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Rank-One Perturbations of Matrices

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Dedication

This thesis is proudly dedicated to my beloved small family. My husband, Jalal Khalil, you taught me that there is always a light shining in the skyline. Thank you for your endless love, support and encouragement. My little daughter, Salma, I am sorry for the time I left you alone. But, you were my inspiration necessary to complete this work. I am really grateful for having you both in my life. I love you!

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I would like to appreciate my supervisor, Dr. Iyad Abu-Jeib, for his expert guidance, support and patience.

I am highly indebted to my dear family: my parents, my sisters and my brothers. Thanks for their great love, help and encouragement.

My deep gratitude also goes for my husband and my daughter.

Abstract

We study rank-one perturbations of centrosymmetric matrices and we discuss three applications of rank-one perturbations of matrices. We also prove new results about the eigen structure of rank-one perturbations of 2 x 2 centrosymmetric matrices and the structure of the eigenpair of rank-one perturbations of Laplacian matrices.

ملخص

التعديل من الدرجة الأولى للمصفوفات

تتناولُ هذه الرسالة دراسةَ التعديلات من الدرجة الأولى للمصفوفات مركزية التماثل و كذلك عرض لثلاثة تطبيقات للتعديلات من الدرجة الأولى للمصفوفات في مجالات مختلفة. كما تتضمن برهنة نتائج جديدة حول القيم الذاتية و المتجهات الذاتية لتعديلات من الدرجة الأولى للمصفوفات مركزية التماثل من الرتبة ٢ x ٢، و أيضاً تشمل وصف للقيم والمتجهات الذاتية لمصفوفة لابلاس، و المصفوفات المعدلة منها.

Contents

Chapter 1
Introduction

It is rarely true that the determinant of a sum of matrices is equal to the sum of determinants. This is one of the facts that perturbation theory of matrices focuses on. So, what is perturbation theory? It is a set of mathematical procedures used to find the solution to a problem using the solution of a simpler related one (see [**7**]). The idea is to break the problem into two parts: solvable and perturbation. A perturbation theory technique describes the desired solution in terms of perturbation series which is a formal power series in a small parameter. The leading term of this series is the solution of the solvable part. On the other hand, by truncating the series, an approximate perturbation solution can be obtained. If we keep the first two terms only, the problem is called then "first-order" perturbation problem.

Perturbation theory was early used to solve intractable problems in the calculation of the motions of planets in the solar system. These problems are known, in physics and classical mechanics, as *n*-body problems. Perturbation theory methods were developed and generalized by the 18th and 19th centuries mathematicians due to the increasing need for the accuracy of solutions to Newton's gravitational equations. This came as a result of the accuracy of the astronomical observations. The name "perturbation theory" is attributed to Lagrange and Laplace. They were the first to develop the view that the motion of a planet is perturbed by the motion of other planets and vary as a function of time. Gauss and Poisson also investigated the perturbation theory. In the 20^{th} century, perturbation theory of matrices, especially rank-one perturbations, underwent a gradual expansion and evolution. It has been studied by many mathematicians such as J. H. Wilkinson, G. H. Golub, A. Cantoni, P. Butler and A. L. Andrew (see [**7**,**13**,**8**,**19**,**6**,**23**].)

Our thesis discusses rank-one perturbations of special types of matrices. We review previous studies about the effects of rank-one perturbation of matrices on determinants and inverses. We also review the effect of perturbation of centro-symmetric matrices on eigenvalues and eigenvectors. In addition, we present some applications. Then, we state the relationship between the eigenvalues, the eigenvectors and the determinant of 2 x 2 centrosymmetric matrices and the eigenvalues, the eigen-vectors and the determinant of a special rank-one perturbation of them. Also, we state the relationship between the eigenpairs of Laplacian matrices and their rank-one updates.

In Chapter 2, we list definitions and notations we use in the next chapters. In Chapter 3, we list basic properties including determinants and inverses of rank-one perturbations of general matrices. Then, we discuss, in brief, previous studies of rank-one perturbed centrosymmetric matrices. We also list three applications of rank-one updates of matricesin the fields of linear algebra, recreational mathematics, and computer science. In Chapter 4, we study the effect of a specific rank-one perturbation on the eigenvalues, eigenvectors and determinants of 2 x 2 centrosymmetric matrices. Moreover, we show how rank-one perturbation affects many properties of Laplacian matrices. In Chapter 5, we write MATLAB programs and algorithms for rank-one updates of matrices. Finaly, in Chapter 6, we list some concluding remarks and suggested future work.

Chapter 2

Preliminaries

First, we state the followings theorem about block determinants.

Theorem 2.1. (Block Determinants)[**12**] *If A and D are square matrices, then*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{cases} \det(A)\det(D - CA^{-1}B) & \textit{when } A^{-1} \textit{ exists} \\ \det(D)\det(A - BD^{-1}C) & \textit{when } D^{-1} \textit{ exists} \end{cases} .$$

We note that the matrices $D - CA^{-1}B$ and $A - BD^{-1}C$ are called Schur complements of *A* and *D* respectively.

Proof. If A^{-1} exists, then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

Hence,

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(\mathrm{A})\det(D - CA^{-1}B).$$

Theorem 2.2. Let ω be a nonzeroreal number, let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ be a nonsingular diagonal matrix, let $s = \sum_{i=1}^{n} \frac{1}{d_i}$, and let $A_\omega = [\omega] + D$. Then

(i) *If* $s = 0$, *then* $\det(A_\omega) = \det(D)$.

(ii) *If* $s \neq 0$, *then* $\det(A_\omega) = \left(1 + \omega \sum_{k=1}^{n} \frac{1}{d_k}\right)\left(\prod_{j=1}^{n} d_j\right)$

$$= (1 + \omega s)\left(\prod_{j=1}^{n} d_j\right).$$

Next we list below definitions we use in the following chapters.

Definition 2.3. The *counterdiagonal* of the n x n matrix $A = (a_{ij})$ is the collection of entries $a_{i,n-i+1}$, $i = 1, 2, \ldots, n$.

Definition 2.4. The *counteridentity matrix* is the matrix in which all of its entries are equal to zero except those on the counterdiagonal which are all ones. We denote the *counter- identity* matrix by J.

Definition 2.5. A vector x is called *symmetric* if $Jx = x$ and it is *skew-symmetric* if $Jx = - x$.

Definition 2.6. An n x n matrix $A = (a_{ij})$ is called *centrosymmetric* if $JAJ = A$. That is A satisfies

$$a_{i,j} = a_{n-i+1,n-j+1}, \ \forall\ i \text{ and } j.$$

Definition 2.7. An n x n matrix $A = (a_{ij})$ is called *skew-centrosymmetric* if $JAJ = - A$.

Definition 2.8. An n x n matrix C is said to be *counterdiagonal* if all of its entries are zero except those on the counterdiagonal.

Definition 2.9. A *Markov chain* is a sequence of random variables X_1, X_2, X_3, … with the Markov property. The possible values of the X_i's are called states and the *Markov property* is a specific kind of memorylessness property in which the probability distribution of the next state depends only on the current state and not on the sequence of events that preceded it.

Definition 2.10. An n x n *stochastic* matrix $P = (p_{ij})$ is a nonnegative real matrix that describes the transitions of a finite Markov chain. The entry p_{ij} represents the probability of moving from state i to state j in one time step. The stochastic matrix is also known as probability matrix, transition matrix, substitution matrix and Markov matrix.

Definition 2.11. An n x n stochastic matrix P is called *row-stochastic* if the sum of each row is 1 and it is called *column-stochastic* if the sum of each column is 1.

Defenition 2.12. A *magic square* of order n is an n x n matrix whose elements are the integers 1 through n^2, with the property that each row and column, the main diagonal and the main counterdiagonal have the sum $\mu = \frac{n^3+n}{2}$. If a magic square $A = (a_{ij})$ of order n satisfies $a_{i,j} + a_{n-i+1,n-j+1} = n^2 + 1$, $i = 1, \ldots, n$, $j = 1, \ldots, n$, then it is called a *regular magic square.*

Definition 2.13. The *singular values* of a square matrix A are equal to the square roots of the eigenvalues of A^*A, where A^* is the conjugate transpose of A. We note that the singular values are nonnegative real numbers.

Definition 2.14. Let $A = (a_{ij})$ be an n x n matrix. If the off-diagonal entries of A are nonpositive and the sum of each row is zero, then A is called *Laplacian*.

We denote the transpose of a matrix A by A^T, the trace of a square matrix A by tr(A) and the characteristic polynomial of a square matrix A by $p(\lambda)$. We denote the column vector whose entries are all equal to 1 by e and the matrix ωee^T by $[\omega]$, where ω is a nonzero real number. We write (λ, x) is an eigenpair of a square matrix A if λ is an eigenvalue of A with a corresponding eigenvector x. We denote the multiset of eigenvalues of A by $\sigma(A)$. If D is an n x n diagonal matrix with $d_1, d_2, \ldots, d_n$ as its diagonal elements, then we write $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

Chapter 3

Survey of Rank-One Perturbations of Matrices

In Section 3.1, we start with some introductory theorems about determinant and inverse of rank-one updated matrices. In Section 3.2, we state properties of rank-one perturbations of centrosymmetric matrices. In Section 3.3, we list applications of rank-one updates used in different fields. Such applications include perturbed linear system, regular magic squares and Google matrix.

3.1 Rank-One Perturbation of Basic Matrices

In this section, we state some theorems and facts about rank-one perturbations of some basic matrices such as the identity matrix I (see [**16**].)

Let us consider the following result regarding determinant of a special type of sums.

Theorem 3.1.1. (Determinants of Rank-One Updates)

If A is n x n nonsingular matrix, and if u and v are n x 1 vectors, then

(i) det $(I + uv^T) = 1 + v^T u$,

(ii) det $(A + uv^T) = (1 + v^T A^{-1} u)$ det (A).

Proof. The proof of (i) follows from applying the product rule to

$$\begin{pmatrix} I & 0 \\ v^T & 1 \end{pmatrix} \begin{pmatrix} I + uv^T & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -v^T & 1 \end{pmatrix} = \begin{pmatrix} I & u \\ 0 & 1 + v^T u \end{pmatrix}.$$

In order to prove (ii), we write

$$A + uv^T = A(I + A^{-1}\, uv^T).$$

Thus,

$$\det\big(A(I + A^{-1}\, uv^T)\big) = \det(A)\det(I + A^{-1}\, uv^T)$$

$$= (1 + v^T A^{-1} u)\det(A). \qquad \square$$

A MATLAB program for this theorem is in Chapter 5.

Theorem 3.1.2. (Sherman-Morrison Formula)

If A is an n x n nonsingular matrix and if u and v are n x 1 vectors such that $1+ v^T A^{-1} u \neq 0$, *then the sum* $A+ u\, v^t$ *is nonsingular and*

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}\, u\, v^T A^{-1}}{1+ v^T A^{-1}\, u}.$$

Proof. We can write

$$A+ u\, v^T = A(I + A^{-1} u\, v^T).$$

Thus,

$$(A + uv^T)^{-1} = [\, A(I + A^{-1} u\, v^T)\,]^{-1}$$

$$= (I + A^{-1}u\, v^T)^{-1}(A^{-1}).$$

But,

$$(I + u\, v^T)^{-1} = I - \frac{u\, v^T}{1+v^T u}.$$

This yields

$$(I + A^{-1}u\, v^T)^{-1} = I - \frac{A^{-1}u\, v^T}{1+v^T A^{-1}u}.$$

Hence,

$$(A + uv^T)^{-1} = \left(I - \frac{A^{-1}u\, v^T}{1+v^T A^{-1}u} \right)(A^{-1})$$

$$= A^{-1} - \frac{A^{-1}u\, v^T A^{-1}}{1+v^T A^{-1}u}.$$

A MATLAB program for this theorem can be found in Chapter 5.

3.2 Rank-One Perturbations of Centrosymmetric Matrices

Centrosymmetric matrices and their rank-one perturbations have applications in different fields such as statistics, differential equations, numerical analysis, engineering, physics, communication theory and pattern recognition. For more details about these applications, see [**15**,**14**,**10**,**9**].

We mention below results and properties of matrices resulting from updating centrosymmetric matrices by special rank-one matrices. This perturbation is obtained using symmetric and skew-symmetric vectors (see [**4**,**14**,**6**,**8**,**5**].)

Theorem 3.2.1. *Let H be an n* x *n centrosymmetric matrix, let* $\delta = \frac{n}{2}$, *and let* $\xi = \frac{n-1}{2}$. *Then,*

(i) *if n is even, then H can be written as*

$$H = \begin{bmatrix} A & JCJ \\ C & JAJ \end{bmatrix},$$

where A, J and C are δ x δ *matrices. If n is odd, then H can be written as*

$$\begin{bmatrix} A & x & JCJ \\ y^T & q & y^T J \\ C & Jx & JAJ \end{bmatrix},$$

where A, J and C are ξ x ξ *matrices, x and y are* ξ x 1 *vectors, and q is a scalar.*

(ii) *If n is even, then H is similar to*

$$\begin{bmatrix} A - JC & 0 \\ 0 & A + JC \end{bmatrix}.$$

If n is odd, then H is similar to

$$\begin{bmatrix} A - JC & 0 & 0 \\ 0 & q & \sqrt{2}\, y^T \\ 0 & \sqrt{2}x & A + JC \end{bmatrix}.$$

(iii) *If n is even, then the eignevalues of H are the eigenvalues of* $F_1 := A - JC$ *and the eigenvalues of* $G_1 := A + JC$. *Moreover, the eigenvectors corresponding to the eigenvalues of* F_1 *can be chosen to be skew-symmetric of the form* $(u^T, -u^T J)^T$, *where u is an eigenvector of* F_1, *while the eigenvectors corresponding to the eigenvalues of* G_1 *can be chosen to be symmetric of the form* $(u^T, u^T J)^T$, *where u is an eigenvector of* G_1. *If n is odd, then the eigenvalues of H are the eigenvalues of* F_1 *and the eigenvalues of*

$$G_2 := \begin{bmatrix} q & \sqrt{2}\, y^T \\ \sqrt{2}x & A + JC \end{bmatrix}.$$

Moreover, the eigenvectors corresponding to the eigenvalues of F_1 *can be chosen to be skew-symmetric of the form* $(u^T, 0, -u^T J)^T$, *where u is an eigenvector of* F_1, *while the eigenvectors corresponding to the eigenvalues of* G_2 *can be chosen to be symmetric of the form* $(u^T, \sqrt{2}\,\alpha, u^T J)^T$, *where* $(\alpha, u^T)^T$ *is an eigenvector of* G_2.

For the proof of the previous theorem, see [**8**,**6**]. Also, the proof of the following theorems can be found in [**2**].

Proposition 3.2.2. *Let u be an n x 1 symmetric vector, let w be an n x 1 vector, let H be an n x n nondefective centrosymmetric matrix, and let $M = H + wu^T$. Then, H and M share at least $\lfloor n/2 \rfloor$ eigenvalues and $\lfloor n/2 \rfloor$ linearly independent eigenvectors.*

Lemma 3.2.3. *Let $\delta = \frac{n}{2}$ and let $\xi = \frac{n-1}{2}$.*

(i) *Let n be even, let u be an n x 1 symmetric/skew-symmetric vector, and let $S = uu^T$. Then u and S can be written as*

$$u = \begin{bmatrix} v \\ p \end{bmatrix}$$

and

$$S = \begin{bmatrix} R & LJ \\ JL & JRJ \end{bmatrix},$$

where v is δ x 1, J is δ x δ, $R = vv^T$, $p = Jv$ if u is symmetric and $p = -Jv$ if u is skew-symmetric, and $L = R$ if u is symmetric and $L = -R$ if u is skew-symmetric.

(ii) *Let n be odd, let u be an n x 1 symmetric/skew-symmetric vector and let $S = uu^T$. Then u and S can be written as*

$$u = \begin{bmatrix} v \\ c \\ p \end{bmatrix}$$

and

$$S = \begin{bmatrix} R & cv & LJ \\ cv^T & c^2 & cv^T J \\ JL & cJv & JRJ \end{bmatrix},$$

where u is ξ x 1, *J is ξ* x *ξ*, *R* = vv^T, *c = k (for same scalar k) if u is symmetric and c =* 0 *if us is skew-symmetric, p = Jv if u is symmetric and p =* - *Jv if u is skew-symmetric, and L = R if u is symmetric and L =* - *R if u is skew-symmetric.*

Proof. It is sufficient to prove only one case and the other cases can be proved in a similar way. Let us prove the case when *u* is skew-symmetric and *n* is odd. The form of *u* follows from the definition of skew-symmetric vectors. Also, note that $J^T = J$. Since

$$u = \begin{bmatrix} v \\ 0 \\ -Jv \end{bmatrix},$$

then

$$\begin{aligned} S &= uu^T \\ &= \begin{bmatrix} v \\ 0 \\ -Jv \end{bmatrix} \begin{bmatrix} v^T & 0 & -v^T J \end{bmatrix} \\ &= \begin{bmatrix} vv^T & 0 & -vv^T \\ 0 & 0 & 0 \\ -Jvv^T & 0 & -Jvv^T J \end{bmatrix}. \end{aligned}$$

□

Theorem 3.2.4. *Let u be an n x 1 symmetric vector, let H be an n x n centrosymmetric matrix, let $M = uu^T+H$, let H be decomposed as in* Theorem 3.2.1, *let R be as in* Lemma 3.2.3, *and let $\delta = \frac{n}{2}$.*

(i) *If n is even, then the eigenvalues of M are the eigenvalues of $F_1 := A-JC$ and the eigenvalues of $G_3 := A+JC+2R$, and the eigenvectors of M can be determined from the eigenvectors of F_1 and the eigenvector of G_3. Moreover, the shared eigenvalues between H and M are the eigenvalues of F_1 and the shared eigenvectors are the eigenvectors determined from the eigenvectors of F_1. If, in addition, M is nondefective, then δ eigenvalues and δ skew-symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$F_1 f_i = \lambda_i f_i ,$$

and δ eigenvalues and δ symmetric linearly independent eigenvectors of M can be determined from solving the equation

$$G_3 g_i = \mu_i g_i.$$

(ii) *If n is odd, then the eigenvalues of M are the eigenvalues of F_1 and the eigenvalues of G_4, and the eigenvectors of M can be determined from the eigenvectors of F_1 and the eigenvectors of G_4, where*

$$G_4 := \begin{bmatrix} q+k^2 & \sqrt{2}\,(y+kv)^T \\ \sqrt{2}(x+kv) & A+JC+2R \end{bmatrix}.$$

Moreover, the shared eigenvalues between H and M are the eigenvalues of F_1 *and the shared eigenvectors are the eigenvectors determined from the eigenvectors of* F_1*. If, in addition, M is nondefective, then* $\lfloor n/2 \rfloor$ *eigenvalues and* $\lfloor n/2 \rfloor$ *skew-symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$F_1 f_i = \lambda_i f_i ,$$

and $\lceil n/2 \rceil$ *eigenvalues and* $\lceil n/2 \rceil$ *symmetric linearly independent eigenvectors of M can be determind from solving the equation*

$$G_4 g_i = \mu_i g_i.$$

Proposition 3.2.5. *Let u be an n* x 1 *skew-symmetric vector, let w be an n* x 1 *vector, let H be n* x *n nondefective centrosymmetric matrix, and let* $M = uu^T + H$*. Then, H and M share at least* $\lceil n/2 \rceil$ *and* $\lceil n/2 \rceil$ *linearly independent eigenvectors.*

Proof. See [**2**].

Theorem 3.2.6. *Let u be n* x 1 *skew-symmetric vector, let H be an n* x *n centrosymmetric matrix, let* $M = uu^T + H$ *, let H be decomposed as in* Theorem 3.2.1, *let R be as* Lemma 3.2.3, *and let* $\delta = \frac{n}{2}$.

(i) *If n is even, then eigenvalues of M are the eigenvalues of* $F_2 := A - JC + 2R$ *and the eigenvalues of* $G_1 := A + JC$*, and the eigenvectors of M can be determined from the eigenvectors of* F_2 *and the eigenvectors of* G_1*. Moreover, the shared eigenvalues* *between H and M are the eigenvalues of* G_1 *and the shared eigenvectors are the eigenvectors determined from the eigenvectors of* G_1*. If, in addition, M is nondefective, then* δ *eigenvalues and* δ *skew symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$F_2 f_i = \lambda_i f_i ,$$

and δ *eigenvalues and* δ *symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$G_1 g_i = \mu_i g_i .$$

(ii) *If n is odd, then the eigenvalues of M are the eigenvalues of* F_2 *and the eigenvalues of* G_2*, and the eigenvectors of M can be determined from the eigenvectors of* F_2 *and the eigenvectors of* G_2*, where*

$$G_2 = \begin{bmatrix} q & \sqrt{2}\, y^T \\ \sqrt{2}x & A + JC \end{bmatrix}.$$

Moreover, the shared eigenvalues between H and M are the eigenvalues of G_2 *and the shared eigenvectors are the eigenvectors determined from the eigenvectors of* G_2*. If, in addition, M is nondefective, then* $\lfloor n/2 \rfloor$ *eigenvalues and* $\lfloor n/2 \rfloor$ *skew-symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$F_2 f_i = \lambda_i f_i ,$$

and $\lceil n/2 \rceil$*eigenvalues and* $\lceil n/2 \rceil$ *symmetric linearly independent eigenvectors of M can be determined from solving the equation*

$$G_2\, g_i = \mu_i\, g_i\,.$$

3.3 Applications of Rank-One Perturbations of Matrices

Perturbation theory have applications in many fields such as physics, engineering, chemistry, computer science, etc. The beginning studies were in the planetary motion as mentioned previously in Chapter 1. However, modern applications are focused on quantum field theory like quantum mechanics and quantum chemistry. For more details, see [**7**,**20**,**21**].

3.3.1 Application I : Updating a Linear System

If the right-hand side of the linear system $Ax = b$ or the left-hand side is changed, then the new system must be solved. Sometimes the solution of the new system depends on the solution x_0 of the original system.

We consider here the case when the change is on the right-hand side of the system $Ax = b$ to get the new system $Ax = c$. See [**12**]

If the change is a small perturbation of b, e.g., $c = b + \alpha b$, then, in order to solve this new system, we use an iterative method with the solution x_0 as the starting point. Moreover, the same idea is applied when the coefficient matrix A is changed to become $A + \alpha A$. Meanwhile, by applying the rank-one perturbation of A, the system becomes $\tilde{A}x = (A + uv^T)x = b$. Solving this system requires finding $\tilde{A}^{-1}$ if A is nonsingular. From Theorem 3.1.2, we can write

$$\tilde{A}^{-1} = A^{-1} - \frac{A^{-1}\, uv^T A^{-1}}{1 + v^T A^{-1}\, u}.$$

Now, the new solution y_0 can be expressed in terms of x_0 as follows

$$\begin{aligned} y_0 &= \tilde{A}^{-1} b \\ &= A^{-1}b - \frac{A^{-1}uv^T A^{-1}b}{1+v^T A^{-1}u} \\ &= x_0 - \frac{A^{-1}uv^T A^{-1}b}{1+ v^T A^{-1}u}. \end{aligned}$$

3.3.2 Application II: Regular Magic Squares

Magic squares first appeared in Chinese culture. They were early used in artworks and had been known to many other cultures. Mathematicians have studied magic squares and their properties. Recently, magic squares have many applications in games. We introduce below a quick overview of some properties of magic squares. See [**1**,**22**].

Proposition 3.3.2.1. *Let S be a skew-centrosymmetric matrix. If* (λ,x) *is an eigenpair of S, then* ($-\lambda$,Jx) *is an eigenpair of S. Moreover, λ and $-\lambda$ have the same multiplicity*.

Proof. Assume that (λ,x) is an eigenpair of S. But, S is skew-centrosymmetric. That is, $JSJ = -S$. Thus, $(JSJ)J = -SJ$. Also, $(JSJ)J = JS(JJ) = JS$. Hence, $JS = -SJ$.

Since (λ,x) is an eigenpair of S, then $Sx = \lambda x$. Therefore,

$$JSx = J(\lambda x) \quad \rightarrow \quad -SJx = \lambda Jx \quad \rightarrow \quad S(Jx) = -\lambda(Jx)$$

This implies that ($-\lambda$,Jx) is an eigenpair of S. □

The following theorem shows that a regular magic square is a type of rank-one perturbation of matrices.

Theorem 3.3.2.2. *Let A be a regular magic square of order n and let* $\mu = \frac{n^3+n}{2}$ *. Then,*

(i) *A can be written as* $A = Z + \frac{\mu}{n} ee^T$*, where Z is skew-centrosymmetric.*

(ii) (μ,e) *is an eigenpair of A and* (0,e) *is an eigenpair of Z. Moreover, μ is the largest eigenvalue of A in magnitude and it is simple.*

(iii) *All eigenpairs of A are the same as those of Z except that* (0,e) *is replaced by* (μ,e).

Proof. The proof can be found in [**1**].

Here below we state a theorem which shows that the eigenvalues of a skew-centrosymmetric matrix after reversing the rows/columns are the same eigenvalues multiplied by *i*.

Theorem 3.3.2.3. *Let S be a skew-centrosymmetric matrix. Then,*

$evals(JS) = i \,.\, evals(S)$.

Proof. The proof follows from Proposition 3.3.2.1 and the fact that $(JS)^2 = - S^2$. □

Now, we state the effect of reversing the rows/columns of a regular magic square on its eigenvalues.

Theorem 3.3.2.4. *Let A be a regular magic square of order n and let B = JA. Then*

(i) *μ is a simple eigenvalue of B and it is the largest eigenvalue in magnitude.*

(ii) *$\lambda \neq \mu$ is an eigenvalue of A if and only if $i\lambda$ is an eigenvalue of B.*

(iii) *$(0,x)$ is an eigenpair of A if and only if $(0,x)$ is an eigenpair of B.*

Proof. First recall that $A = Z + \frac{\mu}{n} ee^T$. Then, $B = JA = JZ + \frac{\mu}{n} ee^T$ since $Jee^T = ee^T$.

(i) Note that B is also a regular magic square and $\mathrm{tr}(B) = \mu$. Thus, μ is a simple eigenvalue of B and it is the largest eigenvalue in magnitude.

(ii) Note that from Theorem 3.3.2.2, the eigenvalues of A are the same as the eigenvalues of Z , except that the zero eigenvalue corresponding to the eigenvaector e, is replaced by μ. Note also that $B = Z' + \frac{\mu}{n} ee^T$, where $Z' = JZ$. Thus, the eigenvalues of B are the same as the eigenvalues of Z', except that the zero eigenvalue corresponding to the eigenvector e is replaced by μ. But, JZ is skew-centrosymmetric. Thus, by Theorem 3.3.2.3, evals(Z') = i evals(Z).

(iii) Note that $B = JA$ and $A = JB$. □

Therefore, the eigenvalues of B are $\{\mu, i\lambda_1, \ldots, i\lambda_{n-1}\}$ if the eigenvalues of A are $\{ \mu, \lambda_1, \ldots, \lambda_{n-1}\}$.

Corollary 3.3.2.5. *Every regular magic square of even order is singular.*

Proof. Let A be a regular magic square of even order n, let $B = JA$, and let the eigenvalues of A be $\{ \mu, \lambda_1, \ldots, \lambda_{n-1}\}$.

First, note that det(J) = 1 if n mod 4 = 0 or 1 and det(J) = -1 if n mod 4 = 2 or 3. Now, the eigenvalues of B are $\{\mu, i\lambda_1, \ldots, i\lambda_{n-1}\}$. Thus, $\det(A) = \mu \cdot \prod_{j=1}^{n-1} \lambda_j$ and $\det(B) = \pm\, i \cdot \mu \cdot \prod_{j=1}^{n-1} \lambda_j$. Therefore, det($B$) = ±$i$ det(A).

But, det(B) = det(JA) = ± det(A). Hence, det(A) = 0.

Now, the following theorem shows that n – 1 singular values of a regular magic square A of order n are the same as n – 1 singular vales of Z.

Theorem 3.3.2.6. *Let A be a regular magic square of order n and let $B = JA$. Then*

(i) *The singular values of B are the same as the singular values of A.*

(ii) *The singular values of A are the same as the singular values of Z, except that one of the zero singular values of Z is replaced by μ. Moreover, the eigenvectors of A^TA are the same as the eigenvectors of Z^TZ.*

Proof.

(i) $B^TB = (JA)^T(JA) = A^TJJA = A^TA$.

(ii) It is easy to prove that (μ^2, e) is an eigenpair of A^TA. Here is the proof

$$A^TAe = A^T\mu e = \mu A^Te$$

Now note that A^T is a regular magic square and note also that the trace of A^T is μ. Thus, (μ,e) is an eigenpair of A^T. Hence, $A^TAe = \mu^2$e. Now

$$A^TA = (Z^T + \frac{\mu}{n}ee^T)(Z + \frac{\mu}{n}ee^T)$$

$$= ZZ^T + \frac{\mu}{n}ee^TZ + \frac{\mu}{n}Z^Tee^T + \frac{\mu^2}{n^2}nee^T.$$

Now let (λ,x) be an eigenpair of A^TA, where $\lambda \neq \mu^2$. Then $e^Tx = 0$, and hence,

$$A^TAx = Z^TZx + \frac{\mu}{n}ee^TZx = Z^TZx$$

3.3.3 Application III: Google Matrix

There are many applications of rank-one perturbation of matrices. Google matrix is one of these applications (see [**17**,**18**]). It is used in the computation of the Page-Rank for the Google Web search engine. The Google matrix, denoted by G, is obtained from a stochastic matrix S after modifying it by a special rank-one perturbation. The matrix S is an n x n nonnegative column-stochastic matrix satisfying $e^T S = e^T$, where e is the n-dimensional vector whose entries are all ones. First choose an arbitrary c, where $0 < c < 1$, and an arbitrary n-dimensional probability vector u, i.e., u is a positive vector normalized by $u^T e = 1$. Then, G is defined as

$$G = cS + (1 - c)\, ue^T .$$

The Google matrix has a dominant eigenvalue of modulus 1. We list below lemmas, propositions and theorems that are used in proving a spectral perturbation theorem for rank-one perturbed matrices.

Theorem 3.3.3.1. *Let u and v be two n- dimensional column vectors such that u is an eigenvector of A associated with eigenvalue λ_1. Then, the eigenvalues of $A + uv^T$ are $\{\lambda_1 + v^T u, \lambda_2, \ldots, \lambda_n\}$, counting algebraic multiplicity.*

Proof. *Let* $\lambda \notin \sigma(A)$ be any complex number. Then, by applying Theorem 3.1.2 to the equality

$$\lambda I - (A + uv^T) = (\lambda I - A) - uv^T,$$

we have

$$\det[\lambda I - (A + uv^T)] = [1 - v^T(\lambda I - A)^{-1} u] \det(\lambda I - A). \tag{1}$$

The condition $Au = \lambda_1 u$ implies that

$$(\lambda I - A)^{-1} u = \frac{1}{\lambda - \lambda_1} u. \tag{2}$$

So (1) becomes

$$\det[\lambda I - (A + uv^T)] = \left(1 - \frac{v^T u}{\lambda - \lambda_1}\right) \det(\lambda I - A)$$

$$= \frac{[\lambda - (\lambda_1 + v^T u)](\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)}{\lambda - \lambda_1}$$

$$= \left(\lambda - (\lambda_1 + v^T u)\right)(\lambda - \lambda_2) \dots (\lambda - \lambda_n).$$

Since the equality is true for all $\lambda \notin \sigma(A)$, the theorem is proved. □

Remark 3.3.3.2. Depending on Theorem 3.3.3.1, it is easy to show that the characteristic polynomial of $(A + uv^T)$ *is*

$$p(\lambda) = [\lambda - (\lambda_1 + v^T u)](\lambda - \lambda_2) \dots (\lambda - \lambda_n).$$

For the second eigenvalue λ_2 of the Google matrix, its modulus can analytically be determined. We mention below two theorems that show the relationship between $|\lambda_2|$ and the constant c. But first, we need the following preliminaries.

Recall that the Google matrix G is written as $G = cS + (1 - c) \cdot ue^T$, where u is an n x 1 vector represents a probability distribution.

We can choose the eigenvectors (x_i's) of G such that $||x_i||_1 = 1$. On the other hand, since G is column-stochastic, its eigenvalues satisfy $\lambda_1 = 1$, $|\lambda_2| \geq \ldots \geq |\lambda_n| \geq 0$. Similarly, if y_i is an eigenvector of S with corresponding eigenvalue γ_i, then $\gamma_1=1$, $|\gamma_2| \geq \ldots \geq |\gamma_n| \geq 0$ because S is column-stochastic.

If we look at the matrix $E^T = ue^T$, we notice that it is rank-one and column-stochastic. Thus, the eigenvalues of E^T are $\mu_1 = 1$, $\mu_2 = \ldots = \mu_n = 0$.

The transition matrix for a Markov chain with n states is an n x n row-stochastic matrix M and this matrix M satisfies $Me = e$. Therefore, we can conclude that $(1,e)$ is an eigenpair for G^T.

A set of states B is said to be a *closed subset* of the Markov chain corresponding to M if and only if $i \in B$ and $j \notin B$ implies that $M_{ij} = 0$.

An *irreducible closed subset* of states, B, of the Markov chain corresponding to M is a closed subset, and no proper subset of B is a closed subset, i.e. all states in B are mutually reachable.

We state below a theorem about the modulus of the second eigenvalue of the Google matrix G.

Theorem 3.3.3.3. *Let G be the Google matrix, Then, its second eigenvalue,* λ_2,*satisfies the inequality* $|\lambda_2| \leq c$.

Proof. We have the following three cases:

Case 1: $c = 0$.

If $c = 0$, then $G = ue^T = E^T$. Therefore, $\lambda_2 = 0$ because E^T is a rank-one matrix. Thus, Theorem 3.3.3.3 is proved for $c = 0$.

Case 2: $c = 1$.

If $c = 1$, then $G = S$. Therefore, $|\lambda_2| \leq |\lambda_1| = 1$ because S is a column-stochastic matrix. Thus, Theorem 3.3.3.3 is proved for $c = 1$.

Case 3: $0 < c < 1$.

This case will be proved by the following lemmas.

Lemma 3.3.3.4. *The second eigenvalue of G has modulus* $|\lambda_2| < 1$.

Proof. See [17] for the proof.

Lemma 3.3.3.5. *The second eigenvector* x_2 *of G is orthogonal to e. I.e.,* $e^T x_2 = 0$.

Proof. First, recall that $(1,e)$ is an eigenpair of G^T because G^T is a row-stochastic. Then, apply the theorem that states ‘‘If x_i is an eigenvector of A corresponding to the eigenvalue λ_i, and y_j is an eigenvector of A^T corresponding to λ_j, then $x_i^T y_j = 0$ for $\lambda_i \neq \lambda_j$’’ on G and G^T.

Since $|\lambda_2| < |\lambda_1|$, then, by Lemma 3.3.3.4, we get that the second eigenvector of G is orthogonal to the first eigenvector of G^T which is e. Therefore, $e^T x_2 = 0$.

Thus, $E^T x_2 = 0$. That is obvious because, by definition, $E^T = ue^T$. Hence, $E^T x_2 = ue^T x_2 = 0$.

Lemma 3.3.3.6. The second eigenvector x_2 of G must be an eigenvector y_i of S and the corresponding eigenvalue is $\gamma_i = \lambda_2 / c$.

Proof. From the definition of G

$$Gx_2 = cSx_2 + (1- c) \cdot ue^T x_2 = \lambda_2 x_2.$$

But, from Lemma 3.3.3.5, we have $e^T x_2 = 0$. Thus,

$$Gx_2 = cSx_2 = \lambda_2 x_2.$$

I.e., $cSx_2 = \lambda_2 x_2$. Dividing by c yields

$$Sx_2 = \frac{\lambda_2}{c} x_2.$$

This implies that $\left(\frac{\lambda_2}{c}, x_2\right)$ is an eigenpair of S. By letting $y_i = x_2$ and $\gamma_i = \frac{\lambda_2}{c}$, Lemma 2.3.3.6 is proved.

Now, since S is a column-stochastic matrix, $|\gamma_i| \leq 1$. But, $|\gamma_i| = \frac{|\lambda_2|}{c}$. Therefore, $|\lambda_2| \leq c$ and Theorem 3.3.3.3 is proved.

The PageRank is an estimate determines the importance of web pages of search results. As we mentioned before, the Google matrix computes the PageRank by using algorithms based on eigenvector computation.

The main idea of algorithms is to compute the principal eigenvectors of the stochastic matrix that represents the web link graph.

Although, in the area of numerical linear algebra, there are a lot of fast algorithms for computing the eigenvectors, many of such algorithms require matrix inversion which makes the operation highly cost for a web scale matrix. Then, developing PageRank-computing techniques becomes a necessary need.

The aim of the development of these techniques is to accelerate the convergence of the iterative PageRank computation. The importance of that can be explained by the limitation of the lag time, where the lag time is the time from where a new crawl is completed to when that crawl can be made available for searching.

Now, we turn to the mathematical section in computing the PageRank. Also, we present below the power method as one of many methods used by PageRank algorithms.

In order to understand the mathematical definition of the PageRank, consider the following.

Let $u,w \in W$ web graph W be two web pages such that there is a link from u to w. This means that v is an important page. But, how much is the importance of the page w depends on the importance of u (with proportional relation) and on the number of pages u has links to (with inversely proportional relation). This implies that an iterative fixed-point computation must be applied in order to determine the importance of any web page $i \in$ web since the importance of u itself is unknown.

Let $u \to w$ denotes the existence of a link from u to w in W and let $\deg(u)$, the outdegree or the outlinks, denotes the number of links out of u. Now we can define the PageRank.

The PageRank of a page i is the probability that at some certain time step $k > K$, the surfer of the web is at page i. Assuming that the surfer is at the web page i at time step k. In the next time step, the surfer will uniformly choose one of the outlink of i in a random way.

This implies that for sufficiently large K it follows the uniform distribution with probability $\frac{1}{\deg(i)}$.

Now, the Markov chain of this process consists of the nodes in the web graph W as states and P as the stochastic transition matrix with $P_{ij} = \frac{1}{\deg(i)}$ describing the transition from page i to page j.

In order to build a valid transition probability matrix P, P must not have any zero rows. That requires the minimum number of outdegree of each node to be 1. Indeed, it is not always possible to have a graph with no nodes of outdegree 0. Thus, P is converted into a valid transition matrix S by special modification. The construction of S is as follows

$$D = d \,.\, v^T,$$

$$S^T = P + D,$$

where v is an n-dimensional column vector whose all entries are $\frac{1}{n}$ and d is an n-dimensional column vector that identifies the zero-outdegree nodes as

$$d_i = \begin{cases} 1 & if \deg(i) = 0 \\ 0 & otherwise \end{cases}.$$

Adding D modify the transition probabilities so that a surfer visiting a dangling page, i.e., a page with no outlinks, randomly jumps to another page in the next time step, using the uniform distribution (the distribution of v).

The irreducible Markov matrix is constructed as

$$E = ev^T$$

$$A = cS^T + (1 - c)\,E$$

Now, let $G = A^T$ for simplicity and consistency and assume that the probability distribution over the surfer's location at time 0 is given by $x^{(0)} = v$. Then, the probability distribution over the surfer's location at time k is given by $x^{(k)} = Gx^{(k-1)} = G^k x^{(0)}$. Thus, the PageRank is the principal eigenvector of the matrix G, then the PageRank is simply $\lim_{k \to \infty} G^k x^{(0)}$.

We discuss below the basic method used to compute the PageRank which is the Power method. This method is the oldest method. It finds the principal eigenvector of G by computing successive iterates $x^{(k)} = Gx^{(k-1)}$ starting with $x^{(0)} = v$ until convergence.

Let's turn to show the convergence of the Power method. If we assume that the starting vector $x^{(0)}$ belongs to the subspace spanned by the eigenvectors of G, then $x^{(0)}$ can be written as a linear combination of the eigenvectors of G as

$x^{(0)} = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$.

Choose $\alpha_1 = 1$. Then,

$$x^{(0)} = u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n.$$

Thus,

$$\begin{aligned} x^{(1)} &= Gx^{(0)} \\ &= Gu_1 + \alpha_2 G u_2 + \ldots + \alpha_n G u_n \\ &= u_1 + \alpha_2 \lambda_2 u_2 + \ldots + \alpha_n \lambda_n u_n \end{aligned}$$

and

$$x^{(k)} = G^k x^{(0)} = u_1 + \alpha_2 \lambda_2^k u_2 + \ldots + \alpha_n \lambda_n^k u_n.$$

Since $\lambda_1 = 1 > \lambda_2 \geq \ldots \geq \lambda_n$, then

$$\begin{aligned} \lim_{k \to \infty} x^{(k)} &= \lim_{k \to \infty} (u_1 + \alpha_2 \lambda_2^k u_2 + \cdots + \alpha_n \lambda_n^k u_n) \\ &= \lim_{k \to \infty} u_1 \\ &= u_1. \end{aligned}$$

This implies that Power method converges to the principal eigenvector of the Markov matrix G.

Note that for faster convergence of the Power method, λ_2 should not be so close to 1.

That is because, for $\lambda_2^n \rightarrow 0$, n must be very large.

Chapter 4

New Results

This chapter includes several new results. We prove, in Section 4.1, three properties of a special rank-one perturbation of 2 x 2 centrosymmetric matrices. These properties include the eigenvalues, the eigenvectors and the determinant. In Section 4.2, we study new results about perturbations of Laplacian matrices.

4.1 Centrosymmetric Matrices

Theorem 4.1.1. *Let* $u = (r,r)^T$ *be a* 2 x 1 *symmetric vector, let* $v = (p,-p)^T$ *be a* 2 x 1 *skewsymmetric vector, let* H *be a* 2 x 2 *centrosymmetric matrix,and let* $M = H+uv^T$. *Then,*

(i) *M and H share the same eigenvalues which they are* $\lambda_1 = h_{11} + h_{12}$ *and* $\lambda_2 = h_{11} - h_{12}$.

(ii) *if H is nonsingular, then the shared eigenvector between H and M is the symmetric eigenvector v of H corresponding to* λ_1 *and the second eigenvector of M can be determined from the skew-symmetric eigenvector of H corresponding to* λ_2.

(iii) det (M) = det(H) = $h_{11}^2 - h_{12}^2$.

Proof. Since H is centrosymmetric, it can be written as

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{11} \end{bmatrix}.$$

Let $a = h_{11}$ and $b = h_{12}$.

Part (i): The eigenvalues of H are the solutions of the characteristic polynomial $P_\lambda\,(H)$ of H where

$$P_\lambda\,(H) = (a - \lambda)^2 - b^2.$$

It can be easily shown that $\lambda_1 = a + b$ and $\lambda_2 = a - b$ are the solutions of $P_\lambda\,(H)$.

Now,

$$M = H + uv^T$$

$$= \begin{bmatrix} a + rp & b - rp \\ b + rp & a - rp \end{bmatrix}.$$

The characteristic polynomial of M is

$$P_\lambda\,(M) = (a + rp - \lambda)(a - rp - \lambda) - (b - rp)(b + rp)$$

$$= (\lambda^2 - 2a\lambda + a^2) - b^2$$

$$= (\,(\lambda - a) - b)\,(\,(\lambda - a) + b).$$

This implies that the eigenvalues of M are $\lambda_1 = a + b$ and $\lambda_2 = a - b$.

Part (ii): Suppose that H is nonsingular, i.e., $|a| \neq |b|$. Then, an eigenvector $x^{(1)}$ of H corresponding to λ_1 is $x^{(1)} = (1,1)^T$. This is can be easily proved by solving $(\mathrm{H}- \lambda_1)x^{(1)} = 0$ for $x^{(1)}$. I.e., solve the following linear system

$$- b\, x_1^{(1)} + b\, x_2^{(1)} = 0,$$

$$b\, x_1^{(1)} - b\, x_2^{(1)} = 0.$$

Similarly, we can show that the second eigenvector $x^{(2)}$ of H corresponding to λ_2 is $x^{(2)} = (1,-1)^T$.

Note that $x^{(1)}$ is symmetric vector while $x^{(2)}$ is skewsymmetric.

For M, the first eigenvector $y^{(1)}$ of M can be determined from the solution of the following system

$$(rp - b)\ y_1^{(1)} + (b - rp)\, y_2^{(1)} = 0,$$

$$(b + rp)\, y_1^{(1)} - (b + rp)\, y_2^{(1)} = 0.$$

Solving the above equation we get $y^{(1)} = (1,1)^T$. Similarly, solving the equation $(M - \lambda_2 I)\, y^{(2)} = 0$ which is equivalent to solving the system

$$(b + rp)\, y_1^{(2)} + (b - rp)\, y_2^{(2)} = 0,$$

$$(b + rp)\, y_1^{(2)} + (b - rp)\, y_2^{(2)} = 0,$$

we can write $y^{(2)}$ in terms of $(1,-1)^T$.

Thus, $y^{(1)} = x^{(1)}$, i.e., M and H share the same eigenvector corresponding to λ_1. Also $y^{(2)}$ is written in terms of $x^{(2)}$.

Part (iii): Trivial.

Theorem 4.1.2. *Let $u = (r,- r)^T$ be* 2 x 1 *skewsymmetric vector, let $v = (p,p)^T$ be* 2 x 1 *symmetric vector, let H be* 2 *x* 2 *centrosymmetric matrix and let $M = H + uv^T$. Let $a = h_{11}$ and $b = h_{12}$. Then,*

(i) *M and H share the same eigenvalues which are $\lambda_1 = a + b$ and $\lambda_2 = a - b$.*

(ii) *if H is nonsingular and M is nondefective, then the shared eigenvector between M and H is the eigenvector $x^{(2)} = (1,-1)^T$, corresponding to λ_2. Moreover, the eigenvector of M corresponding to λ_1 can be determined from the eigenvector $x^{(1)}$ of H corresponding to λ_1.*

(iii) $\det(M) = \det(H) = a^2 - b^2$.

Proof. This theorem can be proved using a proof similar to that of Theorem 4.1.1.

4.2 Laplacian Matrices

Now we consider several properties of rank-one perturbed Laplacian matrix. The following theorem is a generalization of a theorem that can be found in [**16**].

Theorem 4.2.1. *Let* $A = (a_{ij})$ *be an n x n Laplacian matrix,* ω *a nonzero real number, and* $A_\omega = [\omega] + A$. *If* (λ, x) *is an eigenpair of A, where* $\lambda \neq 0$ *and* $\lambda \neq \omega n$*, then either* (λ, x) *or* $(\lambda, x+\alpha e)$ *is an eigenpair of* A_ω*, where* $\alpha = \frac{\omega \sum_{i=1}^{n} x_i}{\lambda - \omega n}$. *Moreover, if* (λ, x) *is an eigenpair of* A^T *and* $\lambda \neq 0$*, then* (λ, x) *is an eigenpair of* $A^T + [\omega]$. *Also, the sum of the entries of* $A^T x$ *is zero for all x, and hence, whenever it is not zero,* $A^T x$ *is an eigenvector of* $[\omega]$.

Proof. For the first part, it suffices to note

(1) $Ae = 0$, and $[\omega]x = \omega(\sum_{i=1}^{n} x_i)e$.

(2) α is well-defined because $\lambda \neq \omega n$ (from the assumptions).

(3) If $x + \alpha e = 0$, then $x_i = -\alpha$, i = 1, 2, …, n. This implies $\sum_{i=1}^{n} x_i = \frac{-n\omega \sum_{i=1}^{n} x_i}{\lambda - \omega n}$. Thus, either $\sum_{i=1}^{n} x_i = 0$, which implies $\alpha = 0$, or $\sum_{i=1}^{n} x_i \neq 0$, which implies $\lambda = 0$, which cannot be the case because we are assuming $\lambda \neq 0$.

Chapter 5

Algorithms and MATLAB Programs

This chapter consists of two sections. In Section 5.1, we include two algorithms. In Section 5.2, we write MATLAB programs for techniques studied in this thesis.

5.1 Algorithms

The following algorithms can be found in [**18**].

Algorithm 5.1.1 : Computing *y = Ax*

$y = cP^Tx;$

$w = \|x\|_1 - \|y\|_1;$

$y = y + wv;$

Algorithm 5.1.2: Power Method

function $x^{(n)}$ = PowerMethod() {
$x^{(0)} = v;$
$k = 1;$
repeat
$x^{(k)} = Ax^{(k-1)}$;
$\delta = \|x^{(k)} - x^{(k-1)}\|_1$;
$k = k + 1;$
until $\delta < \varepsilon$;
}

5.2 MATLAB Programs

5.2.1 Determinant of Rank-One Updates

The following function computes the determinant of a rank-one perturbation of an n x n random matrix A.

```
function [A, detA, B, detB] = det_of_perturbation(n)
      % Description: Choose an arbitrary n x n matrix A,
      %        find a rank-one perturbation matrix B of A,
      %        calculate the determinant of both A and B and
      %        display A, B, det(A)and det(B).
      % Usage: [A, det(A), B, det(B)]= det_of_pertirbation(n),
      %          where n is any  positive integer.

        A = randn(n,n);
        u = randn(n,1);
        v = randn(n,1);

        if  det(A)== 0
            disp('A is singular')
        else
            detA = det(A);
            B = A + u*v';
        end

        detB = (1 + v'*inv(A)*u)*detA;
```

5.2.2 Sherman-Morrison Formula

```
function [invA, invB]= Sherman_Morrison_Formula(A, n)
      %Description: Calculate the inverse of a given n x n matrix A,
      %       find an arbitrary rank-one matrix B of A,
      %       calculate the inversr of B from the inverse of A using
      %       Sherman-Morrison formula.
      %Usage: For n=4, [invA, invB]= Sherman_Morrison_Formula(A, 4)

      invA = inv(A);  % n is the dimension of A and Ainv
                        is the inverse of A
      u = randn(n,1);
      v = randn(n,1);

      if  det(A) == 0
          disp('A is singular')
      else
          B = A + u*v';
          invB = invA - ((invA*u*v'*invA)/(1 + (v'*invA*u)));
      end
```

Example:

```
>> A = [5 -3 7; 3 9 -0.8; 0 5 2]


A =
```

```
    5.0000   -3.0000    7.0000
    3.0000    9.0000   -0.8000
         0    5.0000    2.0000

>> [invA,invB] = Sherman_Morrison_Formula(A,3)

Ainv =

    0.0944    0.1760   -0.2601
   -0.0258    0.0429    0.1073
    0.0644   -0.1073    0.2318



Binv =

    0.1012    0.1886   -0.2653
   -0.0254    0.0437    0.1070
    0.0645   -0.1071    0.2317
```

5.2.3 Centrosymmetric and Skew-Centrosymmetric Matrices

The following MATLAB programs checks whether a matrix is centrosymmetric/ skew-centrosymmetric or not respectively.

(i) Centrosymmetric Matrices

```
function [] = isCentrosymmetric(A,n)
   % Description: This function checks if an nxn matrix A is whether
   %              centrosymmetric or not.
   % Usage: []= isCentrosymmetric(A,4), for n=4, for example.

    j = 1;
    for i = 1:n
            if A(i,j)== A(n-i+1,n-j+1)
```

```
                j = j+1;
            else
                disp('The matrix is not centrosymmetric')
                break
          end
    end

    if j == n+1
        disp('The matrix is centrosymmetric')
    end
```

(ii) Skew-Centrosymmetric Matrices

```
function []= isSkewCentrosymmetric(A,n)
   % Description: This function checks if an nxn matrix A is whether
   %              skew-centrosymmetric or not.
   % Usage: []= isSkewCentrosymmetric(A,4), for n=4, for example.

   j = 1;
    for i = 1:n
            if A(i,j)== -A(n-i+1,n-j+1)
                j = j+1;
            else
                disp('The matrix is not  skew-centrosymmetric')
                break
            end
    end

    if j == n+1
        disp('The matrix is skew-centrosymmetric')
    end
```

5.2.4 Solution to an Updated Linear System

If a linear system $Ax = b$ is updated to $\breve{A}x = b$, where $\breve{A}$ is a rank-one updated matrix of A, then the solution of the new system can be derived from the oirignal one.

```
function y = Updated_Linear_System_Solution(n,A,b)

          %A is the coefficient matrix of dimension n
          %b is the constant vector
          %We call "Sherman_formula(A,n)" function to compute
          %the inverse of the updated coefficient matrix

          nn = n;
```

```
    invA = Sherman_formula(A, nn);
    y = invA * b;
```

5.2.5 Regular Magic Squares

The following MATLAB programs check if a matrix is magic square or regular magic square, respectively.

(i) Magic Square

```
function []= isMagicSquare(A,n)
    % Description: This function checks if an nxn matrix A is whether
    %              magic square or not.
    % Usage: []= isMagicSquare(A,3), for n=3, for example.

    i = 1; j = 1;
    rowSum = 0; columnSum = 0; diagSum = 0; cdiagSum = 0;
    u = (n^3 + n)/2;

    for k = 1:n
        diagSum = diagSum + A(k,k);
        cdiagSum = cdiagSum + A(k,n-k+1);
    end

    if diagSum == u & cdiagSum == u
       for i = 1:n
           for j = 1:n
               rowSum = rowSum + A(i,j);
               columnSum = columnSum + A(j,i);
           end
               if rowSum ~= u | columnSum ~= u
                  disp('The matrix is not magic square')
                  break
               end
           rowSum = 0; columnSum = 0;
       end
    else
        disp('The matrix is not magic square')
    end

    if i == n;
        disp('The matrix is magic square')
    end
```

(ii) Regular Magic Square

```
function []= isRegularMagicSquare(A,n)
    % Description: This function checks if an nxn matrix A is whether
    %              regular magic square or not.
    % Usage: []= isRegularMagicSquare(A,3), for n=3, for example.

    i = 1; j = 1;
    rowSum = 0; columnSum = 0; diagSum = 0; cdiagSum = 0;
    u = (n^3 + n)/2;

    for k = 1:n
        diagSum = diagSum + A(k,k);
        cdiagSum = cdiagSum + A(k,n-k+1);
    end

    if diagSum == u & cdiagSum == u
       for i = 1:n
         for j = 1:n
             rowSum = rowSum + A(i,j);
             columnSum = columnSum + A(j,i);
         end
             if rowSum ~= u | columnSum ~= u
                disp('The matrix is not regular magic square.')
                break
             end
         rowSum = 0; columnSum = 0;
       end
    else
        disp('The matrix is not regular magic square.')
    end

    if i == n
        for i = 1:n
            for j = 1:n
                if A(i,j) + A(n-i+1,n-j+1) ~= n^2 + 1
                   disp('The matrix is not regular magic square.')
                   break
                end
            end
        end
    end

    if i == n
        disp('The matrix is regular magic square.')
    end
```

5.2.6 Power Method

```
function x = PowerMethod(A,x0,error)
%Description: This function returns an approximation of the
eigenvector
%              corresponding to the dominant eigenvalue within an error
%              bound.
%Usage: x = PowerMethod(A,x0,error)

    [N] = size(A); n = N(1,1);
    y = x0; normdif = 0;
    for i = 1:n
        x = y;
        xi = A*y;
        y = xi;
        norm_of_difference = norm(y,1)-norm(x,1);
        if lt(norm_of_difference,error)
            x = y
            break
        end
    end
```

Chapter 6
Conclusions and Future Work

This thesis focused on rank-one perturbations of matrices and their effects on centrosymmetric and Laplacian matrices. Properties of perturbed matrices that have been studied include determinants, eigenvalues and eigenvectors. The thesis also incl-uded MATLAB programs for some techniques that were studied and applications of rank-one perturbations.

Recently, the importance of rank-one perturbed matrices and their applications have increased significantly. That is why we will continue our work and try to gener-alize theorem stated in the thesis.

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