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Multi-Item Inventory Model with Shortage Limitations

By

Amani Akram Muhanna

Supervisor

Dr. Iyad Suwan

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Committee Decision

Multi -Item Inventory Model with Shortage Limitations

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Amani Akram Muhanna

This thesis was defended successfully on .... and approved by

Committee Member	**Signature**
Dr. Iyad Suwan (Supervisor)	....
Dr. Mahmoud Almanassra (Internal Examiner)	....
Dr. Inad Nawajah (External Examiner)	....

DECLARATION

I, Amani Akram Muhanna declare this work provided in this thesis, unless otherwise referenced, that this dissertation is my original work and that it has not been presented, and will not be presented, to any university for similar or any other degree award.

Signature ....

DEDICATION

To my husband Nader Shalabi,

To my children Osama, Nadyn and Yasmeen,

To memory of my father,

To my mother (Amal Afif) the origin of my success,

To my brother Ameer who help me,

To my dear sister Enas and brothers Ahmad, Osama and Ibrahim,

To my friends, Amani and Mona,

My special dedication to Prof. Dr Hala Fergany, Dr. Osama Hollah and Dr. Magda Farag.

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ABSTRACT

The shortages in inventory models are a mixture of backorder and lost sales that result from the unmet demands of customers during the shortage period. In this thesis, the multi-item inventory model is presented with two shortage limitations. The first limitation depends on the expected varying backorder cost and the second one depends on the expected lost sales cost. Our model is formulated in both crisp and fuzzy cases to analyze how to conclude the optimal values of order quantity and the reorder point for each item. As a result, the minimum expected total cost is achieved where the Lagrange Multipliers technique is used for this purpose. In the presented model, the demand during leading time is considered as a random variable that follows the normal distribution. As an illustration, numerical examples are applied and the results of fuzzy and crisp models are compared.

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INTRODUCTION

Inventory is the stock of goods and materials of any resource that is used to satisfy current demand or future need. The control of inventories of physical goods is a common problem to all projects of any sector of economy [22]. Inventories must be maintained, therefore, customers will not be allowed to wait until their orders are filled. Inventory is used on daily basis in life as equipment inventory, inventory of raw materials, spare parts, inventory of vehicles, finished goods or semi-finished, etc. These goods and materials have an economic value which can be sold to the customers in the form of items. These items are the basic building block of the inventory system.

There are several reasons for classifying inventory. But, the most common reasons are counting purposes, such as determining how much finished goods inventory the owners of stores have, or for manufacturing plants; to determine how much materials are in process. Inventories have been classified into direct and indirect. Direct inventories include items which are directly used for production, which are categorized for three types as follows [8]: The first type is the raw materials which are needed in primary production. Then the firms transform them into semi-finished goods or finished goods. The second type is work in process (WIP); the product that partially finished goods are waiting for accomplishing the production. The third type is finished goods after their production is complicated but not yet delivered stock according to the varying market demand.

Indirect inventories include such items which are required for manufacturing. They are classified into six classes as follows:

1- The transit inventory which is also called transit pipe line inventory; goods are transported from one location to another. For example, finished goods are being shipped from production centers to distribution centers depending on the demand; therefore, the inventory needs much time to be moved.
2- The buffer inventories or safety stock which indicates the extra inventory in case there are future demand. This buffer will protect the inventory system from shortage or stock out situations, thus it will give better customer services. The importance of verifying the value of safety stock related to what happens to demand when there is shortage.
3- The decoupling inventories which allows the different stages of production to progress independently from any previous operation. Also, decoupling inventories may be viewed as stocks that decouple the customer from the manufactures.
4- The seasonal inventories in which demands for many cases are seasonal. For example, millions of kids go back to school at the beginning of the academic year and retailers depend on large sales of book bags, notebooks, pens, etc. Hence, inventories have to be kept satisfy high demand.
5- The lot size inventories which are held to take advantages of discounts and they are usually available for purchase of large quantities.
6- The anticipation inventories which are stocked for the demand that are expected in the future and that will avoid long customer wait times [8].

Another classification of inventory item is the ABC one. It is according to some measures of importance and annual usage, usually annual dollar value [33]. Typically, three classes of items are used: A (very important), B (moderately important) and C (least important). The procedure begins by arranging all inventory items from highest to lowest value of annual usage which can be computed by multiplying the annual quantity used by its unit cost. The specific number of items on shelves and storage areas of the store must be known by owners to control placing specific orders of items on their shelves and storage area in order to place order on control. The objectives of inventory control are to decide the level of customer service in order to have the right items, insufficient quantity at the right time to avoid shortages, and taking costs in consideration to minimize investments in inventory. All organizations in economics world should control inventories so the goods have to arrive in a specific system exactly when a demand for them occurs. Hence, customers would not have to wait for filling their orders from a source, even so, customers will not be allowed to wait for long periods of time. However; the control of inventory has to minimize investment in inventories and verifying that production process or sales do not suffer at the same time. In addition, the inventory model is needed to answer two basic questions in supervising the inventory of any physical goods are; when to refill the inventory and how much to order for refilling [20]. There exist certain types of inventory problems such as those concerns with storage area, number of orders, the budget available for inventory, customer service level... etc.

Many mathematical models have been developed for controlling inventory system, and these models are applied to the solutions of inventory problems. However, the history of inventory models goes back to (1915) when Ford, N. Harries of Westinghouse corporations developed a simple model, which is called the simple lot of size formula [20]. The same formula is developed referred to as the Wilson formula because it is also obtained by R.H. Wilson as an integral part of the inventory control formula.

The approach of inventory models has attracted a large number of works based on mathematical analysis. Hence, no single model can take into account all possible situations of real life and suggest how much to be ordered and when to order [1]. Over the years, hundreds of books, papers and survey papers have been published to present inventory models with a wide selection of conditions and assumptions.

Hadley and Whitin [20] are considered to be the first researchers who have deeply discussed the analysis of inventory system. The mathematical models they formulated depend on the costs associated with purchasing the items stock, the costs of holding the items in inventory, the costs of customer's orders and the costs of shortage which are associated with demands happening when the stock is consumed or expended. These costs incurred in an inventory system are associated with determination of inventory policy. The different inventory models must decide how much to order (the optimal order quantity Q^*) in each time period (the optimal reorder point (r^*)) to meet demand. The basis for answering the two basic questions; how much to order and when to order, is minimizing the total relevant costs. Hence, the inventory cost function is defined by:

(Total inventory Cost) = (Purchase Cost) + (Order Cost) + (Holding Cost) + (Shortage cost).

Where,

- Purchasing Cost (PC) is the price per unit of an inventory item.
- Order Cost (OC) represents the fixed charge sustained when an order is placed.
- Holding Cost (HC) represents the cost of maintaining inventory in stock.

- Shortage Cost (SC) is the penalty sustained when stock is depleted.

The purchasing cost is supposed to be constant in most models. Hence, the purchasing cost per unit will not be considered as a part of the total cost in some models since the order quantity will be unaffected by the size of order quantity.

Many authors have studied the inventory models and have deduced the optimal solution for the order quantity and the reorder point, with various assumptions and conditions. These assumptions and conditions are represented in constraints with given limitations, and costs where they constant or varying. For example, Kasthuri et al. [24] presented inventory model under three constraints; the warehouse floor space, upper limit on the number of orders and on investment amount on total production cost. These mathematical inventory models with different constraints can be categorized into deterministic models and probabilistic of demand. The demand for an item in inventory is the number of units that may be predictably during a particular period. When the demand in future period can be forecasted with considerable prediction, it is realistic to use an inventory policy that supposes accuracy of forecasting. This state of known demand when a deterministic inventory model is used. In spite of that, when demand cannot be predicted, it will become necessity to use a probabilistic inventory model where the demand is a random variable for any period [25]. These cases were dealt by Hadely and Whitin. Several researchers studied the probabilistic inventory models with different distributions. For example, Fergany and Al-Saadani [17] presented constrained probabilistic inventory models with varying holding cost. They obtained the optimal solution for the order quantity and the reorder point when the lead time demand follows the exponential or the Laplace distributions. El-wakeel [10] derived a probabilistic inventory model with uniform distribution. Also, Abou-El-Ata and Fergany [1] introduced a multi-item inventory model with probabilistic demand and the varying order cost is considered with two restrictions by using a geometric programming approach. Teng and yang [35] generalized the Economic Order Quantity (EOQ) model to deterministic inventory lot size models with varying demand. They also generalized the holding costs varying with time. They showed that total relevant cost is a convex function of the number of replenishments. Jung and kelin [23] presented the optimal inventory policies under decreasing cost functions via geometric programming. Many inventory models were presented for a single item but most inventory system in real economic life, stock a large number of items or multi- items. Abou-El-Ata and Kotb [2] provided a simple method to deduce the inventory policy of multiple items. In this method, varying holding cost is considered to be a continuous function of order quantity and the EOQ inventory model is derived with two constraints, the first for holding cost and the other is the number of orders. Other related inventory models were written by Gezahegan [18] and Lenard [26]. Febrycky and Banks [11] presented the multi-item and the probabilistic single item, single source inventory system using the classical optimization

Most researchers have presented models with assumption of allowable shortages, which the demand cannot be filled from store immediately then, there will be patient customers who want to wait and receive their orders at the end of the shortage period (backorder), while the other customers are not and want to satisfy their demand from other sources (lost sales). Several inventory models with mixture or combination of backorders and lost sales were proposed by Oyang [29] and Montogomery [27]. Also, Zipkin [28] showed that demands occur during stock out are lost sales and backorders. El-wakeel [10] derived a probabilistic inventory backorders model with uniform distribution. Bhunia and Maiti [7] represented two deterministic inventory models for variable production depending on the inventory level without allowing shortages for the first model and with shortages for the second one. Moreover, Fergany and El-wakeel [14] presented constrained probabilistic continuous review inventory system with mixture shortage and stochastic demand. Park [30] presented a deterministic inventory models with partial backorders. The inventory costs are usually imprecise in real situations due to the effects of various uncontrollable factors for example, exchange rates from domestic money to foreign money. Therefore, these costs parameters are detailed as approximately equal some specific amount and are characterized as fuzzy. Hence, many researchers are dealing with inventory problems with different shortage cases where the cost components are considered as crisp or fuzzy values. For example, probabilistic periodic review inventory model with varying shortage for crisp and fuzzy cost with limitations was introduced by Fergany and Hassanein [15]. Recently, Fergany [13] analyzed and solved probabilistic multi-item inventory model with varying mixture shortage cost with limitations in crisp environment.

In this thesis, multi-item inventory model with shortage limitations in both crisp and fuzzy cases is presented. The model will be developed on two limitations of mixture shortages. One of the limitations depends on the expected varying backorder cost and the other limitation depends on the varying lost sales cost. This model is a probabilistic multi-item with a demand that follows normal distribution, which is formulated to analyze how to get the optimal order quantity Q^* and the optimal reorder point r^* for each item to obtain the main objective of minimizing the expected total cost (E(TC)) by using Lagrange method. And an illustration numerical examples for three items will be presented. The fuzziness in the cost components are represented by using trapezoidal fuzzy number. Comparison between the multi-item probabilistic inventory models in crisp and fuzzy environment is shown according to optimal Q^* and r^* and (min E(TC)). The structure of thesis is organized into three chapters; chapter 1 gives a solid background of inventory models for single item. Chapter 2 offers the necessary preliminaries. In chapters 3, the fuzzy multi-item inventory model is presented and numerically verified.

Chapter 1

INVENTORY MODELS

The inventory models are mathematical equations or formulas that assist a firm or any business organization to decide their inventory policy for ordering the economic quantity and retaining the frequency ordering. So the flowing of goods to the customer will continue without breaking off or lagging time in delivery. In this chapter we present the inventory model in depth by means of the following sections.

1.1 Economic Order Quantity (EOQ) Model

The most common and the simplest inventory model is the Economic Order Quantity (EOQ), this model is widely used by firms and retailers. This model is applied when the stock levels are depleted or decreased and then replenished again by the arrival of a batch of new units. For this basic EOQ model, the considered costs are [38]:

C_o: Order cost per unit item, which is the expenses incurred to place an order to a supplier.

C_h : Holding cost per unit item, which is the cost that associated with storing inventory that still unsold.

The basic assumptions of the EOQ model

1. D: Demand rate, which is a rate based on the maximum quantity that a customer requires, then it will be kept available for use, it will be considered in this known and constant (unit per unit time).
2. Q: Order quantity of the item per unit time.

3. Shortages are not allowed, assuming that the required quantity can be satisfied from the available stock.

4. No discount is considered; the discount is assumed in some models to encourage the customers to buy the product in large batches because it results a reduction in cost.

5. All model's parameters are unchanging over time.

6. The length of planning horizon is infinite, which is the amount of time an organization or firm will look into future when preparing a strategic plan.

The order quantity Q replenished or increased the inventory level whenever it drops to 0. Usually there is a delay between ordering items and their arrival [8]. The time between the placement of an order and its receipt is called lead time (L). The inventory level at which the order placed is called the reorder point (r), Fig. (1.1) illustrates the model [33].

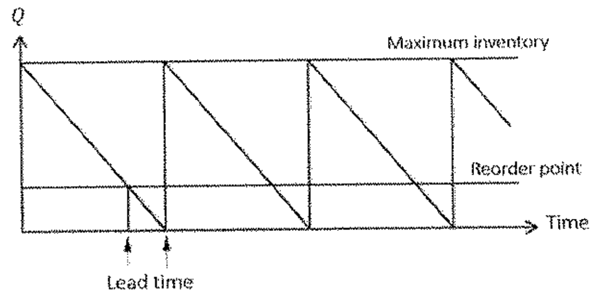

Figure 1.1: The inventory model.

We assume that the lead time of the basic EOQ model is constant, and the reorder point to be set as follows [33]

$$\textit{Reorder point } (r) = \textit{demand rate} \times \textit{Lead time}$$

Also, the order costs are depicted by the formula [22]:

$$Order\cos t = \frac{D}{Q} \times C_o \qquad (1.1)$$

Because the number of orders is $\frac{D}{Q}$ decrease as Q increases, the order cost is inversely related to order quantity. Therefore, order cost is a nonlinear function that inventory related to order quantity Q, as shown in Fig. (1.2).

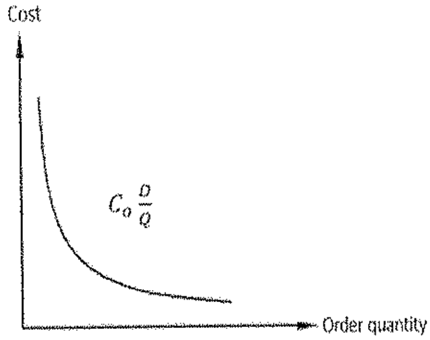

Figure 1.2: Order costs are inversely and non-linearly related to order quantity.

The time between consecutive replenishments of inventory is referred to as a cycle. The average inventory level during a cycle is $\frac{Q+0}{2} = \frac{Q}{2}$ units.

So, the holding cost are obtained by the formula [4]:

$$Holding\ cost = C_h \frac{Q}{2}.$$

Therefore,

$$Holding\ \cos t\ \ per\ cycle = C_h \frac{Q}{2}\frac{Q}{D} = C_h \frac{Q^2}{2D} \qquad (1.2)$$

Holding cost is a linear function in direct proportion to changes in the order quantity Q, as shown in Fig. (1.3).

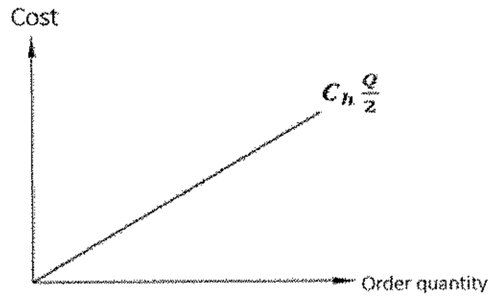

Figure 1.3: Holding costs are linearly related to order quantity.

The total annual cost (TC) is related with holding and ordering inventory when Q units are ordered each time is [22]:

$$Total\ cost = C_o \frac{D}{Q} + C_h \frac{Q}{2} \tag{1.3}$$

Fig. (1.4) shows that the total cost curve is U-shaped which is convex with one minimum and that it reaches its minimum at the quantity where holding and order costs are equal [34].

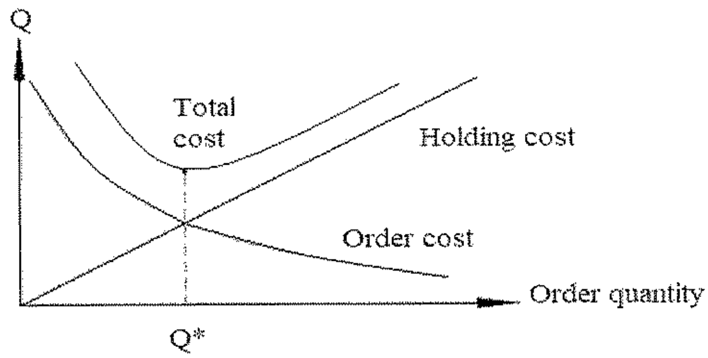

Figure 1.4: The total cost curve is U-Shaped.

To find the optimal value of Q denoted by Q^* that minimizes the total cost (TC), by setting the corresponding first derivatives of Eq. (1.3) equal to zero [20].

$$\frac{\partial(TC)}{\partial Q} = 0.$$

Then we obtain:

$$-C_o \frac{D}{Q^2} + \frac{C_h}{2} = 0,$$

which leads to:

$$Q^* = \sqrt{\frac{2C_o D}{C_h}}, \quad (1.4)$$

which is a common EOQ formula [4], the corresponding cycle time is:

$$t^* = \frac{Q^*}{D} = \sqrt{\frac{2C_o}{DC_h}}. \quad (1.5)$$

We notice, when order cost increases, both optimal values Q^* and t^* increase. But, these optimal values decrease when the unit holding cost increases. Also, the increased demand rate will yield increasing in Q^* and decreasing in t^*.

1.2 The EOQ Model with Shortages

The inventory shortage (stock out) occurs when the demand of the product exceeds the available quantity, subsequently, the demand will not have filled immediately. There are two main cases of the shortage:

1. Complete backorders, this case occurs when the inventory is out of stock. The customers choose to wait until the next arrival of stock and the product becomes available again.

2. Complete lost sales, this case happens when the customers are impatient and choose to satisfy their demand immediately from another source during the temporary shortage period.

In most practical situations, there is a combination or mixture of backorders and lost sales. Therefore, the shortage cost can be computed as:

Shortage cost = backorder cost + lost sales cost

Fig. (1.5) presents the pattern of the EOQ model with shortage [22]. Symbols in Fig. (1.5) are summarized as follows:

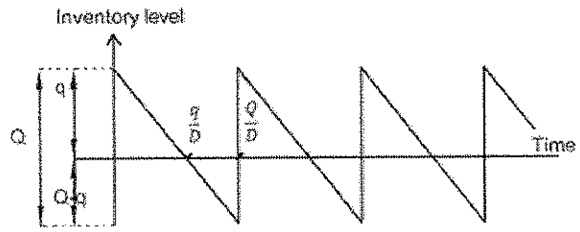

Figure 1.5: Inventory level as a function of time for the EOQ model with shortage.

C_S = Shortage cost per unit time.

q = Inventory level after a batch of Q units is added to inventory.

$Q - q$ = Shortage in inventory before a batch of Q units is added.

The average inventory level; during time $\frac{q}{2}$ units the corresponding cost is $C_h \frac{q}{2}$ per unit time [22]. Hence, holding cost per cycle $= \frac{C_h q}{2}\left(\frac{q}{D}\right) = \left(\frac{C_h q^2}{2D}\right)$, shortage occurs for a time $= \frac{Q-q}{D}$

The average amount of shortage during this time $= \left(\frac{Q-q}{2}\right)$ units. The corresponding cost

$= C_s \frac{Q-q}{2}$

$$Shortage\ cost\ per\ cycle\ = C_h \frac{(Q-q)^2}{2D}$$

Therefore,

$$Total\ \text{cost}\ per\ unit\ time = \frac{D}{Q} C_o + C_h \frac{q^2}{2Q} + C_S \frac{(Q-q)^2}{2Q} \qquad (1.6)$$

The optimal values for two decisions variables (q^*, Q^*) are found by setting the first partial

derivatives $\frac{\partial(TC)}{\partial q}$ and $\frac{\partial(TC)}{\partial Q}$ equal to zero .

Thus,

$$\frac{\partial(TC)}{\partial q} = \frac{C_h q}{Q} - \frac{C_s(Q-q)}{Q} = 0 \qquad (1.7)$$

$$\frac{\partial(TC)}{\partial q} = \frac{-DC_o}{Q^2} - \frac{C_h q^2}{2Q^2} + \frac{C_s(Q-q)}{Q} - \frac{C_s(Q-q)^2}{2Q^2} = 0 \qquad (1.8)$$

By solving these equations, we obtain:

$$q^* = \sqrt{\frac{2DC_o}{C_h}} \sqrt{\frac{C_s}{C_h + C_s}} \qquad (1.9)$$

$$Q^* = \sqrt{\frac{2DC_o}{C_h}} \sqrt{\frac{C_h + C_s}{C_s}} \qquad (1.10)$$

The optimal cycle length t^* is given by:

$$t^* = \frac{Q^*}{D} = \sqrt{\frac{2C_o}{DC_h}}\sqrt{\frac{C_s + C_h}{C_s}} \quad (1.11)$$

The maximum shortage is:

$$Q^* - q^* = \sqrt{\frac{2DC_o}{C_h}}\sqrt{\frac{C_h}{C_s + C_h}} \quad (1.12)$$

The fraction of time that no shortage exists is given by [33]:

$\frac{q^* / D}{Q^* / D} = \frac{C_s}{C_s + C_h}$, which is independent of C_o.

When $C_s \rightarrow \infty$ with C_h constant, $Q^* - q^* \rightarrow 0$, where both Q^* and t^* converge to their values for the basic EOQ model. On the other hand, when $C_h \rightarrow \infty$ with Cs constant, $q^* \rightarrow 0$, so it just removes the shortage cost in inventory and gives high holding cost which is uneconomical situation.

1.3 Periodic Review Model $< Q_m, N >$

The main two types of inventory model that widely used are the continuous and the periodic review model. Periodic review is also called Fixed Order Interval model.

In the periodic review, a periodic checking of physical counting and documenting inventory at specified times (e.g., weekly or monthly) is done. The time between the placements of two successive orders is called a period and is denoted by N. The policy for the periodic review model $< Q_m, N >$ is that every N units of time, a sufficient quantity is ordered to raise the inventory position to level Q_m, where Q_m is the maximum inventory level. The decision of how much to order of each item is made at the beginning of each period.

The Q_m or base stock level, as known in some literatures, must satisfy the demand during protection interval $(N + LT)$,where N is the lead time between the placement of an order and its receipt, plus safety stock that is carried to reduce the risk of stock out during lead time. The maximum inventory level becomes [2]:

Q_m = Expected demand during protection interval + Safety stock.

Fig. (1.6) depicts the typical behavior of this model [33].

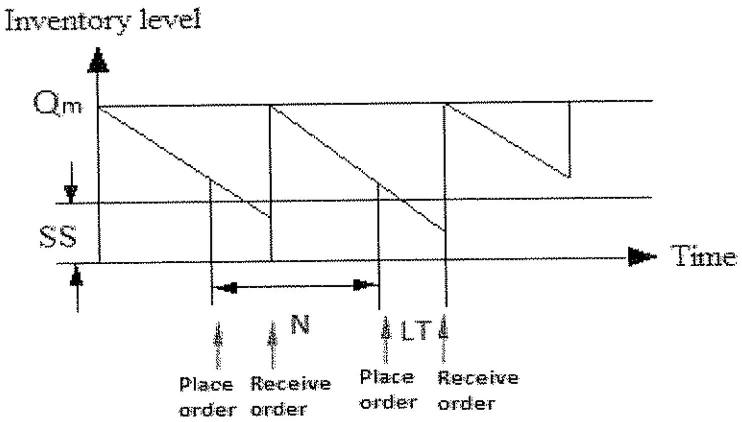

Figure 1.6: The behavior of the periodic review model.

The expected total cost is the sum of the associated costs including; the expected review cost, expected order cost, expected holding cost, expected shortage cost including expected back ordered and expected lost sales respectively [16] i.e.,

$$E(TC) = E(RC) + E(OC) + E(HC) + E(BC) + E(LC) \tag{1.13}$$

Where,

1. $E(RC) = \frac{C_r}{N}, C_r$ the cost of making a review.

2. $E(OC)=\frac{C_o}{N}, C_o$ the cost of placing an order per period.

3. $E(HC)=\frac{C_h\bar{I}}{N}, C_h$ the holding cost of the item per period and $\bar{I}$ is the average inventory level per period.

4. $E(BC)=\frac{C_b}{N}\gamma\bar{S}$, C_b the backorder cost per period. $\bar{S}$: is the expected shortage (backorder in this case) incurred per period, γ : is the backorder fraction of the item.

5. $E(LC)=\frac{C_l}{N}(1-\gamma)\ \bar{S}$, C_l the lost sale cost per period.

The expected total cost in Eq. (1.13) is developed as follows:

$$E(TC)=\frac{C_r}{N}+\frac{C_o}{N}+\frac{C_h\bar{I}}{N}+\frac{C_b}{N}\gamma\bar{S}+\frac{C_l}{N}(1-\gamma)\ \bar{S} \tag{1.14}$$

Fergany and Hollah [15] presented a similar model with varying backorder and lost sales cost and proposed a limitation on the backorder constraint. They derived the estimate of expected total cost under crisp and fuzzy costs. Periodic review model is formulated by Hadley and Whitin [20] in which the demand is deterministic and probabilistic. Ouyang and Chuang [29] have formulated a periodic review inventory control system with variable lead time. Fergany and Hefnaway [16] used Lagrange technique to analyze the probabilistic periodic inventory model. An economic advantage in shipping or processing orders of a model is that new orders are placed for many items at the same time from a particular supplier at the start of each time. The disadvantage of a periodic review is the need to protect against shortage between review periods by carrying extra stock [32].

A gas station is a good example of periodic review since the deliveries of gas quantities reach the station at the beginning of each week.

1.4 Continuous Review Model $< Q, r >$:

The continuous review model also called Fixed Reorder Quantity model since the fixed quantity Q is ordered whenever the inventory position decreases to reorder point (r). In this thesis, we concern with continuous review model. This system provides a continuous monitoring and information on the present level of inventory of each item. Continuous review is suitable for high cost items where constant review is desirable. This is especially suitable for class A items which have a significant amount of inventory [33]. A fixed quantity Q is ordered whenever the inventory position decreases to reorder (r) or lower. The main advantage of continuous review is providing the same level of customer service. It requires less safety stock than what can be provided by the periodic review model. The disadvantage of this model is the extra cost of record keeping.

1.4.1 Two - bin System

Two-bin system where continuous review is not necessary because of low activity or low unit cost. This system is suitable for organization with a large number of class C items. A two-bin system is used to prevent interruption of the flowing of items to customers by using two containers, one of them is the usage bin which items are withdrawn from it until its contents are finished. The other is the preserve bin which contains stock to meet expected demand until the order is filled, plus an extra stock to reduce the chance of stock out, if the order is late or the demand is more than expected. The two-bin system is a cheap system since there is no need to monitor the withdrawn item from inventory [33].

However, the disadvantages of this system is that the store owners must have more quantity of stock than needed. It is not recommended for expensive product or for products with short life times.

1.4.2 Continuous Review Model with Stochastic Demand

When there is a significant uncertainty about future demand, the stochastic inventory models are needed. The inventory policy: is whenever the inventory level of the item decreases to (r) units, place an order for Q more units to refill the inventory. The order cost and holding cost are sustained each time an order is placed. When stock is distributed to customer to satisfy the demand, the inventory drops until it reaches the reorder point (r). At this point, the order is placed and then the order is received. The difference between placed and received order is the lead time. Demand during lead time is denoted by D_{LT} which expressed as [33]:

$r = D_{LT} = D \times LT$, as shown in Fig. (1.7)

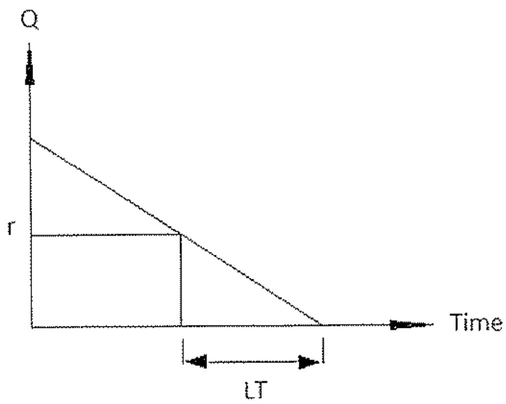

Figure 1.7: the reorder point.

During the lead time, if the demand is greater than expected $(D_{LT} > r)$, then shortage or unmet demand will occur. Moreover, asserted shortage cost is sustained for each unit. This model is closely associated to the EOQ model with shortage introduced in previous section. But this model has uncertain demand. So, some safety stock should be added. Safety stock (SS) can be carried to guard against shortage because of uncertainty of demand during lead time (D_{LT}). Hence, the reorder point (r) becomes [25]:

$$r = D_{LT} + SS \tag{1.15}$$

Fig. (1.8) shows the new value of reorder point.

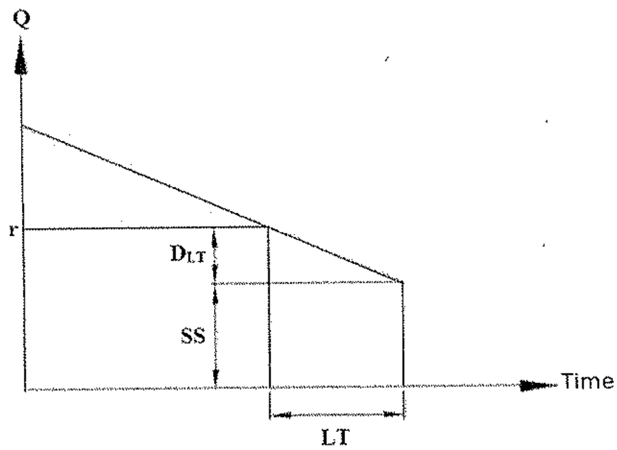

Figure 1.8: Safety stock during lead time.

If the demand follows any distribution with certain mean, then the expected demand during lead time is considered to decide the reorder point with the safety stock at shortage level α. The reorder point can be expressed as:

$$r = E(D_{LT}) + SS_{\alpha} \tag{1.16}$$

Where, SS_{α} is the safety stock at shortage level α. Now, the probability of shortage is expressed as [22]:

$$P\ (D_{LT} > r) = \alpha \ = \text{shortage level.} \tag{1.17}$$

Furthermore, the cases for inventory policies according to the nature of the demand and lead time are demonstrated in Table 1.1 [33]:

Table 1.1: The cases for inventory policies according to the nature of demand and lead time.

Inventory Policy	**When to order**	**How much to order**
1. Constant rate demand *(D)* 2. Constant lead time*(LT)*	$r = D \times LT$	$Q = DT$ T= inventory period
generalized stochastic *(r)* equation	$r = \overline{D} + Z_{\alpha}\ \sigma$ α = shortage level	$Q = \overline{D}T$
1. Stochastic demand $D \sim N(\mu_D, \sigma^2{}_D)$ for normal distribution 2.Constant *(LT)*	$r = \mu_D(LT) + Z_{\alpha}\sigma_D\sqrt{LT}$	$Q = \mu_D T$
1. Stochastic demand 2. Stochastic lead time $LT \sim N(\mu_{LT}, \sigma^2{}_{LT})$	$r = \mu_D\mu_{LT} + Z_{\alpha}\sqrt{\mu_{LT}\sigma^2{}_D + \mu^2{}_D\sigma_{LT}}$	$Q = \mu_D T$

Where,

- μ_{LT}: Mean of lead time demand.
- $\sigma^2{}_{LT}$: Variance of lead time demand.
- μ_D: Mean of demand.
- $\sigma^2{}_D$: Variance of demand.
- $\overline{D}$: The expected demand rate.

- Z_α : The Safety factor, Fig. (1.9) depicts the value of Z based on the normal distribution.

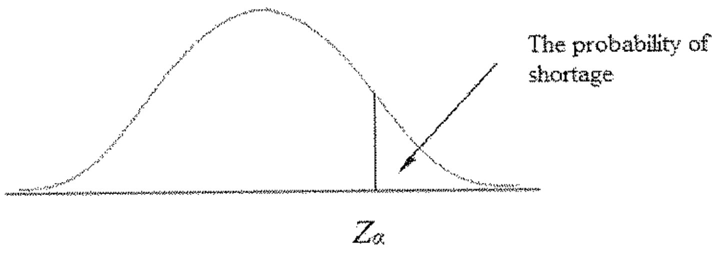

Figure 1.9: The safety factor based on a normal distribution.

In the following chapters we present the continuous review inventory model, when the demand is a random variable follows the normal distribution and lead time is constant.

Chapter 2

MULTI-ITEM INVENTORY MODEL WITH SHORTAGE LIMITATIONS IN CRISP CASE

2.1 Multi-item Inventory Model

Most of the classical inventory models interest with single-item model. Actually, such this model rarely happens. The multi-item inventory models are frequent and closer to reality than the single item models. So, this thesis concerns with three items inventory model. Many inventory models were presented for a single-item, single source (SISS), but most inventory systems stock a large number of items. It is common to find thousands of items stocked, so there is neediness to develop inventory models of such a large number of items. The goal stays the same for single item as well as for multi-item inventory. The analysis for a single item inventory is almost corresponding to that of multi-item one. Also, the results achieved are almost parallel to single item and in multi-item inventory. Multi-item classical inventory models with constraints that imposed from resources can be found in common books [8, 19, 20]. Ben-Daya and Raouf [5] have improved an approach to solve a more practical and realistic inventory problem. They examine a multi-item inventory model with budget and available space constraints. Their model was with stochastic demand that follows uniform probability distribution. Bhattachary [6] has discussed a two-item inventory model for deteriorating items constraint space and investment and they obtained some inventory results.

Lenard and Roy [25] have defined another approach for the specification of inventory policies depended on the theory of efficient policy surface. They expand this theory to multi-item inventory control. Rosenblatt [31] has proposed multi-item inventory system with budgetary constraint using both the fixed cycle approach and the Lagrangian. In the next section, we represent the probabilistic multi-item inventory model with shortage limitations and derive the optimal control of this model.

2.2 Probabilistic Continuous Review $< Q, r >$ Multi-item Inventory Model

The multi-item single source inventory model (MISS) is the most general procurements system that can be characterized as follows; an inventory of n-items (n >1) is retained to satisfy the demand rates assigned $\overline{D}_1$, $\overline{D}_2$,....., $\overline{D}_n$, where $\overline{D}_i$, is the expected demand rate of the *ith* item per period [13]. The goal is to conclude, when to order each item and how much of each item to be ordered, taking into consideration the system and the cost parameters. The multi-item inventory model is deterministic when each influencing factor is known completely. But, in real life the certainty seldom occurs. There are states of inventory problems with uncertainty for some factors, such as price demand and lead time. In the inventory models where only demand is probabilistic or random, the demand pattern maybe have discrete probability distribution or continuous distribution. Therefore, it will be modeled by a defined distribution density function that will get better model in reality [38]. The continuous review $< Q, r >$ multi-item model has to answer the two basic questions that have been previously mentioned. When the demand is probabilistic, levels of inventory; the inventory position and net inventory, should be categorized. This model in which the replenishment occurs whenever the inventory level drops sufficiently low. Hence, a fixed quantity Q is ordered whenever the inventory position drops to reorder point (r) or lower. The inventory position (available stock) is defined here by the following formula [32]:

(Inventory position) = (On Hand) + (On order) – (backorders) – (committed).

Where, on hand stock is stock that is physically on the shelf; it determines whether a certain customer's demand can be satisfied directly or not. The on-order stock is stock which is demanded but has not received yet. The committed quantity is desired if such stock cannot be used for other objectives in the short run.

(Net inventory) = (On hand) - (backorders). Net inventory can be negative, if there are backorders.

2.3 Mathematical Model of Probabilistic Continuous Review Model $< Q, r >$ with Varying Mixture Shortage Cost with Shortage Limitations in Crisp Case

Sometimes, there are some customers do not mind waiting until their orders are available once the next arrival of the stock (the backorder case), and the remaining customers may be impatient to wait and they would to satisfy their demand from other source immediately (the lost sales case). However, inventory models which include both cases are known as mixture shortage. A similar model with fixed reorder point and a variable lead time was derived by Ouyang et al. [29]. Hariga et al. [21] represented both periodic and continuous review models with mixture of backorders and lost sales when the information of demand is full and partial. Also, Fergany et al. [17] solved constrained probabilistic lost sales inventory system with normal distribution and varying order cost. Moreover, Abou- El-Ata et al. [1] discussed probabilistic multi-item inventory model with varying order cost with two limitations. The first is the expected order cost and the second is the expected holding cost. Fergany et al. [12] described periodic review model with zero lead time with limitations and varying order cost. In this section, the mathematical model of a probabilistic continuous review multi -item, single source (MISS) inventory model in crisp case is presented with varying mixture shortage cost with two limitations. One of them is on the expected varying backorder cost and the other on the expected varying lost sales cost. Moreover, this model is presented and verified by a numerical example. Fig. (2.1) shows the behavior of this model [13].

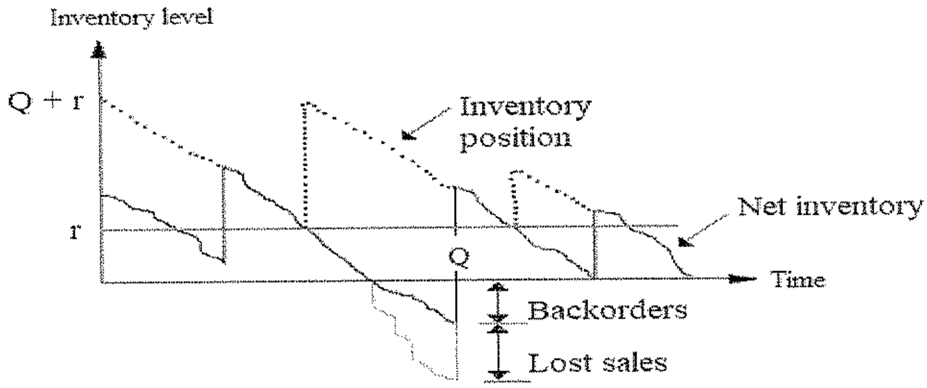

Figure 2.1: The behavior of the continuous review system with backorders and lost sales.

The optimal order quantity Q_i^*, the optimal reorder point r_i^* and the minimum expected total cost [min E(TC)] are obtained.

The following notations are adopted for developing the model

The stock is reviewed continuously $< Q, r >$. The shortages are allowed [1]:

D_i : The demand rate of the *ith* item per period.

$\overline{D}_i$: The expected demand rate of the *ith* item per period.

Q_i : The order quantity of the *ith* item per period.

Q^*_i : The optimal order quantity of the *ith* item per period.

r_i : The reorder point of the *ith* item per period.

r^*_i : The optimal reorder point of the *ith* item per period.

L_i : The lead time between the placing of an order and its receipt of *ith* item.

X_i : The random variable represents the lead time demand of the *ith* item per period.

$f(x_i)$: the probability density function of the lead time demands.

$E(X_i)$: The expected value of x_i.

$r_i - x_i$: The random variable illustrates the net inventory when the procurement quantity arrives if lead time demand $x \leq r$.

$R(r)$: $P(X_i > r)$ = the reliability function = The probability of shortage.

$\overline{S}(r_i)$: The expected shortage quantity per period.

C_{oi} : the order cost per unit of the *ith* item per period.

C_{hi} : the holding cost per unit of the *ith* item per period.

C_{si} : the shortage cost per unit of the *ith* item per period.

C_{bi} : the backorder cost per unit of the *ith* item per period.

C_{li} : the lost sales cost per unit of the *ith* item per period.

β : A real number selected to give the best fit estimation for expected cost function.

γ_i : The backorder fraction of the *ith* item, $0 < \gamma_i < 1$.

$E(OC)$: The expected order (procurement) cost per period.

$E(HC)$: The expected holding (carrying) cost per period.

$E(SC)$: The expected shortage cost per period.

$E(BC)$: The expected back order cost per period.

$E(LC)$: The lost sales cost per period.

$E(TC)$: The expected total cost function.

$MinE(TC)$: The minimum expected total cost function.

K_{bi}: The limitation on the expected varying backorder cost for the backorder model of the *ith* item.

K_{li}: The limitation on the expected varying lost sales cost for the lost sale model of th *ith* item.

We present the mathematical model with varying mixture shortage cost with two limitations where the demand is a continuous random variable, the lead time is constant and the distribution of the demand during the lead time is known. The development of the expected total cost, where it consisted of three components: the expected order cost, the expected holding cost and the expected varying shortage cost, can be expressed simply by:

$$E(TotalCost) = \sum_{i=1}^{m} \left[E(orederCost) + E(HoldingCost) + E(shortageCost)\right]$$

$$= \sum_{i=1}^{m} \left[E(OC) + E(HC) + E(BC) + E(LC)\right]. \qquad (2.1)$$

Where

$E(OrderCost) = E(OC) = C_o \dfrac{\overline{D}_i}{Q_i}$, $\dfrac{\overline{D}_i}{Q_i}$ the number of orders of the *ith* item per period.

Then, we find the average on hand inventory per period ($\bar{I}$)which is obtained by [20] :

$$\bar{I} = \frac{(Max.\ on\ hand + Min.\ on\ hand)}{2} = \left[\frac{(Q_i + E(r_i - x_i)) + E(r_i - x_i)}{2}\right]$$

Therefore;

$$\bar{I} = \frac{Q_i}{2} + r_i - E(x) + (1-\gamma_i)\bar{S}(r_i)$$

which yields,

$$E(Holdingcost) \;=\; E(HC) = C_{hi}\bar{I} = C_{hi}(\frac{Q}{2} + r_i - E(x_i) + (1-\gamma_i)\bar{S}(r_i)) \,. \qquad (2.2)$$

Let S be the shortage quantity which is given by the formula [14]:

$$S = \begin{cases} x - r & , \ x > r \\ 0 & , \ otherwise \end{cases}$$

Then, the expected shortage cost is obtained by:

$$\bar{S}_{(ri)} = \int_0^\infty S_i f(x) dx_i = \int_r^\infty (x_i - r_i) f(x_i) dx_i \,. \qquad (2.3)$$

We assume the varying backorder cost function, where the backorder cost is an increasing function of the number of orders. Then, the expected backorder cost is given by:

$$E(BC) = C_{bi}\gamma_i(\frac{\bar{D}_i}{Q_i})^{\beta+1}\bar{S}_{(ri)} = C_{bi}\gamma_i(\frac{\bar{D}_i}{Q_i})^{\beta+1}\int_r^\infty (x_i - r_i) f(x_i) dx_i \,. \qquad (2.4)$$

Let $1-\gamma_i = \gamma'_i$, where γ'_i is lost sales fraction of the *ith* item.

Also, we assume the varying lost sales cost function, where the lost sales cost is an increasing function of the number of orders. Then, the expected lost sales cost is given by:

$$E(LC) = C_{bi}\gamma'_i(\frac{\bar{D}_i}{Q_i})^{\beta+1}\bar{S}_{(ri)} = C_{bi}\gamma'_i(\frac{\bar{D}_i}{Q_i})^{\beta+1}\int_r^\infty (x_i - r_i) f(x_i) dx_i \,. \qquad (2.5)$$

Therefore, the expected total cost per period is:

$$E[TC<Q,r>]=\sum_{i=1}^{m}\ [C_{oi}(\frac{\overline{D}_i}{Q_i})+C_{hi}\ (\frac{Q_i}{2}+r_i-E(x_i))$$
$$+C_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\int_{r}^{\infty}\ (x_i-r_i)f(x_i)dx_i$$
$$+(C_{li}(\frac{\overline{D}_i}{Q_i})^{\beta+1}+C_{hi})\gamma_i'\int_{r}^{\infty}\ (x_i-r_i)f(x_i)dx_i\ ] \qquad (2.6)$$

To minimize the expected total cost $E[TC<Q,r>]$ with two limitations:

$$C_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i)-K_{bi}\leq 0$$
$$C_{li}\gamma_i'(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i)-K_{li}\leq 0. \qquad (2.7)$$

We rewrite the function (2.7) in the following form:

$$E[TC<Q,r>]=\sum_{i=1}^{m}\ [C_{oi}(\frac{\overline{D}_i}{Q_i})+C_{hi}\ (\frac{Q_i}{2}+r_i-E(x_i))$$
$$+C_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i)+(C_{li}(\frac{\overline{D}_i}{Q_i})^{\beta+1}+C_{hi})\gamma_i'\overline{S}(r_i) \qquad (2.8)$$

Subject to:

$$C_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i)\leq K_{bi}$$
$$C_{li}\gamma_i'(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i)\leq K_{li}. \qquad (2.9)$$

We get the optimal values Q_i^*, r_i^* which minimize Eq. (2.8) by using the Lagrange multiplier technique under constraints [11].

$$L(Q_i,\ r_i\ ,\ \lambda_{1i}\ ,\ \lambda_{2i}) = \sum_{i=1}^{m}\ [C_{oi}(\frac{\overline{D}_i}{Q_i}) + C_{hi}\ (\frac{Q_i}{2} + r_i + E(x_i))$$

$$+ C_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i)$$

$$+ (C_{li}(\frac{\overline{D}_i}{Q_i})^{\beta+1} + C_{hi})\gamma_i'\overline{S}(r_i)$$

$$+ \lambda_{1i}\ (C_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i) - K_{bi})$$

$$+ \lambda_{2i}\ (C_{li}\gamma_i'(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}(r_i) - K_{li})\ ]\ , \tag{2.10}$$

where λ_{1i}, λ_{2i} are the Lagrange multipliers. To calculate the optimal values Q_i^* and r_i^*, set each of the corresponding first partial derivatives of Eq. (2.10) equal to zero [15].

$$\frac{\partial L}{\partial Q_i} = 0 \quad , \quad \frac{\partial L}{\partial r_i} = 0\,.$$

We obtain:

$$C_{hi}Q_i^{*\beta+2} - 2C_{oi}Q_i^{*\beta}\overline{D} - 2A(\beta+1)\overline{S}(r_i) = 0 \tag{2.11}$$

$$R(r_i^*) = \frac{C_{hi}Q_i^{*\beta+1}}{A + C_{hi}\gamma_i'Q_i^{*\beta+1}}\ , \tag{2.12}$$

where $A = \overline{D}_i^{\beta+1}[\gamma_i C_{hi}(1+\lambda_{1i}) + \gamma_i' C_{li}(1+\lambda_{2i})\ ]$.

Clearly, there is no closed form solution of Eq. (2.11) and Eq. (2.12).

The demand D follows the normal distribution with parameters μ and σ^2 [21]:

$$f(D) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{D-\mu}{\sigma}\right]^2},\ -\ \infty < \infty, -\ \infty < \mu < \infty, \sigma > 0\,.$$

Where μ is the mean and σ is the standard deviation. In the case that the lead time demand follows the normal distribution with parameters $L\mu$ *and* $L\sigma^2$.

$$f(x)=\frac{1}{\sigma\sqrt{2\pi L}}e^{-\frac{1}{2L}\left[\frac{x-\mu L}{\sigma}\right]^2},\ -\infty<x<\infty,\ -\infty<\mu L<\infty,\ L\sigma>0.$$

The reliability function $R(r)$ gives the probability of shortage that given by [25]:

$$R(r)=\int_r^{\infty} f(x)dx.$$

Also, $R(r)=1-\phi\left[\frac{r-\mu L}{\sigma\sqrt{L}}\right]=\varphi\left[\frac{r-\mu L}{\sigma\sqrt{L}}\right].$

Where

$$\phi\left[\frac{r-\mu L}{\sigma\sqrt{L}}\right]=\int_{-\infty}^{r} f(x)dx.$$

And $$\varphi\left[\frac{r-\mu L}{\sigma\sqrt{L}}\right]=\int_r^{\infty} f(x)dx.$$

If we let $z=\frac{r-\mu L}{\sigma\sqrt{L}} \quad \Rightarrow \quad r=z\sigma\sqrt{L}+\mu L.$

The expected shortage is obtained by:

$$\bar{S}=\int_r^{\infty}(x-r)f(x)dx$$

$$=\int_r^{\infty} xf(x)dx-r\int_r^{\infty} f(x)dx$$

$$=\frac{1}{\sigma\sqrt{2\pi L}}\int_z^{\infty}((z\sigma\sqrt{L}+\mu L)e^{-\frac{z^2}{2}}\sigma\sqrt{L}).dz-r(1-\phi(\frac{r-\mu L}{\sigma\sqrt{L}}))$$

$$=\frac{\sigma\sqrt{L}}{\sqrt{2\pi L}}\int ze^{\frac{-z^2}{2}}.dz+\frac{\mu L}{\sqrt{2\pi}}\int e^{\frac{-z^2}{2}}.dz-r(1-\phi(\frac{r-\mu L}{\sigma\sqrt{L}})) \qquad (2.13)$$

where $$\psi(\frac{r-\mu L}{\sigma\sqrt{L}}) = \frac{1}{\sqrt{2\pi}}\int_{z}^{\infty} y e^{-\frac{y^2}{2}} dy$$

Then,

$$\bar{S} = \sigma\sqrt{L}\psi(\frac{r-\mu L}{\sigma\sqrt{L}}) + \mu L\varphi(\frac{r-\mu L}{\sigma\sqrt{L}}) - r\varphi(\frac{r-\mu L}{\sigma\sqrt{L}}) \ . \qquad (2.14)$$

So, Eq. (2.13) becomes:

$$\bar{S}(r) = \sigma\sqrt{L}\psi(\frac{r-\mu L}{\sigma\sqrt{L}}) + (\mu L - r)\ \varphi(\frac{r-\mu L}{\sigma\sqrt{L}}) \qquad (2.15)$$

Therefore, the expected total cost can be minimized by substitution Eq. (2.15) into Eq. (2.11) to obtain the following equations:

$$C_{hi}Q_i^{*\beta+2} - 2C_{oi}Q_i^{*\beta}\bar{D} - 2A(\beta+1)[\ \sigma\sqrt{L}\psi(\frac{r-\mu L}{\sigma\sqrt{L}}) + (\mu L - r)\ \varphi(\frac{r-\mu(L)}{\sigma\sqrt{L}})\ ] = 0 \qquad (2.16)$$

$$R(r_i^*) = \varphi(\frac{r-\mu L}{\sigma\sqrt{L}}) = \left[\frac{C_{hi}Q_i^{*\beta+1}}{A + C_{hi}(1-\gamma_i)Q^{*\beta+1}}\right] \qquad (2.17)$$

Special Cases

Two special cases of the presented model are concluded as follows [2]:

Case 1: let $\gamma_i = 0\ ,\ \beta = 0\ ,\ K_{bi} \to\ \infty\ and\ \lambda_i = 0$. The Eqs. (2.16) and (2.17) become:

$$Q^* = \sqrt{\frac{2\bar{D}(C_o + C_l\bar{S}(r))}{C_h}}\ \ and\ R(r^*) = \frac{C_h Q^*}{C_h Q^* + C_l\bar{D}}.$$

This case represents that all the shortage is lost sales. Notice that the unconstrained lost sales continuous review inventory model with constant units of the cost are the same result as in [4].

Case 2: let $\gamma_i = 1,\ \beta = 0,\ K_{li} \to\ \infty\ and\ \lambda_i = 0$. The Eqs. (2.16) and (2.17) become:

$$Q^* = \sqrt{\frac{2\overline{D}(C_o + C_b\overline{S}(r))}{C_h}}\ and\ R(r^*) = \frac{C_h Q^*}{C_b\overline{D}}.$$

This case represents that all the shortage is backorders. This is the unconstrained backorders continuous review inventory model with constant units of the cost, which correspond with the result of [4].

Numerical Example:

Let the demand for three items is fitted to normal distribution. Table (2.1) shows the cost of units for three items. Table (2.2) represents the limitations for both backorder, lost sales and their fractions. The optimal values Q^* and r^* for three items can be obtained using Eq. (2.16) and Eq. (2.17) respectively. The following iterative procedure will be used to solve the equations [15].

Use the following procedure:

- **Step 1**: Assume that $\overline{S} = 0\ and\ r = E(X)$, then from Eq. (2.16) we get:

$$Q_o = \sqrt{\frac{2C_{oi}\overline{D}_i}{C_{hi}}}$$

- **Step 2**: Substituting Q_0 into Eq. (2.17) we obtain r_0.
- **Step 3**: Substituting by r_0 from step 2 into Eq. (2.16) we can obtain Q_1.
- **Step 4**: the procedure is to change the values of λ_i in step 2 and step 3 until the smallest value of $\lambda_i > 0$ is found such that the constraints varying shortage for the different values of β.

The numerical computations are done by using Mathematica program for three items of different values of β. Tables (2.3), (2.4) and (2.5) show the optimal values Q^*, r^* and $\min E(TC)$ at different values. Therefore, we draw the optimal graphs of Q^*, r^* and $\min E(TC)$ versus β. For all three items as depicted in Fig. (2.2), Fig. (2.3) and Fig. (2.4).

Table 2.1: The mean, the standard deviation and the costs of the items.

	item 1	item 2	Item 3
Mean	10000	8500	11000
SD	2500	2800	2250
Costs			
C_o	25	35	32
C_h	3	11	8
Shortage costs			
C_b	3	10	5
C_l	12	28	20

Table 2.2: The limitations for backorder, lost sales and their fractions.

Items	K_b	K_l	γ	$1-\gamma$
Item 1	310	970	0.56	0.44
Item 2	2255	2830	0.67	0.33
Item 3	1280	2200	0.70	0.30

Table 2.3: The optimal value of Q^*, r^* and $\min E(TC)$ at the different values of β for item 1.

β	λ_1^*	λ_2^*	r^*	Q^*	Min E(TC)
0.1	0.0310	0.0312	16317	1428.8	23050
0.2	0.0312	0.0313	16647.5	1440	23983.2
0.3	0.0510	0.0520	16975	1487	24968.5
0.4	0.0530	0.0540	17275	1541	25891.27
0.5	0.060	0.062	17600	1560.75	26849

Table 2.4: The optimal value of Q^*, r^* and $\min E(TC)$ at the different values of β for item 2.

β	λ_1^*	λ_2^*	r^*	Q^*	Min E(TC)
0.1	0.0010	0.0012	15808	1269.27	93877
0.2	0.0013	0.0014	16228	1296.65	98357
0.3	0.0013	0.0015	16648	1359.07	103153
0.4	0.0015	0.0017	16984	1377.77	107838
0.5	0.0020	0.0028	17376	1428.35	113070.8

Table 2.5: The optimal value of Q^*, r^* and min $E(TC)$at the different values of β for item 3.

β	λ_1^*	λ_2^*	r^*	Q^*	Min E(TC)
0.1	0.11	0.12	16861	1533.26	58374.17
0.2	0.13	0.16	17232.5	1539.72	61005.6
0.3	0.10	0.14	17592.5	1568	63885
0.4	0.11	0.17	17863	1643	66477.14
0.5	0.12	0.19	18177	1702	68851.38

The optimal routes of Q^*, r^* and min $E(TC)$ versus $\beta$for all items is exhibited by Fig. (2.2), Fig. (2.3) and Fig. (2.4), respectively.

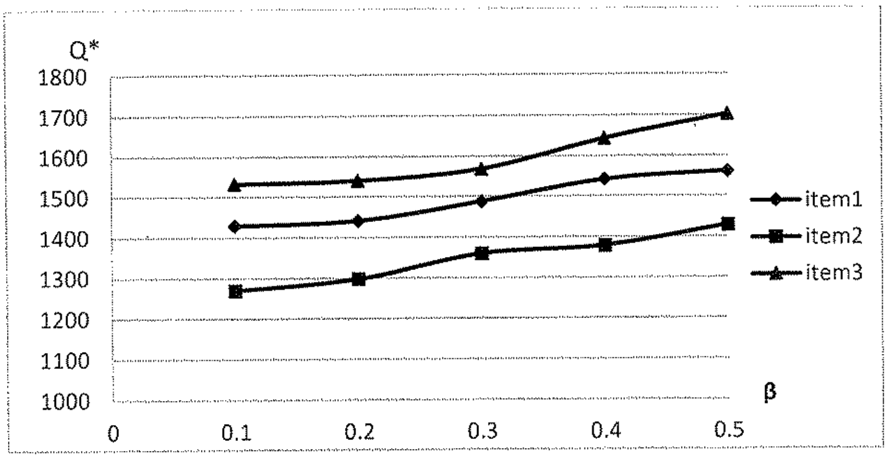

Figure 2.2: The optimal values of Q^* versus β.

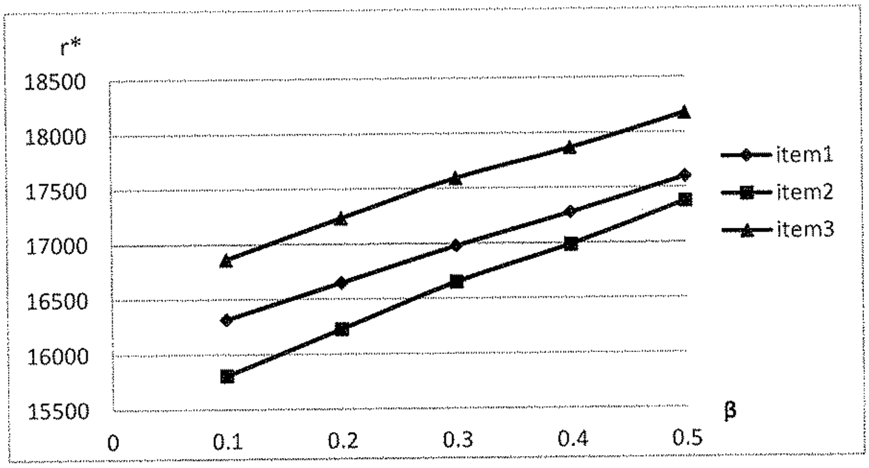

Figure 2.3: The optimal values of r^* versus β.

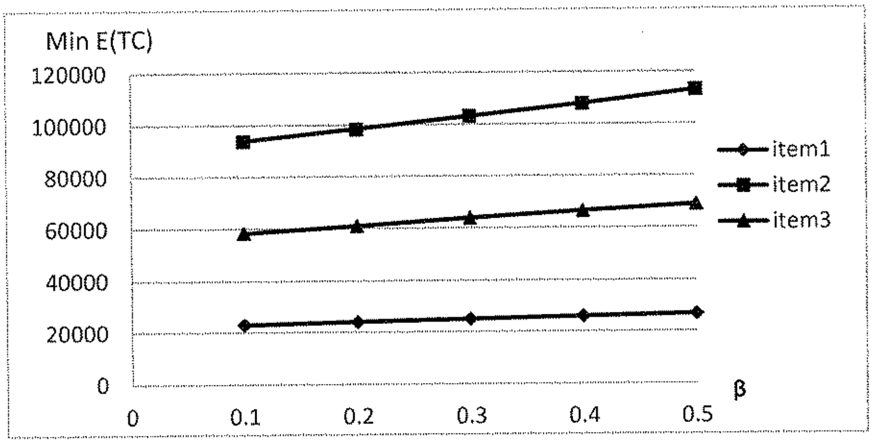

Figure 2.4: The optimal values of min E(TC) versus β.

Chapter 3

MULTI-ITEM INVENTORY MODEL WITH SHORTAGE LIMITATIONS IN FUZZY CASE

In the most of the existing literature, usually the inventory costs are assumed to be deterministic and designated as real numbers. However, in real situations the inventory costs are often imprecise or vague in nature because of the influence effect of different uncontrollable factors. Such as, costs may depend on some foreign monetary units, such as exchange rates from domestic to foreign money, the costs will not give precise values. Moreover, the shortage cost is mostly difficult to give a precise value in the case when it expresses a loss of customer's will. Hence, these cost parameters are described as approximately equal some certain amount, and it is reasonable to characterize these parameters as fuzzy. However, fuzzy set theory can be used in the formulation of inventory models. The same assumptions and notifications will be considered in this chapter during represented model with varying mixture shortage cost with two limitations when the demand is continuous random variable, the lead time demand has a known distribution and the lead time is constant.

3.1 Fuzzy Set

The fuzzy set theory can be used in wide range of domains in which is incomplete or imprecise. We introduce some definitions in order to define the fuzzy set.

Definition 3.1.1 Membership function [37]

For a set A, we define a membership function such that:

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \qquad (3.1)$$

Definition 3.1.2 Fuzzy Set

A fuzzy set is a nonempty set which maps each element belongs to a nonempty set; *X* to [0, 1] by membership function:

$$\mu_A : X \rightarrow [\,0\,,1\,]$$

We can say that the function μ_A maps the elements in the universal set X to the set {0, 1} i.e. μ_A (x) : X → {0, 1}.

Definition 3.1.3 Normal Set: *A* fuzzy set *A* is called normal if there is at least one point x belongs to *X* such that $\mu_A(x) = 1$.

Definition 3.1.4 Convex Set: A fuzzy set A is convex if for any x_1, x_2 belong to X and $\lambda \in [0, 1]$ we have:

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min \;\; \mu_A(x_1),\;\; \mu_A(x_2)$$

Now, we will use the previous definitions to define the fuzzy number.

3.2 Fuzzy Numbers

Definition 3.2.1 Fuzzy Number [37]

A fuzzy set A is a fuzzy number if it is satisfying the following conditions:

1. Convex fuzzy set.
2. Normalized fuzzy set.
3. The membership function is piecewise continuous.
4. The membership function defined on the set of real numbers.

Definition 3.2.2 Trapezoidal Fuzzy Number

A trapezoidal fuzzy number $A=(a,b,c,d)$ is a fuzzy number which has a membership function as:

$$\mu_A(x)=\begin{cases} 0 & x\le a\ ,\ x\ge d \\ \dfrac{x-a}{b-a}\ , & a<x<b \\ 1\ , & b\le x\le c \\ \dfrac{x-d}{c-d} & c<x<d \end{cases} \tag{3.2}$$

Definition 3.2.3 The alpha cut or alpha level set of a fuzzy set is a crisp set defined as follows:

$A_\alpha=\{x:\forall x\in X,\mu_A(x)\ge\alpha\}$

The α-cut method of the fuzzy numbers is used, which is:

Let $A=(a,b,c,d)$ a trapezoidal fuzzy number, then: $[A_\alpha=a+\alpha(b-a),d-\alpha(d-c)]$ is the alpha-cut of A.

3.3 Mathematical Model in Fuzzy Case

Consider continuous review inventory model similar to the model in the crisp environment, but assuming that the cost components C_o, C_h, C_b and C_l are all fuzzy numbers, to control various uncertainties from different effects, where there may be effect on the cost components. We represent these costs by trapezoidal fuzzy number as given below:

$$\widetilde{C}_o=(C_o-\delta_1,\ C_o-\delta_2,\ C_o+\delta_3,C_o+\delta_4)$$

$$\widetilde{C}_h=(C_h-\delta_5,\ C_h-\delta_6,\ C_h+\delta_7,C_h+\delta_8)$$

$$\widetilde{C}_b=(C_b-\theta_1,\ C_b-\theta_2,\ C_b+\theta_3,C_b+\theta_4)$$

$$\widetilde{C}_l=(C_l-\theta_5,\ C_l-\theta_6,\ C_l+\theta_7,C_l+\theta_8)\ , \tag{3.3}$$

where δ_i and, θ_i, $i = 1, 2, \ldots, 8$ are arbitrary positive numbers and should satisfy the following constraints [28]:

$$C_o > \delta_1 , \delta_2$$

$$\delta_3 < \delta_4$$

$$C_h > \delta_5 > \delta_6$$

$$\delta_7 < \delta_8$$

$$C_b > \theta_1 > \theta_2$$

$$\theta_3 < \theta_4$$

$$C_l > \theta_5 > \theta_6$$

$$\theta_7 < \theta_8 .$$

We represent the order cost as a trapezoidal fuzzy number as shown in Fig. (3.1) and similarly for the remaining costs.

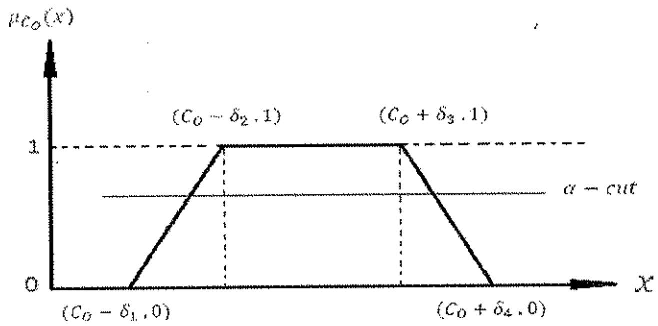

Figure 3.1: Order cost as a trapezoidal fuzzy number.

Note that the membership function of C_o is 1 at points $C_o - \delta_2$ and $C_o + \delta_3$ decreases as the point deviate from $C_o - \delta_2$ and $C_o + \delta_3$, and reaches the zero at the endpoints $C_o - \delta_1$ and $C_o + \delta_4$ [37].

The left and right limits of $\alpha - cut\ of\ C_o, C_h, C_b$ and C_l are given by:

$$\widetilde{C}_{ov}(\alpha) = C_o - \delta_1 + (\delta_1 - \delta_2)\alpha$$

$$\widetilde{C}_{ou}(\alpha) = C_o + \delta_4 - (\delta_4 - \delta_3)\alpha$$

$$\widetilde{C}_{hv}(\alpha) = C_h - \delta_5 + (\delta_5 - \delta_6)\alpha$$

$$\widetilde{C}_{hu}(\alpha) = C_h + \delta_8 - (\delta_8 - \delta_7)\alpha$$

$$\widetilde{C}_{bv}(\alpha) = C_b - \theta_1 + (\theta_1 - \theta_2)\alpha$$

$$\widetilde{C}_{bu}(\alpha) = C_b + \theta_4 - (\theta_4 - \theta_3)\alpha$$

$$\widetilde{C}_{lv}(\alpha) = C_l - \theta_5 + (\theta_5 - \theta_6)\alpha$$

$$\widetilde{C}_{lu}(\alpha) = C_l + \theta_8 - (\theta_8 - \theta_7)\alpha \ . \qquad (3.4)$$

The expected total cost for the fuzzy case with two limitations; the expected backorder and the expected lost sales, can be expressed simply by:

$$\begin{aligned}\widetilde{E}\left[\widetilde{C}_{oi},\ \widetilde{C}_{hi},\ \widetilde{C}_{bi},\ \widetilde{C}_{li}\right] = \sum[\widetilde{C}_{oi}(\frac{\overline{D}_i}{Q_i}) + \widetilde{C}_{hi}(\frac{Q_i}{2} + r_i - E(x)) \\ + \ \widetilde{C}_{bi}\gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\ \overline{S}(r_i) \\ + \ (\widetilde{C}_{li}\ (\frac{\overline{D}_i}{Q_i})^{\beta+1} + \widetilde{C}_{hi})\ \gamma_i'\ \overline{S}(r_i)\ ]\end{aligned} \qquad (3.5)$$

Subject to:

$$\begin{aligned}\widetilde{C}_{bi}\ \gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\ \overline{S}(r_i) \le\ K_{bi} \\ \widetilde{C}_{li}\ \gamma_i'(\frac{\overline{D}_i}{Q_i})^{\beta+1}\ \overline{S}(r_i) \le\ K_{li}\end{aligned} \qquad (3.6)$$

To find the optimal values $Q^*\ and\ r^*$ which minimize Eq. (3.5) with two limitations Eq. (3.6), the Lagrange multiplier technique is used as follows:

$$\widetilde{L}\left[\widetilde{C}_{oi}\ ,\ \widetilde{C}_{hi}\ ,\ \widetilde{C}_{bi}\ ,\ \widetilde{C}_{li}\right]=\widetilde{C}_{oi}(\frac{\overline{D}_i}{Q_i})+\widetilde{C}_{hi}(\frac{Q_i}{2}+r_i-E(x_i))+\ \widetilde{C}_{bi}\ \gamma_i(\frac{D_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)$$
$$+\ (\widetilde{C}_{li}\ (\frac{D_i}{Q_i})^{\beta+1}+\ \widetilde{C}_{hi})\ \gamma_i'\ \overline{S}\ (r_i)$$
$$+\ \lambda_{1i}(\widetilde{C}_{bi}\ \gamma_i(\frac{D_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)-K_{bi})$$
$$+\ \lambda_{2i}(\widetilde{C}_{li}\ \gamma_i'\ (\frac{D_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)-K_{li}) \tag{3.7}$$

The left and right $\alpha - cut$ of the fuzzified cost function (3.7) can be obtained respectively by:

$$\widetilde{L}\left[\widetilde{C}_{oi}\ ,\ \widetilde{C}_{hi}\ ,\ \widetilde{C}_{bi}\ ,\ \widetilde{C}_{li}\right]_v=\widetilde{C}_{ovi}(\frac{\overline{D}_i}{Q_i})+\widetilde{C}_{hvi}(\frac{Q_i}{2}+r_i-E(x_i))+\ \widetilde{C}_{bvi}\ \gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)$$
$$+\ (\widetilde{C}_{lvi}\ (\frac{\overline{D}_i}{Q_i})^{\beta+1}+\ \widetilde{C}_{hvi})\ \gamma_i'\ \overline{S}\ (r_i)$$
$$+\ \lambda_{1i}(\widetilde{C}_{bvi}\ \gamma_i(\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)-K_{bi})$$
$$+\ \lambda_{2i}(\widetilde{C}_{lvi}\ \gamma_i'\ (\frac{\overline{D}_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)-K_{li}) \tag{3.8}$$

$$\widetilde{L}\left[\widetilde{C}_{oi}\ ,\ \widetilde{C}_{hi}\ ,\ \widetilde{C}_{bi}\ ,\ \widetilde{C}_{li}\right]_u=\widetilde{C}_{oui}(\frac{\overline{D}_i}{Q_i}))+\widetilde{C}_{hui}(\frac{Q_i}{2}+r_i-E(x_i))+\ \widetilde{C}_{bui}\ \gamma_i(\frac{D_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)$$
$$+\ (\widetilde{C}_{lui}\ (\frac{D_i}{Q_i})^{\beta+1}+\ \widetilde{C}_{hui})\ \gamma_i'\ \overline{S}\ (r_i)$$
$$+\ \lambda_{1i}(\widetilde{C}_{bui}\ \gamma_i(\frac{D_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)-K_{bi})$$
$$+\ \lambda_{2i}(\widetilde{C}_{lui}\ \gamma_i'\ (\frac{D_i}{Q_i})^{\beta+1}\overline{S}\ (r_i)-K_{li})\ . \tag{3.9}$$

Since $(\widetilde{L}_v)(\alpha)$ *and* $(\widetilde{L}_u)(\alpha)$ exist and are integrable for $\alpha \in [0,\ 1]$ as in Yao and Wu [36], we have:

$$d(\widetilde{L},\ 0)=\frac{1}{2}\int_0^1\ ((\widetilde{L}_v(\alpha)+\widetilde{L}_u(\alpha))d\alpha\ . \tag{3.10}$$

First we get $\widetilde{L}_v(\alpha)+\widetilde{L}_u(\alpha)$ as follows:

$$\widetilde{L}_v(\alpha)+\widetilde{L}_u(\alpha)=\widetilde{C}_{ov}(\frac{\overline{D}}{Q})+\widetilde{C}_{hv}(\frac{Q}{2}+r-E(x))$$

$$+\ \widetilde{C}_{bv}\gamma(\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\ +\ (\widetilde{C}_{lv}(\frac{\overline{D}}{Q})^{\beta+1}+\widetilde{C}_{hv})\gamma'\overline{S}$$

$$+\ \lambda_1\ (\widetilde{C}_{bv}\gamma(\frac{\overline{D}}{Q})^{\beta+1}\overline{S}-K_b)\ +\ \lambda_2\ (\widetilde{C}_{lv}(\frac{\overline{D}}{Q})^{\beta+1}\gamma'\overline{S}-K_l)$$

$$+\ \widetilde{C}_{ou}(\frac{\overline{D}}{Q})+\widetilde{C}_{hv}(\frac{Q}{2}+r-E(x))$$

$$+\ \widetilde{C}_{bu}\gamma(\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\ +\ (\widetilde{C}_{lu}(\frac{\overline{D}}{Q})^{\beta+1}+\widetilde{C}_{hu})\gamma'\overline{S}$$

$$+\ \lambda_1\ (\widetilde{C}_{bu}\gamma(\frac{\overline{D}}{Q})^{\beta+1}\overline{S}-K_b)\ +\ \lambda_2\ (\widetilde{C}_{lu}\gamma(\frac{\overline{D}}{Q})^{\beta+1}\gamma'\overline{S}-K_l)$$

$$=(C_o-\delta_1+(\delta_1-\delta_2)\alpha)(\frac{\overline{D}}{Q})\ +\ (C_h-\delta_5+(\delta_5-\delta_6)\ \alpha)\ (\frac{Q}{2}+r-E(x))$$

$$+(C_b-\theta_1+(\theta_1-\theta_2)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma$$

$$+\ [(C_l-\theta_5+(\theta_5-\theta_6)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}+(C_h-\delta_5+(\delta_5-\delta_6)\ \alpha)\ ]\ \gamma'\ \overline{S}$$

$$+\ \lambda_1((C_b-\theta_1+(\theta_1-\theta_2)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma-K_b)$$

$$+\ \lambda_2((C_l-\theta_5+(\theta_5-\theta_6)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma'-K_l)$$

$$+\ (C_o+\delta_4-(\delta_4-\delta_3)\alpha)(\frac{\overline{D}}{Q})+(C_h+\delta_8-(\delta_8-\delta_7)\ \alpha)\ (\frac{Q}{2}+r-E(x))$$

$$+(C_b+\theta_4-(\theta_4-\theta_3)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma$$

$$+\ [\ (C_l+\theta_8-(\theta_8-\theta_7)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1})+(C_h+\theta_8-(\theta_8-\theta_7)\ \alpha)\ ]\ \gamma'\overline{S}$$

$$+\ \lambda_1((C_b+\theta_4-(\theta_4-\theta_3)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma-K_b)$$

$$+\ \lambda_2((C_l+\theta_8-(\theta_8-\theta_7)\ \alpha)\ (\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma'-K_l)\,.$$

Then,

$$d\ (\widetilde{L},\ 0)=\frac{1}{2}\int_0^1\ ((\widetilde{L}_v(\alpha)+\widetilde{L}_u(\alpha))d\alpha$$

$$=(\frac{1}{2})\ (C_o-\delta_1+\frac{(\delta_1-\delta_2)}{2}(\frac{\overline{D}}{Q})$$

$$+\ (\frac{1}{2})\ [\ C_h-\delta_5+\frac{(\delta_5-\delta_6)}{2}\ \ ]\ (\frac{Q}{2}+r-E(x))$$

$$+\ (\frac{1}{2})\ [\ C_b-\theta_1+\frac{(\theta_1-\theta_2)}{2}(\frac{\overline{D}}{Q})^{\beta+1}\gamma\overline{S}\ \ ]$$

$$+\ (\frac{1}{2})\ [\ (C_l-\theta_5+\frac{(\theta_5-\theta_6)}{2}(\frac{\overline{D}}{Q})^{\beta+1}+(C_h-\delta_5+\frac{(\delta_5-\delta_6)}{2}\ \ ]\ \gamma'\overline{S}$$

$$+\ (\frac{\lambda_1}{2})\ [\ C_b-\theta_1+\frac{(\theta_1-\theta_2)}{2}(\frac{\overline{D}}{Q})^{\beta+1}\gamma\overline{S}-K_b\ \ ]$$

$$+\ (\frac{\lambda_2}{2})\ [\ C_l-\theta_5+\frac{(\theta_5-\theta_6)}{2}(\frac{\overline{D}}{Q})^{\beta+1}\ \gamma'\overline{S}-K_l\ \ ]$$

$$+\ (\frac{1}{2})\ [\ C_o+\delta_4-\frac{(\delta_4-\delta_3)}{2}\ \ ]\ (\frac{\overline{D}}{Q})+\ (\frac{1}{2})\ [\ C_h+\delta_8-\frac{(\delta_8-\delta_7)}{2}\ \ ]\ (\frac{Q}{2}+r-E(x))$$

$$+\ (\frac{1}{2})\ [\ C_b+\theta_4-\frac{(\theta_4-\theta_3)}{2}(\frac{\overline{D}}{Q})^{\beta+1}\gamma\overline{S}\ \ ]$$

$$+\ (\frac{1}{2})\ [\ (C_l+\theta_8-\frac{(\theta_8-\theta_7)}{2}(\frac{\overline{D}}{Q})^{\beta+1}+(C_h-\theta_8+\frac{(\theta_8-\theta_7)}{2}\ \ ]\ \gamma'\overline{S}$$

$$+\ (\frac{\lambda_1}{2})\ [\ C_b+\theta_4-\frac{(\theta_4-\theta_3)}{2}(\frac{\overline{D}}{Q})^{\beta+1}\overline{S}\gamma-K_b\ \ ]$$

$$+\ (\frac{\lambda_2}{2})\ [\ C_l+\theta_8-\frac{(\theta_8-\theta_7)}{2}(\frac{\overline{D}}{Q})^{\beta+1}\gamma'\overline{S}-K_l\ \ ].$$

From the previous resulting equation, we notice the value of the costs as follows:

$$Order\ cost = \left(\frac{\overline{D}}{Q}\right) \left[\ C_o - \delta_1 + \frac{(\delta_1 - \delta_2)}{2}\ +\ C_o + \delta_4 - \frac{(\delta_4 - \delta_3)}{2}\ \right]$$

$$= \left(\frac{4C_o - \delta_1 - \delta_2 + \delta_3 + \delta_4}{4}\right) \left(\frac{\overline{D}}{Q}\right)$$

$$= G_1\ \frac{\overline{D}}{Q}$$

$$Holding\ cost = \left(\frac{1}{2}\right) \left(\frac{Q}{2} + r - E(x)\right) \left[\ C_h - \delta_5 + \frac{(\delta_5 - \delta_6)}{2} + C_h + \delta_8 - \frac{(\delta_8 - \delta_7)}{2}\ \right]$$

$$= \left(\frac{Q}{2} + r - E(x)\right) \left[\ \left(\frac{4C_h - \delta_5 - \delta_6 + \delta_7 + \delta_8}{4}\right)\ \right]$$

$$= G_2\left(\frac{Q}{2} + r - E(x)\right)$$

$$Backorder\ cost = \left(\frac{\overline{D}}{Q}\right)^{\beta+1} \left(\frac{\overline{S}}{2}\right) \left[C_b - \theta_1 + \frac{(\theta_1 - \theta_2)}{2} + C_b + \theta_4 - \frac{(\theta_4 - \theta_3)}{2}\ \right]$$

$$= \gamma\ \left(\frac{\overline{D}}{Q}\right)^{\beta+1}\ \overline{S}\ \left[\ \left(\frac{4C_b - \theta_1 - \theta_2 + \theta_3 + \theta_4}{4}\right)\ \right]$$

$$= \gamma\ G_3\ \left(\frac{\overline{D}}{Q}\right)^{\beta+1}\ \overline{S}$$

$$Lost\ sales\ cost = \left(\frac{\gamma'\overline{S}}{2}\right) \left[\ \left(\frac{\overline{D}}{Q}\right)^{\beta+1} \left[\ (C_l - \theta_5 + \frac{(\theta_5 - \theta_6)}{2} + (C_l - \theta_8 + \frac{(\theta_8 - \theta_7)}{2}\ \right] + G_2\ \right]$$

$$= \gamma'\overline{S}\ \left[\ \left(\frac{\overline{D}}{Q}\right)^{\beta+1} G_4 + G_2\ \right]$$

Since:

$$G_1 = \left(\frac{4C_o - \delta_1 - \delta_2 + \delta_3 + \delta_4}{4}\right)$$

$$G_2 = \left(\frac{4C_h - \delta_5 - \delta_6 + \delta_7 + \delta_8}{4}\right)$$

$$G_3 = \left(\frac{4C_b - \theta_1 - \theta_2 + \theta_3 + \theta_4}{4}\right)$$

$$G_4 = \left(\frac{4C_l - \theta_5 - \theta_6 + \theta_7 + \theta_8}{4}\right)$$

By using the Lagrange multiplier technique similarly to the previous model in crisp model to get the optimal values Q^* *and* r^*; by setting each of the corresponding first partial derivatives of Eq. (3.5) equal to zero.

$$i.e \quad \frac{\partial \widetilde{L}}{\partial Q_i} = 0 \qquad , \qquad \frac{\partial \widetilde{L}}{\partial r_i} = 0$$

Then we obtain:

$$\overline{D}_i^{\beta+1}(\beta+1)\ (G_3\gamma_i\ (1+\lambda_{1i}) + G_4\gamma_i'\ (1+\lambda_{2i})\ \overline{S}(r_i)) + G_1\overline{D}_i Q_i^{*\beta} - \frac{G_2}{2} Q^{*\beta+2} = 0 \tag{3.11}$$

$$R(r_i^*) = \frac{G_2 Q^{\beta+1}}{\overline{D}^{\beta+1} G_3 (1+\lambda_{1i}) + \overline{D}^{\beta+1} G_4 \gamma' (1+\lambda_{2i}) + G_2 \gamma'\ Q_i^{*\beta+1}} \tag{3.12}$$

Clearly, there is no closed form solution of Eq. (3.11) and Eq. (3.12). These two equations can be solved by using the same manner as in previous chapter.

3.4 Numerical example and Results

Consider the following for the previous model in chapter two of constrained fuzzy probabilistic multi-item continuous review $< Q,\ r >$ with varying mixture shortage:

Let:

$\delta_1 = 24$	$\delta_2 = 23$	$\delta_3 = 1$	$\delta_4 = 2$
$\delta_5 = 2$	$\delta_6 = 1$	$\delta_7 = 1$	$\delta_8 = 2$
$\theta_1 = 2$	$\theta_2 = 1$	$\theta_3 = 1$	$\theta_4 = 2$
$\theta_5 = 11$	$\theta_6 = 10$	$\theta_7 = 2$	$\theta_8 = 3$

By using the previous procedure in chapter two, the numerical computations are done by using Mathematica program for three items at different values of $\beta = 0.1, 0.2, \ldots, 0.5$. We get the following data:

Table 3.1: The optimal value of Q^*, r^* and min $E(TC)$ at the different values of β for item1.

β	λ_1^*	λ_2^*	r^*	Q^*	$Min\ E(TC)$
0.1	0.0002	0.0552	16350	1220.91	22581
0.2	0.0553	0.0553	16700	1293.56	23550
0.3	0.0555	0.0556	17050	1296.45	24631.8
0.4	0.060	0.061	17350	1350.96	25578.3
0.5	0.0662	0.0663	17625	1438.9	26577

Table 3.2: The optimal value of Q^*, r^* and $\min E(TC)$ at the different values of β for item 2.

β	λ_1^*	λ_2^*	r^*	Q^*	*Min E(TC)*
0.1	0.002	0.0022	15920	1140.51	92625
0.2	0.0030	0.0034	16354	1158.88	97145
0.3	0.0035	0.0040	16760	1212.23	102185
0.4	0.0041	0.0043	17152	1252.94	106825
0.5	0.0045	0.0046	17530	1287.77	111468

Table 3.3: The optimal value of Q^*, r^* and $\min E(TC)$ at the different values of β for item 3.

β	λ_1^*	λ_2^*	r^*	Q^*	*Min E(TC)*
0.1	0.21	0.23	17018.75	1264.9	57360
0.2	0.22	0.25	17367.5	1380.19	60180
0.3	0.30	0.32	17738.75	1417.38	63141
0.4	0.32	0.34	18065	1477.14	65867.7
0.5	0.33	0.35	18368.75	1559	68258

The optimal routes of Q^*, r^* and $\min E(TC)$ versus β for all items is exhibited by Fig. (3.2), Fig. (3.3) and Fig. (3.4), respectively.

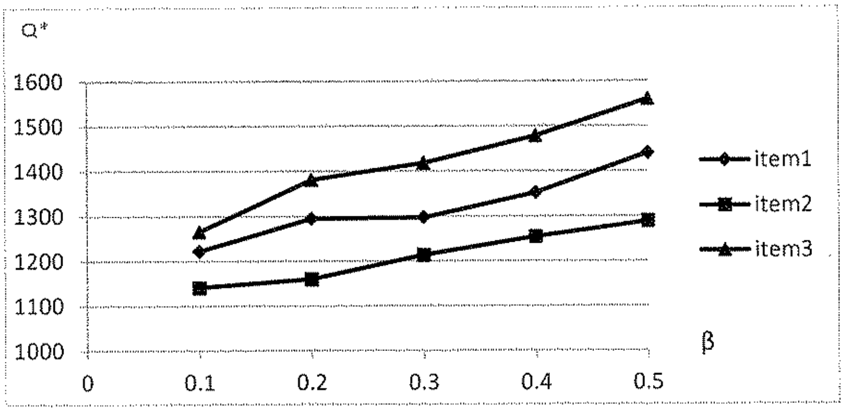

Figure 3.2: The optimal values of Q^* versus β.

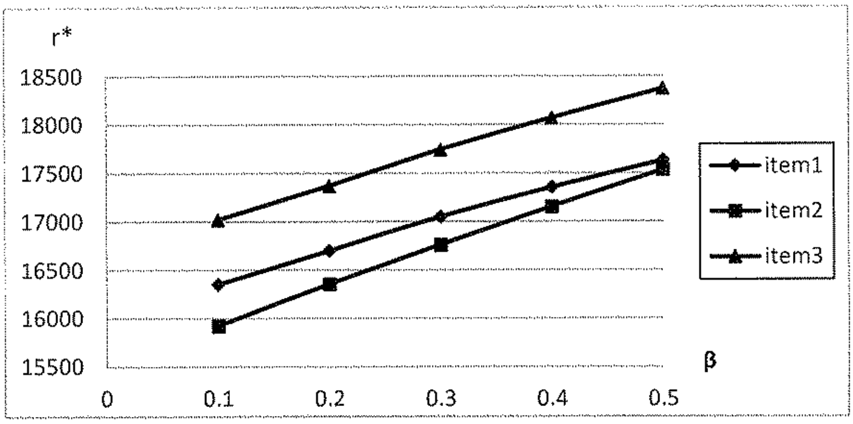

Figure 3.3: The optimal values of r^* versus β.

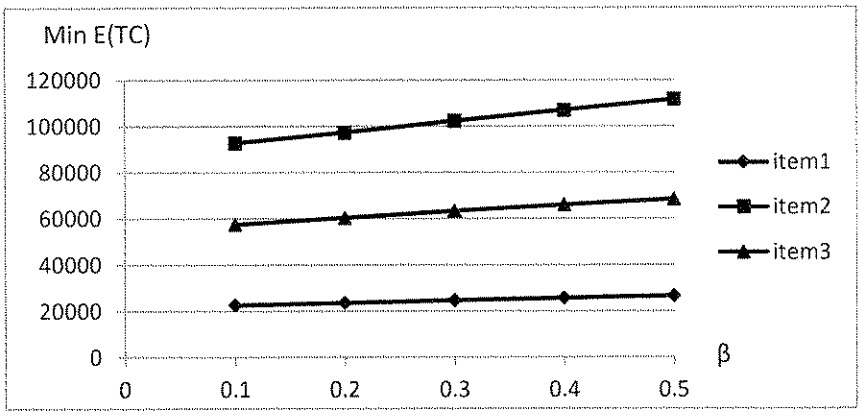

Figure 3.4: The optimal values of $\min E(TC)$ versus β.

By comparison within the expected total cost crisp and fuzzy cases when using different values of β. We can conclude that the expected total cost is sensitive of fuzziness in the cost components which indicates that the fuzziness is very realistic and gets minimum expected total cost less than the crisp. This is clarified in Fig. (3.5), Fig. (3.6), and Fig. (3.7).

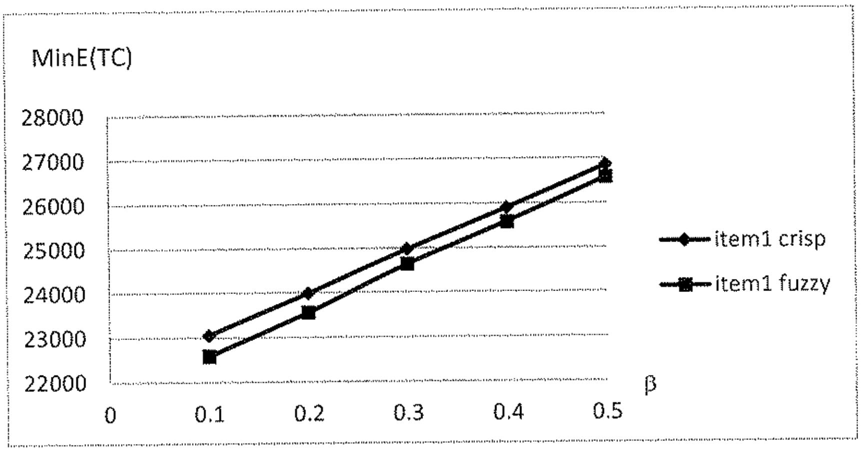

Figure 3.5: The comparison between the crisp and fuzzy cases for item 1.

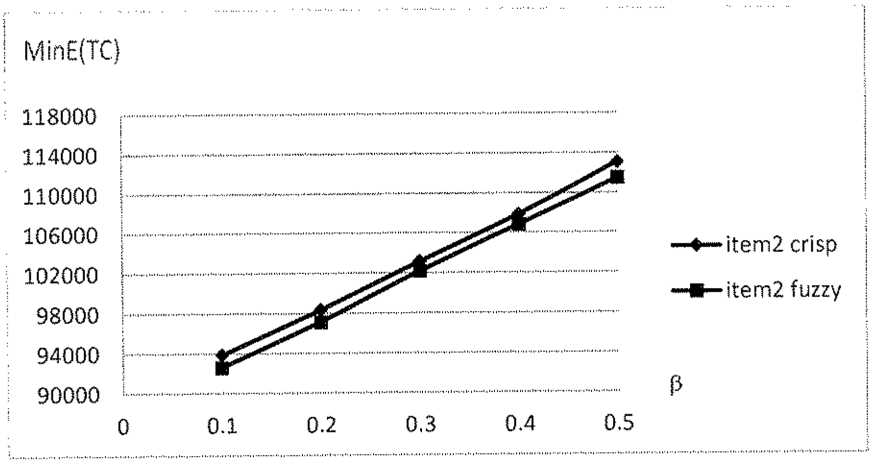

Figure 3.6: The comparison between the crisp and fuzzy cases for item 2.

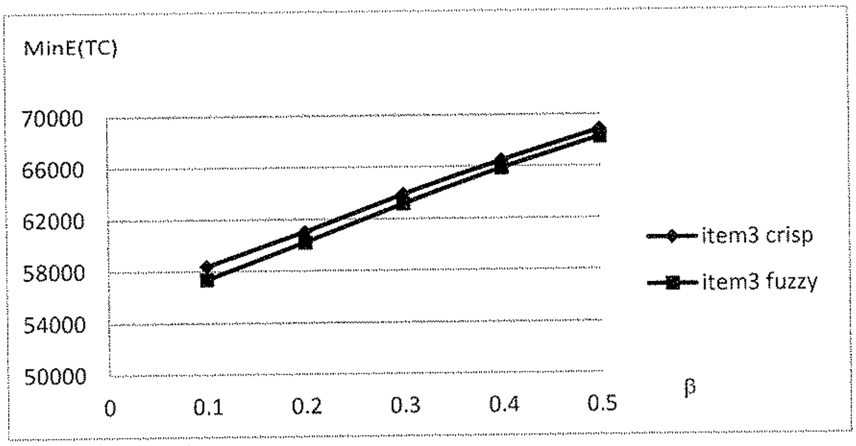

Figure 3.7: The comparison between the crisp and fuzzy cases for item 3.

CONCLUSIONS

Multi-item inventory model with two limitations on shortage cost is presented. These limitations include; the expected backorder cost and the lost sales cost with varying values. The limitations are identified by assuming a fraction of demand is backordered and the remaining fraction is lost sales during the shortage period. This model is a probabilistic multi-item, single source with a demand that follows normal distribution, which is formulated as a continuous review to analyze how to get the optimal order quantity Q^* and the optimal reorder point r^*. This is done to obtain the main objective of minimizing the expected total cost $E(TC)$. We have developed this model in fuzzy case. The fuzziness in the costs is presented by using trapezoidal fuzzy number. For the results of the illustrative numerical examples, we have deduced the minimum expected total cost by using Lagrange multiplier technique and the solutions of optimal values of Q^* and r^* are evaluated for each value of β and the Lagrange multipliers; $\lambda^*{}_1$, $\lambda^*{}_2$ for three different items are obtained. Our work shows that when β increases, r^* increases and thus Q^* increases which indicates that $\min E(TC)$ increases. By comparing between the minimum expected total cost for crisp case and fuzzy case, a sensitive to the level of fuzziness in the cost components is revealed. Three curves for Q^*, r^* and $\min E(TC)$ are displayed to illustrate them for each item versus the different values of β. Finally, the $\min E(TC)$ is achieved at minimum value of β.

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MATHEMATICA COMPUTATIONS

item1(crisp)
B=0.1

```
In[19]:= (3*(816)^1.1)÷(((40000^1.1)((1.68*1.0310)+(5.28*1.0312)))+(3*0.44*(816^1.1)))
Out[19]= 0.00576342

In[4]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.527, ∞}]
Out[4]= 0.0164216

In[16]:= FindInstance[3*Q^2.1- (2*25*40000*Q^0.1)- 2*1.1*4.68*((1.68*1.0310)
        +(5.28*1.0312))*(40000^1.1)==0, Q]
Out[16]= {{Q→1428.81}}

In[17]:= 1.68*((40000÷1428.8)^1.1)*4.68
Out[17]= 307.151

In[18]:= 5.28*((40000÷1428.8)^1.1)*4.68
Out[18]= 965.333
```

item1(crisp)
B=0.2

```
In[23]:= (3*(816)^1.2)÷(((40000^1.2)((1.68*1.0312)+(5.28*1.0313)))+(3*0.44*(816^1.2)))
Out[23]= 0.00390792

In[24]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.659, ∞}]
Out[24]= 0.0116619

In[42]:= FindInstance[3*Q^2.2- (2*25*40000*Q^0.2)- 2*1.2*3.15063*((1.68*1.0312)
        +(5.28*1.0313))*(40000^1.2)==0, Q]
Out[42]= {{Q→1440.00}}

In[43]:= 1.68*((40000÷1440)^1.2)*3.15063
Out[43]= 285.854

In[44]:= 5.28*((40000÷1440)^1.2)*3.15063
Out[44]= 898.397
```

```
item1(crisp)
B=0.4
In[8]:= (3*(816)^1.4)+(((40000^1.4)((1.68*1.0530)+(5.28*1.0540)))+(3*0.44*(816^1.4)))

Out[8]= 0.00175759

In[9]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.91, ∞}]

Out[9]= 0.00579743

In[11]:= FindInstance[3*Q^2.4- (2*25*40000*Q^0.4)- 2*1.4*1.696*((1.68*1.0530)
           +(5.28*1.0540))*(40000^1.4)==0, Q]

Out[11]= {{Q→1541.09}}

In[12]:= 1.68*((40000+1541.09)^1.4)*1.696

Out[12]= 272.057

In[13]:= 5.28*((40000+1541.09)^1.4)*1.696

Out[13]= 855.036
```

```
item1(crisp)
B=0.3
In[1]:= (3*(816)^1.3)+(((40000^1.3)((1.68*1.0510)+(5.28*1.0520)))+(3*0.44*(816^1.3)))

Out[1]= 0.00259788

In[2]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.79, ∞}]

Out[2]= 0.00816139

In[5]:= FindInstance[3*Q^2.3- (2*25*40000*Q^0.3)- 2*1.3*2.267*((1.68*1.0510)
           +(5.28*1.0520))*(40000^1.3)==0, Q]

Out[5]= {{Q→1487.03}}

In[6]:= 1.68*((40000+1487)^1.3)*2.267

Out[6]= 275.064

In[7]:= 5.28*((40000+1487)^1.3)*2.267

Out[7]= 864.488
```

```
item1(crisp)
B = 0.5

In[14]:= (3*(816)^1.5)+(((40000^1.5)((1.68*1.060)+(5.28*1.0620)))+(3*0.44*(816^1.5)))

Out[14]= 0.00118251

In[15]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.03, ∞}]

Out[15]= 0.00405932

In[17]:= FindInstance[3*Q^2.5- (2*25*40000*Q^0.5)- 2*1.5*1.1826*((1.68*1.060)
            +(5.28*1.062))*(40000^1.5)==0, Q]

Out[17]= {{Q→1560.75}}

In[18]:=
   1.68*((40000+1560.75)^1.5)*1.1826

Out[18]= 257.773

In[19]:= 5.28*((40000+1560.75)^1.5)*1.1826

Out[19]= 810.143

item2(crisp)

B = 0.1

In[44]:= (11*(618)^1.1)+((11*0.33*(618^1.1))+((60000^1.1)((6.7*1.0010)+(9.24*1.0012))))
Out[44]= 0.00448639

In[55]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.61, ∞}]
Out[55]= 0.0132688

In[56]:= FindInstance[11*Q^2.1- (2*35*60000*Q^0.1)- 2*1.1*4.3661*((6.7*1.0010)
            +(9.24*1.0012))*(60000^1.1)==0, Q]
Out[56]= {{Q→1269.35}}

In[57]:= 6.7*((60000+1269.35)^1.1)*4.3661
Out[57]= 2033.27

In[58]:= 9·24*((60000+1269·35)^1·1)*4·3661
Out[58]= 2804.09
```

```
item2(crisp)

B=0.2

In[48]:= (11*(618)^1.2)÷((11*0.33*(618^1.2))+((60000^1.2)((6.7*1.0013)+(9.24*1.0014))))

Out[48]= 0.00283996

In[22]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.76, ∞}]

Out[22]= 0.00886991

In[59]:= FindInstance[11*Q^2.2- (2*35*60000*Q^0.2)- 2*1.2*2.8884*((6.7*1.0013)
    +(9.24*1.0014))*(60000^1.2)==0, Q]

Out[59]= {{Q→1296.65}}

In[60]:= 6.7*((60000+1296.65)^1.2)*2.8884

Out[60]= 1928.09

In[61]:= 9.24*((60000+1296.65)^1.2)*2.8884

Out[61]= 2659.03


item2(crisp)

B=0.3

In[80]:= (11*(618)^1.3)÷((11*0.33*(618^1.3))+((60000^1.3)((6.7*1.0013)+(9.24*1.0015))))

Out[80]= 0.00179771

In[66]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.905, ∞}]

Out[66]= 0.00588232

In[81]:= FindInstance[11*Q^2.3- (2*35*60000*Q^0.3)- 2*1.3*2.05168*((6.7*1.013)
    +(9.24*1.015))*(60000^1.3)==0, Q]

Out[81]= {{Q→1359.07}}


In[82]:= 6.7*((60000+1359.07)^1.3)*2.05168

Out[82]= 1890.46

In[83]:= 9.24*((60000+1359.07)^1.3)*2.05168

Out[83]= 2607.14
```

```
item2(crisp)

B=0.4

In[112]:= (11*(618)^1.4)÷((11*0.33*(618^1.4))+((60000^1.4)((6.7*1.0015)+(9.24*1.0017))))
Out[112]= 0.00113766


In[108]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.04, ∞}]
Out[108]= 0.00393797

In[113]:= FindInstance[11*Q^2.4- (2*35*60000*Q^0.4)- 2*1.4*1.3744*((6.7*1.0016)
             +(9.24*1.0017))*(60000^1.4)==0, Q]
Out[113]= {{Q→1377.77}}

In[114]:= 6.7*((60000+1377.77)^1.4)*1.3744
Out[114]= 1814.48

In[115]:= 9.24*((60000+1377.77)^1.4)*1.3744
Out[115]= 2502.36



item2(crisp)

B=0.5

In[116]:= (11*(618)^1.5)÷(((60000^1.5)((6.7*1.0020)+(9.24*1.0028)))+(11*0.33*(618^1.5)))
Out[116]= 0.00071943


In[117]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.17, ∞}]
Out[117]= 0.00263008

In[118]:= FindInstance[11*Q^2.5- (2*35*60000*Q^0.5)- 2*1.5*0.97856*((6.7*1.0020)
             +(9.24*1.0028))*(60000^1.5)==0, Q]
Out[118]= {{Q→1428.35}}

In[119]:= 6.7*((60000+1428.35)^1.5)0.97856
Out[119]= 1785.00

In[120]:= 9.24*((60000+1428.35)^1.5)*0.97856
Out[120]= 2461.7
```

```
item3(crisp)
B=0.1
In[6]:= (8*(848.5)^1.1)+(((90000^1.1)((3.5*1.11)+(6*1.12)))+(8*0.3*(848.5^1.1)))
Out[6]= 0.00445501

In[7]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.605, ∞}]
Out[7]= 0.0134429

In[9]:= FindInstance[8*Q^2.1- (2*32*90000*Q^0.1)- 2*1.1*4.135*((3.5*1.11)
          +(6*1.12))*(90000^1.1)==0, Q]
Out[9]= {{Q→1533.26}}

In[10]:= 3.5*((90000+1533.26)^1.1)*4.135
Out[10]= 1276.54

In[11]:= 6*((90000+1533.26)^1.1)*4.135
Out[11]= 2188.35
```

```
item3(crisp)
B=0.2
In[24]:= (8*(848.5)^1.2)+(((90000^1.2)((3.5*1.13)+(6*1.16)))+(8*0.3*(848.5^1.2)))
Out[24]= 0.00271646

In[20]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.77, ∞}]
Out[20]= 0.00862802

In[21]:= FindInstance[8*Q^2.2- (2*32*90000*Q^0.2)- 2*1.2*2.482745*((3.5*1.13)
          +(6*1.16))*(90000^1.2)==0, Q]
Out[21]= {{Q→1539.72}}

In[22]:= 3.5*((90000+1539.72)^1.2)*2.482745
Out[22]= 1145.94

In[23]:= 6*((90000+1539.72)^1.2)*2.482745
Out[23]= 1964.46
```

item3(crisp)
B=0.3

```
In[66]:= (8*(848.5)^1.3)+(((90000^1.3)((3.5*1.10)+(6*1.14)))+(8*0.3*(848.5^1.3)))
Out[66]= 0.00174027

In[75]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.91, ∞}]
Out[75]= 0.00579743

In[76]:= FindInstance[8*Q^2.3- (2*32*90000*Q^0.3)- 2*1.3*1.65*((3.5*1.10)
          +(6*1.14))*(90000^1.3)==0, Q]
Out[76]= {{Q→1568.06}}

In[77]:= 3.5*((90000+1568.06)^1.3)*1.65
Out[77]= 1117.11

In[78]:= 6*((90000+1568.06)^1.3)*1.65
Out[78]= 1915.04
```

item3(crisp)
B=0.4

```
In[82]:= (8*(848.5)^1.4)+(((90000^1.4)((3.5*1.11)+(6*1.17)))+(8*0.3*(848.5^1.4)))
Out[82]= 0.00107028

In[83]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.05, ∞}]
Out[83]= 0.00381986

In[22]:= FindInstance[8*Q^2.4- (2*32*90000*Q^0.4)- 2*1.4*1.1619*((3.5*1.11)
          +(6*1.17))*(90000^1.4)==0, Q]
Out[22]= {{Q→1643.}}

In[23]:= 3.5*((90000+1643)^1.4)*1.253
Out[23]= 1191.42

In[24]:= 6*((90000+1643)^1.4)*1.253
Out[24]= 2042.44
```

```
In[97]:= item3(crisp)
      B=0.5

In[90]:= (8*(848.5)^1.5)÷(((90000^1.5)((3.5*1.12)+(6*1.19)))+(8*0.3*(848.5^1.5)))
Out[90]= 0.000662007

In[93]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 3.19, ∞}]
Out[93]= 0.00246802

In[94]:= FindInstance[8*Q^2.5- (2*32*90000*Q^0.5)- 2*1.5*0.803*((3.5*1.12)
          +(6*1.19))*(90000^1.5)==0, Q]
Out[94]= {{Q→1702.7}}

In[95]:= 3.5*((90000+1702.7)^1.5)*0.803
Out[95]= 1080.04

In[96]:= 6*((90000+1702.7)^1.5)*0.803
Out[96]= 1851.5
```

```
      item1(fuzzy)
      B=0.1
      (3*(611)^1.1)÷((3*0.44*(611^1.1))+((40000^1.1)((1.68*1.0550)+(3.52*1.0552))))
Out[12]= 0.00548453

In[13]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.54, ∞}]
Out[13]= 0.0158896

In[25]:= FindInstance[1.1*4.89723*((1.68*1.0550)+(3.52*1.0552))*(40000^1.1)
          + (14*40000*Q^0.1)-(3*Q^2.1÷2)==0, Q]

Out[25]= {{Q→1220.91}}

In[15]:= 1.68((40000+1220.91)^1.1)*4.8972
Out[15]= 382.095

In[16]:= 3.52((40000+1220.91)^1.1)*4.8972
Out[16]= 800.58
```

item1(fuzzy)

B = 0.2

```
In[19]:= (3*(611)^1.2)+((3*0.44*(611^1.2))+((40000^1.2)((1.68*1.0553)+(3.52*1.0554))))
Out[19]= 0.00361244

In[20]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.68, ∞}]
Out[20]= 0.0110261

In[21]:= FindInstance[1.2*3.3619*((1.68*1.0553)+(3.52*1.0554))*(40000^1.2)
          + (14*40000*Q^0.2)-(3*Q^2.2+2)==0, Q]

Out[21]= {{Q→1293.56}}

In[22]:= 1.68((40000+1293.56)^1.2)*3.3619
Out[22]= 346.914

In[23]:= 3.52((40000+1293.56)^1.2)*3.3619
Out[23]= 726.868
```

item1(fuzzy)

B = 0.3

```
In[44]:= (3*(611)^1.3)+((3*0.44*(611^1.3))+((40000^1.3)((1.68*1.0555)+(3.52*1.0556))))
Out[44]= 0.00237877

In[45]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.82, ∞}]
Out[45]= 0.00750271

In[56]:= FindInstance[1.3*2.456*((1.68*1.0555)+(3.52*1.0556))*(40000^1.3)
          + (14*40000*Q^0.3)-(3*Q^2.3+2)==0, Q]

Out[56]= {{Q→1296.45}}

In[57]:= 1.68((40000+1296.45)^1.3)*2.456
Out[57]= 356.15

In[58]:= 3.52((40000+1296.45)^1.3)*2.456
Out[58]= 746.219
```

item1(fuzzy)

B=0.5

```
In[64]:= (3*(611)^1.5)+((3*0.44*(611^1.5))+((40000^1.5)((1.68*1.0662)+(3.52*1.0663))))
Out[64]= 0.00102101

In[83]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.06, ∞}]
Out[83]= 0.00370493

In[84]:= FindInstance[1.5*1.4516*((1.68*1.0662)+(3.52*1.0663))*(40000^1.5)
          + (14*40000*Q^0.5)-(3*Q^2.5+2)==0, Q]

Out[84]= {{Q → 1438.99}}

In[85]:= 1.68*((40000+1439)^1.5)*1.4516
Out[85]= 357.4

In[86]:= 3.52*((40000+1439)^1.5)*1.4516
Out[86]= 748.839
```

item1(fuzzy)

B=0.4

```
In[59]:= (3*(611)^1.4)+((3*0.44*(611^1.4))+((40000^1.4)((1.68*1.060)+(3.52*1.061))))
Out[59]= 0.00155887

In[60]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.94, ∞}]
Out[60]= 0.00531039

In[62]:= FindInstance[1.4*1.8182*((1.68*1.060)+(3.52*1.061))*(40000^1.4)
          + (14*40000*Q^0.4)-(3*Q^2.4+2)==0, Q]

Out[62]= {{Q → 1350.96}}

In[63]:= 1.68*((40000+1350.96)^1.4)*1.8182
Out[63]= 350.7

In[64]:= 3.52*((40000+1350.96)^1.4)*1.8182
Out[64]= 734.799
```

```
item2(fuzzy)
B = 0.1
In[2]:= (11*(511.68)^1.1)÷(((60000^1.1)((6.7*1.0020)+(7.92*1.0022)))+(11*0.33*(511.68^1.
Out[2]= 0.00397094

In[3]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.65, ∞}]
Out[3]= 0.0119438

In[26]:= FindInstance[1.1*3.9760*((6.7*1.0020)+(7.92*1.0022))*(60000^1.1)
    + (24*60000*Q^0.1)-(11*Q^2.1÷2) == 0, Q]

Out[26]= {{Q → 1140.51}}

In[6]:= 6.7*((60000÷1140.51)^1.1)*3.9760
Out[6]= 2082.95

In[16]:= 7.92*((60000÷1140.51)^1.1)*3.9760
Out[16]= 2462.23


item2(fuzzy)

B = 0.2
In[11]:= (11*(511.68)^1.2)÷(((60000^1.2)((6.7*1.0030)+(7.92*1.0034)))+(11*0.33*(511.68^1.
Out[11]= 0.00246442

In[12]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 2.805, ∞}]
Out[12]= 0.007826

In[13]:= FindInstance[1.2*2.5572*((6.7*1.0030)+(7.92*1.0034))*(60000^1.2)
    + (24*60000*Q^0.2)-(11*Q^2.2÷2) == 0, Q]

Out[13]= {{Q → 1158.88}}

In[14]:= 6.7*((60000÷1158.88)^1.2)*2.5572
Out[14]= 1953.33

In[15]:= 7.92*((60000÷1158.88)^1.2)*2.5572
Out[15]= 2309.01
```

```
In[17]:= item2(fuzzy)
         B = 0.3

In[20]:= (11*(511.68)^1.3)+(((60000^1.3)((6.7*1.0035)+(7.92*1.0040)))+(11*0.33*(511.68^1.

Out[20]= 0.00153

In[21]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.95, ∞}]

Out[21]= 0.00515628

In[22]:= FindInstance[1.3*1.8*((6.7*1.0035)+(7.92*1.0040))*(60000^1.3)
           + (24*60000*Q^0.3)-(11*Q^2.3+2)==0, Q]

Out[22]= {{Q → 1212.23}}

In[23]:= 6.7*((60000+1212.23)^1.3)*1.8

Out[23]= 1924.35

In[24]:= 7.92*((60000+1212.23)^1.3)*1.8

Out[24]= 2274.75
```

```
item2(fuzzy)
B = 0.4

In[49]:= (11*(511.68)^1.4)+(((60000^1.4)((6.7*1.0041)+(7.92*1.0043)))+(11*0.33*(511.68^1.

Out[49]= 0.000949886

In[48]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.09, ∞}]

Out[48]= 0.00337844

In[50]:= FindInstance[1.4*1.2412*((6.7*1.0041)+(7.92*1.0043))*(60000^1.4)
           + (24*60000*Q^0.4)-(11*Q^2.4+2)==0, Q]

Out[50]= {{Q → 1252.94}}

In[51]:= 6.7*((60000+1252.94)^1.4)*1.2412

Out[51]= 1871.66

In[52]:= 7.92*((60000+1252.94)^1.4)*1.2412

Out[52]= 2212.46
```

```
item2(fuzzy)
B=0.5

In[55]:= (11*(511.68)^1.5)+(((60000^1.5)((6.7*1.0045)+(7.92*1.0046)))+(11*0.33*(511.68^1.
Out[55]= 0.000589737

In[68]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 3.225, ∞}]
Out[68]= 0.00220594

In[69]:= FindInstance[1.5*0.85133*((6.7*1.0045)+(7.92*1.0046))*(60000^1.5)
   + (24*60000*Q^0.5)-(11*Q^2.5+2)==0, Q]

Out[69]= {{Q→1287.77}}

In[70]:= 6.7*((60000+1287.77)^1.5)*0.85133
Out[70]= 1814.02

In[71]:= 7.92*((60000+1287.77)^1.5)*0.85133
Out[71]= 2144.33


item3(fuzzy)
B=0.1

In[97]:= (8*(687.38)^1.1)+(((90000^1.1)((3.5*1.21)+(4.8*1.23)))+(8*0.3*(687.38^1.1)))
Out[97]= 0.00369711

In[100]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.675, ∞}]
Out[100]= 0.0111747

In[1]:= FindInstance[1.1*2.933*((3.5*1.21)+(4.8*1.23))*(90000^1.1)
   + (21*90000*Q^0.1)-(8*Q^2.1+2)==0, Q]
Out[1]= {{Q→1264.9}}

In[102]:= 3.5*((90000+1264.9)^1.1)*2.933
Out[102]= 1118.88

In[103]:= 4.8*((90000+1264.9)^1.1)*2.933
Out[103]= 1534.47
```

```
item3(fuzzy)
B=0.2

In[104]:= (8*(687·38)^1·2)+(((90000^1·2)((3·5*1·24)+(4·8*1·25)))+(8*0·3*(687·38^1·2)))
Out[104]= 0.00222753

In[109]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.83, ∞}]
Out[109]= 0.00729373

In[2]:= FindInstance[1.2*2.225*((3.5*1.24)+(4.8*1.25))*(90000^1.2)
     + (21*90000*Q^0.2)-(8*Q^2.2+2)==0, Q]

Out[2]= {{Q→1380.19}}

In[111]:= 3.5*((90000+1380.19)^1.2)*2.225
Out[111]= 1171.01

In[112]:= 4.8*((90000+1380.19)^1.2)*2.225
Out[112]= 1605.96


item3(fuzzy)
B=0.3

In[113]:= (8*(687.38)^1.3)+(((90000^1.3)((3.5*1.30)+(4.8*1.32)))+(8*0.3*(687.38^1.3)))
Out[113]= 0.00129985

In[121]:= Integrate[0.4*y*Exp[-(y^2)+2], {y, 2.995, ∞}]
Out[121]= 0.0045107

In[123]:= FindInstance[1.3*1.389*((3.5*1.30)+(4.8*1.32))*(90000^1.3)
     + (21*90000*Q^0.3)-(8*Q^2.3+2)==0, Q]

Out[123]= {{Q→1417.38}}

In[124]:= 3.5*((90000+1417.38)^1.3)*1.389
Out[124]= 1072.39

In[125]:= 4.8*((90000+1417.38)^1.3)*1.389
Out[125]= 1470.71
```

```
item3(fuzzy)
B=0.4

In[127]:= (8*(687.38)^1.4)÷(((90000^1.4)((3.5*1.32)+(4.8*1.34)))÷(8*0.3*(687.38^1.4)))
Out[127]= 0.000786471

In[128]:= Integrate[0.4*y*Exp[-(y^2)÷2],{y,3.14,∞}]
Out[128]= 0.00289118

In[129]:= FindInstance[1.4*0.9488*((3.5*1.32)+(4.8*1.34))*(90000^1.4)
            + (21*90000*Q^0.4)-(8*Q^2.4÷2)==0, Q]

Out[129]= {{Q→1477.14}}

In[130]:= 3.5*((90000÷1477.14)^1.4)*0.9488
Out[130]= 1047.11

In[131]:=
          4.8*((90000÷1477.14)^1.4)*0.9488

Out[131]= 1436.04


item3(fuzzy)
B=0.5

In[132]:= (8*(687.38)^1.5)÷(((90000^1.5)((3.5*1.33)+(4.8*1.35)))÷(8*0.3*(687.38^1.5)))
Out[132]= 0.000479478

In[9]:= Integrate[0.4*y*Exp[-(y^2)÷2], {y, 3.275, ∞}]
Out[9]= 0.00187508

In[10]:= FindInstance[1.5*0.6857*((3.5*1.33)+(4.8*1.35))*(90000^1.5)
           + (21*90000*Q^0.5)-(8*Q^2.5÷2)==0, Q]

Out[10]= {{Q→1558.99}}

In[11]:= 3.5*((90000÷1500.96)^1.5)*0.6857

Out[11]= 1114.33

In[12]:= 4.8*((90000÷1500.96)^1.5)*0.6857
Out[12]= 1528.22
```

الملخص

نموذج المخزون متعدد العناصر مع القيود على النقص

في هذه الأطروحة تم عرض نموذج المخزون متعدد العناصر مع وجود قيدين على النقص في المخزون وعدم تلبية الطلب للزبائن. أحد القيدين وضع على التكلفة المتوقعة للمبيعات المؤجلة، والقيد الآخر وضع على التكلفة المتوقعة لفقدان المبيعات، مع الأخذ بالاعتبار أن القيدين يتم حسابهم في حالتين لقيم التكلفة، وهما قيم واضحة وقيم ضبابية. تمت صياغة هذا النموذج في حالة ضبابية التكاليف لتحليل القيم المثلى لكمية الطلب ونقطة إعادة الطلب لكل عنصر من عناصر المخزون لتقليل التكلفة الإجمالية المتوقعة. تم تطوير النموذج باعتبار الطلب خلال المهلة الزمنية للنقص كمتغير عشوائي يتبع التوزيع الطبيعي. تم تطبيق مثال رقمي و مقارنة نتائج النموذجين في كلا الحالتين لقيم التكلفة. تم الاستعانة بتقنية المضاعف لاغرانج، بينت النتائج أن تقليل التكلفة الإجمالية المتوقعة تم عند أقل قيمة من قيم β .