractional Fourier series of two dimension and expand some fractal functions into local fractional Fourier series of two dimension. Moreover, the local fractional Fourier series of one and two dimensions will be used to solve the local fractional Fredholm integral equation of the first and second kind (1.2) and (1.3) by writing the local fractional Fourier series of the functions $u(x)$, $f(x)$, and $K(x,t)$ and then substituting the corresponding series into the integral equation.

Chapter 2

Local Fractional Calculus

In this chapter, we present the concept of local fractional calculus that are needed in this work. The chapter consists of four sections. In section one, we give the definition of the gamma function and its properties. In section two, three and four, the definitions and some properties of the local fractional complex function, the local fractional derivative and the local fractional integration are presented respectively.

2.1 Gamma Function

The Gamma function plays an important role in the theory of local fractional theory. In this section, the definition and some properties of this functions are to be presented.

Definition 2.1.1 *[27]The Gamma function* $\Gamma(z)$ *is defined by:*

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt. \tag{2.1.1}$$

which converges in the right half of the complex plane $Re(z) > 0$. We have

$$\begin{aligned}\Gamma(x+iy) &= \int_0^1 e^{-t} t^{x-1+iy} dt = \int_0^1 e^{-t} t^{x-1} e^{iy\log(t)} dt \\ &= \int_0^1 e^{-t} t^{x-1} (\cos(y\log(t) + i\sin(y\log(t))dt\end{aligned}$$

Below is a list of some properties of the Gamma function where n is a natural number [27–29]

1. $\Gamma(z+1) = z\Gamma(z)$
2. $\Gamma(z)\Gamma(z-1) = \pi/sin(\pi z)$
3. $\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$

4. $\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-(1/2)} \prod_{k=0}^{n-1} \Gamma(z + \frac{k}{n})$

5. $\Gamma(n + \frac{1}{2}) = \dfrac{\pi^{\frac{1}{2}}(2n)!}{2^{2n} n!}$

6. $\dfrac{\Gamma(n+\alpha)}{\Gamma(\alpha)} = (\alpha)_n$, where $\alpha, \beta \in \mathbb{C},\ \alpha \notin \mathbb{Z}^-$, $(\alpha)_0 = 1$,$(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1)$

7. $\dfrac{\Gamma(\alpha - n)}{\Gamma(\alpha)} = \dfrac{(-1)^n}{(1-\alpha)_n}$

8. $(\beta^\alpha) = \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} \Gamma(\beta+1)$

9. The Binomial coefficient is defined by the gamma function

$$(a+b)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} a^{\alpha-n} b^n. \tag{2.1.2}$$

$$\binom{\alpha}{n} = \frac{\Gamma(1+\alpha)}{\Gamma(n+1)\Gamma(\alpha-n+1)},\ \binom{\alpha}{n} = 0, for\ ,\alpha \in\ \mathbb{C}, \alpha \notin \mathbb{Z}^-$$

and n$\in \mathbb{N}$.

2.2 Local Fractional of Complex Function

The present section introduces the Mittag Leffler function and the Mittag Leffler representation of the trigonometric functions.

Definition 2.2.1 *[18, 30](Local Fractional Order of Complex Number)*
The fractional order of complex number is defined by $I^\alpha = x^\alpha + i^\alpha y^\alpha$, *where* $x, y \in \mathbb{R}$ *and* $0 < \alpha \leq 1$.

The conjugate of the complex number is $\bar{I^\alpha} = x^\alpha - i^\alpha y^\alpha$ and the modulus is given by the expression $\mid I^\alpha \mid = I^\alpha \bar{I^\alpha} = \bar{I^\alpha} I^\alpha = \sqrt{x^{2\alpha} + y^{2\alpha}}$

Definition 2.2.2 *[3](Complex function on fractal set)*
The complex function on fractal set can be written by $w^\alpha = f(z)$ *and in general* $w^\alpha = f(z) = u(x,y) + i^\alpha v(x,y)$, *where* u *and* v *are real valued functions of* x *and* y.

Definition 2.2.3 *[3](Complex Mittag Leffler Function)*
The complex Mittag Leffler function $E_\alpha : F \rightarrow R^\alpha$ *is defined on fractal set of dimention* α *by* $E_\alpha(Z^\alpha) = \sum_{k=0}^{\infty} \dfrac{z^{\alpha k}}{\Gamma(1+k\alpha)}$ *for* $z \in \mathbb{C}$ *and* $0 < \alpha \leq 1$.

For Mittag Leffler function the following rules are hold [3, 6, 9, 11, 18, 30]:

1. $E_\alpha(z_1^\alpha)E_\alpha(z_2^\alpha) = E_\alpha(z_1+z_2)^\alpha$
2. $E_\alpha(z_1^\alpha)E_\alpha(-z_2^\alpha) = E_\alpha(z_1-z_2)^\alpha$
3. $E_\alpha(i^\alpha z_1^\alpha)E_\alpha(i^\alpha z_2^\alpha) = E_\alpha(i^\alpha(z_1+z_2)^\alpha)$
4. $i^{2\alpha} = -1$
5. $E_\alpha\left(0\right)^\alpha = 1$

For $x \in \mathbb{R}$ *and* $0 < \alpha \leq 1$, the sine and cosine functions in fractal space are given by:

$$\sin_\alpha x^\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k x^{\alpha(2k+1)}}{\Gamma(1+\alpha(2k+1))}. \tag{2.2.1}$$

$$\cos_\alpha x^\alpha = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2\alpha k}}{\Gamma(1+2k\alpha)}. \tag{2.2.2}$$

Here are some fractal functions which can be expressed by Mittag Leffler function

1. $\sin_\alpha x^\alpha = \dfrac{E_\alpha(i^\alpha x^\alpha) - E_\alpha(-i^\alpha x^\alpha)}{2i^\alpha}.$
2. $\cos_\alpha x^\alpha = \dfrac{E_\alpha(i^\alpha x^\alpha) + E_\alpha(-i^\alpha x^\alpha)}{2}.$
3. $\sinh_\alpha x^\alpha = \dfrac{E_\alpha(x)^\alpha + E_\alpha(-x)^\alpha}{2}.$
4. $\cosh_\alpha x^\alpha = \dfrac{E_\alpha(x)^\alpha - E_\alpha(-x)^\alpha}{2},$

Similar to the usual trigonometric identities, we have the following rules, where $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ [11]:

1. $\cos_\alpha(-x)^\alpha = \cos_\alpha x^\alpha.$
2. $\sin_\alpha(-x)^\alpha = -\sin_\alpha x^\alpha.$
3. $\cos_\alpha^2 x^\alpha + \sin_\alpha^2 x^\alpha = 1.$
4. $\sin_\alpha^2 x^\alpha = \dfrac{1-\cos_\alpha(2x)^\alpha}{2}.$
5. $\cos_\alpha^2 x^\alpha = \dfrac{1+\cos_\alpha(2x)^\alpha}{2}.$
6. $\sin_\alpha x^\alpha \sin_\alpha y^\alpha = \dfrac{-\cos_\alpha(x+y)^\alpha - \cos_\alpha(x-y)^\alpha}{2}.$
7. $\sin_\alpha x^\alpha \cos_\alpha y^\alpha = \dfrac{\sin_\alpha(x+y)^\alpha + \sin_\alpha(x-y)^\alpha}{2}.$

8. $\sin_\alpha(mx)^\alpha \sin_\alpha(nx)^\alpha = \dfrac{\cos_\alpha((m-n)x)^\alpha - \cos_\alpha((m+n)x)^\alpha}{2}$

9. $\cos_\alpha(nx)^\alpha \sin_\alpha(mx)^\alpha = \dfrac{\sin_\alpha((m+n)x)^\alpha - \sin_\alpha((m-n)x)^\alpha}{2}.$

2.3 Local Fractional Derivative

In the current section, the local fractional continuity and the local fractional derivative of a function are introduced. Then we present a general theorem on a local fractional derivative of functions under multiplication, summation, division and composition.

Definition 2.3.1 *[1, 31] A function $f(x)$ is called local fractional continuous at $x = x_0$ if for every $\epsilon > 0$, there is $\delta > 0$ such that whenever $| x - x_0 |< \delta$, we have $| f(x) - f(x_0) |< \varepsilon^\alpha$.*

A function f is called a local fractional continuous on (a, b), if it is continuous at each point in the interval (a, b). The space of all local fractional continuous functions on (a, b) is denoted by $C_\alpha(a, b)$.

Definition 2.3.2 *[14] In fractal space, let f(x) $\in C_\alpha(a, b)$. The local fractional derivative of f(x) of order α at x=x_0 is defined by*

$$D_x^\alpha f(x_0) = f^\alpha(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0} = \lim_{x\to x_0} \frac{\triangle^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{2.3.1}$$

where $\triangle^\alpha(f(x) - f(x_0)) \cong \Gamma(1 + \alpha)\triangle(f(x) - f(x_0))$ and $0 < \alpha < 1$
If the limit exists then the function $f(x)$ is said to be local fractional analytic .

Definition 2.3.3 *[4, 32] Let f(x) $\in C_\alpha[a, b]$ or $C_1[a, b]$, the modified fractional derivative of the function f(x) is defined by*

$$D^\alpha f\,(x) = \lim_{h\to 0} \frac{\triangle^\alpha f(x)}{h^\alpha}, \tag{2.3.2}$$

where $\triangle^\alpha f(x) = \sum\limits_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)h).$ $0 < \alpha < 1$

The local fractional derivative of higher order is given by [1, 15]

$$f^{(k\alpha)}\,(x) = \underbrace{D_x^\alpha D_x^\alpha ......D_x^\alpha}_{k-times} f(x), \tag{2.3.3}$$

and the local fractional partial derivative is defined as [6, 18]

$$\frac{d^{(\alpha k)}}{dx^{k\alpha}} f(x,y) = \underbrace{\frac{d^{\alpha}}{dx^{\alpha}} \frac{d^{\alpha}}{dx^{\alpha}} \ldots\ldots \frac{d^{\alpha}}{dx^{\alpha}}}_{k-times} f(x,y). \tag{2.3.4}$$

Definition 2.3.4 *[29] (Riemann - Liouville Derivative)*
The Rieman - Liouville derivative of a fractal function f(x) is defined by

$$D_{a^{+}}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dx^{n}} \int\limits_{a}^{x} (x-t)^{n-\alpha-1} f(t) dt. \tag{2.3.5}$$

$$D_{b^{-}}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{-d^{n}}{dx^{n}} \int\limits_{x}^{b} (t-x)^{n-\alpha-1} f(t) dt. \tag{2.3.6}$$

where $(n = [\alpha] + 1, x < b)$

To simplify the task of finding the local derivatives we need the following theorem.

Theorem 2.3.1 *[3,4] Suppose that $f(z)$ and $g(z)$ are analytic functions , then following are valid:*

1. $\dfrac{d^{\alpha}\left(f(z) \pm g(z)\right)}{dz^{\alpha}} = \dfrac{d^{\alpha} f(z)}{dz^{\alpha}} \pm \dfrac{d^{\alpha} g(z)}{dz^{\alpha}}$
2. $\dfrac{d^{\alpha}\left(f(z).g(z)\right)}{dz^{\alpha}} = g(z)\dfrac{d^{\alpha} f(z)}{dz^{\alpha}} + f(z)\dfrac{d^{\alpha} g(z)}{dz^{\alpha}}$
3. $\dfrac{d^{\alpha}\dfrac{f(z)}{g(z)}}{dz^{\alpha}} = \dfrac{g(z)\dfrac{d^{\alpha} f(z)}{dz^{\alpha}} + f(z)\dfrac{d^{\alpha} g(z)}{dz^{\alpha}}}{\left(g\left(z\right)\right)^{2}}$
4. $\dfrac{d^{\alpha}\left(cf(z)\right)}{dz^{\alpha}} = c\dfrac{d^{\alpha} f(z)}{dz^{\alpha}}$ where c is constant
5. If $y(z) = \left(f \circ u\right)(z)$, then $\dfrac{d^{\alpha} y(z)}{dz^{\alpha}} = f^{\alpha}\left(u\left(z\right)\right)\left(u^{(1)}(z)\right)^{\alpha}$

For a special case, the local fractional derivative of the function $f(t) = t^{\beta}$ is given by the formula [4]

$$D^{\alpha} t^{\beta} = \frac{\Gamma\left(\beta+1\right)}{\Gamma\left(\beta-\alpha+1\right)} t^{\beta-\alpha}. \tag{2.3.7}$$

In the following examples, we derive the local fractional derivative of some functions using the definition of fractional derivative.

Example 2.3.1 *(Fractional Derivative of Exponential Function)*

$$\begin{aligned} D^{\alpha}e^{ax} &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x+(\alpha-k)h) \\ &= \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{a(x+(\alpha-k)h)} \\ &= e^{ax}\lim_{h\to 0}\frac{1}{h^{\alpha}}[\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{ah(\alpha-k)}]. \end{aligned}$$

Using the Binomial expression (2.1.2)

$$\left(e^{ah}-1\right)^{\alpha} = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}e^{ah(\alpha-k)}.$$

So, the derivative becomes

$$\begin{aligned} D^{\alpha}e^{ax} &= e^{ax}\lim_{h\to 0}\frac{1}{h^{\alpha}}(e^{ah}-1)^{\alpha} \\ &= e^{ax}\left(\lim_{h\to 0}\left(\frac{e^{ah}-1}{h}\right)\right)^{\alpha} \\ &= e^{ax}\left(f'(0)\right)^{\alpha} = a^{\alpha}e^{ax} \end{aligned}$$

Therefore, $D^{\alpha}e^{ax} = a^{\alpha}e^{ax}$

Example 2.3.2 *(Local fractional derivative of sine and cosine)*

From formulas (2.2.1)and (2.2.2),

$$\begin{aligned} D^{\alpha}\left(\sin_{\alpha}(nt)^{\alpha}\right) &= \sum_{k=0}^{\infty}\frac{(-1)^k n^{\alpha(2k+1)}}{\Gamma(1+\alpha(2k+1))}\frac{\Gamma\left(\alpha\left(2k+1\right)+1\right)}{\Gamma\left(\alpha\left(2k+1\right)-\alpha+1\right)}t^{\alpha(2k+1)-\alpha} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k n^{\alpha(2k+1)}}{\Gamma(1+\alpha(2k+1)-\alpha)}t^{\alpha(2k)} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k n^{2\alpha k}n^{\alpha}}{\Gamma\left(2\alpha k+1\right)}t^{2\alpha k} \\ &= n^{\alpha}\sum_{k=0}^{\infty}\frac{(-1)^k (nt)^{2\alpha k}}{\Gamma(1+2\alpha k)} \\ &= n^{\alpha}\cos_{\alpha}(nt)^{\alpha} \end{aligned}$$

Hence,

$$D^{\alpha}\left(\sin_{\alpha}(nt)^{\alpha}\right) = n^{\alpha}\cos_{\alpha}(nt)^{\alpha}\,. \tag{2.3.8}$$

Similarly, since $\cos_\alpha (nt)^\alpha = \sin_\alpha \left(\frac{\pi}{2} - (nt)^\alpha\right)$, we get

$$D^\alpha \cos_\alpha (nt)^\alpha = -n^\alpha \sin_\alpha (nt)^\alpha \tag{2.3.9}$$

2.4 Local Fractional Integration

This section introduces some definitions, properties and examples on the local fractional integration.

Definition 2.4.1 *[30, 31, 33] Let $f(x) \in C_\alpha[a,b]$, the local fractional integral of the function f(x) of order α in the interval $[a,b]$is given by :*

$$_aI_b^\alpha f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)\,(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\triangle t \to 0} \sum_{j=0}^{j=N-1} f(t_j)\,(\triangle t_j)^\alpha , \tag{2.4.1}$$

where $\triangle t_j = t_{j+1} - t_j$, $\triangle t = \max\{\triangle t_1, \triangle t_{2,} \triangle t_3, ..., \triangle t_j, ...\}$ and $[t_j, t_{j+1}]$, $j = 0, 1, 2, 3, ..., N-1, t_0 = a, t_N = b$ is a partition of the interval $[a,b]$.

The multiple local fractional integrals of $f(x)$ is written in the form

$$_{x_0}I_x^{(k\alpha)} f(x) = \underbrace{_{x_0}I_x^{(\alpha)} ... \; _{x_0}I_x^{(\alpha)}}_{k-times} f(x)$$

For convenience, we assume that

$$_aI_a^{(\alpha)} f(x) = 0 \text{ if } a = band\; if\; a < b,_a I_b^{(\alpha)} f(x) = -_bI_a^{(\alpha)} f(x).$$

Here we present some properties of local fractional integral [4, 27]:

Property 2.4.1 *Suppose that $f(x), g(x) \in C_\alpha[a,b]$, then*

$$_aI_b^{(\alpha)}[f(x) \pm g(x)] =_a I_b^{(\alpha)} f(x) \pm_a I_b^{(\alpha)} g(x) \tag{2.4.2}$$

Property 2.4.2 *Suppose that $f(x) \in C_\alpha[a,b]$ and c is constant, then*

$$_aI_b^{(\alpha)}[cf(x)] = c[_aI_b^{(\alpha)} f(x)] \tag{2.4.3}$$

Property 2.4.3 *Suppose that $f(x) = 1$, then*

$$_aI_b^{(\alpha)} 1 = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \tag{2.4.4}$$

Property 2.4.4 *Suppose that* $f(x) \in C_\alpha[a,b]$ *and* $f(x) \geq 0$ *,then*

$$ {}_aI_b^{(\alpha)} f(x) \geq 0 \tag{2.4.5} $$

with $b - a > 0$

Property 2.4.5 *Suppose that* $g(x), f(x) \in C_\alpha[a,b]$ *and* $f(x) \geq g(x)$ *then*

$$ {}_aI_b^{(\alpha)} f(x) \geq {}_aI_b^{(\alpha)} g(x) \tag{2.4.6} $$

with $b - a > 0$

Property 2.4.6 *Suppose that* $f(x) \in C_\alpha[a,b]$ *. Let* M *and* m *be the maximum and minimum values of* $f(x)$ *over the interval* $[a,b]$ *respectively , then*

$$ M\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \geq {}_aI_b^{(\alpha)} f(x) \geq m\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} \tag{2.4.7} $$

with $b - a > 0$

Property 2.4.7 *Suppose that* $f(x) \in C_\alpha[a,b]$ *and* $a < c < b$ *, then*

$$ {}_aI_b^{(\alpha)} f(x) = {}_a I_c^{(\alpha)} f(x) + {}_c I_b^{(\alpha)} f(x) \tag{2.4.8} $$

Property 2.4.8 *The composition of fractional derivative and fractional integral*

$$ {}_aD_t^q \left({}_aD_t^p f(t)\right) = {}_a D_t^{p+q} f(t). \tag{2.4.9} $$

So, we conclude that

$$ {}_aI_b^{(\alpha)} \left(D^{(\alpha)} f(t)\right) = f(t) \mid_a^b \tag{2.4.10} $$

In the following examples, we derive explicitly the local fractional integral for some main functions that are needed in this thesis.

Example 2.4.1 $\int\limits_{-\pi}^{\pi} \cos_\alpha (nt)^\alpha (dt)^\alpha = 0$

In fact, from the first part of(2.4.1), we have

$$ {}_{-\pi}I_\pi^{(\alpha)} \cos_\alpha (nt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \int\limits_{-\pi}^{\pi} \cos_\alpha (nt)^\alpha (dt)^\alpha . $$

Thus,

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos_\alpha (nt)^\alpha (dt)^\alpha &= \Gamma(1+\alpha)_{-\pi} I_\pi^{(\alpha)} \cos_\alpha (nt)^\alpha \\
&= \Gamma(1+\alpha)_{-\pi} I_\pi^{(\alpha)} \left(\frac{1}{n^\alpha} D^\alpha \sin_\alpha (nt)^\alpha \right) \\
&= \frac{\Gamma(1+\alpha)}{n^\alpha} \left({}_{-\pi} I_\pi^{(\alpha)} D^\alpha \sin_\alpha (nt)^\alpha \right) \\
&= \frac{\Gamma(1+\alpha)}{n^\alpha} \sin_\alpha (nt)^\alpha \Big|_{-\pi}^{\pi} = 0.
\end{aligned}
$$

Example 2.4.2 $\int_{-\pi}^{\pi} t^\alpha (dt)^\alpha = 0$

Using the first of formula (2.4.1)

$$
\begin{aligned}
\int_{-\pi}^{\pi} t^\alpha (dt)^\alpha &= \Gamma(1+\alpha)_{-\pi} I_\pi^{(\alpha)} t^\alpha \\
&= \Gamma(1+\alpha) \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} {}_{-\pi} I_\pi{}^{(\alpha)} \left(D^\alpha t^{(2\alpha)} \right) \\
&= \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} t^{2\alpha} \Big|_{-\pi}^{\pi} = 0
\end{aligned}
$$

Example 2.4.3 $\int_{-\pi}^{\pi} t^\alpha \cos_\alpha (nt)^\alpha (dt)^\alpha = 0$

From the first part of formula (2.4.1), the local fractional integral of $t^\alpha \cos_\alpha (nt)^\alpha$ is

$$
\int_{-\pi}^{\pi} t^\alpha \cos_\alpha (nt)^\alpha (dt)^\alpha = \Gamma(1+\alpha)_{-\pi} I_\pi^{(\alpha)} t^\alpha \cos_\alpha (nt)^\alpha .
$$

On the other hand, the local fractional derivative of $t^\alpha \sin_\alpha (nt)^\alpha$.

$$
D^\alpha t^\alpha \sin_\alpha (nt)^\alpha = t^\alpha n^\alpha \cos_\alpha (nt)^\alpha + \Gamma(1+\alpha) \sin_\alpha (nt)^\alpha .
$$

So, from (2.4.1) and (2.4.10) we have

$$\begin{aligned}
\int_{-\pi}^{\pi} t^{\alpha}\cos_{\alpha}(nt)^{\alpha}(dt)^{\alpha} &= \frac{\Gamma(1+\alpha)}{n^{\alpha}}\,{}_{-\pi}I_{\pi}^{\alpha}\left(D^{\alpha}t^{\alpha}\sin_{\alpha}(nt)^{\alpha}-\Gamma(1+\alpha)\sin_{\alpha}(nt)^{\alpha}\right)\\
&= \frac{\Gamma(1+\alpha)}{n^{\alpha}}\left(t^{\alpha}\sin_{\alpha}(nt)^{\alpha}\Big|_{-\pi}^{\pi}-\Gamma(1+\alpha)_{-\pi}I_{\pi}^{\alpha}\sin_{\alpha}(nt)^{\alpha}\right)\\
&= \frac{\Gamma(1+\alpha)}{n^{\alpha}}\left(0+\frac{\Gamma(1+\alpha)}{n^{\alpha}}{}_{-\pi}I_{\pi}{}^{\alpha}D^{\alpha}\cos_{\alpha}(nt)^{\alpha}\right)\\
&= \frac{\Gamma(1+\alpha)}{n^{\alpha}}\left(\frac{\Gamma(1+\alpha)}{n^{\alpha}}\cos_{\alpha}(nt)^{\alpha}\Big|_{-\pi}^{\pi}\right)=0.
\end{aligned}$$

Example 2.4.4 $\int_{-\pi}^{\pi} t^{\alpha}\sin_{\alpha}(jt)^{\alpha}(dt)^{\alpha}=\dfrac{-2\pi^{\alpha}(-1)^{j}\Gamma(1+\alpha)}{j^{\alpha}}$

Since $D^{\alpha}t^{\alpha}\cos_{\alpha}(jt)^{\alpha}=-t^{\alpha}j^{\alpha}\sin_{\alpha}(jt)^{\alpha}+\Gamma(1+\alpha)\cos_{\alpha}(jt)^{\alpha}$ and from (2.4.1), we get

$$\begin{aligned}
\int_{-\pi}^{\pi} t^{\alpha}\sin_{\alpha}(jt)^{\alpha}(dt)^{\alpha} &= \frac{-\Gamma(1+\alpha)}{j^{\alpha}}{}_{-\pi}I_{\pi}{}^{(\alpha)}\left(D^{\alpha}t^{\alpha}\cos_{\alpha}(jt)^{\alpha}-\Gamma(1+\alpha)\cos_{\alpha}(jt)^{\alpha}\right)\\
&= \frac{-\Gamma(1+\alpha)}{j^{\alpha}}\left(t^{\alpha}\cos_{\alpha}(jt)^{\alpha}\right)_{-\pi}^{\pi}+\frac{\Gamma^{2}(1+\alpha)}{j^{\alpha}}{}_{-\pi}I_{\pi}{}^{(\alpha)}\cos_{\alpha}(jt)^{\alpha}\\
&= \frac{-\Gamma(1+\alpha)(-1)^{j}}{j^{\alpha}}\left(\pi^{\alpha}-(-\pi)^{\alpha}\right)+\frac{\Gamma(1+\alpha)}{j^{\alpha}}\int_{-\pi}^{\pi}\cos_{\alpha}(jt)^{\alpha}(dt)^{\alpha}\\
&= \frac{-2\pi^{\alpha}(-1)^{j}\Gamma(1+\alpha)}{j^{\alpha}}.
\end{aligned}$$

Example 2.4.5 $\int_{-\pi}^{\pi} t^{2\alpha}\sin_{\alpha}(jt)^{\alpha}(dt)^{\alpha}=0.$

Since $D^{\alpha}t^{2\alpha}\cos_{\alpha}(jt)^{\alpha}=-t^{2\alpha}j^{\alpha}\sin_{\alpha}(jt)^{\alpha}+\dfrac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}t^{\alpha}\cos_{\alpha}(jt)^{\alpha}$ and from (2.4.1) we have:

$$\begin{aligned}
\int_{-\pi}^{\pi} t^{2\alpha}\sin_{\alpha}(jt)^{\alpha}(dt)^{\alpha} &= \frac{-\Gamma(1+\alpha)}{j^{\alpha}}{}_{-\pi}I_{\pi}{}^{(\alpha)}\left(D^{\alpha}t^{2\alpha}\cos_{\alpha}(jt)^{\alpha}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}t^{\alpha}\cos_{\alpha}(jt)^{\alpha}\right)\\
&= \frac{-\Gamma(1+\alpha)}{j^{\alpha}}\left(t^{2\alpha}\cos_{\alpha}(jt)^{\alpha}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}{}_{-\pi}I_{\pi}{}^{(\alpha)}t^{\alpha}\cos_{\alpha}(jt)^{\alpha}\right)\\
&= \frac{-\Gamma(1+\alpha)(-1)^{j}}{j^{\alpha}}\left(\pi^{2\alpha}-\pi^{2\alpha}\right)+\frac{\Gamma(1+2\alpha)}{j^{\alpha}\Gamma(1+\alpha)}\int_{-\pi}^{\pi}t^{\alpha}\cos_{\alpha}(jt)^{\alpha}(dt)^{\alpha}\\
&= 0+0=0.
\end{aligned}$$

Example 2.4.6 $\int\limits_{-\pi}^{\pi} t^{2\alpha}\cos_\alpha(jt)^\alpha(dt)^\alpha = \dfrac{2\Gamma(1+2\alpha)\pi^\alpha(-1)^j}{j^{2\alpha}}$.

Since $D^\alpha t^{2\alpha}\sin_\alpha(jt)^\alpha = t^{(2\alpha)}j^\alpha\cos_\alpha(jt)^\alpha + \dfrac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}t^\alpha\sin_\alpha(jt)^\alpha$, and from the formula (2.4.1)

$$\begin{aligned}
\int\limits_{-\pi}^{\pi} t^{2\alpha}\cos(jt)^\alpha(dt)^\alpha &= \frac{\Gamma(1+\alpha)}{j^\alpha}{}_{-\pi}I_\pi{}^\alpha\left(D^\alpha t^{2\alpha}\sin_\alpha(jt)^\alpha - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}t^\alpha\sin_\alpha(jt)^\alpha\right)\\
&= \frac{\Gamma(1+\alpha)}{j^\alpha}\left(t^{2\alpha}\sin_\alpha(jt)^\alpha\right)_{-\pi}^{\pi} - \frac{\Gamma(1+2\alpha)}{j^\alpha\Gamma(1+\alpha)}\int\limits_{-\pi}^{\pi} t^\alpha\sin_\alpha(jt)^\alpha(dt)^\alpha\\
&= 0 - \frac{\Gamma(1+2\alpha)}{j^\alpha\Gamma(1+\alpha)}\left(\frac{-2\pi^\alpha(-1)^j\Gamma(1+\alpha)}{j^\alpha}\right)\\
&= \frac{2\Gamma(1+2\alpha)\pi^\alpha(-1)^j}{j^{2\alpha}}.
\end{aligned}$$

Example 2.4.7 $\int\limits_0^1 x^\alpha e^{3x}(dx)^\alpha = \dfrac{\Gamma(1+\alpha)}{3^\alpha}e^3 - \frac{\Gamma^2(1+\alpha)}{3^{2\alpha}}(e^3-1)$

From (2.4.1), the local fractional integral of $x^\alpha e^{3x}$ is

$$\int\limits_0^1 x^\alpha e^{3x}(dx)^\alpha = \Gamma(1+\alpha)_0\, I_1^{(\alpha)}x^\alpha e^{3x}$$

On the other hand, the local fractional derivative of $x^\alpha e^{3x}$

$$D^\alpha x^\alpha e^{3x} = 3^\alpha x^\alpha e^{3x} + \Gamma(1+\alpha)e^{3x}$$

So, from (2.4.1) and (2.4.10) we have

$$\begin{aligned}
\int\limits_0^1 x^\alpha e^{3x}(dx)^\alpha &= \Gamma(1+\alpha)_0\, I_1^{(\alpha)}\left(D^\alpha x^\alpha e^{3x} - \Gamma(1+\alpha)e^{3x}\right)\\
&= \frac{\Gamma(1+\alpha)}{3^\alpha}\left(x^\alpha e^{3x}\right)_0^1 - \frac{\Gamma^2(1+\alpha)}{3^{2\alpha}}e^{3x}\\
&= \frac{\Gamma(1+\alpha)}{3^\alpha}e^3 - \frac{\Gamma^2(1+\alpha)}{3^{2\alpha}}\left(e^3-1\right)
\end{aligned}$$

Chapter 3

Methods for Solving Local Fractional Fredholm Integral Equation of the Second Kind

This chapter focuses on two methods for solving the local fractional Fredholm integral equation of the second kind (1.3), namely the Neumann Series Method and the Adomain Decomposition Method (ADM).

3.1 Neumann Series Method

The theory of this section can be found in [25]. The strategy of the Neumann series method is based on assuming that the initial solution is

$$u_0(x) = f(x)$$

and for $k > 1$,

$$u_k(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u_{k-1}(t)\, (dt)^\alpha .$$

Let $\Psi_1(x) = \dfrac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t)\, u_0(t)\, (dt)^\alpha$, then $u_1(x) = f(x) + \lambda^\alpha \Psi_1(x)$

The second iteration is given by:

$$\begin{aligned} u_2(x) &= f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u_1(t)\, (dt)^\alpha \\ &= f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t)(f(t) + \lambda^\alpha \Psi_1(t))\, (dt)^\alpha \\ &= f(x) + \lambda^\alpha \Psi_1(x) + \frac{\lambda^{2\alpha}}{\Gamma(1+\alpha)} \int_a^b K(x,t)\, \Psi_1(t)\, (dt)^\alpha \end{aligned}$$

Therefore, $u(x)$ can be written as

$$u(x) = f(x) + \sum_{n=1}^{\infty} \lambda^{n\alpha} \Psi_n(x) \tag{3.1.1}$$

such that

$$\Psi_n(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t)\, \Psi_{n-1}(t)\, (dt)^\alpha\,, \text{ for } n \geq 1 \tag{3.1.2}$$

Now, we apply the Neumann method on two examples.

Example 3.1.1 *Consider the following local fractional Fredholm integral equation*

$$u(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} x^\alpha t^\alpha u(t)\, (dt)^\alpha$$

The initial solution is $u_0(x) = \Gamma(1+\alpha)$. Then:

$$\begin{aligned}
u_1(x) &= \Gamma(1+\alpha) + \frac{x^\alpha}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} t^\alpha \Gamma(1+\alpha)\, (dt)^\alpha \\
&= \Gamma(1+\alpha) + \frac{x^\alpha}{\Gamma(1+\alpha)} \left(\Gamma^2(1+\alpha) \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} t^{2\alpha} \right)_{-\pi}^{\pi} \\
&= \Gamma(1+\alpha) \\
u_2(x) &= \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} x^\alpha t^\alpha \Gamma(1+\alpha)\, (dt)^\alpha \\
&= \Gamma(1+\alpha) \\
u_3(x) &= \Gamma(1+\alpha) \\
&\vdots \\
u_n(x) &= \Gamma(1+\alpha) \\
u(x) &= \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \Gamma(1+\alpha) \\
&= \Gamma(1+\alpha)
\end{aligned}$$

Example 3.1.2 *Consider the following local fractional Fredholm integral equation*

$$u(x) = 1 + x^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^\alpha \sin_\alpha (2\pi x)^\alpha\, u(t)\, (dt)^\alpha$$

Using Neumann series method, $u_0(x) = 1 + x^\alpha$

$$
\begin{aligned}
u_1(x) &= 1+x^{\alpha}+\frac{\sin_{\alpha}(2\pi x)^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}(1+t^{\alpha})(dt)^{\alpha} \\
&= 1+x^{\alpha}+\frac{\sin_{\alpha}(2\pi x)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}t^{2\alpha}+\frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}t^{3\alpha}\right)_0^1 \\
&= 1+x^{\alpha}+\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right) \\
u_2(x) &= 1+x^{\alpha}+\frac{\sin_{\alpha}(2\pi x)^{\alpha}}{\Gamma(1+\alpha)}\int_0^1 t^{\alpha}\left(1+t^{\alpha}+\sin_{\alpha}(2\pi t)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\right)(dt)^{\alpha} \\
&= 1+x^{\alpha}+\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right) \\
&\quad +\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\frac{-1}{(2\pi)^{\alpha}}\right) \\
&\quad \vdots \\
u_n(x) &= 1+x^{\alpha}+\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\ \sum_{i=1}^{n}\left(\frac{-1}{(2\pi)^{\alpha}}\right)^{(i-1)}\ \right)
\end{aligned}
$$

Therefore, the solution of the integral equation is

$$
\begin{aligned}
u(x) &= \lim_{n\to\infty} u_n(x)=\lim_{n\to\infty} 1+x^{\alpha}+\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\sum_{i=0}^{n-1}\left(\frac{-1}{(2\pi)^{\alpha}}\right)^{i} \\
&= 1+x^{\alpha}+\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\frac{1}{\left(1+\frac{1}{(2\pi)^{\alpha}}\right)} \\
&= 1+x^{\alpha}+\sin_{\alpha}(2\pi x)^{\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\frac{(2\pi)^{\alpha}}{1+(2\pi)^{\alpha}}\right)
\end{aligned}
$$

3.2 Local Fractional Adomian Decomposition Method

This section describes the Local fractional Adomain's decomposition method for solving local fractional Fredholm integral equation of the second kind (1.3). For more details, see [26].

Suppose that the solution exists, the Adomain decomposition method assumed that the solution has the series form:

$$
u(x)=\sum_{n=0}^{\infty} u_n(x)
$$

such that $u_0(x)=f(x)$ and for $n\geq 1$

$$u_n(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u_{n-1}(t) (dt)^\alpha . \tag{3.2.1}$$

As an application on this method, we solve the following two examples.

Example 3.2.1 *Solve the following local fractional Fredholm integral equation*

$$u(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} x^\alpha t^\alpha (dt)^\alpha$$

The initial solution is

$$u_0(x) = \Gamma(1+\alpha)$$

then

$$\begin{aligned} u_1(x) &= \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} x^\alpha t^\alpha \Gamma(1+\alpha) (dt)^\alpha \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} (0) = 0 \\ u_2(x) &= \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} x^\alpha t^\alpha (0) (dt)^\alpha = 0 \\ &\vdots \\ u_n(x) &= 0 \\ u(x) &= \Gamma(1+\alpha) \end{aligned}$$

Example 3.2.2 *Solve the following local fractional Fredholm integral equation*

$$u(x) = 1 + x^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^1 t^\alpha \sin_\alpha (2\pi x)^\alpha u(t) (dt)^\alpha$$

Let

$$u_0(x) = 1 + x^\alpha$$

By applying the relation recursively we can reach to the n^{th} term of $u_n(x)$ as follows:

$$
\begin{aligned}
u_1\left(x\right) &= \frac{\sin_\alpha\left(2\pi x\right)^\alpha}{\Gamma\left(1+\alpha\right)}\int_0^1 t^\alpha\left(1+t^\alpha\right)\left(dt\right)^\alpha \\
&= \frac{\sin_\alpha\left(2\pi x\right)^\alpha}{\Gamma\left(1+\alpha\right)}\left(\frac{\Gamma^2\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}t^{2\alpha}+\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}t^{3\alpha}\right)_0^1 \\
&= \sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right) \\
u_2\left(x\right) &= \frac{\sin_\alpha\left(2x\right)^\alpha}{\Gamma\left(1+\alpha\right)}\int_0^1 t^\alpha\left(\sin_\alpha\left(2\pi t\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\right)\left(dt\right)^\alpha \\
&= \sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\left(\frac{-1}{\left(2\pi\right)^\alpha}\right) \\
u_3\left(x\right) &= \frac{\sin_\alpha\left(2\pi x\right)^\alpha}{\Gamma\left(1+\alpha\right)}\int_0^1 t^\alpha\left(\sin_\alpha\left(2\pi t\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\left(\frac{-1}{\left(2\pi\right)^\alpha}\right)\ \right)\left(dt\right)^\alpha \\
&= \sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\left(\frac{-1}{\left(2\pi\right)^\alpha}\right)\left(\frac{-1}{\left(2\pi\right)^\alpha}\right) \\
&\vdots \\
u_n\left(x\right) &= \sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\left(\left(\frac{-1}{\left(2\pi\right)^\alpha}\right)^{(n-1)}\right) \\
u\left(x\right) &= \sum_{i=0}^{\infty}u_i\left(x\right) = 1+x^\alpha+\sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\sum_{i=0}^{\infty}\left(\frac{-1}{\left(2\pi\right)^\alpha}\right)^i \\
&= 1+x^\alpha+\sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\frac{1}{\left(1+\frac{1}{\left(2\pi\right)^\alpha}\right)} \\
&= 1+x^\alpha+\sin_\alpha\left(2\pi x\right)^\alpha\left(\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}+\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\right)\left(\frac{\left(2\pi\right)^\alpha}{1+\left(2\pi\right)^\alpha}\right)
\end{aligned}
$$

Chapter 4

Local Fractional Fourier Series

In this chapter, the local fractional Fourier series of functions of one variable will be presented, and then some examples will be discussed. Moreover, we derive a formula of the local fractional Fourier series of functions of two variables, which are needed in chapter 5. Finally, local fractional Fourier series of two dimensions will be applied on some examples.

4.1 One-Dimensional Local Fractional Fourier Series

In the present section, we introduce the generalized Hilbert space and then the generalized local fractional Fourier series.

Definition 4.1.1 *[10, 14](Generalized inner product space)*

Let V be a complex or real vector space, a generalized inner product space is a function $\langle .,. \rangle_\alpha$ on pairs (x^α, y^α) of vectors in $V \times V$ satisfying the following axioms [9, 10]

Axiom 4.1.1 *Positive definiteness*

$$\langle x^\alpha, y^\alpha \rangle_\alpha \geq 0, \ \forall x^\alpha, y^\alpha \in V \ \text{and} \ \langle x^\alpha, x^\alpha \rangle_\alpha = 0 \ \text{if and only if} \ x = 0 \tag{4.1.1}$$

Axiom 4.1.2 *Conjugate symmetry*

$$\langle x^\alpha, y^\alpha \rangle_\alpha = \overline{\langle y^\alpha, x^\alpha \rangle}_\alpha, \ for \ all \ x^\alpha \ and \ y^\alpha \in V \tag{4.1.2}$$

Axiom 4.1.3 *For all $x^\alpha, y^\alpha, z^\alpha \in V$ and $a, b \in \mathbb{R}$, we have*

$$\langle a^\alpha x^\alpha + b^\alpha y^\alpha, z^\alpha \rangle_\alpha \ = \ a^\alpha \langle x^\alpha, z^\alpha \rangle_\alpha + b^\alpha \langle y^\alpha, z^\alpha \rangle_\alpha \tag{4.1.3}$$

Remark 4.1.1 L_2 *– norm on a generalized inner product space is defined by [10, 14]*

$$\|x^\alpha\|_\alpha = \langle x^\alpha, x^\alpha \rangle_\alpha^{\frac{1}{2}} = \sqrt{\sum_{k=1}^{\infty} |x_k^\alpha|^2} \tag{4.1.4}$$

Definition 4.1.2 *[10]A scalar (or dot) product of two T-periodic functions f (t) and g(t) is*

$$\langle f, g \rangle_\alpha = \int_0^T f(t)\overline{g(t)} \, (dt)^\alpha \tag{4.1.5}$$

Suppose $\{e_i^\alpha\}_{i=1}^n$ is an orthonormal system in an inner product space X, then following are equivalent [2]

1. Span $\{e_1^\alpha, ..., e_n^\alpha\} = X$, i.e. $\{e_i\}_{i=1}^n$ is a basis.
2. Pythagorean theorem in fractal space

   The equation

$$\sum_{k=1}^{\infty} |a_k^\alpha|^2 = \|f\|_\alpha^2, \quad for\ all\ f \in X \text{ and } a_k^\alpha = \langle f, e_k^\alpha \rangle_\alpha \tag{4.1.6}$$

3. Generalized Pythagorean theorem in fractal space

$$\langle f, g \rangle_\alpha = \sum_{k=1}^{n} a_k^\alpha \overline{b_k^\alpha} \tag{4.1.7}$$

   where $a_k^\alpha = \langle f, e_k^\alpha \rangle_\alpha and b_k^\alpha = \langle g, e_k^\alpha \rangle_\alpha$

4. $f = \sum\limits_{k=1}^{n} a_k^\alpha e_k^\alpha$ with a sum convergent in X for all $f \in X$

Let $\{\phi_k\}$, k= 1, 2, 3..., be any sequence of T- periodic local fractional continuous functions that are [1, 10]:

1. Orthogonal

$$\langle \phi_k, \phi_j \rangle_\alpha = \int_0^T \phi_k(t) \overline{\phi_j(t)} (dt)^\alpha = 0 \forall k \neq j$$

2. Normalized

$$\langle \phi_k, \phi_k \rangle_\alpha = \int_0^T \phi_k^2(t) (dt)^\alpha = 1 \tag{4.1.8}$$

3. Complete

   If a function x (t) is such that

$$\langle x, \phi_k \rangle_\alpha = \int_0^T x(t) \phi_k(t) (dt)^\alpha = 0 \tag{4.1.9}$$

   for all k, then $x(t) = 0$.

Then $\{\phi_k\}$ is called a complete orthonormal set of functions.

Definition 4.1.3 *[1] A generalized Hilbert space is a complete generalized inner product space.*

Definition 4.1.4 *[35]Let $\{\phi_k(t)\}_{k=1}^{\infty}$ be a complete orthonormal set of functions , then any T - periodic fractal signal $f(t)$ can be uniquely represented as the infinite series*

$$f(t) = \sum_{k=0}^{\infty} \varphi_k \phi_k(t). \tag{4.1.10}$$

The series in (4.1.10) is called the local fractional Fourier series representation of $f(t)$ in the generalized Hilbert space, where φ_k are the local fractional Fourier series coefficients of $f(t)$.

In order to find the coefficients φ_k, we can use the fact that $\{\phi_k(t)\}$ form a complete orthonormal system and the partial sum of the local fractional Fourier series converges to $f(t)$. Therefore, to approximate the function $f(t)$, we can use the partial sums

$$f(t) = \sum_{k=0}^{N} \varphi_k \phi_k(t). \tag{4.1.11}$$

The sequence of T - periodic function in fractal space $\{\phi_k(t)\}_{k=0}^{\infty}$ defined by [14]:

$$\phi_0(t) = \left(\frac{1}{T}\right)^{\frac{\alpha}{2}}$$

$$\text{and } \phi_k(t) = \left\{ \begin{array}{ll} \left(\frac{2}{T}\right)^{\frac{\alpha}{2}} \sin_\alpha (k w_0 t)^\alpha, & \text{if k} \geq 1 \text{ is odd} \\ \left(\frac{2}{T}\right)^{\frac{\alpha}{2}} \cos_\alpha (k w_0 t)^\alpha, & \text{if k} > 1 \text{ is even} \end{array} \right\}$$

is a complete and orthonormal sequence, where $w_0 = \dfrac{2\pi}{T}$

Definition 4.1.5 *[1]The local fractional trigonometric Fourier series of f(t) is given by:*

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos_\alpha (k w_0 t)^\alpha + \sum_{k=1}^{\infty} b_k \sin_\alpha (k w_0 t)^\alpha \tag{4.1.12}$$

where the local fractional Fourier coefficients are

$$
\begin{aligned}
a_0 &= \left(\frac{1}{T}\right)^{\alpha} \int_0^T f(t)\,(dt)^{\alpha} \\
a_k &= \left(\frac{2}{T}\right)^{\alpha} \int_0^T f(t)\cos_{\alpha}(kw_0t)^{\alpha}\,(dt)^{\alpha} \\
b_k &= \left(\frac{2}{T}\right)^{\alpha} \int_0^T f(t)\sin_{\alpha}(kw_0t)^{\alpha}\,(dt)^{\alpha}
\end{aligned}
$$

Remark 4.1.2 *[2, 11, 20]If a function $f(t)$ is $2l-$ periodic, then the local fractional Fourier series is*

$$
f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha} + b_k \sin_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}\right) \tag{4.1.13}
$$

where the local fractional Fourier coefficients are

$$
\begin{aligned}
a_k &= \left(\frac{1}{l}\right)^{\alpha} \int_{-l}^{l} f(t)\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha} \\
b_k &= \left(\frac{1}{l}\right)^{\alpha} \int_{-l}^{l} f(t)\sin_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha}
\end{aligned}
$$

The weight forms of the coefficients in the local fractional Fourier series are [2]

$$
\begin{aligned}
a_k &= \frac{\left(1/\Gamma(1+\alpha)\int_{-l}^{l} f(x)\cos_{\alpha}(\pi kx/l)^{\alpha}(dx)^{\alpha}\right)}{\left(1/\Gamma(1+\alpha)\int_{-l}^{l} \cos_{\alpha}^{2}(\pi kx/l)^{\alpha}(dx)^{\alpha}\right)} \\
b_k &= \frac{\left(1/\Gamma(1+\alpha)\int_{-l}^{l} f(x)\sin_{\alpha}(\pi kx/l)^{\alpha}(dx)^{\alpha}\right)}{\left(1/\Gamma(1+\alpha)\int_{-l}^{l} \sin_{\alpha}^{2}(\pi kx/l)^{\alpha}(dx)^{\alpha}\right)}
\end{aligned}
$$

Remark 4.1.3 *[10]The local fractional Fourier series can be represented by using the Mittag - Leffler function by*

$$
f(x) = \sum_{k=-\infty}^{\infty} C_k E_{\alpha}\left(\frac{\pi^{\alpha} i^{\alpha}(kx)^{\alpha}}{l^{\alpha}}\right) \tag{4.1.14}
$$

where the local fractional Fourier coefficient is

$$C_k = \frac{1}{(2l)^\alpha} \int_{-l}^{l} f(x) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) (dx)^\alpha \quad , \quad \text{and } k \in \mathbb{Z} \tag{4.1.15}$$

The weights of the Mittag Leffler functions are written in the form

$$C_k = \frac{\frac{1}{(2l)^\alpha} \int_{-l+t_0}^{l+t_0} f(x) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) (dx)^\alpha}{\frac{1}{(2l)^\alpha} \int_{-l+t_0}^{l+t_0} E_\alpha \left(\frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) E_\alpha \left(\frac{-\pi^\alpha i^\alpha (kx)^\alpha}{l^\alpha} \right) (dx)^\alpha} \tag{4.1.16}$$

Some Properties of Local Fractional Fourier Series [37, 38] are listed below.

Property 4.1.1 *(Linearity) Suppose that the local fractional Fourier coefficients of $f(x)$ and $g(x)$ are f_n and g_n, respectively, then we has for two constants a and b*

$$af(x) + bg(x) \leftrightarrow af_n + bgf_n \tag{4.1.17}$$

Property 4.1.2 *(Conjugation). Suppose that C_n is Fourier coefficients of $f(x)$. Then we have*

$$\overline{f(x)} \longleftrightarrow \overline{C_{-n}} \tag{4.1.18}$$

Property 4.1.3 *(Shift in time). Suppose that C_n is Fourier coefficients of $f(x)$. Then we have*

$$f(x - x_0) \leftrightarrow E_\alpha \left(-i^\alpha (nx)^\alpha\right) C_n \tag{4.1.19}$$

Property 4.1.4 *(Time reversal). Suppose that C_n is Fourier coefficients of $f(x)$. Then we have*

$$f(-x) \leftrightarrow C_{-n}$$

In the following we present some core theorems on the local fractional Fourier series [37, 38].

Theorem 4.1.1 *(Local fractional Bessel inequality).*

Suppose that $f(t)$ is 2π-periodic, bounded and local fractional integral on $[-\pi, \pi]$. If both a_n and b_n are Fourier coefficients of $f(t)$, then there exists the inequality

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) \leq \frac{1}{\pi^\alpha} \int_{-\pi}^{\pi} f^2(t) (dt)^\alpha . \tag{4.1.20}$$

Theorem 4.1.2 *(Local fractional Riemann–Lebesgue Theorem).*

Suppose that $f(x)$ is 2π - periodic, bounded and local fractional integrable on $[-\pi\ ,\ \pi]$. Then we have

$$\lim_{n\to+\infty}\frac{1}{(2\pi)^{\alpha}}\int_{-\pi}^{\pi} f(t)\sin_{\alpha}(nt)^{\alpha}\,(dt)^{\alpha}=0, \tag{4.1.21}$$

and

$$\lim_{n\to+\infty}\frac{1}{(2\pi)^{\alpha}}\int_{-\pi}^{\pi} f(t)\cos_{\alpha}(nt)^{\alpha}\,(dt)^{\alpha}=0. \tag{4.1.22}$$

Theorem 4.1.3 *Suppose that $f(t)$ is 2π–periodic, bounded and a local fractional integrable on $[-\pi,\pi]$, if*

$$f(t)\sim\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos_{\alpha}(kt)^{\alpha}+\sum_{k=1}^{\infty}b_k\sin_{\alpha}(kt)^{\alpha}.$$

Then

$$\frac{1}{\pi^{\alpha}}\int_{-\pi}^{\pi} f^2(t)\,(dt)^{\alpha}=\frac{a_0^2}{2}+\sum_{k=0}^{\infty}\left(a_k^2+b_k^2\right). \tag{4.1.23}$$

Theorem 4.1.4 *(Convergence Theorem for Local Fractional Fourier Series).*

Suppose that $f(t)$ is 2π– periodic, bounded and a local fractional integrable on $[-\pi,\pi]$. The local fractional series of $f(t)$ converges to $f(t)$ at $t\in[-\pi,\pi]$, and

$$\frac{f(t+0)+f(t-0)}{2}=\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos_{\alpha}(kt)^{\alpha}+\sum_{k=1}^{\infty}b_k\sin_{\alpha}(kt)^{\alpha}, \tag{4.1.24}$$

where

$$\begin{aligned}
a_0 &= \frac{1}{\pi^{\alpha}}\int_{-\pi}^{\pi} f(t)\,(dt)^{\alpha},\\
a_k &= \left(\frac{1}{\pi}\right)^{\alpha}\int_{-\pi}^{\pi} f(t)\cos_{\alpha}(nt)^{\alpha}\,(dt)^{\alpha},\\
b_k &= \left(\frac{1}{\pi}\right)^{\alpha}\int_{-\pi}^{\pi} f(t)\sin_{\alpha}(nt)^{\alpha}\,(dt)^{\alpha}.
\end{aligned}$$

4.2 Applications on Local Fractional One-dimensional Fourier Series

In this section, we give the local fractional Fourier series of some solved examples in literature, and then we solve some own examples explicitly.

Example 4.2.1 *[1] The local fractional fourier series expansion of the fractal signal* $x(t) = t^\alpha + 1$ *where* $(-\pi \leq t \leq \pi)$ *is:*

$$x(t) = 1 + \sum_{k=1}^{\infty} \frac{2\Gamma(1+\alpha)(-1)^{k+1}}{k^\alpha} \sin_\alpha (kt)^\alpha$$

Example 4.2.2 *[21] The local fractional fourier series expansion of the fractal signal* $x(t) = t^\alpha$ *where* $(-l \leq t \leq l)$ *is*

$$x(t) = \sum_{k=1}^{\infty} \frac{2\Gamma(1+\alpha)\, l^\alpha (-1)^{k+1}}{k^\alpha \pi^\alpha} \sin_\alpha (\frac{\pi k t}{l})^\alpha$$

Example 4.2.3 *(Local Fractional Fourier Cosine)*

Consider the fractal signal $f(t) = t^{2\alpha}$ where $-l \leq t \leq l$. The coefficients can be calculated

using formula 4.1.13

$$\begin{aligned} a_0 &= \frac{1}{l^\alpha} \int_{-l}^{l} f(t)\,(dt)^\alpha = \frac{1}{l^\alpha} \int_{-l}^{l} t^{2\alpha}\,(dt)^\alpha = \left(\frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{l^\alpha \Gamma(1+3\alpha)} t^{3\alpha} \right)_{-l}^{l} \\ &= \frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{l^\alpha \Gamma(1+3\alpha)} \left(l^{3\alpha} - (-l)^{3\alpha} \right) = \frac{2 l^{3\alpha} \Gamma(1+\alpha)\Gamma(1+2\alpha)}{l^\alpha \Gamma(1+3\alpha)}. \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{l^\alpha} \int_{-l}^{l} t^{2\alpha} \cos_\alpha \left(\frac{k\pi t}{l} \right)^\alpha (dt)^\alpha \\ &= \frac{\Gamma(1+\alpha)}{l^\alpha} \left(\frac{l}{k\pi} \right)^\alpha \left(t^{2\alpha} \sin_\alpha \left(\frac{k\pi t}{l} \right)^\alpha \right)_{-l}^{l} - \frac{\Gamma(1+2\alpha)}{(k\pi)^\alpha \Gamma(1+\alpha)} \int_{-l}^{l} t^\alpha \sin_\alpha \left(\frac{k\pi t}{l} \right)^\alpha (dt)^\alpha \\ &= 0 + \frac{\Gamma(1+2\alpha)\Gamma(1+\alpha)}{(k\pi)^\alpha \Gamma(1+\alpha)} \left(\frac{l}{k\pi} \right)^\alpha \left(\left(t^\alpha \cos_\alpha \left(\frac{k\pi t}{l} \right)^\alpha \right)_{-l}^{l} - \int_{-l}^{l} \cos_\alpha \left(\frac{k\pi t}{l} \right)^\alpha (dt)^\alpha \right) \\ &= \frac{\Gamma(1+2\alpha)\, l^\alpha}{(k\pi)^{2\alpha}} \left(l^\alpha (-1)^k - (-l)^\alpha (-1)^k - 0 \right) \\ &= \frac{2(-1)^k \Gamma(1+2\alpha)\, l^{2\alpha}}{(k\pi)^{2\alpha}} \end{aligned}$$

$$
\begin{aligned}
b_k &= \frac{1}{l^\alpha}\int_{-l}^{l} t^{2\alpha}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \left(\frac{1}{k\pi}\right)^\alpha\left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}\int_{-l}^{l} t^\alpha\cos_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha - \Gamma(1+\alpha)\left(t^{2\alpha}\cos_\alpha\left(\frac{k\pi t}{l}\right)^\alpha\right)_{-l}^{l}\right) \\
&= \frac{\Gamma(1+2\alpha)}{(k\pi)^\alpha}\left(\frac{l}{k\pi}\right)^\alpha\left(\left(t^\alpha\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha\right)_{-l}^{l} - \int_{-l}^{l}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha\right) - 0 = 0.
\end{aligned}
$$

So, we have

$$
x^{2\alpha} = \frac{l^{2\alpha}\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + \sum_{k=1}^{\infty}\frac{2(-1)^k\Gamma(1+2\alpha)\,l^{2\alpha}}{(k\pi)^{2\alpha}}\cos_\alpha\left(\frac{k\pi x}{l}\right)^\alpha.
$$

Example 4.2.4 *Expand the fractal signal $f(x) = e^{2x}$ where $(-l \leq x \leq l)$.*

Using the formula 4.1.13

$$
\begin{aligned}
a_0 &= \frac{1}{l^\alpha}\int_{-l}^{l} e^{2t}(dt)^\alpha = \frac{\Gamma(1+\alpha)}{2^\alpha l^\alpha}\left(e^{2t}\right)_{-l}^{l} \\
&= \frac{\Gamma(1+\alpha)}{2^\alpha l^\alpha}\left(e^{2l} - e^{-2l}\right)
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{l^\alpha}\int_{-l}^{l} e^{2t}\cos_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \frac{\Gamma(1+\alpha)}{l^\alpha}\left(\frac{l}{k\pi}\right)^\alpha\left(e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha\right)_{-l}^{l} - \frac{2^\alpha}{(k\pi)^\alpha}\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= 0 - \frac{2^\alpha}{(k\pi)^\alpha}\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{\alpha}\Gamma(1+\alpha)}{(k\pi)^{\alpha}}\left(\frac{l}{k\pi}\right)^{\alpha}\left(e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}\right)_{-l}^{l} - \frac{2^{2\alpha}l^{\alpha}}{(k\pi)^{2\alpha}}\int_{-l}^{l} e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha} \\
&= \frac{2^{\alpha}\Gamma(1+\alpha)\,l^{\alpha}(-1)^{k}\left(e^{2l}-e^{-2l}\right)}{(k\pi)^{2\alpha}} - \frac{2^{2\alpha}l^{\alpha}}{(k\pi)^{2\alpha}}\int_{-l}^{l} e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha} \\
&= \frac{1}{l^{\alpha}}\int_{-l}^{l} e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha}
\end{aligned}
$$

so we have

$$
\left(\frac{1}{l^{\alpha}} + \frac{2^{2\alpha}l^{\alpha}}{(k\pi)^{2\alpha}}\right)\int_{-l}^{l} e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha} = \frac{2^{\alpha}l^{\alpha}\Gamma(1+\alpha)(-1)^{k}\left(e^{2l}-e^{-2l}\right)}{(k\pi)^{2\alpha}}
$$

$$
\int_{-l}^{l} e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha} = \frac{2^{\alpha}l^{\alpha}\Gamma(1+\alpha)(-1)^{k}\left(e^{2l}-e^{-2l}\right)}{(k\pi)^{2\alpha}}\left(\frac{l^{\alpha}(k\pi)^{2\alpha}}{(k\pi)^{2\alpha}+2^{2\alpha}l^{2\alpha}}\right)
$$

From this we have

$$
\begin{aligned}
a_k &= \frac{1}{l^{\alpha}}\int_{-l}^{l} e^{2t}\cos_{\alpha}\left(\frac{k\pi t}{l}\right)^{\alpha}(dt)^{\alpha} \\
&= \frac{1}{l^{\alpha}}\frac{2^{\alpha}l^{\alpha}\Gamma(1+\alpha)(-1)^{k}\left(e^{2l}-e^{-2l}\right)}{(k\pi)^{2\alpha}}\left(\frac{l^{\alpha}(k\pi)^{2\alpha}}{(k\pi)^{2\alpha}+2^{2\alpha}l^{2\alpha}}\right) \\
&= \frac{2^{\alpha}l^{\alpha}\Gamma(1+\alpha)(-1)^{k}\left(e^{2l}-e^{-2l}\right)}{(k\pi)^{2\alpha}+2^{2\alpha}l^{2\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
b_k &= \frac{1}{l^\alpha}\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \frac{-\Gamma(1+\alpha)}{l^\alpha}\left(\frac{l}{k\pi}\right)^\alpha \left(e^{2t}\cos_\alpha\left(\frac{k\pi t}{l}\right)^\alpha\right)_{-l}^{l} + \frac{2^\alpha}{(k\pi)^\alpha}\int_{-l}^{l} e^{2t}\cos_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)}{(k\pi)^\alpha} + \frac{2^\alpha\Gamma(1+\alpha)}{(k\pi)^\alpha}\left(\frac{l}{k\pi}\right)^\alpha\left(e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha\right)_{-l}^{l} \\
&- \frac{2^{2\alpha}}{(k\pi)^\alpha}\left(\frac{l}{k\pi}\right)^\alpha\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)}{(k\pi)^\alpha} + 0 - \frac{2^{2\alpha}}{(k\pi)^\alpha}\left(\frac{l}{k\pi}\right)^\alpha\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \frac{1}{l^\alpha}\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha
\end{aligned}
$$

so we have

$$
\left(\frac{1}{l^\alpha}+\frac{2^{2\alpha}l^\alpha}{(k\pi)^{2\alpha}}\right)\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha = \frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)}{(k\pi)^\alpha}
$$

$$
\begin{aligned}
\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha &= \frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)}{(k\pi)^\alpha}\left(\frac{(k\pi)^{2\alpha}l^\alpha}{(k\pi)^{2\alpha}+(2l)^{2\alpha}}\right) \\
&= \frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)(k\pi)^\alpha l^\alpha}{(k\pi)^{2\alpha}+(2l)^{2\alpha}}
\end{aligned}
$$

from this we have

$$
\begin{aligned}
b_k &= \frac{1}{l^\alpha}\int_{-l}^{l} e^{2t}\sin_\alpha\left(\frac{k\pi t}{l}\right)^\alpha (dt)^\alpha \\
&= \frac{1}{l^\alpha}\frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)(k\pi)^\alpha l^\alpha}{(k\pi)^{2\alpha}+(2l)^{2\alpha}} \\
&= \frac{-\Gamma(1+\alpha)(-1)^k\left(e^{2l}-e^{-2l}\right)(k\pi)^\alpha}{(k\pi)^{2\alpha}+(2l)^{2\alpha}}
\end{aligned}
$$

the local fractional Fourier series formula of the function is

$$f(x) = e^{2x} = \frac{\Gamma(1+\alpha)}{2(2l)^{\alpha}}\left(e^{2l}-e^{-2l}\right)+\sum_{k=1}^{\infty}\frac{2^{\alpha}l^{\alpha}\Gamma(1+\alpha)(-1)^{k}\left(e^{2l}-e^{-2l}\right)}{(k\pi)^{2\alpha}+2^{2\alpha}l^{2\alpha}}\cos_{\alpha}\left(\frac{k\pi x}{l}\right)^{\alpha}$$
$$+\sum_{k=1}^{\infty}\frac{-\Gamma(1+\alpha)(-1)^{k}\left(e^{2l}-e^{-2l}\right)(k\pi)^{\alpha}}{(k\pi)^{2\alpha}+(2l)^{2\alpha}}\sin_{\alpha}\left(\frac{k\pi x}{l}\right)^{\alpha}$$

Example 4.2.5 *Expand the fractal signal $f(x) = x^{\alpha}$, $0 < x \leq 1$, into a local fractional Fourier series*

$$a_0 = \frac{1}{T^{\alpha}}\int_0^T f(t)\,(dt)^{\alpha} = \int_0^1 t^{\alpha}\,(dt)^{\alpha} = \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\left(t^{2\alpha}\right)_0^1 = \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)},$$

$a_k = \left(\frac{2}{T}\right)^{\alpha}\int_0^T f(t)\cos_{\alpha}(kw_0t)^{\alpha}(dt)^{\alpha} = 2^{\alpha}\int_0^1 t^{\alpha}\cos_{\alpha}(2\pi kt)^{\alpha}(dt)^{\alpha} = 0$ and

$$b_k = \left(\frac{2}{T}\right)^{\alpha}\int_0^T f(t)\sin_{\alpha}(kw_0t)^{\alpha}(dt)^{\alpha} = 2^{\alpha}\int_0^1 t^{\alpha}\sin_{\alpha}(2\pi kt)^{\alpha}(dt)^{\alpha}$$
$$= \frac{-2^{\alpha}\Gamma(1+\alpha)}{(2\pi k)^{\alpha}}\left(t^{\alpha}\cos_{\alpha}(2\pi kt)^{\alpha}\right)_0^1 + \frac{2^{\alpha}\Gamma(1+\alpha)}{(2\pi k)^{\alpha}}\int_0^1 \cos_{\alpha}(2\pi kt)^{\alpha} = \frac{-\Gamma(1+\alpha)}{(k\pi)^{\alpha}}.$$

So, the local fractional Fourier series formula is :

$$x^{\alpha} = \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} + \sum_{k=1}^{\infty}\left(\frac{-\Gamma(1+\alpha)}{(k\pi)^{\alpha}}\right)\sin_{\alpha}(2\pi kx)^{\alpha}$$

4.3 Two-Dimensional Local Fractional Fourier Series

In this section, we derive the formula of the local fractional Fourier series of functions of two variables similar to the two-dimensional Fourier series presented in [39].

The complex form of the local fractional Fourier series of two dimesion is :

$$f(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} C_{mn}e^{\left(i\pi\left(\frac{mx}{l}+\frac{ny}{h}\right)\right)^{\alpha}}. \quad (4.3.1)$$

where $-l < x^{\alpha} \leq l$, $-h < y^{\alpha} \leq h$ and

$$C_{mn} = \frac{1}{(2l.2h)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h} f(x,y)\,e^{-\left(i\pi\left(\frac{mx}{l}+\frac{ny}{h}\right)\right)^{\alpha}}(dx)^{\alpha}(dy)^{\alpha}.$$

To derive the trigonometric formula :

$$f(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}C_{mn}e^{\left(\frac{i\pi mx}{l}\right)^{\alpha}}e^{\left(\frac{i\pi ny}{h}\right)^{\alpha}}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}C_{mn}\left(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}+i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\right)\left(\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+i^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)$$

$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+i^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}$$

$$+i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}-\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha})$$

$$+\sum_{m=-\infty}^{-1}\sum_{n=-\infty}^{-1}C_{mn}(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}-i^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}$$

$$-i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}-\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha})$$

$$+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{-1}C_{mn}(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}-i^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}$$

$$+i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha})$$

$$+\sum_{m=-\infty}^{-1}\sum_{n=1}^{\infty}C_{mn}(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+i^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}$$

$$-i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha})$$

$$+C_{00}+\sum_{n=1}^{\infty}C_{n}\left(\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+i^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)+\sum_{m=1}^{\infty}C_{m}\left(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}+i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\right)$$

$$+\sum_{n=-\infty}^{-1}C_{n}\left(\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}-i^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)+\sum_{m=-\infty}^{-1}C_{m}\left(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}-i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\right)$$

$$=\sum\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\left(C_{m,n>0}+C_{m,n<0}+C_{\substack{m>0\\n<0}}+C_{\substack{m<0\\n>0}}\right)$$

$$+\sum i^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\left(C_{m,n>0}-C_{m,n<0}-C_{\substack{m>0\\n<0}}+C_{\substack{m<0\\n>0}}\right)$$

$$+\sum i^\alpha \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha \left(C_{m,n>0} - C_{m,n<0} + C_{\substack{m>0\\n<0}} - C_{\substack{m<0\\n>0}}\right)$$

$$+\sum \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha \left(-C_{m,n>0} - C_{m,n<0} + C_{\substack{m>0\\n<0}} + C_{\substack{m<0\\n>0}}\right)$$

$$+C_{00} + \sum \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha (C_{n>0} + C_{n<0}) + \sum \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha (i^\alpha C_{n>0} - i^\alpha C_{n<0})$$

$$+\sum \cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha (C_{m>0} + C_{m<0}) + \sum \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha (i^\alpha C_{m>0} - i^\alpha C_{m<0})$$

Now we find the coefficitients when $n, m \neq 0$

$$C_{m,n>0} = \frac{1}{(2l.2h)^\alpha}\int_{-l}^{l}\int_{-h}^{h} f(x,y)(\cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha - i^\alpha \cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha$$

$$-i^\alpha \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha - \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha)\,(dx)^\alpha\,(dy)^\alpha$$

$$C_{m,n<0} = \frac{1}{(2l.2h)^\alpha}\int_{-l}^{l}\int_{-h}^{h} f(x,y)(\cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha + i^\alpha \cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha$$

$$+i^\alpha \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha - \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha)\,(dx)^\alpha\,(dy)^\alpha$$

$$C_{\substack{m>0\\n<0}} = \frac{1}{(2l.2h)^\alpha}\int_{-l}^{l}\int_{-h}^{h} f(x,y)(\cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha + i^\alpha \cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha$$

$$-i^\alpha \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha + \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha)\,(dx)^\alpha\,(dy)^\alpha$$

$$C_{\substack{m<0\\n>0}} = \frac{1}{(2l.2h)^\alpha}\int_{-l}^{l}\int_{-h}^{h} f(x,y)(\cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha - i^\alpha \cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha$$

$$+i^\alpha \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha + \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha)\,(dx)^\alpha\,(dy)^\alpha$$

when $m = 0$

$$C_{n>0} = \frac{1}{(2l.2h)^\alpha}\int_{-l}^{l}\int_{-h}^{h} f(x,y)(\cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha - i^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha)\,(dy)^\alpha\,(dx)^\alpha$$

$$C_{n<0} = \frac{1}{(2l.2h)^\alpha}\int_{-l}^{l}\int_{-h}^{h} f(x,y)(\cos_\alpha\left(\frac{n\pi y}{h}\right)^\alpha + i^\alpha \sin_\alpha\left(\frac{n\pi y}{h}\right)^\alpha)\,(dy)^\alpha\,(dx)^\alpha$$

when $n=0$

$$C_{m>0}=\frac{1}{(2l.2h)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}-i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha})\,(dy)^{\alpha}\,(dx)^{\alpha}$$

$$C_{m<0}=\frac{1}{(2l.2h)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}+i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha})\,(dy)^{\alpha}\,(dx)^{\alpha}$$

Let $A_{mn}=\left(C_{m,n>0}+C_{m,n<0}+C_{\substack{m>0\\n<0}}+C_{\substack{m<0\\n>0}}\right)$, then

$$\begin{aligned}A_{mn} &= \frac{1}{(2l.2h)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)\left(4^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)(dy)^{\alpha}\,(dx)^{\alpha}\\ &= \frac{1}{(lh)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)\left(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)(dy)^{\alpha}\,(dx)^{\alpha}\end{aligned}$$

$B_{mn}=i^{\alpha}\left(C_{m,n>0}-C_{m,n<0}-C_{\substack{m>0\\n<0}}+C_{\substack{m<0\\n>0}}\right)$, then

$$\begin{aligned}B_{mn} &= \frac{i^{\alpha}}{(2l.2h)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y\left(-4^{\alpha}i^{\alpha}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)(dy)^{\alpha}\,(dx)^{\alpha}\\ &= \frac{1}{(lh)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)\left(\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)(dy)^{\alpha}\,(dx)^{\alpha}\end{aligned}$$

$$C_{mn}=i^{\alpha}\left(C_{m,n>0}-C_{m,n<0}+C_{\substack{m>0\\n<0}}-C_{\substack{m<0\\n>0}}\right)$$

$$=\frac{i^{\alpha}}{(2l.2h)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)\left(-4^{\alpha}i^{\alpha}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)(dy)^{\alpha}\,(dx)^{\alpha}$$

$$=\frac{1}{(lh)^{\alpha}}\int_{-l}^{l}\int_{-h}^{h}f(x,y)\left(\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)(dy)^{\alpha}\,(dx)^{\alpha}$$

$$D_{mn}=\left(-C_{m,n>0}-C_{m,n<0}+C_{\substack{m>0\\n<0}}+C_{\substack{m<0\\n>0}}\right)$$

$$= \frac{1}{(2l.2h)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(4^{\alpha} \sin_{\alpha} \left(\frac{m\pi x}{l} \right)^{\alpha} \sin_{\alpha} \left(\frac{n\pi y}{h} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

$$= \frac{1}{(lh)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(\sin_{\alpha} \left(\frac{m\pi x}{l} \right)^{\alpha} \sin_{\alpha} \left(\frac{n\pi y}{h} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

when $m = 0$

$E_n = C_{n>0} + C_{n<0}$

$$= \frac{1}{(2l.2h)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(2^{\alpha} \cos_{\alpha} \left(\frac{n\pi y}{h} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

$$= \frac{1}{(2lh)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(\cos_{\alpha} \left(\frac{n\pi y}{h} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

$F_n = i^{\alpha} C_{n>0} - i^{\alpha} C_{n<0}$

$$= \frac{1}{(2lh)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(\sin_{\alpha} \left(\frac{n\pi y}{h} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

when $n = 0$

$G_m = C_{m>0} + C_{m<0}$

$$= \frac{1}{(2lh)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(\cos_{\alpha} \left(\frac{m\pi x}{l} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

$H_m = i^{\alpha} C_{m>0} - i^{\alpha} C_{m<0}$

$$= \frac{1}{(2lh)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) \Big(\sin_{\alpha} \left(\frac{m\pi x}{l} \right)^{\alpha} \Big) (dy)^{\alpha} (dx)^{\alpha}$$

$$A_{00} = \frac{1}{(2l.2h)^{\alpha}} \int_{-l}^{l} \int_{-h}^{h} f(x,y) (dy)^{\alpha} (dx)^{\alpha}$$

Hence , the formula of the local fractional Fourier series two dimension when x on the period $[-l, l]$ and y on the period $[-h, h]$ is

$$
\begin{aligned}
f(x,y) &= \sum_{m,n=1}^{\infty}\left(A_{mn}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+B_{mn}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)\\
&+ \sum_{m,n=1}^{\infty}\left(C_{mn}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+D_{mn}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}\right)\\
&+ A_{00}+\sum_{n=1}^{\infty}E_{n}\cos_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+\sum_{n=1}^{\infty}F_{n}\sin_{\alpha}\left(\frac{n\pi y}{h}\right)^{\alpha}+\sum_{m=1}^{\infty}G_{m}\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}\\
&+ \sum_{m=1}^{\infty}H_{m}\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha} \qquad (4.3.2)
\end{aligned}
$$

We can define the local fractional fourier series on the periods $[0,T_1]$ and $[0,T_2]$as :

$$
\begin{aligned}
f(x,y) &= \sum_{m,n=1}^{\infty}\left(A_{mn}\cos_{\alpha}(mw_1x)^{\alpha}\cos_{\alpha}(nw_2y)^{\alpha}+B_{mn}\cos_{\alpha}(mw_1x)^{\alpha}\sin_{\alpha}(nw_2y)^{\alpha}\right)\\
&+ \sum_{m,n=1}^{\infty}\left(C_{mn}\sin_{\alpha}(mw_1x)^{\alpha}\cos_{\alpha}(nw_2y)^{\alpha}+D_{mn}\sin_{\alpha}(mw_1x)^{\alpha}\sin_{\alpha}(nw_2y)^{\alpha}\right)\\
&+ A_{00}+\sum_{n=1}^{\infty}E_{n}\cos_{\alpha}(nw_2y)^{\alpha}+\sum_{n=1}^{\infty}F_{n}\sin_{\alpha}(nw_2y)^{\alpha}+\sum_{m=1}^{\infty}G_{m}\cos_{\alpha}(mw_1x)^{\alpha}\\
&+ \sum_{m=1}^{\infty}H_{m}\sin_{\alpha}(mw_1x)^{\alpha} \qquad (4.3.3)
\end{aligned}
$$

where $w_1=\frac{2\pi}{T_1}$ and $w_2=\frac{2\pi}{T_2}$
and the coefficients will be :
when $n,m\neq 0$

$$A_{mn}=\left(\frac{2.2}{T_1.T_2}\right)^{\alpha}\int_0^{T_1}\int_0^{T_2}f(x,y)(\cos_{\alpha}(mw_1x)^{\alpha}\cos_{\alpha}(nw_2y)^{\alpha})\,(dy)^{\alpha}\,(dx)^{\alpha}$$

$$B_{mn}=\left(\frac{2.2}{T_1.T_2}\right)^{\alpha}\int_0^{T_1}\int_0^{T_2}f(x,y)(\cos_{\alpha}(mw_1x)^{\alpha}\sin_{\alpha}(nw_2y)^{\alpha})\,(dy)^{\alpha}\,(dx)^{\alpha}$$

$$C_{mn}=\left(\frac{2.2}{T_1.T_2}\right)^{\alpha}\int_0^{T_1}\int_0^{T_2}f(x,y)(\sin_{\alpha}(mw_1x)^{\alpha}\cos_{\alpha}(nw_2y)^{\alpha})\,(dy)^{\alpha}\,(dx)^{\alpha}$$

$$D_{mn}=\left(\frac{2.2}{T_1.T_2}\right)^{\alpha}\int_0^{T_1}\int_0^{T_2}f(x,y)(\sin_{\alpha}(mw_1x)^{\alpha}\sin_{\alpha}(nw_2y)^{\alpha})\,(dy)^{\alpha}\,(dx)^{\alpha}$$

when $m = 0$

$$E_n = \left(\frac{2}{T_1.T_2}\right)^{\alpha} \int_0^{T_1} \int_0^{T_2} f(x,y) \cos_\alpha (nw_2 y)^\alpha (dy)^\alpha (dx)^\alpha$$

$$F_n = \left(\frac{2}{T_1.T_2}\right)^{\alpha} \int_0^{T_1} \int_0^{T_2} f(x,y) \sin_\alpha (nw_2 y)^\alpha (dy)^\alpha (dx)^\alpha$$

when $n = 0$

$$G_m = \left(\frac{2}{T_1.T_2}\right)^{\alpha} \int_0^{T_1} \int_0^{T_2} f(x,y) \cos_\alpha (mw_1 x)^\alpha (dy)^\alpha (dx)^\alpha$$

$$H_m = \left(\frac{2}{T_1.T_2}\right)^{\alpha} \int_0^{T_1} \int_0^{T_2} f(x,y) \sin_\alpha (mw_1 x)^\alpha (dy)^\alpha (dx)^\alpha$$

when $n = m = 0$

$$C_{00} = \left(\frac{1}{T_1.T_2}\right)^{\alpha} \int_0^{T_1} \int_0^{T_2} f(x,y) (dy)^\alpha (dx)^\alpha$$

4.4 Applictions on Local Fractional Two-dimensional Fourier Series

Example 4.4.1 *Expand the function $k(x,t) = x^\alpha t^\alpha$ into local fractional Fourier series where$(-\pi < x^\alpha \leq \pi, -\pi < t^\alpha \leq \pi)$*

By computing the local fractional Fourier series coefficients

when $n, m \neq 0$

$$A_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^\alpha t^\alpha \cos_\alpha (mx)^\alpha \cos_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$B_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^\alpha t^\alpha \cos_\alpha (mx)^\alpha \sin_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$C_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^\alpha t^\alpha \sin_\alpha (mx)^\alpha \cos_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$D_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^\alpha t^\alpha \sin_\alpha (mx)^\alpha \sin_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha$$

$$= \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} x^{\alpha} \sin_{\alpha}(mx)^{\alpha} \left(\frac{-2\pi^{\alpha}\Gamma(1+\alpha)(-1)^{n}}{n^{\alpha}} \right) (dx)^{\alpha}$$

$$= \left(\frac{-2\Gamma(1+\alpha)(-1)^{n}}{\pi^{\alpha} n^{\alpha}} \right) \left(\frac{-2\pi^{\alpha}\Gamma(1+\alpha)(-1)^{m}}{m^{\alpha}} \right)$$

$$= \frac{4\Gamma^{2}(1+\alpha)(-1)^{n}(-1)^{m}}{n^{\alpha} m^{\alpha}}$$

when $m = 0$

$$E_n = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{\alpha} t^{\alpha} \cos_{\alpha}(nt)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

$$F_n = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{\alpha} t^{\alpha} \sin_{\alpha}(nt)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

when $n = 0$

$$G_m = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{\alpha} t^{\alpha} \cos_{\alpha}(mx)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

$$H_m = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{\alpha} t^{\alpha} \sin_{\alpha}(mx)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

when $m = n = 0$

$$A_{00} = \frac{1}{4^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{\alpha} t^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

so $K(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4\Gamma^{2}(1+\alpha)(-1)^{n}(-1)^{m}}{n^{\alpha} m^{\alpha}} \sin_{\alpha}(mx)^{\alpha} \sin_{\alpha}(nt)^{\alpha}$

Example 4.4.2 *Expand the function $k(x,t) = x^{2\alpha}t^{2\alpha}$ into local fractional Fourier series where* $(-\pi < x^{\alpha} \leq \pi, -\pi < t^{\alpha} \leq \pi)$

By computing the local fractional Fourier series coefficients

when $n, m \neq 0$

$$
\begin{aligned}
A_{mn} &= \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \cos_\alpha (mx)^\alpha \cos_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha \\
&= \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} x^{2\alpha} \cos_\alpha (mx)^\alpha \int_{-\pi}^{\pi} t^{2\alpha} \cos_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha \\
&= \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} x^{2\alpha} \cos_\alpha (mx)^\alpha \left(\frac{2\Gamma\left(1+2\alpha\right)\pi^\alpha \left(-1\right)^n}{n^{2\alpha}} \right) (dx)^\alpha \\
&= \frac{2\Gamma\left(1+2\alpha\right)\left(-1\right)^n}{\pi^\alpha n^{2\alpha}} \left(\frac{2\Gamma\left(1+2\alpha\right)\pi^\alpha \left(-1\right)^m}{m^{2\alpha}} \right) \\
&= \frac{4\Gamma^2\left(1+2\alpha\right)\left(-1\right)^m \left(-1\right)^n}{n^{2\alpha} m^{2\alpha}}
\end{aligned}
$$

$$
B_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \cos_\alpha (mx)^\alpha \sin_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0
$$

$$
C_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \sin_\alpha (mx)^\alpha \cos_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0
$$

$$
D_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \sin_\alpha (mx)^\alpha \sin_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0
$$

when $m = 0$

$$
\begin{aligned}
E_n &= \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \cos_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha \\
&= \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} x^{2\alpha} \left(\frac{2\Gamma\left(1+2\alpha\right)\pi^\alpha \left(-1\right)^n}{n^{2\alpha}} \right) (dx)^\alpha \\
&= \frac{2\Gamma\left(1+2\alpha\right)\left(-1\right)^n}{2^\alpha n^{2\alpha} \pi^\alpha} \left(\frac{\Gamma\left(1+\alpha\right)\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)} \right) \left(x^{3\alpha}\right)_{-\pi}^{\pi} \\
&= \frac{4\Gamma^2\left(1+2\alpha\right)\Gamma\left(1+\alpha\right)\left(-1\right)^n \pi^{2\alpha}}{2^\alpha n^{2\alpha} \Gamma\left(1+3\alpha\right)}
\end{aligned}
$$

$$F_n = \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \sin_\alpha (nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

when $n = 0$

$$\begin{aligned}
G_m &= \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \cos_\alpha (mx)^\alpha (dt)^\alpha (dx)^\alpha \\
&= \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} x^{2\alpha} \cos_\alpha (mx)^\alpha \left(\frac{2\pi^{3\alpha} \Gamma(1+\alpha) \Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right) (dx)^\alpha \\
&= \frac{2\pi^\alpha \Gamma(1+\alpha) \Gamma(1+2\alpha)}{2^\alpha \Gamma(1+3\alpha)} \left(\frac{2\pi^\alpha \Gamma(1+2\alpha)(-1)^m}{m^{2\alpha}} \right) \\
&= \frac{4\pi^{2\alpha} (-1)^m \Gamma(1+\alpha) \Gamma^2(1+2\alpha)}{2^\alpha \Gamma(1+3\alpha) m^{2\alpha}}
\end{aligned}$$

$$H_m = \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} \sin_\alpha (mx)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$\begin{aligned}
A_{00} &= \frac{1}{4^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{2\alpha} t^{2\alpha} (dt)^\alpha (dx)^\alpha \\
&= \frac{1}{4^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi} x^{2\alpha} \left(\frac{\Gamma(1+\alpha) \Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} t^{3\alpha} \right)_{-\pi}^{\pi} (dx)^\alpha \\
&= \frac{2\pi^\alpha \Gamma(1+\alpha) \Gamma(1+2\alpha)}{4^\alpha \Gamma(1+3\alpha)} \int_{-\pi}^{\pi} x^{2\alpha} (dx)^\alpha \\
&= \frac{2\pi^\alpha \Gamma(1+\alpha) \Gamma(1+2\alpha)}{4^\alpha \Gamma(1+3\alpha)} \left(\frac{2\pi^{3\alpha} \Gamma(1+\alpha) \Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right) \\
&= \frac{4\pi^{4\alpha} \Gamma^2(1+\alpha) \Gamma^2(1+2\alpha)}{4^\alpha \Gamma^2(1+3\alpha)}
\end{aligned}$$

the general solution is :

$$k(x,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\Gamma^2(1+2\alpha)(-1)^m (-1)^n}{n^{2\alpha} m^{2\alpha}} \cos_\alpha (mx)^\alpha \cos_\alpha (nt)^\alpha$$

$$+ \sum_{n=1}^{\infty} \frac{4\Gamma^2(1+2\alpha) \Gamma(1+\alpha)(-1)^n \pi^{2\alpha}}{2^\alpha n^{2\alpha} \Gamma(1+3\alpha)} \cos_\alpha (nt)^\alpha$$

$$+\sum_{m=1}^{\infty}\frac{4\pi^{2\alpha}\left(-1\right)^{m}\Gamma\left(1+\alpha\right)\Gamma^{2}\left(1+2\alpha\right)}{2^{\alpha}\Gamma\left(1+3\alpha\right)m^{2\alpha}}\cos_{\alpha}\left(mx\right)^{\alpha}$$

$$+\frac{4\pi^{4\alpha}\Gamma^{2}\left(1+\alpha\right)\Gamma^{2}\left(1+2\alpha\right)}{4^{\alpha}\Gamma^{2}\left(1+3\alpha\right)}$$

Example 4.4.3 *Expand the function $k(x,t) = x^{2\alpha}t^{\alpha}$ into a local fractional Fourier series where $(-\pi < x^{\alpha} \leq \pi, -\pi < t^{\alpha} \leq \pi)$*

By computing the local fractional Fourier series coefficients

$$A_{mn} = \frac{1}{\pi^{2\alpha}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha}t^{\alpha}\cos_{\alpha}(mx)^{\alpha}\cos_{\alpha}(nt)^{\alpha}(dt)^{\alpha}(dx)^{\alpha} = 0$$

$$B_{mn} = \frac{1}{\pi^{2\alpha}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha}t^{\alpha}\cos_{\alpha}(mx)^{\alpha}\sin_{\alpha}(nt)^{\alpha}(dt)^{\alpha}(dx)^{\alpha}$$

$$= \frac{1}{\pi^{2\alpha}}\int_{-\pi}^{\pi} x^{2\alpha}\cos_{\alpha}(mx)^{\alpha}\left(\frac{-2\Gamma\left(1+\alpha\right)\left(-1\right)^{n}\pi^{\alpha}}{n^{\alpha}}\right)(dx)^{\alpha}$$

$$= \frac{-2\Gamma\left(1+\alpha\right)\left(-1\right)^{n}}{n^{\alpha}\pi^{\alpha}}\left(\frac{2\pi^{\alpha}\Gamma\left(1+2\alpha\right)\left(-1\right)^{m}}{m^{2\alpha}}\right)$$

$$= \frac{-4\Gamma\left(1+\alpha\right)\Gamma\left(1+2\alpha\right)\left(-1\right)^{n}\left(-1\right)^{m}}{n^{\alpha}m^{2\alpha}}$$

$$C_{mn} = \frac{1}{\pi^{2\alpha}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha}t^{\alpha}\sin_{\alpha}(mx)^{\alpha}\cos_{\alpha}(nt)^{\alpha}(dt)^{\alpha}(dx)^{\alpha} = 0$$

$$D_{mn} = \frac{1}{\pi^{2\alpha}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha}t^{\alpha}\sin_{\alpha}(mx)^{\alpha}\sin_{\alpha}(nt)^{\alpha}(dt)^{\alpha}(dx)^{\alpha} = 0$$

when $m = 0$

$$E_{n} = \frac{1}{2^{\alpha}\pi^{2\alpha}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha}t^{\alpha}\cos_{\alpha}(nt)^{\alpha}(dt)^{\alpha}(dx)^{\alpha} = 0$$

$$F_{n} = \frac{1}{2^{\alpha}\pi^{2\alpha}}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha}t^{\alpha}\sin_{\alpha}(nt)^{\alpha}(dt)^{\alpha}(dx)^{\alpha}$$

$$= \frac{1}{2^{\alpha}\pi^{2\alpha}}\int_{-\pi}^{\pi} x^{2\alpha}\left(\frac{-2\Gamma\left(1+\alpha\right)\pi^{\alpha}\left(-1\right)^{n}}{n^{\alpha}}\right)(dx)^{\alpha}$$

$$= \frac{-2\Gamma(1+\alpha)(-1)^n}{2^\alpha n^\alpha \pi^\alpha}\left(\frac{2\pi^{3\alpha}\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)$$

$$= \frac{-4\pi^{2\alpha}\Gamma^2(1+\alpha)\Gamma(1+2\alpha)(-1)^n}{2^\alpha n^\alpha \Gamma(1+3\alpha)}$$

when $n = 0$

$$G_m = \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^\alpha \cos_\alpha(mx)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$H_m = \frac{1}{2^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^\alpha \sin_\alpha(mx)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

when $n = m = 0$

$$A_{00} = \frac{1}{4^\alpha \pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^{2\alpha} t^\alpha (dt)^\alpha (dx)^\alpha = 0$$

The local fractional Fourier series formula is :

$$\begin{aligned} k(x,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{-4\Gamma(1+\alpha)\Gamma(1+2\alpha)(-1)^n(-1)^m}{n^\alpha m^{2\alpha}} \cos_\alpha(mx)^\alpha \sin_\alpha(nt)^\alpha \\ &+ \sum_{n=1}^{\infty} \frac{-4\pi^{2\alpha}\Gamma^2(1+\alpha)\Gamma(1+2\alpha)(-1)^n}{2^\alpha n^\alpha \Gamma(1+3\alpha)} \sin_\alpha(nt)^\alpha \end{aligned}$$

Example 4.4.4 *Expand the function $k(x,t) = x^\alpha t^{2\alpha}$ into local fractional Fourier series where $(-\pi < x^\alpha \le \pi, -\pi < t^\alpha \le \pi)$*

Solution

By computing the local fractional Fourier series coefficients

when $n, m \neq 0$

$$A_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^\alpha t^{2\alpha} \cos_\alpha(mx)^\alpha \cos_\alpha(nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$B_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^\alpha t^{2\alpha} \cos_\alpha(mx)^\alpha \sin_\alpha(nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$C_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^\alpha t^{2\alpha} \sin_\alpha(mx)^\alpha \cos_\alpha(nt)^\alpha (dt)^\alpha (dx)^\alpha$$

$$= \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} x^{\alpha} \sin_{\alpha}(mx)^{\alpha} \left(\frac{2\Gamma(1+2\alpha)\pi^{\alpha}(-1)^{n}}{n^{2\alpha}} \right) (dx)^{\alpha}$$

$$= \frac{2\Gamma(1+2\alpha)(-1)^{n}}{\pi^{\alpha} n^{2\alpha}} \left(\frac{-2\Gamma(1+\alpha)\pi^{\alpha}(-1)^{m}}{m^{\alpha}} \right)$$

$$= \frac{-4\Gamma(1+2\alpha)\Gamma(1+\alpha)(-1)^{n}(-1)^{m}}{m^{\alpha} n^{2\alpha}}$$

$$D_{mn} = \frac{1}{\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{\alpha} t^{2\alpha} \sin_{\alpha}(mx)^{\alpha} \sin_{\alpha}(nt)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

when $m = 0$

$$E_{n} = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{\alpha} t^{2\alpha} \cos_{\alpha}(nt)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

$$F_{n} = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{\alpha} t^{2\alpha} \sin_{\alpha}(nt)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

when $n = 0$

$$G_{m} = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{\alpha} t^{2\alpha} \cos_{\alpha}(mx)^{\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

$$H_{m} = \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{\alpha} t^{2\alpha} \sin_{\alpha}(mx)^{\alpha} (dt)^{\alpha} (dx)^{\alpha}$$

$$= \frac{1}{2^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi} x^{\alpha} \sin_{\alpha}(mx)^{\alpha} \left(\frac{2\pi^{3\alpha}\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right) (dx)^{\alpha}$$

$$= \frac{2\pi^{\alpha}\Gamma(1+\alpha)\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+3\alpha)} \left(\frac{-2\pi^{\alpha}(-1)^{m}\Gamma(1+\alpha)}{m^{\alpha}} \right)$$

$$= \frac{-4\pi^{2\alpha}\Gamma^{2}(1+\alpha)\Gamma(1+2\alpha)(-1)^{m}}{2^{\alpha}\Gamma(1+3\alpha)m^{\alpha}}$$

$$A_{00} = \frac{1}{4^{\alpha}\pi^{2\alpha}} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^{\alpha} t^{2\alpha} (dt)^{\alpha} (dx)^{\alpha} = 0$$

so we have

$$k(x,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{-4\Gamma(1+2\alpha)\Gamma(1+\alpha)(-1)^n(-1)^m}{m^\alpha n^{2\alpha}} \sin_\alpha(mx)^\alpha \cos_\alpha(nt)^\alpha + \sum_{m=1}^{\infty} \frac{-4\pi^{2\alpha}\Gamma^2(1+\alpha)\Gamma(1+2\alpha)(-1)^m}{2^\alpha\Gamma(1+3\alpha)m^\alpha} \sin_\alpha(mx)^\alpha$$

Example 4.4.5 *Expand the function $k(x,t) = x^\alpha$ into local fractional Fourier series where* $(0 < x^\alpha \leq 1)$

By computing the local fractional Fourier series coefficients

when $n, m \neq 0$

$$A_{mn} = \int_0^1\int_0^1 x^\alpha \cos_\alpha(2\pi mx)^\alpha \cos_\alpha(2\pi nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$B_{mn} = \int_0^1\int_0^1 x^\alpha \cos_\alpha(2\pi mx)^\alpha \sin_\alpha(2\pi nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$C_{mn} = \int_0^1\int_0^1 x^\alpha \sin_\alpha(2\pi mx)^\alpha \cos_\alpha(2\pi nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$D_{mn} = \int_0^1\int_0^1 x^\alpha \sin_\alpha(2\pi mx)^\alpha \sin_\alpha(2\pi nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

when $m = 0$

$$E_n = 2^\alpha \int_0^1\int_0^1 x^\alpha \cos_\alpha(2\pi nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$F_n = 2^\alpha \int_0^1\int_0^1 x^\alpha \sin_\alpha(2\pi nt)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

when $n = 0$

$$G_m = 2^\alpha \int_0^1\int_0^1 x^\alpha \cos_\alpha(2\pi mx)^\alpha (dt)^\alpha (dx)^\alpha = 0$$

$$H_m = 2^\alpha \int_0^1\int_0^1 x^\alpha \sin_\alpha(2\pi mx)^\alpha (dt)^\alpha (dx)^\alpha$$

$$= \frac{-2^\alpha\Gamma(1+\alpha)}{(2m\pi)^\alpha} \left(x^\alpha \cos_\alpha(2m\pi x)^\alpha\right)_0^1 + \frac{2^\alpha\Gamma(1+\alpha)}{(2m\pi)^\alpha} \int_0^1 \cos_\alpha(2m\pi x)^\alpha (dx)^\alpha$$

$$= \frac{-\Gamma(1+\alpha)}{(m\pi)^{\alpha}}$$

when m = n = 0

$$A_{00} = \int_0^1 \int_0^1 \; x^{\alpha}(dt)^{\alpha}(dx)^{\alpha}$$

$$= \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}$$

so we have :

$$K(x,t) = x^{\alpha} = \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} + \sum_{m=1}^{\infty} \frac{-\Gamma(1+\alpha)}{(m\pi)^{\alpha}} \sin_{\alpha}(2m\pi x)^{\alpha}$$

Chapter 5

Solving Local Fractional Fredholm Integral Equation using Local Fractional Fourier Series

In this chapter, the solution of the first and second kind of the local fractional Fredholm integral equation will be studied using the one and two-dimensional local fractional Fourier series of the functions $u(x)$, $f(x)$ and $k(x,t)$.

5.1 Local Fractional Fredholm Integral Equation of the First Kind

In this section, we deal with the Fredholm integral equation of the first kind (1.2), where $f(x)$ is a given function, and the kernel $k(x,t)$ is a function of two variables, $u(x)$ is the unknown function and λ^α is a parameter.

The integral equation (1.2), will be solved using local fractional Fourier series. From the paragraph that followed definition 4.1.4, we assume that

$$f(x) = f_0 + \sum_{i=1}^{m} f_i^* \sin_\alpha\left(\frac{i\pi x}{l}\right)^\alpha + \sum_{i=1}^{m} f_i \cos_\alpha\left(\frac{i\pi x}{l}\right)^\alpha \tag{5.1.1}$$

$$\begin{aligned} k(x,t) \;&=\; A_{00} + \sum_{i,j=1}^{m,n}\left(A_{ij}\cos_\alpha\left(\frac{i\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{j\pi t}{h}\right)^\alpha + B_{ij}\cos_\alpha\left(\frac{i\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{j\pi t}{h}\right)^\alpha\right) \\ &+\; \sum_{i,j=1}^{m,n}\left(C_{ij}\sin_\alpha\left(\frac{i\pi x}{l}\right)^\alpha \cos_\alpha\left(\frac{j\pi t}{h}\right)^\alpha + D_{ij}\sin_\alpha\left(\frac{i\pi x}{l}\right)^\alpha \sin_\alpha\left(\frac{j\pi t}{h}\right)^\alpha\right) \\ &+\; \sum_{i=1}^{n}\left(E_i\cos_\alpha\left(\frac{i\pi t}{h}\right)^\alpha + F_i\sin_\alpha\left(\frac{i\pi t}{h}\right)^\alpha\right) \\ &+\; \sum_{i=1}^{m}\left(G_i\cos_\alpha\left(\frac{i\pi x}{l}\right)^\alpha + H_i\sin_\alpha\left(\frac{i\pi x}{l}\right)^\alpha\right) \end{aligned} \tag{5.1.2}$$

$$u(t) = u_0 + \sum_{i=1}^{n} u_i^* \sin_\alpha\left(\frac{i\pi t}{h}\right)^\alpha + \sum_{i=1}^{n} u_i \cos_\alpha\left(\frac{i\pi t}{h}\right)^\alpha \tag{5.1.3}$$

Now, insert equations (5.1.1), (5.1.2) and (5.1.3) into the integral equation, and after integrating each term we get :

$$\int_{-h}^{h} k(x,t)u(t)(dt)^{\alpha} = \frac{\lambda^{\alpha}(2h)^{\alpha}}{\Gamma(1+\alpha)}(A_{00}u_0 + \frac{E_1u_1}{2} + \ldots + \frac{E_nu_n}{2} + \frac{F_1u_1^*}{2} + \ldots + \frac{F_nu_n^*}{2})$$

$$+\frac{\lambda^{\alpha}(2h)^{\alpha}}{\Gamma(1+\alpha)}[\sin_{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}(\frac{C_{11}u_1}{2} + \ldots + \frac{C_{1n}u_n}{2} + \frac{D_{11}u_1^*}{2} + \ldots + \frac{D_{1n}u_n^*}{2} + H_1u_0)$$

$$+\sin_{\alpha}\left(\frac{2\pi x}{l}\right)^{\alpha}(\frac{C_{21}u_1}{2} + \ldots + \frac{C_{2n}u_n}{2} + \frac{D_{21}u_1^*}{2} + \ldots + \frac{D_{2n}u_n^*}{2} + H_2u_0)$$

$$\vdots$$

$$+\sin_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}(\frac{C_{m1}u_1}{2} + \ldots + \frac{C_{mn}u_n}{2} + \frac{D_{m1}u_1^*}{2} + \ldots + \frac{D_{mn}u_n^*}{2} + H_mu_0)]$$

$$+\frac{\lambda^{\alpha}(2h)^{\alpha}}{\Gamma(1+\alpha)}[\cos_{\alpha}\left(\frac{\pi x}{l}\right)^{\alpha}(\frac{A_{11}u_1}{2} + \ldots + \frac{A_{1n}u_n}{2} + \frac{B_{11}u_1^*}{2} + \ldots + \frac{B_{1n}u_n^*}{2} + G_1u_0)$$

$$+\cos_{\alpha}\left(\frac{2\pi x}{l}\right)^{\alpha}(\frac{A_{21}u_1}{2} + \ldots + \frac{A_{2n}u_n}{2} + \frac{B_{21}u_1^*}{2} + \ldots + \frac{B_{2n}u_n^*}{2} + G_2u_0)$$

$$\vdots$$

$$+\cos_{\alpha}\left(\frac{m\pi x}{l}\right)^{\alpha}(\frac{A_{m1}u_1}{2} + \ldots + \frac{A_{mn}u_n}{2} + \frac{B_{m1}u_1^*}{2} + \ldots + \frac{B_{mn}u_n^*}{2} + G_mu_0)$$

Equate the coefficients of both sides in the integral equation to get the following system:

$$F_{((2m+1)\times 1)} = K_{((2m+1)\times(2n+1))}U_{((2n+1)\times 1)} \tag{5.1.4}$$

where $F = \begin{bmatrix} f_0 & f_1 & \ldots & f_m & f_1^* & \ldots & f_m^* \end{bmatrix}^T$, $U = \begin{bmatrix} u_0 & u_1 & \ldots & u_m & u_1^* & \ldots & u_m^* \end{bmatrix}^T$

$$K = \frac{(2h\lambda)^{\alpha}}{2\Gamma(1+\alpha)}\begin{bmatrix} 2A_{00} & E_1 & E_2 & \ldots & E_n & F_1 & F_2 & \ldots & F_n \\ 2G_1 & A_{11} & A_{12} & \ldots & A_{1n} & B_{11} & B_{12} & \ldots & B_{1n} \\ 2G_2 & A_{21} & A_{22} & \ldots & A_{2n} & B_{21} & B_{22} & \ldots & B_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2G_m & A_{m1} & A_{m2} & \ldots & A_{mn} & B_{m1} & B_{m2} & \ldots & B_{mn} \\ 2H_1 & C_{11} & C_{12} & \ldots & C_{1n} & D_{11} & D_{12} & \ldots & D_{1n} \\ 2H_2 & C_{21} & C_{22} & \ldots & C_{2n} & D_{21} & D_{22} & \ldots & D_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2H_m & C_{m1} & C_{m2} & \ldots & C_{mn} & D_{m1} & D_{m2} & \ldots & D_{mn} \end{bmatrix}$$

As an application on solving local fractional Fredholm integral equation of the First Kind, we present the following examples.

Example 5.1.1 *Consider the following local fractional Fredholm integral equation*

$$\Gamma(1+\alpha) = \frac{1}{\Gamma(1+\alpha)}\int_0^1 u(t)(dt)^\alpha$$
$$= \frac{1}{\Gamma(1+\alpha)}\int_0^1 (\sum_{j=1}^{n} u_j^* \sin_\alpha(2j\pi t)^\alpha + \sum_{j=0}^{n} u_j \cos_\alpha(2j\pi t)^\alpha)(dt)^\alpha$$

so we have an infinite number of solutions such that $u_0 = \Gamma^2(1+\alpha)$

Hence,

$$u(x) = \Gamma^2(1+\alpha) + \sum_{k=1}^{\infty} u_k^* \sin_\alpha(2\pi kx)^\alpha + \sum_{k=1}^{\infty} u_k \cos_\alpha(2\pi kx)^\alpha$$

Example 5.1.2 *Consider the following local fractional Fredholm integral equation*

$$\sin_\alpha(x)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\pi}^{\pi} x^\alpha t^\alpha u(t)(dt)^\alpha$$

from example (4.4.1) :

$$K(x,t) = x^\alpha t^\alpha = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4\Gamma^2(1+\alpha)(-1)^m(-1)^n}{m^\alpha n^\alpha} \sin_\alpha(mx)^\alpha \sin_\alpha(nt)^\alpha$$

for $n, m = 2$

$$\sin_\alpha(x)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{-\pi}^{\pi} (\sum_{i,j=1}^{2} k_{ij}\sin_\alpha(ix)^\alpha \sin_\alpha(jt)^\alpha)(\sum_{i=1}^{2} u_i^* \sin_\alpha(it)^\alpha + \sum_{i=0}^{2} u_i \cos_\alpha(it)^\alpha)(dt)^\alpha$$

$$\sin_\alpha(x)^\alpha = \frac{(2\pi)^\alpha}{\Gamma(1+\alpha)}\left(\sin_\alpha x^\alpha\left(\frac{k_{11}u_1^*}{2} + \frac{k_{12}u_2^*}{2}\right) + \sin_\alpha(2x)^\alpha\left(\frac{k_{21}u_1^*}{2} + \frac{k_{22}u_2^*}{2}\right)\right)$$

where

$k_{11} = 4\Gamma^2(1+\alpha)$, $k_{12} = \dfrac{-4\Gamma^2(1+\alpha)}{2^\alpha}$, $k_{21} = \dfrac{-4\Gamma^2(1+\alpha)}{2^\alpha}$, $k_{22} = \dfrac{4\Gamma^2(1+\alpha)}{2^{2\alpha}}$

Thus, we have the following linear system:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{(2\pi)^\alpha}{2\Gamma(1+\alpha)} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\Gamma^2(1+\alpha) & \frac{-4\Gamma^2(1+\alpha)}{2^\alpha} \\ 0 & 0 & 0 & \frac{-4\Gamma^2(1+\alpha)}{2^\alpha} & \frac{4\Gamma^2(1+\alpha)}{2^{2\alpha}} \end{bmatrix} \times \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_1^* \\ u_2^* \end{bmatrix}$$

since the determinant of the matrix K is 0, the problem has no solution or infinitely many solutions.

From the above system we have the following two equations

$$\begin{aligned}
1 &= \frac{(2\pi)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{k_{11}u_1^*}{2}+\frac{k_{12}u_2^*}{2}\right)\\
0 &= \frac{(2\pi)^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{k_{21}u_1^*}{2}+\frac{k_{22}u_2^*}{2}\right)
\end{aligned}$$

Simple calculations on the above system will lead to the contradiction as follows:

$$\begin{aligned}
1 &= \frac{(2\pi)^{\alpha}4\Gamma^2(1+\alpha)}{2\Gamma(1+\alpha)}u_1^*-\frac{(2\pi)^{\alpha}4\Gamma^2(1+\alpha)}{2.2^{\alpha}\Gamma(1+\alpha)}u_2^*\\
1 &= 2(2\pi)^{\alpha}\Gamma(1+\alpha)u_1^*-\frac{2(2\pi)^{\alpha}\Gamma(1+\alpha)}{2^{\alpha}}u_2^*\\
0 &= \frac{-2(2\pi)^{\alpha}\Gamma(1+\alpha)}{2^{\alpha}}u_1^*+\frac{2(2\pi)^{\alpha}\Gamma(1+\alpha)}{2^{2\alpha}}u_2^*\\
1 &= 0.
\end{aligned}$$

So, the problem has no solution .

Example 5.1.3 *Consider the following local fractional Fredholm integral equation*

$$e^{2x}=\frac{1}{\Gamma(1+\alpha)}\int_{-\pi}^{\pi}\left(x^{\alpha}+x^{2\alpha}\right)\left(t^{\alpha}+t^{2\alpha}\right)u(t)\,(dt)^{\alpha}$$

First we write the kernel as follows:
$k(x,t)=(x^{\alpha}+x^{2\alpha})(t^{\alpha}+t^{2\alpha})=(xt)^{\alpha}+x^{\alpha}t^{2\alpha}+x^{2\alpha}t^{\alpha}+(xt)^{2\alpha}$. So, from examples (4.4.1), (4.4.2), (4.4.3), (4.4.4) and (4.2.5) the local fractional Fourier series of $k(x,t)$ is

$$\begin{aligned}
k(x,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}D_{mn}\sin_{\alpha}(mx)^{\alpha}\sin_{\alpha}(nt)^{\alpha}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos_{\alpha}(mx)^{\alpha}\cos_{\alpha}(nt)^{\alpha}\\
&+\sum_{n=1}^{\infty}E_n\cos_{\alpha}(nt)^{\alpha}+\sum_{m=1}^{\infty}G_m\cos_{\alpha}(mx)^{\alpha}+A_{00}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}C_{mn}\sin_{\alpha}(mx)^{\alpha}\cos_{\alpha}(nt)^{\alpha}\\
&+\sum_{m=1}^{\infty}H_m\sin_{\alpha}(mx)^{\alpha}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\cos_{\alpha}(mx)^{\alpha}\sin_{\alpha}(nt)^{\alpha}+\sum_{n=1}^{\infty}F_n\sin_{\alpha}(nt)^{\alpha}.
\end{aligned}$$

Where $D_{mn}=\dfrac{4\Gamma^2(1+\alpha)(-1)^{n+m}}{(nm)^{\alpha}}$, $A_{mn}=\dfrac{4\Gamma^2(1+2\alpha)(-1)^{n+m}}{(nm)^{2\alpha}}$,
$E_n=\dfrac{4\Gamma^2(1+2\alpha)\Gamma(1+\alpha)(-1)^n\pi^{2\alpha}}{2^{\alpha}n^{2\alpha}\Gamma(1+3\alpha)}$, $G_m=\dfrac{4\pi^{2\alpha}(-1)^m\Gamma(1+\alpha)\Gamma^2(1+2\alpha)}{2^{\alpha}\Gamma(1+3\alpha)m^{2\alpha}}$,

$$A_{00}=\frac{4\pi^{4\alpha}\Gamma^2(1+\alpha)\Gamma^2(1+2\alpha)}{4^{\alpha}\Gamma^2(1+3\alpha)},\ C_{mn}=\frac{-4\Gamma(1+2\alpha)\Gamma(1+\alpha)(-1)^{n+m}}{m^{\alpha}n^{2\alpha}},$$
$$H_m=\frac{-4\pi^{2\alpha}\Gamma^2(1+\alpha)\Gamma(1+2\alpha)(-1)^{m}}{2^{\alpha}\Gamma(1+3\alpha)m^{\alpha}},\ B_{mn}=\frac{-4\Gamma(1+\alpha)\Gamma(1+2\alpha)(-1)^{n+m}}{n^{\alpha}m^{2\alpha}},$$
and $F_n=\frac{-4\pi^{2\alpha}\Gamma^2(1+\alpha)\Gamma(1+2\alpha)(-1)^{n}}{2^{\alpha}n^{\alpha}\Gamma(1+3\alpha)}$.

Also, the local fractional Fourier series of the exponential function e^{2x} is

$$\frac{\Gamma(1+\alpha)(e^{2\pi}-e^{-2\pi})}{2\pi^{\alpha}2^{\alpha}}+\sum_{k=1}^{\infty}\frac{2^{\alpha}\pi^{\alpha}\Gamma(1+\alpha)(-1)^k(e^{2\pi}-e^{-2\pi})}{(k\pi)^{2\alpha}+2^{2\alpha}\pi^{2\alpha}}\cos_{\alpha}\left(\frac{k\pi t}{\pi}\right)^{\alpha}$$
$$+\sum_{k=1}^{\infty}\frac{-\Gamma(1+\alpha)(-1)^k(e^{2\pi}-e^{-2\pi})(k\pi)^{\alpha}}{(k\pi)^{2\alpha}+(2\pi)^{2\alpha}}\sin_{\alpha}\left(\frac{k\pi t}{\pi}\right)^{\alpha}$$

Substituting the local fractional Fourier series of the functions $f(x), K(x,t)$ and $u(x)$ into the integral equation and integrate term by term to reach system(5.1.4).
Below, we consider different sizes of the coeffiicients matrix and see whether system (5.1.4) has a solution or not.
Using Matlab, different sizes of the coefficients matrices are calculated and the results have been studied .The below results show that the system has no solution. When $m=2$ and $n=2$, the augmented matrix

$$\begin{bmatrix} 17.560091 & -5.926228 & 2.963114 & 5.251983 & -3.713713 & 94.882725 \\ -11.852457 & 4.000000 & -2.000000 & -3.544908 & 1.772454 & -126.510300 \\ 5.926228 & -2.000000 & 1.000000 & 2.506628 & -1.253314 & 94.882725 \\ 10.503966 & -3.544908 & 1.772454 & 3.141593 & -2.221441 & 89.456291 \\ -7.427426 & 2.506628 & -1.253314 & -2.221441 & 1.570796 & -94.882725 \\ -7.427426 & 2.506628 & -1.253314 & -2.221441 & 1.570796 & -94.882725 \end{bmatrix}$$

and the reduced row echelon matrix is $\begin{bmatrix} 1 & -0.337483 & 0.168741 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

When $n=2$, $m=3$ the augmented matrix is

$$\begin{bmatrix} 17.560091 & -5.926228 & 2.963114 & 5.251983 & -3.713713 & 94.882725 \\ -11.852457 & 4.000000 & -2.000000 & -3.544908 & 1.772454 & -126.510300 \\ 5.926228 & -2.000000 & 1.000000 & 2.506628 & -1.253314 & 94.882725 \\ -3.950819 & 1.333333 & -0.666667 & -2.046653 & 1.023327 & -75.906180 \\ 10.503966 & -3.544908 & 1.772454 & 3.141593 & -2.221441 & 89.456291 \\ -7.427426 & 2.506628 & -1.253314 & -2.221441 & 1.570796 & -94.882725 \\ 6.064468 & -2.046653 & 1.023327 & 1.813799 & -1.282550 & 92.965705 \end{bmatrix}$$

and the reduced row echelon matrix is

$$\begin{bmatrix} 1.000000 & -0.337483 & 0.168741 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 1.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 & 0.000000 \end{bmatrix}$$

When $n = 3$, $m = 2$, the reduced row echelon matrix is

$$\begin{bmatrix} 1 & -0.337483 & 0.168741 & -0.112494 & 0 & 00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -0.255776 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.178218 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

When $n = m = 3$, the reduced row echelon matrix of the augmented matrix is

$$\begin{bmatrix} 1 & -0.337483 & 0.168741 & -0.112494 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -0.255776 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.178218 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we conclude that the integral equation has no solution.

5.2 Local Fractional Fredholm Integral Equation of the Second Kind

In this section, the local fractional Fredholm integral equation of the second kind 1.3 which is introduced in [12] will be solved using local fractional Fourier series.

Similar to the case of the first kind local fractional Fredholm integral equation, we substitute the local fractional Fourier series given by equations(4.1.12) and (4.3.2) into the integral equation (1.3), and after integrating each term in the resultant integral equation, we produce the following terms:

$$u_0+u_1^* \sin_\alpha\left(\frac{\pi x}{l}\right)^\alpha+u_2^* \sin_\alpha\left(\frac{2\pi x}{l}\right)^\alpha+\ldots+u_n^* \sin_\alpha\left(\frac{n\pi x}{l}\right)^\alpha+u_1 \cos_\alpha\left(\frac{\pi x}{l}\right)^\alpha+u_2 \cos_\alpha\left(\frac{2\pi x}{l}\right)^\alpha+$$
$$\ldots+u_n \cos_\alpha\left(\frac{n\pi x}{l}\right)^\alpha=$$

$f_0+f_1^* \sin_\alpha\left(\frac{\pi x}{l}\right)^\alpha+f_2^* \sin_\alpha\left(\frac{2\pi x}{l}\right)^\alpha+\ldots+f_m^* \sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha+f_1 \cos_\alpha\left(\frac{\pi x}{l}\right)^\alpha+f_2 \cos_\alpha\left(\frac{2\pi x}{l}\right)^\alpha+ \ldots + f_m \cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha$

$+\frac{\lambda^\alpha (2h)^\alpha}{\Gamma(1+\alpha)}(\cos_\alpha\left(\frac{\pi x}{l}\right)^\alpha(\frac{A_{11}u_1}{2}+\frac{A_{12}u_2}{2}+\ldots+\frac{A_{1n}u_n}{2}+\frac{B_{11}u_1^*}{2}+\frac{B_{12}u_2^*}{2}+\ldots+\frac{B_{1n}u_n^*}{2}+ G_1u_0)$

$+\cos_\alpha\left(\frac{2\pi x}{l}\right)^\alpha(\frac{A_{21}u_1}{2}+\frac{A_{22}u_2}{2}+\ldots+\frac{A_{2n}u_n}{2}+\frac{B_{21}u_1^*}{2}+\frac{B_{22}u_2^*}{2}+\ldots+\frac{B_{2n}u_n^*}{2}+G_2u_0).$

$\vdots$

$+\cos_\alpha\left(\frac{m\pi x}{l}\right)^\alpha(\frac{A_{m1}u_1}{2}+\frac{A_{m2}u_2}{2}+\ldots+\frac{A_{mn}u_n}{2}+\frac{B_{m1}u_1^*}{2}+\frac{B_{m2}u_2^*}{2}+\ldots+\frac{B_{mn}u_n^*}{2}+G_mu_0)$

$+\sin_\alpha\left(\frac{\pi x}{l}\right)^\alpha(\frac{C_{11}u_1}{2}+\frac{C_{12}u_2}{2}+\ldots+\frac{C_{1n}u_n}{2}+\frac{D_{11}u_1^*}{2}+\frac{D_{12}u_2^*}{2}+\ldots+\frac{D_{1n}u_n^*}{2}+H_1u_0)$

$+\sin_\alpha\left(\frac{2\pi x}{l}\right)^\alpha(\frac{C_{21}u_1}{2}+\frac{C_{22}u_2}{2}+......+\frac{C_{2n}u_n}{2}+\frac{D_{21}u_1^*}{2}+\frac{D_{22}u_2^*}{2}+\ldots\frac{D_{2n}u_n^*}{2}+H_2u_0).$

$\vdots$

$+\sin_\alpha\left(\frac{m\pi x}{l}\right)^\alpha(\frac{C_{m1}u_1}{2}+\frac{C_{m2}u_2}{2}+\cdots+\frac{C_{mn}u_n}{2}+\frac{D_{m1}u_1^*}{2}+\frac{D_{m2}u_2^*}{2}+\cdots+\frac{D_{mn}u_n^*}{2}+H_mu_0)$

$+(A_{00}u_0+\frac{E_1u_1}{2}+\frac{E_2u_2}{2}+\ldots+\frac{E_nu_n}{2}+\frac{F_1u_1^*}{2}+\frac{F_2u_2^*}{2}+\ldots+\frac{F_nu_n^*}{2})).$

Below we present three examples on local fractional Fredholm integral equation of the second kind.

Example 5.2.1 *Consider the following local fractional Fredholm integral equation*

$$u(x)=\Gamma(1+\alpha)+\frac{1}{\Gamma(1+\alpha)}\int_0^1 x^\alpha u(t)\,(dt)^\alpha$$

The local fractional fourier series formula of x^α is :

$$x^\alpha=\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}-\sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha)}{(k\pi)^\alpha}\sin_\alpha(2\pi kx)^\alpha$$

let $u(x)=u_0+\sum_{k=1}^{\infty}u_k^*\sin_\alpha(2\pi kx)^\alpha+\sum_{k=1}^{\infty}u_k\cos_\alpha(2\pi kx)^\alpha$

By substituting these series in the integral equation

$$u_0+\sum_{k=1}^{\infty}u_k^*\sin_\alpha(2\pi kx)^\alpha+\sum_{k=1}^{\infty}u_k\cos_\alpha(2\pi kx)^\alpha=$$

$$\Gamma(1+\alpha)+\frac{1}{\Gamma(1+\alpha)}\left(\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}-\sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha)}{(k\pi)^\alpha}\sin_\alpha(2\pi kx)^\alpha\right)$$

$$\times\int_0^1\left(u_0+\sum_{k=1}^{\infty}u_k^*\sin_\alpha(2\pi kt)^\alpha+\sum_{k=1}^{\infty}u_k\cos_\alpha(2\pi kt)^\alpha\right)(dt)^\alpha$$

$$=\Gamma(1+\alpha)+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\sum_{k=1}^{\infty}\frac{1}{(k\pi)^\alpha}\sin_\alpha(2\pi kx)^\alpha\right)(u_0)$$

$$=\left(\Gamma(1+\alpha)+\frac{u_0\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)-\sum_{k=1}^{\infty}\frac{u_0}{(k\pi)^\alpha}\sin_\alpha(2\pi kx)^\alpha$$

By equating the coefficients, we have

$$u_0=\Gamma(1+\alpha)+\frac{u_0\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$$

$$u_0\left(1-\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)=u_0\left(\frac{\Gamma(1+2\alpha)-\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)=\Gamma(1+\alpha)$$

$$u_0=\frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)-\Gamma(1+\alpha)}$$

$$\frac{1}{\pi^\alpha}u_0+u_1^*=0$$

$$u_1^*=\frac{-u_0}{\pi^\alpha}$$

$$\frac{1}{(2\pi)^\alpha}u_0+u_2^*=0$$

$$u_2^*=\frac{-u_0}{(2\pi)^\alpha}$$

$$u_k^*=\frac{-u_0}{(k\pi)^\alpha}$$

$$u_k^*=\frac{-\Gamma(1+\alpha)\Gamma(1+2\alpha)}{(k\pi)^\alpha(\Gamma(1+2\alpha)-\Gamma(1+\alpha))}$$

$$u_1=0$$

$$u_2=0$$

$$u_k=0$$

$$\text{Hence , } u(x) = \frac{\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)-\Gamma(1+\alpha)} + \sum_{k=1}^{\infty} \frac{-\Gamma(1+\alpha)\Gamma(1+2\alpha)}{(k\pi)^{\alpha}(\Gamma(1+2\alpha)-\Gamma(1+\alpha))} \sin_{\alpha}(2\pi kx)^{\alpha}$$

$$u(x) = \Gamma(1+\alpha) + \frac{x^{\alpha}\Gamma(1+2\alpha)}{\Gamma(1+2\alpha)-\Gamma(1+\alpha)}$$

which is the same answer as in [25] example 1 where the problem was solved using the Neumann series method .

Example 5.2.2 *Consider the following local fractional Fredholm integral equation :*

$$u(x) = \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} x^{\alpha} t^{\alpha} u(t) (dt)^{\alpha}$$

The local fractional Fourier series formula of the kernel is :

$$k(x,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_{mn} \sin_{\alpha}(mx)^{\alpha} \sin_{\alpha}(nt)^{\alpha} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{4\Gamma^{2}(1+\alpha)(-1)^{m}(-1)^{n}}{m^{\alpha} n^{\alpha}} \sin_{\alpha}(mx)^{\alpha} \sin_{\alpha}(nt)^{\alpha}$$

let $u(x) = u_0 + \sum_{k=1}^{\infty} u_k^{*} \sin_{\alpha}(kx)^{\alpha} + \sum_{k=1}^{\infty} u_k \cos(kx)^{\alpha}$

By substituting these series in the integral equation for n = 0, 1,2

$$u_0 + u_1^{*} \sin_{\alpha}(x)^{\alpha} + u_2^{*} \sin_{\alpha}(2x)^{\alpha} + u_1 \cos_{\alpha}(x)^{\alpha} + u_2 \cos_{\alpha}(2x)^{\alpha}$$

$$= \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_{-\pi}^{\pi} ((k_{11} \sin_{\alpha}(x)^{\alpha} \sin_{\alpha}(t)^{\alpha} + k_{12} \sin_{\alpha}(x)^{\alpha} \sin_{\alpha}(2t)^{\alpha} + k_{21} \sin_{\alpha}(2x)^{\alpha} \sin_{\alpha}(t)^{\alpha} + k_{22} \sin_{\alpha}(2x)^{\alpha} \sin_{\alpha}(2t)^{\alpha})(u_0 + u_1^{*} \sin_{\alpha}(t)^{\alpha} + u_2^{*} \sin_{\alpha}(2t)^{\alpha} + u_1 \cos_{\alpha}(t)^{\alpha} + u_2 \cos_{\alpha}(2t)^{\alpha})) (dt)^{\alpha}$$

$$= \Gamma(1+\alpha) + \frac{1}{\Gamma(1+\alpha)} \left(\frac{k_{11} u_1^{*}}{2} + \frac{k_{12} u_2^{*}}{2} \right) \sin_{\alpha}(x)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \left(\frac{k_{21} u_1^{*}}{2} + \frac{k_{22} u_2^{*}}{2} \right) \sin_{\alpha}(2x)^{\alpha}$$

By equating the coefficients, we have

$u_0 = \Gamma(1+\alpha)$

$u_1 = 0$

$u_2 = 0$

$u_k = 0 \ , \forall k = 1, 2, 3, 4, ...$

$$u_1^{*} = \frac{1}{\Gamma(1+\alpha)} \left(\frac{k_{11} u_1^{*}}{2} + \frac{k_{12} u_2^{*}}{2} \right)$$

$$u_1^* = \left(\frac{k_{12}}{2\Gamma(1+\alpha) - k_{11}} \right) u_2^*$$

$$u_2^* = \frac{1}{\Gamma(1+\alpha)} \left(\frac{k_{21} u_1^*}{2} + \frac{k_{22} u_2^*}{2} \right)$$

$$= \frac{k_{21}}{2\Gamma(1+\alpha)} \left(\frac{k_{12}}{2\Gamma(1+\alpha) - k_{11}} \right) u_2^* + \frac{k_{22} u_2^*}{2\Gamma(1+\alpha)}$$

$$= \left(\frac{k_{21} k_{12}}{2\Gamma(1+\alpha)(2\Gamma(1+\alpha) - k_{11})} + \frac{k_{22}}{2\Gamma(1+\alpha)} \right) u_2^* = A u_2^*$$

$$u_2^* (1 - A) = 0$$

$$u_2^* = 0$$

$$u_1^* = 0$$

so , $u(x) = \Gamma(1+\alpha)$

Remark 5.2.1 $1 - A \neq 0$

$$k_{11} = 4\Gamma^2(1+\alpha) \quad , \quad k_{12} = \frac{-4\Gamma^2(1+\alpha)}{2^\alpha}$$

$$k_{21} = \frac{-4\Gamma^2(1+\alpha)}{2^\alpha} \quad , \quad k_{22} = \frac{4\Gamma^2(1+\alpha)}{2^{2\alpha}}$$

$$A = \frac{\left(\frac{-4\Gamma^2(1+\alpha)}{2^\alpha} \right) \left(\frac{-4\Gamma^2(1+\alpha)}{2^\alpha} \right)}{2\Gamma(1+\alpha)(2\Gamma(1+\alpha) - 4\Gamma^2(1+\alpha))} + \frac{4\Gamma^2(1+\alpha)}{2^{2\alpha}.2.\Gamma(1+\alpha)}$$

$\neq 1$

Example 5.2.3 *Solve the following local fractional Fredholm integral equation*

$$u(x) = 1 + x^\alpha + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \sin_\alpha (2\pi x + 2\pi t)^\alpha \, u(t) \, (dt)^\alpha$$

let $u(x) = u_0 + u_1^* \sin_\alpha (2\pi x)^\alpha + u_2^* \sin_\alpha (4\pi x)^\alpha + u_1 \cos_\alpha (2\pi x)^\alpha + u_2 \cos_\alpha (4\pi x)^\alpha$

$$x^{\alpha} = \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} - \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{(k\pi)^{\alpha}} \sin_{\alpha}(2\pi kx)^{\alpha}$$

By substituting the local fractional Fourier series formula, we have :

$$u_0 + u_1^* \sin_{\alpha}(2\pi x)^{\alpha} + u_2^* \sin_{\alpha}(4\pi x)^{\alpha} + ... + u_1 \cos_{\alpha}(2\pi x)^{\alpha} + u_2 \cos_{\alpha}(4\pi x)^{\alpha} + ... =$$

$$1 + \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} - \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{(k\pi)^{\alpha}} \sin_{\alpha}(2\pi kx)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 (\sin_{\alpha}(2\pi x)^{\alpha} \cos_{\alpha}(2\pi t)^{\alpha} + \cos_{\alpha}(2\pi x)^{\alpha} \sin_{\alpha}(2\pi t)^{\alpha}) \times (u_0 + u_1^* \sin_{\alpha}(2\pi t)^{\alpha} + u_2^* \sin_{\alpha}(4\pi t)^{\alpha} + ... + u_1 \cos_{\alpha}(2\pi t)^{\alpha} + u_2 \cos_{\alpha}(4\pi t)^{\alpha}) + ...)(dt)^{\alpha}$$

$$= 1 + \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+\alpha)}{\pi^{\alpha}} \sin_{\alpha}(2\pi x)^{\alpha} - \frac{\Gamma(1+\alpha)}{(2\pi)^{\alpha}} \sin_{\alpha}(4\pi x)^{\alpha} - \sum_{k=3}^{\infty} \frac{\Gamma(1+\alpha)}{(k\pi)^{\alpha}} \sin_{\alpha}(2k\pi x)^{\alpha}$$

$$+ \frac{u_1}{2\Gamma(1+\alpha)} \sin_{\alpha}(2\pi x)^{\alpha} + \frac{u_1^*}{2\Gamma(1+\alpha)} \cos_{\alpha}(2\pi x)^{\alpha}$$

By equating the coefficients, we have

$$u_0 = 1 + \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}$$

$$u_1^* = \frac{-\Gamma(1+\alpha)}{\pi^{\alpha}} + \frac{u_1}{2\Gamma(1+\alpha)}$$

$$u_k^* = \frac{-\Gamma(1+\alpha)}{(k\pi)^{\alpha}} \quad \forall k > 1$$

$$u_1 = \frac{u_1^*}{2\Gamma(1+\alpha)} = \frac{\frac{-\Gamma(1+\alpha)}{\pi^{\alpha}} + \frac{u_1}{2\Gamma(1+\alpha)}}{2\Gamma(1+\alpha)}$$

$$= \frac{-1}{2\pi^{\alpha}} + \frac{u_1}{4\Gamma^2(1+\alpha)}$$

$$u_1 = \frac{-4\Gamma^2(1+\alpha)}{2\pi^{\alpha}(4\Gamma^2(1+\alpha)-1)}$$

$$u_1 = \frac{u_1^*}{2\Gamma(1+\alpha)} \text{ and } u_k = 0 \quad \forall k > 1$$

$$u_1^* = 2\Gamma(1+\alpha)\left(\frac{-4\Gamma^2(1+\alpha)}{2\pi^{\alpha}(4\Gamma^2(1+\alpha)-1)}\right) = \frac{-4\Gamma^3(1+\alpha)}{\pi^{\alpha}(4\Gamma^2(1+\alpha)-1)}$$

$$u(x) = 1 + \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{4\Gamma^3(1+\alpha)}{\pi^\alpha\left(4\Gamma^2(1+\alpha)-1\right)}\sin_\alpha(2\pi x)^\alpha - \sum_{k=2}^{\infty}\frac{\Gamma(1+\alpha)}{(k\pi)^\alpha}\sin_\alpha(2\pi k x)^\alpha$$

$$-\frac{4\Gamma^2(1+\alpha)}{2\pi^\alpha\left(4\Gamma^2(1+\alpha)-1\right)}\cos_\alpha(2\pi x)^\alpha$$

Conclusion

In the thesis we study the local fractional Fourier series of one dimension and give new examples, also we study the local fractional Fourier series of two dimension with examples. Then the local fractional Fourier series of one and two-dimensional are used to solve the local fractional Fredholm integral equation of the first and second kind.

Although the complexity in calculations, the method has shown its effectiveness in solving local fractional Fredholm integral equations. The simulation method in section 5.1 can be modified to be valid for the second kind integral equations in which extra calculations are needed to find the coefficients of local fractional Fourier series of the unknown function. If time permits and for future work, using Matlab software, we will design a solver program for any local fractional Fredholm integral equation.

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