The Arab American University

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Numerical Solution of General Higher-Dimensional Nonlinear Fredholm-Volterra Integral Equations Using Chebyshev Approximation

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Committee Decision

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This thesis was defended successfully on April-2018 and approved by

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Declaration

I, the undersigned, the author of the Master thesis entitled Numerical Solution of General Higher-Dimensional Nonlinear Fredholm-Volterra Integral Equations Using Chebyshev Approximation .

Where I submitted to the Arab American University for a master's degree and that it is the result of my own research, except ad indicated, of which none has been offered for a higher degree to any university or other educational institution.

Signature...........................

Date: ....

Dedication

To my dear father and mother who supported me in my educational career and encouraged me to continue my studies and helped me in all aspects, including moral and material.

Acknowledgements

I cordially dedicate this work to my loving family who were always there for me, my sincere friends with whom I shared every single moment of happiness, and who always went the extra mile for me to make me feel special, my great supervisor Dr. Iyad Suwan , who has generously helped me throughout to the research, offering guidance and various support with everything I needed.

Special thanks also goes to my university and particularly my Mathematics Department which hosted me throughout the course of my studies, and provided me with all the means necessary to successfully complete this work.

Abstract

In this work, we study computational methods for solving general one, two and three dimensional nonlinear Fredholm-Volterra Integral Equations of the second kind Using Chebyshev Approximation. The method is based on replacement unknown functions by truncted series of well known Chebyshev expantion function. The final result will be compared with published experimental and theoritical results. Further, in order to find the approximated solution, the Fredholm-Volterra integral Equation of the second kind is converted to a system of non-linear equation using the Chebyshev approximation. Finally, many numerical examples were provided to demonstrate the applicability and the accuracy of the presented method.

Contents

List of Tables

List of Figures

Introduction

The integral equation is the one in which the unknown function $U(x)$ appears inside an integral sign [41]. The most standard form among all the forms of the integral equation is [31,41]:

$$U(x) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t)U(t)dt, \tag{1}$$

λ is considered as constant parameter, g(x) , h(x) are the limits of integration, and k(x, t) is a known function of two variables x and t which is called the kernel of the integral equation.
The unknown function$U(x)$ can be seen inside the integral sign or outside it, where the boundaries may be constants, variables, or mixed.
There are a lot of forms for the integral equations, but there are two distinct forms depend on the integral boundaries. The boundaries of integration can define the type of integral equations as follows [41] :

1. If the limits of integration are fixed numbers, the integral equation is called a Fredholm integral equation FIE [12] :

$$U(x) = f(x) + \lambda \int_{a}^{b} k(x,t)U(t)dt \tag{2}$$

   where a and b are constants.

2. If at least one limit is a variable, the equation is called Volterra Integral Equation (VIE) [41]:

$$U(x) = f(x) + \lambda \int_{a}^{x} k(x,t)u(t)dt \tag{3}$$

   There are two distinct types based on the appearance of the unknown function $U(x)$ [31,41] :

(a) If the unknown function $U(x)$ appears only under the integral sign of Fredholm or Volterra equation, the integral equation is called a first kind Fredholm or Volterra integral equation as:

$$f(x) = \lambda \int_a^b k(x,t)U(t)dt \tag{4}$$

and

$$f(x) = \lambda \int_a^x k(x,t)U(t)dt \tag{5}$$

(b) If the unknown function $U(x)$ appears both inside and outside the integral sign of Fredholm or Volterra equation, the integral equation is called a second kind Fredholm or Volterra equation integral equation as:

$$U(x) = \lambda \int_a^b k(x,t)U(t)dt \tag{6}$$

and

$$U(x) = \lambda \int_a^x k(x,t)U(t)dt \tag{7}$$

Another way to classify integral equations is according to linearity and homogeneity, and these two types play a major role in the form of solutions. Now, we highlight these two types:

If $U(x)$ is linear, the integral equation is called linear,other wise it is nonlinear.

In all Fredholm or Volterra integral equations presented above, if $f(x)$ is identically zero, the resulting equations are:

$$U(x) = \lambda \int_a^b k(x,t)U(t)dt \tag{8}$$

and

$$U(x) = \lambda \int_a^x k(x,t)U(t)dt \tag{9}$$

is called homogeneous equation.

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modelling of the spatio-temporal of an epidemic, and form various physical and biological model [1,30,41] .

The general form of Fredholm-Volterra equations as [1,41]:

$$U(x) = f(x) + \lambda_1 \int_a^x K_1(x,t)U(t)dt + \lambda_2 \int_a^b K_2(x,t)U(t)dt. \tag{10}$$

It's notable that the integral equations contain mixed of Volterra and Fredholm. Moreover, the unknown function $U(x)$ appear inside and outside all of the integral signs.

If $U(x)$is linear, the integral equation is called linear,other wise it is nonlinear. We say that the integral equation of the second kind is homogeneous if the function f(x) equals to zero, otherwise it is non-homogeneous.

There are many ways to solve integral equations including analytical and numarical methods [4,13,23]. These analytical methods include: the Adomian decomposition method, the modified decomposition method, the succcessive approximations, the series solution method and the conversion to initial value problem [36]. Many numerical methods exist to solve the integral equations in the one, two and three dimensions. like the two-dimensional triangular functions and their applications to nonlinear $2D$ Volterra–Fredholm integral equations [6], the numerical solution of two-dimensional Volterra integral equations by collocation and iterated collocation methods [8],extrapolation method of iterated collocation solution for two-dimensional nonlinear Volterra integral equations [20],a new differential transformation approach for two-dimensional Volterra integral equations [21] ,solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method [40] ,applying the modified block-pulse functions to solve the three-dimensional Volterra–Fredholm integral equations [15] , a numerical method for solving Fredholm–Volterra integral equations in two-dimensional spaces using block pulse functions and an operational matrix [5],a computational method for solving a class of two dimensional Volterra integral equations [7], the numerical solution of nonlinear two-dimensional Volterra-Fredholm integral equations of the second kind based on the radial basis functions approximation with error analysis [11],a new numerical method for solving two-dimensional Volterra-Fredholm integral equations [16],applying the modified block-pulse functions to solve the three-dimensional Volterra–Fredholm integral equations [32], two-dimensional legendre wavelets and their applications to integral equations [38],a matrix based method for two dimensional nonlinear Volterra-Fredholm integral equations [24], a new method based on Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations in [27],a computational method based on hybrid of block-pulse functions and Taylor series for solving two-dimensional nonlinear integral equations [33], numerical solution of a class of two-dimensional nonlinear Volterra integral equations using Legendre polynomials [35], application of triangular functions to numerical solution of stochastic Volterra integral equations [28],applications of two-dimensional triangular functions for solving nonlinear class of mixed Volterra-Fredholm integral equations [29].

In this research, we have extended the method of solving nonlinear Fredholm-Volterra integral equations of the second kind using Chebyshev approximation in [17] ,in which the first dimension of the Fredholm-Volterra nonlinear integral equation of the second kind was solved. The general idea of the solution is converting the equation into systems of non-linear equations.

In chapter one, we present general concepts like Clenshaw-Curtis Quadrature, Newton's

Method for System of Nonlinear Equations, and Orthogonal Polynomials. In chapters three, four and five, Chebyshev Polynomial, Approximation with Cebyshev Polynomial , method of solution of one ,two, and three dimensional Fredholm-Volterra integral equations, and numerical example are presented.

Chapter One

Preliminaries

In this chapter, we will present the basic terms that will be used throughout this work:Inner product, Orthognality of Polynomyals,Discrete Chebyshev transforms, The product operational matrix, Newton's Method for system of nonlinear equation,and Clenshaw-curtis quadrature.

1.1 Inner Product

Recall that if u, and v are two vectors in vector space, then the inner product $< u,v >$ is a function satisfying the following conditions $[18,25,43]$:

1. $< u+v,w >=< u,w > + < v,w >$, where w is a vector.
2. $< u,v >=< v,u >$
3. $< u,u >\leq 0$ with equality if and only if $u = 0$
4. $< Ku,v >= K < u,v >$, K is scalar

Throughout this work, we will deal with real valued functions defined on a finite interval $[a,b]$. The inner product of two functions f and g on an interval $[a,b]$ is the number $< f,g >$ given by $[25,43]$:

$$< f,g >= \int_a^b f(x)g(x)dx$$

1.2 Normed spaces

Definition 1.1. [10] Let X be a vector space over $\mathbb{K}$. A norm on X is a map $\|.\| : X \longrightarrow [0,\infty)$ that satisfies the following three properties.

1) $\|x\| = 0$ implies $x = 0$

2) $\|\lambda x\| = |\lambda|.\|x\|$ for $x \in X$ and $\lambda \in \mathbb{K}$.

3) $\|x+y\| \leq \|x\| + \|y\|$.

A normed space is a pair $(X, \|.\|)$ where X is a vector space and $\|.\|$ is a norm on X.

1.3 Chebyshev Equations

Definition 1.2. [3, 39] The Chebyshev equation is given by:

$$(1-x^2)y'' - xy' + \lambda^2 y = 0 \tag{11}$$

of which 0 is an ordinary point.

1.4 Chebyshev Polynomial

In this chapter we give definition and properties of chebyshev polynomials (CHP) in one-dimensional space.

Definition 1.3. [2, 14] If $t = cos(\theta)(0 \leq \theta \leq \pi)$, the function

$$T_n(t) = cos(n\theta) = cos(n\,cos^{-1}t)\,,$$

is a polynomial of t of degree n where $(n = 0, 1, 2, ...)$, T_n is called the Chebyshev Polynomial of degree n when θ in crease from 0 to π, t decreas from 1 to -1,Then the interval $[-1,1]$ is domain of definition of $T_n(t)$.

Also, that (CHP) are derived from the following recursive formula [2,14,17,37]:

$$T_0(t) = 1 \tag{12}$$

$$T_1(t) = t \tag{13}$$

$$T_{(n+1)}(t) = 2tT_n(t) - T_{(n-1)}(t), n = 1,2,3,... \tag{14}$$

Definition 1.4. [31] The Chebyshev polynomial $U_n(x)$ of the second kind is a polynomial of degree n in x defined by:

$$U_n(x) = \frac{sin(n+1)\theta}{sin\theta} \quad when\, x = cos\theta \tag{15}$$

The ranges of x and θ are the same as for $T_n(x)$. Elementary formulae give

$$sin1\theta = sin\theta, \quad sin2\theta = 2sin\theta cos\theta, \quad sin3\theta = sin\theta(4cos^2\theta - 1),$$

$$sin4\theta = sin\theta(8cos^3\theta - 4cos\theta), ...$$

so that we see that the ratio of sine functions (15) is indeed a polynomial in $cos\theta$, and we may immediately deduce that

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_2(x) = 4x2 - 1, \quad U_3(x) = 8x^3 - 4x,$$

Coefficients of all polynomials up to degree n.

1.4.1 Orthogonality of Chebyshev polynomials

Definition 1.5. [19] A locally integrable function on $\mathbb{R}^n$ that takes values in the interval $(0,\infty)$ almost everywhere is called a weight. So by definition a weight function can be zero or infinity only on a set whose Lebesgue measure is zero.

Definition 1.6. [31] Two functions $f(x)$ and $g(x)$ in $L_2[a,b]$ are said to be Orthogonal on the interval $[a,b]$ with respect to a given continuous and non-negative weight function $w(x)$ if:

$$\int_a^b w(x)f(x)g(x)dx = 0 \tag{16}$$

And to make it clear, we use the "inner product" notation,

$$< f,g >= \int_a^b w(x)f(x)g(x)dx = 0 \tag{17}$$

An inner product defines an $L_2 - type$ norm:

$$||f|| = ||f||_2 = \sqrt{< f,f >} \tag{18}$$

We shall adopt the inner product (17) (with various weight functions) and the associated L_2 norm (18). We shall in particular be concerned with families of orthogonal polynomials$\phi_i(x), i = 0,1,2,...$ where ϕ_i is of degree i exactly, defined so that:

$$< \phi_i, \phi_j >= 0, (i \neq j) \tag{19}$$

Clearly, since $w(x)$ is non-negative,

$$< \phi_i, \phi_i >= ||\phi_i||_2 > 0. \tag{20}$$

we say that the family is orthonormal if, in addition to (78), the functions $\phi_i(x)$ satisfy [31]:

$$||\phi_i|| = 1, for\, all\, i \tag{21}$$

1.4.2 Chebyshev polynomials as orthogonal polynomials

If we define the inner product (17) using the interval and weight function [17, 31]:

$$[a,b] = [-1,1], \quad w(x) = \frac{1}{\sqrt{1-x^2}} \tag{22}$$

then we find that the first kind Chebyshev polynomials satisfy [31]:

$$\int_{-1}^{1} T_i(x)T_j(x)w(x)dx = \begin{cases} 0 & \text{if } i \neq j \\ \pi/\gamma_i & \text{if } i = j \end{cases}$$

where,

$$\gamma_i = \begin{cases} 1 & \text{if } i = 0 \\ 2 & \text{if } i \leq 0, \end{cases}$$

In order to prove that,

$$< T_i, T_j >= \int_{-1}^{1} \frac{T_i(x)T_j(x)}{\sqrt{1-x^2}} dx = \int_{0}^{\pi} \cos i\theta \cos j\theta d\theta \tag{23}$$

shown by setting $x = cos\theta$ and using the relations, $T_i = cosi\theta$ and,

$dx = -sin\theta d\theta = -\sqrt{1-x^2}d\theta$

Now, for $i \neq j$,

$$\int_{0}^{\pi} \cos i\theta \cos j\theta d\theta = \frac{1}{2}\int_{0}^{\pi} [cos(i+j)\theta + cos(i-j)\theta] d\theta$$

$$\frac{1}{2}[\frac{sin(i+j)\theta}{i+j}+\frac{sin(i-j)\theta}{i-j}]_0^\pi=0$$

Hence,

$$<T_i,T_j>=0,(i\neq j,) \tag{24}$$

and $T_i(x), i=0,1,...$ forms an orthogonal polynomial system on $[-1,1]$ with respect to the weight $\frac{1}{\sqrt{1-x^2}}$. The norm of T_i is given by [31]

$$||T_i||^2 = <T_i,T_i>$$

$$= \int_0^\pi (\cos i\theta)^2 d\theta$$

$$= \frac{1}{2}\int_0^\pi (1+\cos 2i\theta)d\theta$$

$$= \frac{1}{2}[\theta+\frac{sin2i\theta}{2i}]_0^\pi, \quad (i\neq 0)$$

$$= \frac{\pi}{2} \tag{25}$$

While,

$$||T_0||^2 = <T_0,T_0>=<1,1>=\pi \tag{26}$$

The system T_i is therefore not orthonormal. We could, if we wish, scale the polynomials to derive the orthonormal system [31]

$$\frac{1}{\pi}T_0(x), \frac{2}{\pi}T_i(x), i=1,2,..., \tag{27}$$

but the resulting irrational coefficients usually make this inconvenient. It is simple in partice to adopt the T_i we defined initially, taking note of the values of their norms (25).

1.5 Discrete Chebyshev transforms and the fast Fourier transform

Using the values of a function$f(x)$ at the extrema y_k of $T_n(x)$. given by [31]:

$$y_k = cos(\frac{k\pi}{n}), (k = 0, ..., n) \tag{28}$$

we can define a discrete Chebyshev transform

$$\hat{f}(y_k) = \sqrt{\frac{2}{n}} \sum_{j=0}^{n}{}'' T_k(y_i) f(y_i), (k = 0, ..., n). \tag{29}$$

Where the double prime indicate that the first and the last term of the summation should be halved, values of $\hat{f}(y_k)$ are in fact proportional to the coefficients in the interpolant of $f(y_k)$ by a sum of Chebyshev polynomials.

Using the discrete orthogonality, namely [31]

$$\sum_{k=0}^{n}{}'' T_i(y_k) T_i(y_k) = \begin{cases} 0 & \text{if } i \neq j; i, j \leq n \\ \frac{n}{2} & \text{if } 0 < i = j < n \\ n & \text{if } i = j = 0 \, or \, n \end{cases}$$

We can easily deduce that the inverse transform is given by

$$f(y_i) = \sqrt{\frac{2}{n}} \sum_{k=0}^{n}{}'' T_k(y_i) \hat{f}(y_k), \qquad (j = 0, ..., n). \tag{30}$$

In fact, since

$$T_k(y_j) = \cos\frac{jk\pi}{n} = T_j(y_k) \tag{31}$$

which is symmetric in j and k, it is clear that the discrete Chebyshev transform is self-inverse It is possible to define other forms of discrete Chebyshev transform, based on any of the other discrete orthogonality relations.

The discrete Chebyshev transform defined here is intimately connected with the discrete Fourier (cosine) transform. Defining

$$\phi_k = \frac{k\pi}{n} \tag{32}$$

(the zeros of $sin n\theta$) and

$$g(\theta) = f(cos\theta), \hat{g}(\theta) = \hat{f}(cos\theta) \tag{33}$$

then

$$\hat{g}(\phi_k) = \sqrt{\frac{2}{n}}\sum_{j=0}^{n}{}'' \cos\frac{jk\pi}{n} g(\phi_i), (k = 0, ..., n). \tag{34}$$

Where the double prime indicate that the first and the last term of the summation should be halved.

Since $cos\theta$ and therefore $g(\theta)$ are even and $2\pi - periodic$ functions of ϕ, (34) has alternative equivalent expressions [31]

$$\hat{g}(\frac{k\pi}{n}) = \sqrt{\frac{1}{2n}}\sum_{j=-n}^{n}\cos(\frac{jk\pi}{n})g(\frac{j\pi}{n}), (k = -n, ..., n). \tag{35}$$

or

$$\hat{g}(\frac{k\pi}{n}) = \sqrt{\frac{1}{2n}} \sum_{j=-n}^{n} exp(\frac{ijk\pi}{n}) g(\frac{j\pi}{n}), (k = -n, ..., n). \tag{36}$$

or

$$\hat{g}(\frac{k\pi}{n}) = \sqrt{\frac{1}{2n}} \sum_{j=0}^{2n-1} exp\frac{ijk\pi}{n} g(\frac{j\pi}{n}), (k = 0, ..., 2n-1). \tag{37}$$

The formulas (36) and (37) define the general discrete Fourier transform, applicable to functions $g(\phi)$ that are periodic but not necessarily even.

1.6 Taylor Expansion

Definition 1.7. [9] Suppose that $f(x)$ has a power series expansion at x = a with radius of convergence$R > 0$, then the series expansion of$f(x)$ takes the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ... \tag{38}$$

that is, the coefficient c_n in the expansion of $f(x)$ centered at $x = a$ is precisely $c_n = \dfrac{f^{(n)}(a)}{n!}$. The expansion (38) is called Taylor series.

1.7 Newton's Method for System of nonlinear equations

One of the repetitive methods used to solve nonlinear systems of algebraic equations is the Newton Rphson method. The system of n- nonlinear equations in n unknowns is given by:

$$\begin{aligned} f_1(x_1,x_2,...,x_n) &= 0 \\ f_2(x_1,x_2,...,x_n) &= 0 \\ &\vdots \\ f_n(x_1,x_2,...,x_n) &= 0 \end{aligned}$$

This system can be written using a single form as [25] :

$$F(X)=0$$

$$where \quad X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} and \;\; F=\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}, 0\; is\; the\; zero\; vector. \tag{39}$$

As in newton method of one variable, we need to start with an initial guess X_0. Theoretically, the more variables one has, the harder it is to find the initial guess. In practice, that must be overcomed by reasonable assumptions about the possible values of the solution. Once X_0 is chosen, let $\Delta X = X_1 - X_0$, then it can be approximated around the vector X_0 using Taylor expansion as follows:

$$F(X_0+\Delta X) \approx F(X_0) + J(F(X_0))\Delta X, \tag{40}$$

where

$$J = \frac{\partial(f_1, f_2, ..., f_n)}{\partial(x_1, x_2, ..., x_n)}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix},$$

J is called the Jacobian.

Newton's method is based on constructing of a sequence of vectors that converges to X, such that $F(X) = 0$. Let F be a continuously differentiable function at X_0, and the target is to find X that makes $F(X)$ equal to the zero vector. If $X = X_0$ represents the first guess for the solution, successive approximations to the solution are obtained from [26]

$$X_{n+1} = X_n - J^{-1}(F(X_n))F(X_n)$$

with $X_{n+1} = \Delta X + X_n$.

A convergence criterion of the solution of the system of nonlinear equations could be ,for example, that the maximum of the absolute values of the function $f_i(X_n)$ is smaller that a certain tolerance ε,

$$Max_i|f_i(X_n)| < \varepsilon$$

Another possibility for convergence is that the magnitude of the vector $F(X_n)$ be smaller than the tolerance

$$|F(X_n)| < \varepsilon$$

We can also use the difference consecutive values of the solution [25, 26]

$$Max_i|(X_i)_{n+1} - (X_i)_n| < \varepsilon$$

or

$$|\Delta X| = |X_{n+1} - X_n| < \varepsilon$$

.

1.8 Clenshaw-Curtis Quadrature Formula

In [12], Clenshaw and Curtis set a formula for integration of f as follow:

$$\int_{-1}^{1} f(x)dx \approx \sum_{k=0}^{N}{}'' w_k f(cos\frac{nk\pi}{N}) \tag{41}$$

where

$$w_k = \frac{4}{N}\sum_{\substack{even\\ n}}^{N}{}'' \frac{1}{1-n^2}\cos(\frac{nk\pi}{N}), \tag{42}$$

and the double prime means that the first and the last term of the summation should be halved.

Lemma 1.1 The Clenshaw-Curtis formula of 2D is given by:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \approx \sum_{k=0}^{N}{}'' \sum_{l=0}^{M}{}'' w_k a_l f(\cos(\frac{nk\pi}{N}), \cos(\frac{ml\pi}{M})) \tag{43}$$

where,

$$w_k = \frac{4}{N}\sum_{\substack{even\\n}}^{N}{}'' \frac{1}{1-n^2}\cos(\frac{nk\pi}{N}),$$

$$a_l = \frac{4}{M}\sum_{\substack{even\\n}}^{M}{}'' \frac{1}{1-n^2}\cos(\frac{nl\pi}{M}), \tag{44}$$

Proof:

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = \int_{-1}^{1}\Big(\int_{-1}^{1} f(x,y)dx\Big)dy$$

According to equation (42), we get:

$$\int_{-1}^{1}\Big(\int_{-1}^{1} f(x,y)dx\Big)dy = \int_{-1}^{1}\Big(\sum_{k=0}^{N}{}'' w_k f(\cos(\frac{nk\pi}{N}),y))\Big)dy =$$

$$= \sum_{k=0}^{N}{}'' \sum_{l=0}^{M}{}'' w_k a_l f(\cos(\frac{nk\pi}{N}),\cos(\frac{ml\pi}{M}))$$

where,

$$w_k = \frac{4}{N}\sum_{\substack{even\\n}}^{N}{}'' \frac{1}{1-n^2}\cos(\frac{nk\pi}{N}),$$

$$a_l = \frac{4}{M}\sum_{\substack{even\\n}}^{M}{}'' \frac{1}{1-n^2}\cos(\frac{nl\pi}{M}), \tag{45}$$

Lemma 1.2 The Clenshaw-Curtis formula in 3D is:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)dxdydz \approx \sum_{k=0}^{N}{}'' \sum_{l=0}^{M}{}'' \sum_{m=0}^{L}{}'' w_k a_l b_m f(\cos(\frac{nk\pi}{N}),\cos(\frac{ml\pi}{M}), f(\cos(\frac{nm\pi}{L})) \tag{46}$$

where,

$$w_k = \frac{4}{N}\sum_{\substack{even\\n}}^{N}{}'' \frac{1}{1-n^2}\cos(\frac{nk\pi}{N}),$$

$$a_l = \frac{4}{M} \sum_{\substack{even\\n}}^{M}{}'' \frac{1}{1-n^2} \cos(\frac{nl\pi}{M}), \tag{47}$$

$$b_m = \frac{4}{L} \sum_{\substack{even\\n}}^{M}{}'' \frac{1}{1-n^2} \cos(\frac{nm\pi}{M}), \tag{48}$$

Proof:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)dxdydz) = \int_{-1}^{1}\Big(\int_{-1}^{1}\Big(\int_{-1}^{1} f(x,y,z)dx\Big)dy\Big)dz$$

According to equation (42), we get:

$$\int_{-1}^{1}\Big(\int_{-1}^{1}\Big(\int_{-1}^{1} f(x,y,z)dx\Big)dy\Big)dz \approx \int_{-1}^{1}\Big(\int_{-1}^{1}\Big(\sum_{k=0}^{N}{}'' w_k f(\cos(\frac{nk\pi}{N}),y,z)\Big)dy\Big)dz \tag{49}$$

$$= \int_{-1}^{1}\Big(\sum_{k=0}^{N}{}''\sum_{l=0}^{M}{}'' w_k a_l f(\cos(\frac{nk\pi}{N}),\cos(\frac{ml\pi}{M}),z)\Big)dz \tag{50}$$

$$= \sum_{k=0}^{N}{}''\sum_{l=0}^{M}{}''\sum_{m=0}^{L}{}'' w_k a_l b_m f(\cos(\frac{nk\pi}{N}),\cos(\frac{ml\pi}{M}), f(\cos(\frac{nm\pi}{L})) \tag{51}$$

where,

$$w_k = \frac{4}{N} \sum_{\substack{even\\n}}^{N}{}'' \frac{1}{1-n^2} \cos(\frac{nk\pi}{N}),$$

$$a_l = \frac{4}{M} \sum_{\substack{even\\n}}^{M}{}'' \frac{1}{1-n^2} \cos(\frac{nl\pi}{M}), \tag{52}$$

$$b_m = \frac{4}{L} \sum_{\substack{even\\n}}^{M}{}'' \frac{1}{1-n^2} \cos(\frac{nm\pi}{M}), \tag{53}$$

,

Chapter Two

One-Dimensional Nonlinear Fredholm-Volterra Integral Equation

In this chapter, we present a numerical solution of nonlinear one dimensional Fredholm-volterra Integral Equation of the second kind $1D-FVIE$ of the form [17]

$$U(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)y(t,U(t))dt + \lambda_2 \int_0^1 K_2(x,t)g(t,U(t))dt, 0 \leq x,t \leq 1 \tag{54}$$

where $U(X)$ is an unknown function, $f(x), y(t,U(t))$, and $g(t,U(t))$ are continous functions on $[-1,1]$.

The nonlinear Fredholm-Volterra integral equation $FVIE$ arises from varios phisical and biological models . The essential features of these models are of wide applications [42]. As result of this type of problems, there numerical solution becomes very difficult to reach.

2.1 Approximation With Chebyshev Polynomial

Chebyshev polynomyals are important in numerical anlysis and approximation theory. A function $U(t)$ over $[-1,1]$ may be represented by Chebyshev Polynomials series as [17, 31, 37] :

$$U(t) = \sum_{i=0}^{\infty} C_i T_i(t) \tag{55}$$

If the infinite series in (55) is truncated, then (55) can be written as:

$$U(t) \approx \sum_{i=0}^{N} C_i T_i(t) \tag{56}$$

$$= T(t)^T C, \tag{57}$$

where,

$$T(t) = [T_0(t), T_1(t), ..., T_N(t)]^T \tag{58}$$

and,

$$C = [c_0, c_1, ..., c_N]^T \tag{59}$$

and,

$$c_i = \frac{\gamma_i}{\pi} \int_{-1}^{1} U(t) T_i(t) w(t) dt \tag{60}$$

Also, $w(t)$ is the weight function.

2.2 Method of solution

consider the nonlinear integral equation (54). we start to approximate $U(t)$ as equation (57). then we substitute (57) into equation (54)to get:

$$C^T T(x) = f(x) + \lambda_1 \int_0^x K_1(x,t) y(t, C^T T(t)) dt + \lambda_2 \int_0^1 K_2(x,t) g(t, C^T T(t)) dt, \tag{61}$$

In order to use Gaussian integeration formula for equation (61) , we transfer the intervals $[0, x_i]$ into interval $[-1, 1]$ by transformations :

$$\tau_1 = \frac{2t}{x_i} - 1 \tag{62}$$

$$\tau_2 = 2t - 1,$$

For Chebyshev polynomial we consider the Collocation points:

$$x_i = \cos(\frac{i\pi}{N}) \tag{63}$$

where, $i = 0, 1, ..., N$, Let

$$H_1(x,t) = K_1(x,t)y(t, C^T T(t)),$$

and,

$$H_2(x,t) = K_2(x,t)g(t, C^T T(t)),$$

Now, we use *Colloction points* (63) in transformed equation (61), to get:

$$C^T T(x_i) = f(x_i) + \lambda_1 \frac{x_i}{2} \int_{-1}^{1} H_1(x_i, \frac{x_i(\tau_1+1)}{2}) d\tau_1 + \frac{\lambda_2}{2} \int_{-1}^{1} H_2(x_i, \frac{(\tau_2+1)}{2}) d\tau_2, \tag{64}$$

Now, using Clenshaw-Curtis quadratrure formula, we get [12] [17] :

$$C^T T(x_i) = f(x_i) + \sum_{k=0}^{N}{}'' w_k [\lambda_1 \frac{x_i}{2} H_1(x_i, \frac{x_i(x_k+1)}{2}) + \frac{\lambda_2}{2} H_2(x_i, \frac{(x_k+1)}{2})] \tag{65}$$

for $i = 0, ..., N$, and $k = 0, ..., N$ where

$$w_k = \frac{4}{N} \sum_{\substack{even \\ n}}^{N}{}'' \frac{1}{1-n^2} \cos(\frac{nk\pi}{N}), \tag{66}$$

Where the double prime means that the first and the last term of the summation should be halved [12] .

The resulting system of equation consists of an equation of number (65) $N+1$ in addition to an unknown $N+1$ This system can be solved in the usual by iterative method such as simplex method or Newton Raphson method or any other method.

We can used The Fast Fourier Transform (FFT) techniqueto evaluate the summation part in (66) in $O(NlogN)$ operations. In fact the discrete cosine transformation of the vector v with entries [12] :

$$v_n = \begin{cases} \dfrac{2}{1-n^2} & \text{if } n\, even \\ 0 & \text{if } n\, odd. \end{cases}$$

The weights w_k there fore is computed directly in $O(N\,logN)$ operations,this will be the faster computation when we integrate functions in (64) using the same value of N.

2.3 Numerical Examples

In this section, we will present several illustrative examples and use our suggested method to solve these examples. All our calculations were using the Matlab programming. Our standard of accuracy is the maximum absolute error.

Example 1. Consider the following $1D$ nonlinear Fredholm integral equation given in [17] :

$$U(x) = exp(1)x + 1\int_0^1 (x+t)e^{U(t)}dt, \tag{67}$$

The exact solution is $U(t) = t$.

The numerical solution for this problem is given in the following tables and figures. The following tables show the exact solution, estimation, and the absolute error $||U-U_N||_2$ in some point of $[0,1]$

Table 1: The numerical solution at N=2

t	Estimation	Error
0.1	0.098372	0.001628
0.2	0.19849	0.0015069
0.3	0.29861	0.0013857
0.4	0.39874	0.0012646
0.5	0.49886	0.0011434
0.6	0.59898	0.0010223
0.7	0.6991	0.00090111
0.8	0.79922	0.00077996
0.9	0.89934	0.0006588
1	0.99946	0.00053765

$The\ max\ error = 0.0017492$ at $N = 2$

The exact and numerical solution at N=2

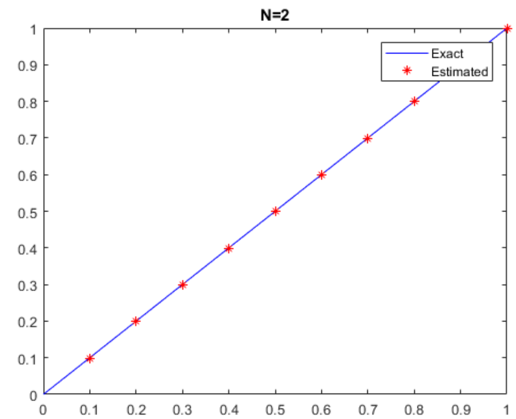


Figure 1: The exact and numerical solution at N=2

Table 2: The numerical solution at N=4

t	Estimation	Error
0.1	0.1	1.4099e-06
0.2	0.2	1.2961e-06
0.3	0.3	1.1823e-06
0.4	0.4	1.0686e-06
0.5	0.5	9.548e-07
0.6	0.6	8.4103e-07
0.7	0.7	7.2726e-07
0.8	0.8	6.1349e-07
0.9	0.9	4.9972e-07
1	1	3.8595e-07

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number (67) at $N = 4$, and $the\ max\ error = 1.5237e - 06$.

The exact and numerical solution at N=4

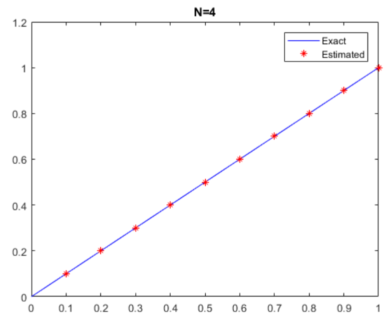


Figure 2: The exact and numerical solution at N=4

Table 3: Thenumerical solution at N=6

t	Estimation	Error
0.1	0.1	5.7158e-10
0.2	0.2	5.6673e-10
0.3	0.3	5.6194e-10
0.4	0.4	5.572e-10
0.5	0.5	5.5252e-10
0.6	0.6	5.4782e-10
0.7	0.7	5.4304e-10
0.8	0.8	5.3803e-10
0.9	0.9	5.3259e-10
1	1	5.2643e-10

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number (67) at $N = 6$, and *the max error* $= 5.57649e - 10$.

The exact and numerical solution at N=6

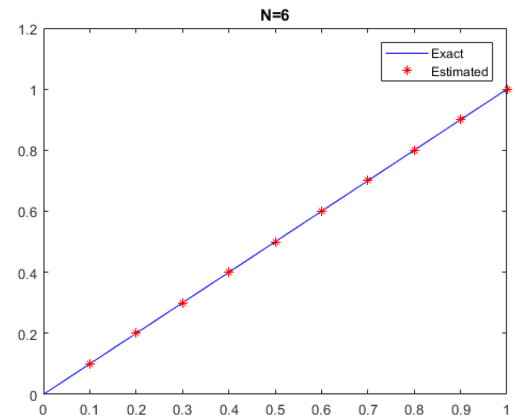


Figure 3: The exact and numerical solution at N=6

Table 4: The numerical solution at N=8

t	Estimation	Error
0.1	0.1	1.1263e-13
0.2	0.2	1.0172e-13
0.3	0.3	9.1704e-14
0.4	0.4	8.3544e-14
0.5	0.5	7.6938e-14
0.6	0.6	7.0499e-14
0.7	0.7	6.2172e-14
0.8	0.8	5.0071e-14
0.9	0.9	3.6859e-14
1	1	3.4639e-14

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number (67) at $N = 8$, and $the\ max\ error = 1.2336e-13$.

The exact and numerical solution at N=8

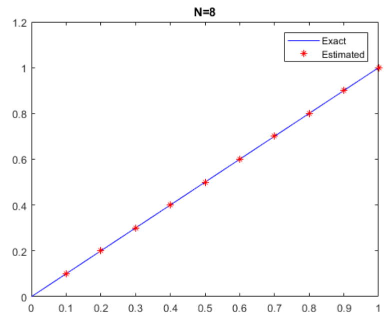


Figure 4: The exact and numerical solution at N=8

Table 5: The numerical solution at N=10

t	Estimation	Error
0.1	0.1	1.1668e-09
0.2	0.2	1.2915e-09
0.3	0.3	1.4172e-09
0.4	0.4	1.5436e-09
0.5	0.5	1.6701e-09
0.6	0.6	1.7956e-09
0.7	0.7	1.9199e-09
0.8	0.8	2.0444e-09
0.9	0.9	2.1706e-09
1	1	2.2943e-09

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number (67) at $N = 10$, and *the max error* $= 2.2943e - 09$.

The exact and numerical solution at N=10

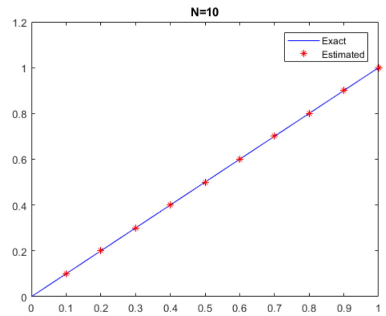


Figure 5: The exact and numerical solution at N=10

Example 2. Consider nonlinear $1D-FVIE$ given in [17]:

$$U(x) = 2cos(x) - 2 + 3\int_0^x (x-t)u^2(x)dt + \frac{6}{7-6cos1}\int_0^1 (1-t)cos^2(x)(t+U(t))dt, \quad (68)$$

The exact solution is $U(x) = cos(x)$.

and,

Fig. presents the approximate solution computed by the for $N = 2,4,6,8,10$. Table shows the numerical results given by the Chebyshev Approximation method.

The exact and numerical solution at N=2

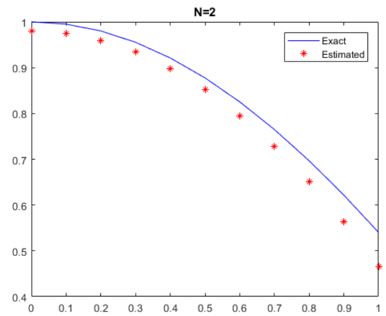


Figure 6: The exact andnumerical solution at N=2

Table 6: The numerical solution at N=2

t	exact	Estimation	Error
0	1	0.98005	0.01995
0.1	0.995	0.97491	0.020093
0.2	0.98007	0.95949	0.020572
0.3	0.95534	0.9338	0.021536
0.4	0.92106	0.89783	0.023232
0.5	0.87758	0.85158	0.026003
0.6	0.82534	0.79505	0.030283
0.7	0.76484	0.72825	0.036593
0.8	0.69671	0.65117	0.04554
0.9	0.62161	0.56381	0.057803
1	0.5403	0.46617	0.074132

$$The\ max\ error = 0.074132 \text{ at } N = 2$$

The exact and numerical solution at N=4

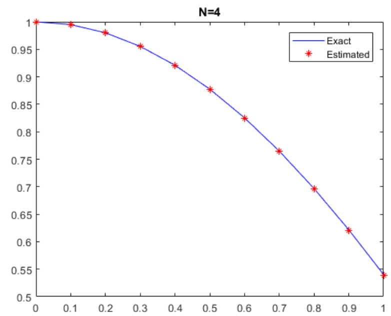


Figure 7: The exact and numerical solution at N=4

Table 7: The numerical solution at N=4

t	exact	Estimation	Error
0	1	0.99968	0.00031679
0.1	0.995	0.99469	0.0003155
0.2	0.98007	0.97975	0.00031433
0.3	0.95534	0.95502	0.00032087
0.4	0.92106	0.92071	0.00034611
0.5	0.87758	0.87718	0.00040199
0.6	0.82534	0.82484	0.00049794
0.7	0.76484	0.76421	0.00063657
0.8	0.69671	0.6959	0.00080846
0.9	0.62161	0.62062	0.00098609
1	0.5403	0.53919	0.0011171

$$The\ max\ error = 0.0011171 \text{ at } N = 4$$

The exact and numerical solution at N=6

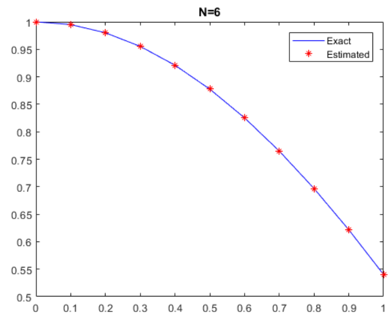


Figure 8: The exact and numerical solution at N=6

Table 8: The numerical solution at N=6

t	exact	Estimation	Error
0	1	1	5.2402e-07
0.1	0.995	0.995	4.8862e-07
0.2	0.98007	0.98007	4.1516e-07
0.3	0.95534	0.95534	3.8546e-07
0.4	0.92106	0.92106	4.8602e-07
0.5	0.87758	0.87758	7.5065e-07
0.6	0.82534	0.82533	1.1147e-06
0.7	0.76484	0.76484	1.4152e-06
0.8	0.69671	0.69671	1.4801e-06
0.9	0.62161	0.62161	1.359e-06
1	0.5403	0.5403	1.7553e-06

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number at $N = 6$, and $the\ max\ error = 1.7553e - 06$.

The exact and numerical solution at N=8

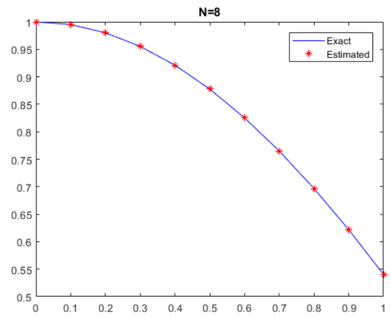


Figure 9: The exact and numerical solution at N=8

Table 9: The numerical solution at N=8

t	exact	Estimation	Error
0	1	1	6.5873e-10
0.1	0.995	0.995	5.1267e-10
0.2	0.98007	0.98007	2.6756e-10
0.3	0.95534	0.95534	3.2755e-10
0.4	0.92106	0.92106	9.0288e-10
0.5	0.87758	0.87758	1.6876e-09
0.6	0.82534	0.82534	1.9752e-09
0.7	0.76484	0.76484	1.3631e-09
0.8	0.69671	0.69671	6.5323e-10
0.9	0.62161	0.62161	1.5749e-09
1	0.5403	0.5403	2.4589e-09

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number 68 at $N = 8$, and $the\ max\ error = 2.4589e - 09$.

Table 10: The numerical solution at N=10

t	exact	Estimation	Error
0	1	1	3.3714e-08
0.1	0.995	0.995	3.4201e-08
0.2	0.98007	0.98007	3.567e-08
0.3	0.95534	0.95534	3.8156e-08
0.4	0.92106	0.92106	4.1709e-08
0.5	0.87758	0.87758	4.6382e-08
0.6	0.82534	0.82534	5.2206e-08
0.7	0.76484	0.76484	5.9159e-08
0.8	0.69671	0.69671	6.7131e-08
0.9	0.62161	0.62161	7.5899e-08
1	0.5403	0.5403	8.5078e-08

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number 68 at $N = 10$, and $the\ max\ error = 8.5075e - 08$.

Chapter Three

Two-Dimensional Nonlinear Fredholm-Volterra Integral Equation

In this chapter, we present a computational method for solving tow-dimensional nonlinear Fredholm-Volterra integral equations, $2D-FVIE$ of the second kind [6] ;

$$\begin{aligned} U(x,t) = f(x,t) + \lambda_1 \int_0^t \int_0^x K_1(x,t,y,z,U(y,z))dydz \\ +\lambda_2 \int_0^1 \int_0^1 K_2(x,t,y,z,U(y,z))dydz, (y,z) \in D \end{aligned} \tag{69}$$

where $U(x,t)$ is an unknown scalar valued function on $D = ([0,1] \times [0,1])$.
The function $K_1(x,t,y,z,U(y,z))$ and $K_2(x,t,y,z,U(y,z))$ are give function define on:
$W = \{(x,t,y,z,U) : 0 \leq x \leq s \leq 1, 0 \leq y \leq t \leq 1\}$,
For convenience, we put:

$$K_1(x,t,y,z,U(y,z)) = K_1(x,t,y,z)[U(y,z)]^p$$

$$K_2(x,t,y,z,U(y,z)) = K_2(x,t,y,z)[U(y,z)]^q$$

Where p and q are positive integer , moreover, $f(x,t)$ is a known function defined on D.

3.1 Two-Dimensional Chebyshev polynomial

we defined an $(N+1)^2$ set of Two-Dimensional Chebyshev polynomial by considering one-dimensional chebyshev polynomial as [37]:

$$T_{ij} = T_i(x)T_j(t), i, j = 0, 1, ..., N \tag{70}$$

Therefore, the Two-Dimensional Chebyshev polynomial basis vector is [37]:

$$T(x,t) = [T_0(x)T_0(t), T_0(x)T_1(t), ..., T_0(x)T_N(t), T_1(x)T_0(t), ..., T_N(x)T_N(t)]^T$$

$$= (A_N \otimes B_N)^T$$

where,

$$A_N = [T_0(x), T_1(x), .., T_N(x)]$$

and,

$$B_N = [T_0(t), T_1(t), .., T_N(t)]$$

are one dimensional chebyshev polynomial vectors. The orthogonanality property for these polynomials with respect to the weight function [37],

$$w(x,t) = \frac{1}{(\sqrt{1-x^2})(\sqrt{1-t^2})}$$

on the interval $[-1,1]\times[-1,1]$ is;

$$(T_{i,j}(x,t), T_{k,i}(x,t))_{w(x,t)} = \int_{-1}^{1}\int_{-1}^{1} \frac{T_{i,j}(x,t)T_{k,i}(x,t)dxdt}{(\sqrt{1-x^2})(\sqrt{1-t^2})}$$

$$= \begin{cases} \pi^2/4 & \text{if } i=k\neq 0, j=l\neq 0 \\ \pi^2/2 & \text{if } i=k=0, j=l\neq 0 \\ \pi^2/2 & \text{if } i=k\neq 0, j=1=0 \\ \pi^2 & \text{if } i=k=0, j=1=0 \\ 0 & \text{if } else \end{cases}$$

3.2 approximation with chebyshev polynomial

In this chapeter, we give definitions and properties of chebyshev polynomial (CHD) in two dimensional space.

Definition 3.8. [2] suppose $U(x,t)$ be a continous function on $[-1,1]\times[-1,1]$ for a given natural number N,M, we set:

$$U(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{ij} T_i(x) T_j(t) \tag{71}$$
$$(x,t) \in [-1,1]\times[-1,1]$$

where,

$$a_{ij} = \frac{< T_i(x), < U(x,t), T_i(t) >>}{< T_i(x), T_i(x) >< T_i(t), T_j(t) >} \tag{72}$$

and $< .,. >$ denotes the inner product space in $L^2([-1,1]\times[-1,1])$.

3.3 Method of solution

suppuse that $U(x,t)$ be a continous function on $[-1,1]\times[-1,1]$ a give natural number N,M we set:

$$U(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} T_i(x) T_j(t), \quad (x,t) \in [-1,1]\times[-1,1] \tag{73}$$

If the infinite seriese in (73) is truncated, then 73 can be written as:

$$U(x,t) \approx \sum_{i=0}^{N}\sum_{j=0}^{M} a_{ij} T_i(x) T_j(t), \quad (x,t) \in [-1,1]\times[-1,1] \tag{74}$$

where N,M is any Natural number,

$$i = 0, \ldots N$$

and

$$j = 0, ..., M$$

now, substitute (74) into (69) to get:

$$\begin{aligned} &\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}T_i(x)T_j(t) = f(x,t) \\ &+\lambda_1 \int_0^t \int_0^x k_1(x,t,y,z,\sum_{i=0}^{N}\sum_{j=0}^{M} C_{ij}T_i(y)T_j(z)))dydz \\ &+\lambda_2 \int_0^1 \int_0^1 k_2(x,t,y,z,\sum_{i=0}^{N}\sum_{j=0}^{M} C_{ij}T_i(y)T_j(z)))dydz \end{aligned} \tag{75}$$

Then, we transfere the intervals $[0,t_m],[0,x_n]$ and $[0,1]$ into interval $[-1,1]$ by transformations:

$$\alpha_1 = \frac{2}{x_n}y - 1 \tag{76}$$

$$\beta_1 = \frac{2}{t_m}z - 1$$

$$\alpha_2 = 2y - 1$$

$$\beta_2 = 2z - 1$$

now, for (CHP) we consider the colloction points:

$$\begin{aligned} x_n = &\quad \cos(\frac{n\pi}{N}), \quad n = 0, ..., N \\ t_m = &\quad \cos(\frac{m\pi}{M}), \quad m = 0, ..., M \end{aligned} \tag{77}$$

then, let $H_1(x,t,y,z) = k_1(x,t,y,z,\sum_{i=0}^{N}\sum_{i=0}^{M} C_{ij}T_i(y)T_j(z))$

and

$H_2(x,t,y,z) = k_2(x,t,y,z,\sum_{i=0}^{N}\sum_{i=0}^{M} C_{ij}T_i(y)T_j(z))$

now, we used transformed equations (76) we obtain

$$\begin{aligned}\sum_{i=0}^{N}\sum_{j=0}^{M} C_{ij}T_i(x)T_j(t) &= f(x,t)\\ &+\frac{\lambda_1 xt}{4}\int_{-1}^{1}\int_{-1}^{1} H_1(x,t,\frac{x}{2}(\alpha_1+1),\frac{1}{2}(\beta_1+1))d\alpha_1 d\beta_1\\ &+\frac{\lambda_2}{4}\int_{-1}^{1}\int_{-1}^{1} H_2(x,t,\frac{1}{2}(\alpha_2+1),\frac{1}{2}(\beta_2+1))d\alpha_2 d\beta_2\end{aligned} \tag{78}$$

using collocation points (77) in transformed equation (78) , we get:

then we used Clenshaw-Curtis formela to get:

$$\sum_{i=0}^{N}\sum_{j=0}^{M} CijT_i(x_n)T_j(t_m) = f(x_n,t_m)$$
$$+\sum_{k=0}^{N}{}''\sum_{l=0}^{M}{}'' w_k a_l[\frac{\lambda_1 x_n t_m}{m}(H_1(x_n,t_m,\frac{x_n}{2}(x_k+1),\frac{t_m(t_l+1)}{2}))$$
$$+\frac{\lambda_2}{4}H_2(x_n,t_m,(\frac{x_k+1}{2}),(\frac{t_l+1}{2})]$$

where,

$$n = 0,...,N$$

,

$$m = 0,...,M$$

$$k = 0,...,N$$

$$l = 0, ..., M$$

and,

$$w_k = \frac{4}{n} \sum_{\substack{even \\ n}}^{N}{}'' \frac{1}{1-n^2} \cos(\frac{nk\pi}{N})$$
$$a_l = \frac{4}{M} \sum_{\substack{even \\ m}}^{M}{}'' \frac{1}{1-m^2} \cos(\frac{mk\pi}{M}) \tag{79}$$

and the double prime indicate that the first and the last term of the summation should be halved.

Note that the obtained system from the previous equations contains$(N+1) \times (M+1)$ of the equations and also contains the same number of unknown, and these equations can be solved by Newton Raphson's method or any other method.

3.4 Numerical Result

In this section, we will present several illustrative examples and use our suggested method to solve these examples.All our calculations were using the Matlab programming. Our standard of accuracy is the maximum absolute error.

Example 3. Consider nonlinear $2D - FVIE$ given in [6]:

$$U(x,t) = f(x,t) - \int_0^1 \int_0^1 (xt + yz^2) U(y,z) dydz - \int_0^x \int_0^t (x+t+y+z)[U(y,z)]^2 dydz,$$

where $f(x,t) = x^2 + \frac{1}{4} + \frac{17}{6} Xt + \frac{7}{9} X^3 t^4 + \frac{29}{18} X^4 t^3 + \frac{6}{5} X^5 t^2 + \frac{11}{30} x^6 t$, which has the exact solution $U(x,t) = x^2 + 2xt$. We solve this example at $N = M = 3,4,6,8$. Consider the following tables:

Table 11: The numerical solution at N=M=3

x_Vals	t_Vals	Est	Tr	Err
0	0	0.99003	1	0.0099657
0.1	0.1	0.97099	0.98007	0.009076
0.2	0.2	0.91449	0.92106	0.0065749
0.3	0.3	0.82239	0.82534	0.0029435
0.4	0.4	0.6978	0.69671	0.0010913
0.5	0.5	0.54497	0.5403	0.0046649
0.6	0.6	0.36928	0.36236	0.0069177
0.7	0.7	0.17713	0.16997	0.0071642
0.8	0.8	-0.024121	-0.0292	0.005079
0.9	0.9	-0.22631	-0.2272	0.00088786
1	1	-0.42059	-0.41615	0.004448

$$The\ max\ error = 0.0099657 \text{ at } N = M = 3$$

Table 12: The numerical solution at N=M=4

N=M=4

x_Vals	t_Vals	Est	Tr	Err
0	0	0.99992	1	8.413e-05
0.1	0.1	0.98005	0.98007	2.0772e-05
0.2	0.2	0.9212	0.92106	0.00013963
0.3	0.3	0.82565	0.82534	0.0003168
0.4	0.4	0.69711	0.69671	0.00040494
0.5	0.5	0.54061	0.5403	0.00031164
0.6	0.6	0.36236	0.36236	5.7556e-06
0.7	0.7	0.16954	0.16997	0.00043125
0.8	0.8	-0.029957	-0.0292	0.00075794
0.9	0.9	-0.22774	-0.2272	0.0005371
1	1	-0.41527	-0.41615	0.00088042

$$The\ max\ error = 0.0011171 \text{ at } N = M = 4$$

Table 13: The numerical solution at N=M=6

N=M=6

x_Vals	t_Vals	Est	Tr	Err
0	0	1	1	3.7795e-07
0.1	0.1	0.98007	0.98007	1.2719e-07
0.2	0.2	0.92106	0.92106	1.1977e-06
0.3	0.3	0.82534	0.82534	1.7805e-06
0.4	0.4	0.69671	0.69671	9.4319e-07
0.5	0.5	0.5403	0.5403	1.2857e-06
0.6	0.6	0.36235	0.36236	3.4578e-06
0.7	0.7	0.16996	0.16997	3.2009e-06
0.8	0.8	-0.029199	-0.0292	7.9287e-07
0.9	0.9	-0.2272	-0.2272	4.9058e-06
1	1	-0.41615	-0.41615	5.2421e-06

$$The\ max\ error = 0.0011171 \text{ at } N = M = 6$$

Table 14: The numerical solution at N=M=8

N=M=8

x_Vals	t_Vals	Est	Tr	Err
0	0	1	1	3.7795e-07
0.1	0.1	0.98007	0.98007	2.5983e-07
0.2	0.2	0.92106	0.92106	1.9297e-08
0.3	0.3	0.82534	0.82534	2.7786e-07
0.4	0.4	0.69671	0.69671	3.3938e-07
0.5	0.5	0.5403	0.5403	1.5484e-07
0.6	0.6	0.36236	0.36236	1.377e-07
0.7	0.7	0.16997	0.16997	2.7873e-07
0.8	0.8	-0.0292	-0.0292	9.6825e-08
0.9	0.9	-0.2272	-0.2272	1.991e-07
1	1	-0.41615	-0.41615	1.8544e-07

$$The\ max\ error = 0.0011171 \text{ at } N = M = 8$$

Example 4. Consider nonlinear $2D-FVIE$ given in [6]:

$$U(x,t)=f(x,t)+16\int_0^1\int_0^1 e^{(x+t+y+z)}[U(y,z)]^3dydz+\int_0^x\int_0^t[U(y,z)]dydz,$$

where, $f(x,t)=2e^{s+t+4}-e^{s+t+8}-e^{s+t}e^s+e^t-1$, with the exact solution $U(x,t)=e^{s+t}$

We solve this example at $N=M=4,5,6,7,8$. Consider the following tables:

Table 15: The numerical solution at N=M=4

x_Vals	t_Vals	Est	Tr	Err
0	0	1.0001	1	8.9559e-05
0.1	0.1	1.2209	1.2214	0.00048994
0.2	0.2	1.4908	1.4918	0.001074
0.3	0.3	1.8206	1.8221	0.001487
0.4	0.4	2.224	2.2255	0.0015334
0.5	0.5	2.7172	2.7183	0.0010483
0.6	0.6	3.3201	3.3201	1.5073e-05
0.7	0.7	4.0566	4.0552	0.0014457
0.8	0.8	4.9556	4.953	0.0025631
0.9	0.9	6.0516	6.0496	0.0019057
1	1	7.3858	7.3891	0.0032144

$$The\ max\ error=0.0032144 \text{ at } N=4$$

Table 16: The numerical solution at N=M=5

N=M=5

x_Vals	t_Vals	Est	Tr	Err
0	0	1.0001	1	8.9559e-05
0.1	0.1	1.2215	1.2214	8.6232e-05
0.2	0.2	1.4919	1.4918	4.6682e-05
0.3	0.3	1.8221	1.8221	2.3203e-05
0.4	0.4	2.2254	2.2255	0.00010191
0.5	0.5	2.7181	2.7183	0.00015322
0.6	0.6	3.32	3.3201	0.00013498
0.7	0.7	4.0552	4.0552	2.1414e-05
0.8	0.8	4.9532	4.953	0.00015311
0.9	0.9	6.0499	6.0496	0.00021725
1	1	7.3888	7.3891	0.00026305

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number 68 at $N = M = 5$, and $the\ max\ error = 0.00026305$.

Table 17: The numerical solution at N=M=6

x_Vals	t_Vals	Est	Tr	Err
0	0	1	1	3.9733e-07
0.1	0.1	1.2214	1.2214	4.2359e-06
0.2	0.2	1.4918	1.4918	7.7045e-06
0.3	0.3	1.8221	1.8221	7.7034e-06
0.4	0.4	2.2255	2.2255	3.0665e-06
0.5	0.5	2.7183	2.7183	4.9076e-06
0.6	0.6	3.3201	3.3201	1.169e-05
0.7	0.7	4.0552	4.0552	1.0552e-05
0.8	0.8	4.953	4.953	2.5183e-06
0.9	0.9	6.0497	6.0496	1.6641e-05
1	1	7.389	7.3891	1.8535e-05

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number 68 at $N = M = 6$, and *the max error* $= 1.8535e-05$.

Table 18: The numerical solution at N=M=7

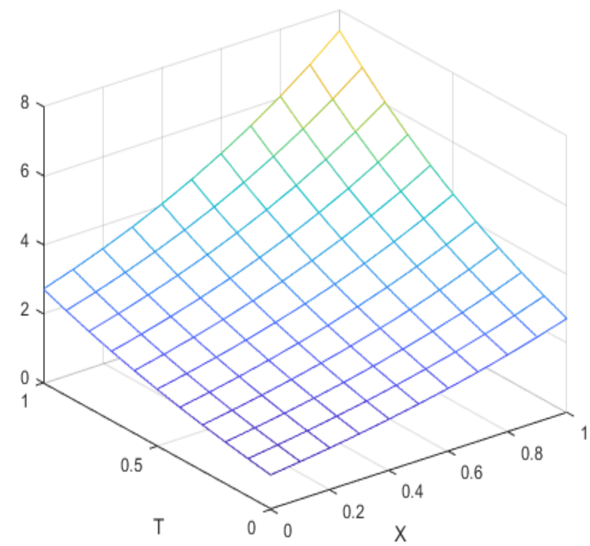
x_Vals	t_Vals	Est	Tr	Err
0	0	1	1	3.9733e-07
0.1	0.1	1.2214	1.2214	3.2478e-07
0.2	0.2	1.4918	1.4918	7.2115e-09
0.3	0.3	1.8221	1.8221	3.9686e-07
0.4	0.4	2.2255	2.2255	6.0421e-07
0.5	0.5	2.7183	2.7183	3.657e-07
0.6	0.6	3.3201	3.3201	2.8531e-07
0.7	0.7	4.0552	4.0552	8.2794e-07
0.8	0.8	4.953	4.953	4.1997e-07
0.9	0.9	6.0496	6.0496	9.087e-07
1	1	7.3891	7.3891	1.1462e-06

This approximations show the accuracy of the Chebyshev Approximation method to solve equation number 68 at $N = M = 7$, and *the max error* $= 1.1462e-05$.

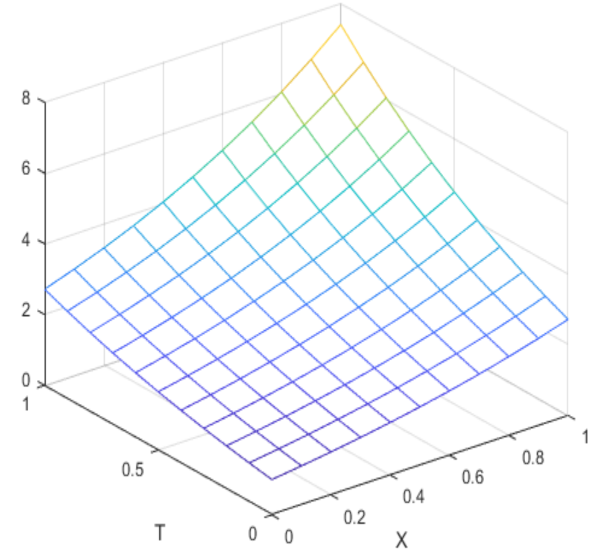
Table 19: The numerical solution at N=M=8

x_Vals	t_Vals	Est	Tr	Err
0	0	1	1	1.0991e-09
0.1	0.1	1.2214	1.2214	1.8428e-08
0.2	0.2	1.4918	1.4918	2.6704e-08
0.3	0.3	1.8221	1.8221	1.3077e-08
0.4	0.4	2.2255	2.2255	1.6552e-08
0.5	0.5	2.7183	2.7183	3.7256e-08
0.6	0.6	3.3201	3.3201	2.1036e-08
0.7	0.7	4.0552	4.0552	2.8175e-08
0.8	0.8	4.953	4.953	4.5807e-08
0.9	0.9	6.0496	6.0496	3.3503e-08
1	1	7.3891	7.3891	6.3137e-08

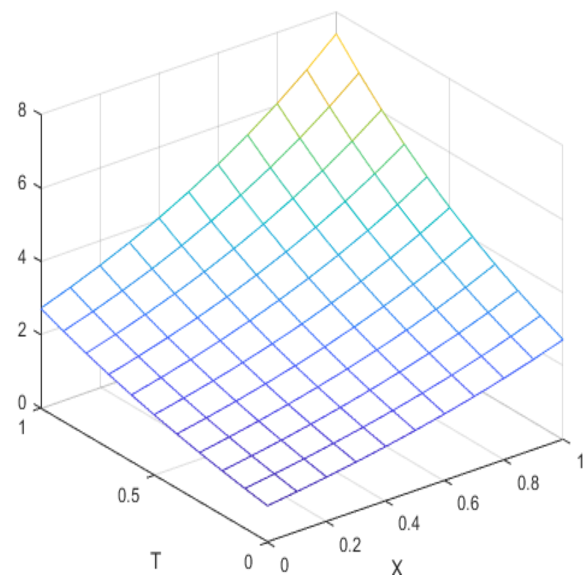
N=M=4



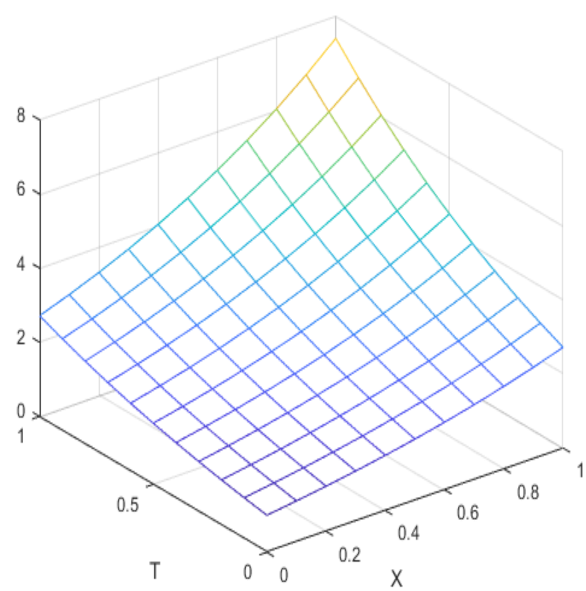
N=M=5



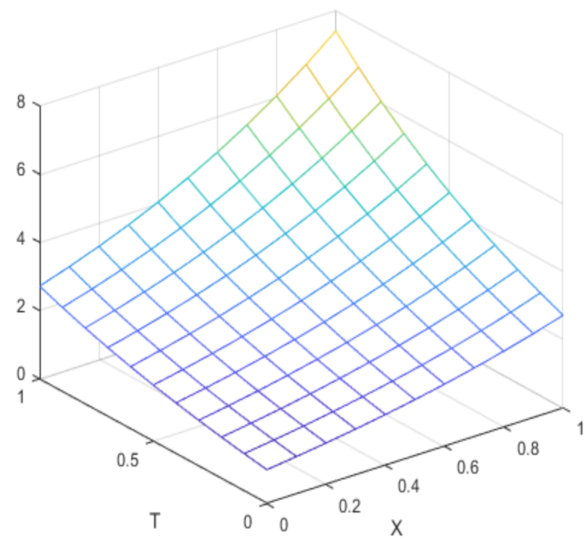
N=M=6



N=M=7



N=M=8



Chapter Four

Three-Dimensional Nonlinear Fredholm-volterra Integral Equation

In this chapter, we present a numerical method for the solution of nonlinear Three-Dimensional Fredholm-Volterra Integral Equation, $3D-FVIE$ of the second kind of the form:

$$U(x,y,z) = f(x,y,z) + \lambda_1 \int_0^z \int_0^y \int_0^z K_1(x,y,z,r,s,t,U(r,s,t))drdsdt$$

$$+\lambda_2 \int_0^1 \int_0^1 \int_0^1 K_2(x,y,z,r,s,t,U(r,s,t))drdsdt \qquad (80)$$

where $U(x,y,z)$ is an unknown scalar valued function on $D = [0,1]\times[0,1]\times[0,1]$. The functions $K_1(x,y,z,r,s,t,U(r,s,t))$ and $K_2(x,y,z,r,s,t,U(r,s,t))$ are given functions we put:

$$K_1(x,y,z,r,s,t,U(r,s,t)) = K_1(x,y,z,r,s,t)[U(r,s,t)]^p$$

.

$$K_2(x,y,z,r,s,t,U(r,s,t)) = K_2(x,y,z,r,s,t)[U(r,s,t)]^q$$

. where p and q are postive integer,moreover, $f(x,y,z)$ is a knowm function define on D.

4.1 Three-Dimensional Chebyshev Polynomial

In this chapter, by considering $2D-CHP$, we define an $(N+1)^3$ set of three-dimensional chebyshev polynomial $3D-CHP$as:

4.2 Approximation With Chebyshev Polynomial

Definition 4.9. [22, 34] suppose that $U(x,y,z)$ be a continious function on $[-1,1]\times[-1,1]\times[-1,1]$ for give natural numbers L,M,N, we set

$$U(x,y,z) \approx \sum_{i=0}^{L}\sum_{j=0}^{M}\sum_{k=0}^{N} C_{ijk}T_i(x)T_j(y)T_k(z) \tag{81}$$

where N is any natural number.

4.3 Method of solution

consider the nonlinear integral equation (80), at first we approximate $U(x,y,z)$ as (88) :

$$U(x,y,z) \approx \sum_{i=0}^{L}\sum_{j=0}^{M}\sum_{k=0}^{N} CijkT_i(x)T_j(y)T_k(z) \tag{82}$$

then we substitute this approximation into equation (80) to get:

$$\sum_{i=0}^{L}\sum_{j=0}^{M}\sum_{k=0}^{N}C_{ijk}T_i(x)T_j(y)T_k(z)=f(x,y,z)$$
$$+\lambda_1\int_0^z\int_0^y\int_0^x k_1(x,y,z,r,s,t,\sum_{i=0}^{L}\sum_{j=0}^{M}\sum_{k=0}^{N}C_{ijk}T_i(r)T_j(s)T_k(t))dsdrdt$$
$$+\lambda_2\int_0^1\int_0^1\int_0^1 k_2(x,y,z,r,s,t,\sum_{i=0}^{L}\sum_{j=0}^{M}\sum_{k=0}^{N}C_{ijk}T_i(r)T_j(s)T_k(t))dsdrdt \quad (83)$$

now, we trasfer the intervals $[0,x_l],[0,y_m],[0,z_n]$ and $[0,1]$ into $[-1,1]$ by transformitions:

$$\alpha_1=\frac{2}{x_l}r-1 \quad (84)$$

$$\beta_1=\frac{2}{y_m}s-1$$

$$\gamma_1=\frac{2}{z_n}t-1$$

$$\alpha_2=2r-1$$

$$\beta_2=2s-1$$

$$\gamma_2=2t-1$$

then for chebyshev polynomils we consider the cocollection points

$$x_l = \cos(\frac{l\pi}{L}), i = 1, ..., L \tag{85}$$
$$y_m = \cos(\frac{m\pi}{M}), m = 0, ..., M$$
$$z_n = \cos(\frac{n\pi}{N}), n = 0, ..., N$$

let,

$$H_1(x,y,z,r,s,t) = k_1(x,,y,z,r,s,t,\sum_{i=0}^{L}\sum_{j=0}^{N}\sum_{z=0}^{M} C_{ijk}T_i(x)T_j(y)T_k(z))$$
$$and, H_2(x,y,z,r,s,t) = k_2(x,,y,z,r,s,t,\sum_{i=0}^{L}\sum_{j=0}^{N}\sum_{z=0}^{M} C_{ijk}T_i(x)T_j(y)T_k(z)) \tag{86}$$
$$\tag{87}$$

using transformed equation (84) we get,

$$\sum_{i=0}^{L}\sum_{j=0}^{N}\sum_{k=0}^{M} C_{ijk}T_i(x)T_j(y)T_k(z) = f(x,y,z)$$
$$+\frac{\lambda_1 xyz}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} H_1(x,y,z,\frac{x(\alpha_1+1)}{2},\frac{y(\beta_1+1)}{2},\frac{x(\gamma_1+1)}{2})d\alpha_1 d\beta_1 d\gamma_1$$
$$+\frac{\lambda_2}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} H_2(x,y,z,\frac{\alpha_2+1}{2},\frac{\beta_2+1}{2},\frac{\gamma_2+1}{2})d\alpha_2 d\beta_2 d\gamma_2 \tag{88}$$

using collocation points (85) in transformed equation (88) , we get:

$$\sum_{i=0}^{L}\sum_{j=0}^{N}\sum_{z=0}^{M} C_{ijk}T_i(x_l)T_j(y_m)T_k(z_m) = f(x_l, y_m, z_n)$$
$$+\frac{\lambda_1 x_l y_m z_n}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} H_1(x_l, y_m, z_n, \frac{x_l(\alpha_1+1)}{2}, \frac{y_m(\beta_1+1)}{2}, \frac{z_n(\gamma_1+1)}{2})d\alpha_1\, d\beta_1\, d\gamma_1$$
$$+\frac{\lambda_2}{8}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} H_1(x_i, y_m, z_n, \frac{\alpha_2+1}{2}, \frac{\beta_2+1}{2}, \frac{\gamma_2+1}{2})d\alpha_2\, d\beta_2\, d\gamma_2 \qquad (89)$$

now, we use clenshaw-curtis quadraturs formela, to get:

$$\sum_{i=0}^{L}\sum_{j=0}^{M}\sum_{k=0}^{N} C_{ijk}T_i(x_l)T_j(y_m)T_k(z_n) = f(x_l, y_m, z_n)$$
$$+\sum_{a=0}^{L}{}''\sum_{b=0}^{M}{}''\sum_{c=0}^{N}{}'' w_a\zeta_b\xi_c[\frac{\lambda_1 x_l y_m z_n}{8} H_i(x_l, y_m, z_n x_l(\frac{x_a+1}{2}), \frac{y_m(y_b+1)}{2}, \frac{z_n(z_c+1)}{2})$$
$$+\frac{\lambda_2}{8} H_2(x_l, y_m, z_n, \frac{x_a+1}{2}, \frac{y_b+1}{2}, \frac{z_c+1}{2})] \qquad (90)$$

$$for \quad i = 0, \ldots, l$$
$$j = 0, \ldots, M$$
$$k = 0, \ldots, N$$

$$a = 0, \ldots, L$$

$$b = 0, \ldots, M$$

$$c = 0, \ldots, N$$

and,

$$\begin{aligned} w_a = & \quad \frac{4}{L}\sum_{n=0}^{L}{}_{even}'' \frac{1}{1-n^2}\cos(\frac{na\pi}{L}) \\ \zeta_b = & \quad \frac{4}{M}\sum_{n=0}^{M}{}_{even}'' \frac{1}{1-n^2}\cos(\frac{nb\pi}{M}) \\ \xi_c = & \quad \frac{4}{N}\sum_{n=0}^{N}{}_{even}'' \frac{1}{1-n^2}\cos(\frac{nc\pi}{M}) \end{aligned} \tag{91}$$

Where the double prime indicate that the first and the last term of the summation should be halved. Note that the obtained system from the previous equations contains $(N+1)\times(M+1)\times(L+1)$ equations and also contains the same number of unknown. These equations can be solved by Newton Raphson's method or any other method

5 Conclusions

In the presented method, one, two, and three-dimensional nonlinear FVIE of the second kind are transformed into a system of nonlinear equation using Chebyshev polynomial of the first-kind. The Chebyshev method used to approximate the solution of the problem. The approximated non-linear FVIE of the second-kind is transformed to a non-linear system of algebraic equations with unknown coefficients which is solved using Newton's method. The applicability and accuracy of the method have been checked by some examples in one and two dimensional FVEI. It was noticed that the method gives more accurate results than the methods presented even when we use a small number of basis functions.

As a future work, the method can be used also for the three dimensional case. In chapter four of this thesis, the theoretical back of the future work has been done.

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ملخص

الحل العددي لمعادلة فريدهولم فولتيرا التكاملية غير الخطية العامة ذات الرتب العليا باستخدام تقريب تشيبيشيف

محمد علان حسني أبو معلا

في هذه الاطروحة تم دراسة الطرق العددية لحل معادلة فريدهولم فولتيرا التكاملية غير الخطية ذات الابعاد العليا باستخدام تقريب تشيبيشيف ، وتستند هذه الطريقة الى استبدال الاقتران المجهول بمتسلسلة تشيبيشيف ، ثم تحويل معادلة فريدهولم فولتيرا التكاملية غير الخطية من النوع الثاني الى نظام من المعادلات غير الخطية باستخدام تقريب تشيبيشيف ، في النهاية قُدمت العديد من الامثلة العددية التي وضحت مدى دقة وفاعلية هذه الطريقة .