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Inverse Integrating Factor Method of Planar Autonomous Polynomial Differential Systems

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Dedication

To my husband Lawyer Seyij Daraghmeh , my children Saif Eddine and Hallaa for their assistance, tolerance and endless support.

To my father Engineer Mjali Jameel and my mother, there is no doubt in my mind that without there continued sopport I could not have completed my study.

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I will never forget my family; my parents and my husband for their infinite support, also my children Saif Eddine and Hallaa . Finally, my great gratitude to Mathematics Department at The Arab American University

List of Symbols

$\mathbb{R}[x, y]$	:	ring of real polynomials in two variables
C^1	:	continuously differentiable functions

Abstract

Inverse Integrating Factor Method of Planar Autonomous Polynomial Differential Systems

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This thesis is a survey study in which the inverse integrating factor for planar autonomous polynomial differential systems has been done and some illustrative examples are presented. Also, a quick survey on the relationship between the inverse integrating factor and limit cycle is handled. Furthermore, some aspects of integrability problem, invariant algebraic curves and their relations with inverse integrating factor for planar differential systems are reviewed.

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Chapter 1

Introduction

System of Differential Equations is a wide field in pure and applied mathematics and physics, as well as engineering, and all of these areas are concentrated on the study of different properties of these systems. For example, pure and applied mathematics focused on the existence and uniqueness of the solution, while applied mathematics also searched for numerical solutions to these systems. So, many researches dealt with differential systems and looked on the methodologies of solving such systems [1, 4, 5, 8–10, 34].

The study of differential system in terms of the existence and uniqueness of the solution, stability, existence of periodic solution and other properties of solution is of great importance in many applications as in physics for example. Some of such interesting questions are related with the so-called ***qualitative theory of differential equations***. The existence of first integral and inverse integrating factor are important tools in the study of planar differential systems [12].

The method of integrating factor based on reducing ordinary differential equation of the first order to an exact one. In the theory of elementary ordinary differential equations, some exact equations can be solved by elementary integration method while most of them can't be solved in such method. the method of integrating factors is one of the useful methods for this purpose. In the last decades, the researchers looked to that problem from another point of view, so the study of the so-called inverse integrating factor arose.

An inverse integrating factor is a very important function and perhaps it is the best way to understand the integrability of two dimensional differential systems, because, $V(x, y) = 0$ contains all limit cycles in the domain of definition of V. In general, the expression of V is simpler than the expression of the associated first integral and its domain of definition is usually larger [1].

Planar autonomous polynomial differential system took a considerable attention by many researchers. To be more specific, the existence of an inverse integrating factor of planar differential system was one of the researches interests. Besides, inverse integrating factor and its relation with many problems in the area of applied mathematics, such as integrability problem, limit cycles, Lie symmetries, center problem, vanishing set of an inverse integrating factor, for example see [19, 20, 23, 29, 40].

In this thesis, we will deal with a planar autonomous differential system of the form

$$\begin{aligned} x' &= \frac{dx}{dt} = P(x, y) \\ y' &= \frac{dy}{dt} = Q(x, y)), \end{aligned} \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are functions of class C^1 in the domain of interest, or elements in the ring of real polynomials of two variables, $\mathbb{R}[x, y]$ with degree m = max(deg P, deg Q). The vector field and divergence associated to system (1.1) will be respectively

$$\tilde{X}(x, y) = P(x, y)\partial x + Q(x, y)\partial y \tag{1.2}$$

$$div\tilde{X} = \frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y} \tag{1.3}$$

,

A function $V : U \to \mathbb{R}$ is said to be an inverse integrating factor of system (1.1)on an open subset U of $\mathbb{R}^2$. if satisfies the differential equation [1]

$$P(x,y)\frac{\partial V(x,y)}{\partial x}+Q(x,y)\ \frac{\partial V(x,y)}{\partial y}=(\frac{\partial P(x,y)}{\partial x}+\frac{\partial Q(x,y)}{\partial y})V(x,y) \tag{1.4}$$

Aldosary [1] showed the relation between inverse integrating factor and integrating factor also they presented some examples P and Q belongs to the ring of real polynomials in two variables $\mathbb{R}[x,y]$. Besides system (1.1) in this thesis we will handle the following system and state explicitly a condition that will guarantee the existence of the inverse integrating factor for such system

$$\begin{aligned} x' &= P_{n1}(x,y)+P_{n2}(x,y)+\cdots\ +P_{nk}(x,y) \\ y' &= Q_{n1}(x,y)+Q_{n2}(x,y)+\cdots+Q_{nk}(x,y) \end{aligned}$$

where $P_i(x,y)$and $Q_i(x,y)$ are homogeneous polynomials in x, y of degree n_i Associate to the system the vector field

$$X(x,y)=X_1(x,y)+X_2(x,y)+\ldots+X_n(x,y) \tag{1.5}$$

then the function

$$V(x,y)=(-y,x)\cdot(C_{n1}X_{n1}+C_{n2}X_{n2}+\cdots+C_{n_k}X_{n_k})$$

is an inverse integrating factor where $X_{ni}=(P_{ni}\ ,Q_{ni})\ ,i=1,...,k.$

Enciso and Salas [11] studied the existence of inverse integrating factor for an analytic planar vector field in a neighborhood of its vanishing set. Berrone and Giacomini [3] analyzed the properties of vanishing set of inverse integrating factor. Ferragut [12] classified all quadratic systems that have a polynomial inverse integrating factors.

The existence of first integral and strategies of finding are in general not easy (Hamiltonian system is one of the easiest planar systems). Many methods have been applied to construct first integrals, such as Lie symmetries, Noether symmetries, the Painlevé analysis, the use of Lax pairs, the Darboux method and the direct method see [2, 38] and references therein.

In 1878 Darboux introduced an easy method to build the first integral and the integrating factor for planar polynomial differential systems. The theory of Draboux has been used to study various problems that are related to planar differential systems, such as limit cycle, center problem, see [32] and refernces therein. In recent years, many authors have extended Draboux theory taking in account some other terms such as exponential factors, multiplicity of invariant algebraic curves or non-autonomous polynomial differential systems in the field of complex numbers [25]. [22]

In fact, Darboux Theory of Integrability deals with the number of algebraic invariants and exponential factors in order to assure that a Darboux first integral exists. Also, he showed how the integrability of a polynomial system could be forced from the abundance of invariant algebraic curves. The idea of Darboux Theorem was to construct an integrating factor and first integral for the system of the form

$$f_1^{\ell_1} \ldots\ldots f_q^{\ell_q}$$

where f_i invariant algebraic curves of the system [36].

Invariant algebraic curves is an algebraic curve $f(x, y) = 0$,where $f(x, y) \in \mathbb{R}[x, y]$, if there exists a polynomial $k(x, y)$ called the cofactor of $f(x, y)$ has at most degree $(m - 1)$ such that [8, 36, 37]

$$P(x, y)\ \frac{\partial f(\text{x,y})}{\partial x} + Q(x, y)\ \frac{\partial f(x, y)}{\partial y} = \text{k(x,y) f(x,y)}. \tag{1.6}$$

A limit cycle of system (1.1) is a periodic orbit isolated in a set of orbits of the system. In 1996, Giacomini, Libre and viano [17] showed the relation between limit cycle of planar differential system and inverse integrating factor. the authors in [11] studied the existence of inverse integrating factors for an analytic (algebraic) planar vector field in a neighborhood of its non wandering sets and proved that there always exists a smooth inverse integrating factor in a neighborhood of a limit cycle. The authors in [6] had given a set of necessary conditions for a system to have an invariant algebraic curve, such conditions are determined from the value of the cofactor at the singular points of the system. Also, they applied the founded results to show the non-Liouvillian integrability of several families of quadratic systems with an algebraic limit cycle. In [18] investigated the non-degenerate center problem for polynomial vector fields as well as the relation between the existence of an inverse integrating factor and the center problem. It is known that if a planar differential systems has an inverse integrating factor, then all the limit cycles [29] contained in the domain of definition of the inverse integrating factor are contained in the zero set of this function. Using this fact they gave some criteria to rule out the existence of limit cycles and presented some applications and examples that illustrate the results.

Besides the introduction, this thesis consists of four chapters. In chapter two, elementary concepts about planar differential systems are introduced and compact formulas of integrating factors of special types are also derived. Basic facts about inverse integrating factors for planar differential system are presented in chapter three. Moreover, some remarks and illustrative examples on the presented theory are discussed. Also, a quick view about the relationship between limit cycle and inverse integrating factor is presented. In chapter four, we studied some result of the integrability problem such as Darboux theorem.

Chapter 2

Planar Differential Systems

In this chapter, some basic concepts that are needed in this thesis are presented. The chapter consists of two sections. In section one, the planar autonomous polynomial differential system is introduced. section two consists of the definition of integrating factor and four formula of the integrating factor, besides examples on each case.

2.1 Planar Autonomous Polynomial Differential Systems

Consider the planar autonomous system which has the following general form

$$x' = P(x, y), y' = Q(x, y). \tag{2.1}$$

The autonomous system is a system of ordinary differential equations that depend explicitly on the state (x, y) and implicitly on the variable t. The word "autonomous" has Greek roots, means "self governing". What makes the system planar is the fact it has two dependent variables, x and y. If $P, Q \in \mathbb{R}[x, y]$, then system (2.1) is called polynomial differential system of degree $m = max\{degP, degQ\}$. The vector field and the divergence of the vector field associated to the differential system (2.1) are given in the following two equations respectively:

$$\tilde{X}(x,y) = P(x,y)\partial x + Q(x,y)\partial y, \tag{2.2}$$

$$div\tilde{X} = \frac{\partial P(x,y)}{\partial x} + \frac{\partial Q(x,y)}{\partial y}. \tag{2.3}$$

Also, system(2.1) can be written as

$$P(x,y)dx + Q(x,y)dy = 0. \tag{2.4}$$

Definition 2.1 *The ordinary Differential equation (2.4) is called exact if there exists a function* $f(x,y)$ *such that* $\frac{\partial f}{\partial x} = P(x,y)$ *and* $\frac{\partial f}{\partial y} = Q(x,y)$ *and the general solution is given by f(x,y) =c.*

So, equation (2.4) is exact if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, otherwise it is not exact. To make it exact we multiply it by a suitable function $\mu(x,y)$ which is called the integrating factor.

2.2 Integrating Factor Method for Ordinary Differential Equations

In the present section, the method of integrating factor is presented and four types of inegrating factors are deduced.

Remark 2.1 *[43]* $\mu(x,y)$*is the integrating factor of equation (2.4) if one of the following conditions is hold:*

1. $\text{div}(\mu(x,y)P(x,y)\ ,\ \mu(x,y)Q(x,y)) = 0.$
2. $\frac{\partial(\mu P)}{\partial x} + \frac{\partial(\mu Q)}{\partial y} = 0.$
3. $\tilde{X}\mu + \mu\ div(P,Q) = 0.$

Below we give exact formulas of the integrating factor $\mu(x,y)$ for equation (2.4):

Case 1: If μ is a function of x only then the integrating factor is

$$\mu(x) = exp\int \frac{P_y - Q_x}{Q} dx. \quad (2.5)$$

Proof: Since $\frac{\partial}{\partial y}[\mu(x)P] = \frac{\partial}{\partial x}[\mu(x)Q]$, we have $\mu(x)\frac{\partial}{\partial y}P = \mu(x)\frac{\partial}{\partial x}Q + \mu' Q$. Therefore, $\frac{\mu'}{\mu} = \frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q}$ and the result follows immediately. □

Example 2.1 *consider the differential equation:*

$(y-3)\partial x + x^2 \partial y = 0$, then $\frac{\partial}{\partial y}[\mu(x)(y-3)] = \frac{\partial}{\partial x}[\mu(x)x^2]$. Therefore, $\frac{\partial \mu(x)}{\mu(x)} = \int (1 - 2x/x^2)dx$, and the integrating factor is $\mu(x) = \exp(-\frac{1}{x} - 2\ln x)$.

Case 2: If μ is a function of y only then the integrating factor function is

$$\mu(y) = exp\Big(\int \frac{Q_x - P_y}{P} dy\Big) \quad (2.6)$$

Example 2.2 *consider the differential equation:*

$(2xy^4 \exp y + 2xy^3 + y)\partial x + (x^2y^4 - x^2y^2 - 3x)\partial y = 0$, then $\frac{\partial \mu(y)}{\mu(y)} = \int (-4/y)dy$, and the integrating factor is $\mu(y) = 1/y^4$.

Case 3: If μ is a function of xy ,i.e $\mu = \mu(u)$ where $u = xy$.Then

$\mu(u)P_y + P\mu_y = \mu Q_x + Q\mu_x$. Also, since $\frac{\partial u}{\partial y} = x$ and $\frac{\partial u}{\partial x} = y$,

$$\mu(u)P_y - \mu(u)Q_x = yQ(x,y)\frac{\partial \mu(u)}{\partial u} - xP(x,y)\frac{\partial \mu(u)}{\partial u}$$

$$\frac{d\mu}{\mu(u)} = \frac{P_y - Q_x}{yQ(x,y) - xP(x,y)} du$$

Therefore, $\mu(u) = \exp\left(\int \frac{P_y - Q_x}{yQ(x,y) - xP(x,y)} du\right)$ is an integrating factor.

Example 2.3 *Consider the differential equation*

$$(y^3 + xy^2 + y)dx + (x^3 + x^2y + x)dy = 0.$$

Since $\frac{\partial}{\partial y}P(x,y) = 3y^2 + 2xy + 1$, $\frac{\partial}{\partial x}Q(x,y) = 3x^2 + 2xy + 1$, the differential equation is not exact. According to the current case, simple calculations give

$\dfrac{\frac{\partial}{\partial y}P(x,y) - \frac{\partial}{\partial x}Q(x,y)}{yQ(x,y) - xP(x,y)} = -3(x^2 - y^2)/(xy(\ (x^2 - y^2)) = -3/(xy) = -3/u$

$\mu(u) = \exp \int(-3/u)du = u^{-3}$.

Case 4: μ is a function of $\frac{x}{y}$, $\mu = \mu(u)$ where u $= \frac{x}{y}$ in this case

$\frac{\partial}{\partial y}[\mu(u)P(x,y)] = \frac{\partial}{\partial x}[\mu(u)Q(x,y)]$

$\mu(u)\frac{\partial}{\partial y}P(x,y) + P(x,y)\frac{\partial}{\partial y}\mu(u) = \mu(u)\frac{\partial}{\partial x}Q(x,y) + Q(x,y)\frac{\partial}{\partial x}\mu(u)$

since $,u = \frac{x}{y}$, $\frac{\partial}{\partial y}\mu(u) = \frac{\partial\mu(u)}{\partial u}\frac{\partial u}{\partial y} = (\frac{-x}{y^2})\frac{\partial\mu(u)}{\partial u}$

similarly , $\frac{\partial}{\partial x}\mu(u) = \frac{\partial\mu(u)}{\partial u}\frac{\partial u}{\partial y} = \frac{1}{y}\frac{\partial\mu(u)}{\partial u}$

$\frac{\partial\mu(u)}{\mu(u)} = \dfrac{y^2\frac{\partial}{\partial y}P(x,y) - \frac{\partial}{\partial x}Q(x,y)}{xP(x,y) + yQ(x,y)}\partial u$

$\frac{\partial\mu(u)}{\mu(u)} = G(u)\partial u$

where $,G(u) = \dfrac{y^2\frac{\partial}{\partial y}P(x,y) - \frac{\partial}{\partial x}Q(x,y)}{xP(x,y) + yQ(x,y)}$

hence, $\mu(u) = \exp \int G(u)du$ is an integrating factor where, $u = \frac{x}{y}$

Tab. 2.1: Error function

Condition for P and Q	Integrating factor $\mu(u)$	Remarks
$P = y\varphi(xy), Q = x\psi(xy)$	$\mu = \dfrac{1}{xP - yQ}$	$xP - yQ \neq 0; \varphi(z), \psi(Z)$ are any functions
$P_x = Q_y, P_y = -Q_x$	$\mu = \frac{1}{P^2+Q^2}$	$P + iQ$ is an analytic function of the complex variable x+iy
$\frac{P_y - Q_x}{P} = \varphi(y)$	$\mu = \exp[\int \psi(y)dy]$	$\psi(y)$is any function
$\frac{P_y - Q_x}{Q} = \varphi(x)$	$\mu = \exp[\int \varphi(x)dx]$	$\varphi(x)$ is any function
$\frac{P_y - Q_x}{Q-P} = \varphi(x+y)$	$\mu = \exp[\int \varphi(z)dz], z = x + y$	$\varphi(z)$ is any function
$\frac{P_y - Q_x}{yQ-xP} = \varphi(xy)$	$\mu = \exp[\int \varphi(z)dz], z = xy$	$\varphi(z) is any function$
$\frac{x^2(P_y - Q_x)}{Q+xP} = \varphi(\frac{y}{x})$	$mu = \exp[\int \varphi(z)dz], z = \frac{y}{x}$	$\varphi(z)$ is any function
$\frac{P_y - Q_x}{xQ-yP} = \varphi(x^2+y^2)$	$\mu = \exp[\frac{1}{2}\int \varphi(z)dz], z = x^2 + y^2$	$\varphi(z)$ is any function
$P_y - Q_x = \varphi(x)Q - \psi(y)P$	$\mu = \exp[\int \varphi(x)dx + \int \psi(y)dy]$	$\varphi(x), \psi(y)$ are any function
$\frac{P_y - Q_x}{QW_x - PW_y} = \varphi(W)$	$\mu = \exp[\int \varphi(W)dW]$	$W = W(x, y)$ is any function of two variables

Chapter 3

Inverse Integrating Factor of Planar Differential System

In this chapter, we drive some formulas for the inverse integrating factor of planar differential system and then give some illustrative examples. The chapter consists of three sections. In the first section, the definition of inverse integrating factor is given. The aim of section two is to deduce formulas of the inverse integrating factor for the system (2.1)) when $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials in x, y of degree i. Section three deals with the case, $P(x, y)$ and $Q(x, y)$ are in C^1.

In system (2.1), let $P(x, y)$ and $Q(x, y)$ be two continuously differentiable functions on an open subset U of $\mathbb{R}^2$.

Definition 3.1 *[1] . Afunction $V : U \to \mathbb{R}$ is said to be an inverse integrating factor of system (2.1) if it is of class $C^1(U)$, it is not locally null and it satisfies the following linear partial differential equation*

$$P(x, y)\frac{\partial V(x, y)}{\partial x} + Q(x, y)\ \frac{\partial V(x, y)}{\partial y} = (\frac{\partial P(x, y)}{\partial x} + \frac{\partial Q(x, y)}{\partial y})V(x, y). \tag{3.1}$$

In other words, $V(x, y)$ is an inverse integrating factor of system (2.1) if it satisfies the relation

$$\tilde{X}(x,y)V(x,y) = V(x,y)div(\tilde{X}(x,y)). \tag{3.2}$$

, where $div\tilde{X}$ stands for the divergence of the vector field $\tilde{X}$

3.1 Inverse Integrating Factor for Homogeneous Polynomial System

The aim of the current section is to state a condition and prove a generalized formula of the fact given in [1] and solve some examples to illustrate the presented theory. Throughout this section, $P(x,y)$ and $Q(x,y)$ are two homogeneous polynomials in $\mathbb{R}[x,y]$.

Definition 3.2 *A polynomial $P \in R[x,y]$ is called homogeneous of degree n if P is a linear combination of $x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n$.*

From the definition above, the polynomial can be written as $P(x,y) = \sum\limits_{i=0}^{n} a_i x^{n-i} y^i$. Also, we will use the notation $P_n(x,y)$ to indicate a polynomial of degree n.

Proposition 3.1 *[27] If $F(x) = F(x_1, \cdots .x_n)$ is a homogeneous polynomial of degree m then $F(\alpha x) = \alpha^m F(x)$ for $\alpha \in R$*

Theorem 3.1 (EulersTheorem) *[42] If $F(x_1, \cdots .x_n)$ is a homogeneous polynomial of degree m then $\sum\limits_{j=1}^{n} x_j \frac{\partial F}{\partial x_j} = mF(x_1, \cdots .x_n)$*

Example 3.1 *: consider the following polynomial*

$$F(x_1, x_2) = x_1^2 - 2x_1x_2 + 3x_2^2$$

is a homogeneous polynomial of degree 2 ,by Euler's Theorem

$$\begin{aligned}\sum_{j=1}^{2} x_j \frac{\partial F}{\partial x_j} &= x_1(2x_1 - 2x_2) + x_2(6x_2 - 2x_1) \\ &= 2(x_1^2 - 2x_1x_2 + 3x_2^2) \\ &= 2F(x_1, x_2)\end{aligned}$$

Corollary 3.1 *[1] If V is an inverse integrating factor for system (2.1)) then* $\frac{1}{V(x,y)}$ *is an integrating factor on u -{ V=0} that is* $u/\{(x,y) \in u : V(x,y) = 0\}$

Theorem 3.2 *[1, 13] Let P and Q be two homogeneous polynomials of degree n, then the function* $V(x,y) = yP(x,y) - xQ(x,y)$ *is the inverse integrating factor for the following system*

$$\begin{aligned} x' &= P(x,y) \\ y' &= Q(x,y). \end{aligned} \tag{3.3}$$

Proof: To prove that the function $V(x,y) = yP(x,y) - xQ(x,y)$ is an inverse integrating factor of system (3.3, it's enough to show that the function satisfies (3.1). The partial derivatives of V with respect to x and y are $V_x = yP_x - xQ_x - Q$ and $V_y = yP_y - xQ_y + P$. Substituting these derivatives into the left hand side of (3.1) to get

$$\begin{aligned} P(x,y)V_x + Q(x,y)V_y &= yPP_x - xPQ_x - QP + yQP_y - xQQ_y + QP \\ &= yPP_x - xPQ_x + yQP_y - xQQ_y, \end{aligned}$$

the right hand side of (3.1) is $(P_x + Q_y)(yP - xQ) = yPP_x - xQP_x + yPQ_y - xQQ_y$. Thus, from Eular theorem

$$\begin{aligned} PV_x + QV_y - (P_x + Q_y)V &= -xPQ_x + yQP_y + xQP_x - yPQ_y \\ &= -P(xQ_x + yQ_y) + Q(yP_y + xP_x) \\ &= -P(nQ(x,y)) + Q(nP(x,y)) = 0. \end{aligned}$$

□

Remark 3.1 *The function $V(x, y) = A(xQ_n - yP_n)$ is an inverse integrating factor for system(3.3), where A is nonzero constant.*

The two follow cases will be taking is consideration:

Case 1: Consider the following planar system:

$$\begin{aligned} x' &= P_n(x,y) + P_m(x,y) \\ y' &= Q_n(x,y) + Q_m(x,y). \end{aligned} \tag{3.4}$$

where $n \neq m$, P_n, P_m, Q_n and Q_m are homogeneous polynomials.

Theorem 3.3 *[1] Let C_m and C_n be two distinct real numbers and $X_i = (P_i, Q_i)$. If P_i and Q_i satisfy the following condition:*

$$\frac{(1+n)C_n - (1+m)C_m}{C_m - C_n}(P_mQ_n - P_nQ_m) = (P_{nx} + Q_{ny})(xQ_m - yP_m) - (P_{mx} + Q_{my})(xQ_n - yP_n) \tag{3.5}$$

then an inverse integrating factor for system (3.3) is

$$V(x, y) = (-y, x) \cdot (C_nX_n + C_mX_m). \tag{3.6}$$

Remark 3.2 *In theorem(3.3), let* $V_n = (-y, x) \cdot (P_n, Q_n)$ *and* $V_m = (-y, x) \cdot (P_n, Q_n)$ *then the function* $V(x, y) = C_nV_n + C_mV_m$ *is an inverse integrating factor of system(3.4).*

Proof: From equation (3.6) we have

$$\begin{aligned}
V(x,y) &= (-y,x)\cdot(C_n(P_n,Q_n)+C_m(P_m,Q_m)) \\
&= (-y,x)\cdot(C_nP_n+C_mP_m, C_nQ_n+C_mQ_m) \\
&= -y(C_nP_n+C_mP_m+x(C_nQ_n+C_mQ_m) \\
&= C_n(-yP_n+xQ_n)+C_m(C_mP_m+xQ_m) \\
&= C_nV_n+C_mV_m
\end{aligned}$$

□

Example 3.2 *: consider the following system*

$$\begin{aligned}
x' &= x+y+\frac{5}{2}x^2+xy-y^2 \\
y' &= x+2y+2x^2+\frac{7}{2}xy-2y^2.
\end{aligned}$$

Take $P_1 = x + y, P_2 = \frac{5}{2}x^2 + xy - y^2, Q_1 = x + 2y$, and $Q_2 = 2x^2 + \frac{7}{2}xy - 2y^2$. Then

$$\begin{aligned}
P_2Q_1 - P_1Q_2 &= (\frac{5}{2}+6x^2y+xy^2-2y^3)-(2x^3=\frac{11}{2}x^2y+\frac{3}{2}xy^2-2y^3) \\
&= \frac{1}{2}x^3+\frac{1}{2}x^2y-\frac{1}{2}xy^2
\end{aligned}$$

Also,

$$\begin{aligned}
(xQ_1-yP_1)(P_{2x}+Q_{2y}) &= \frac{17}{2}x^3+\frac{11}{2}x^2y-\frac{23}{2}xy^2+3y^3 \\
(xQ_2-yP_2)(P_{1x}+Q_{1y}) &= 6x^3+3x^2y-9xy^2+3y^3
\end{aligned}$$

Let $A = \dfrac{2C_1 - 3C_2}{C_2 - C_1}$, from condition (3.5) we have $A = -5$, which implies $2C_2 = 3C_1$. Choose $C_1 = \dfrac{2}{3}$ then $C_2 = 1$. Therefore,

$$\begin{aligned}
V(x,y) &= (-y,x)\cdot(C_1X_1 + C_2X_2) \\
&= (-y,x)(\frac{2}{3}(x+y, x+2y) + (\frac{5}{2}x^2 + xy - y^2, 2x^2 + \frac{7}{2}xy - 2y^2)) \\
&= \frac{2}{3}xy - \frac{2}{3}y^2 + \frac{2}{3}x^2 + 2x^3 + y^3 + x^2y
\end{aligned}$$

Remark 3.3 *In example (3.2), $C_1 = \frac{2}{3}C_2$, so the function*

$$V = C_2(\frac{2}{3}xy - \frac{2}{3}y^2 + \frac{2}{3}x^2 - 3xy^2 + y^3 + 6x^3 + x^2y)$$

is an inverse integrating factor for the system.

Corollary 3.2 *[1] If P_n, P_m, Q_n, and Q_m are homogeneous polynomial satisfying condition $(P_{mx} + Q_{my})(xQ_n - yP_n) - (P_{nx} + Q_{ny})(xQ_m - yP_m) = 0$. Then the function $V(x,y) = (-y,x)\cdot(X_n + \frac{1+n}{1+m}X_m)$ is an inverse integrating factor for system (3.4).*

Proof: From remark (3.2), $V = V_n + \dfrac{1+n}{1+m}V_m$. To show that V is an inverse integrating factor for system (3.3) it must satisfy condition (3.1). Now, $P = P_n + P_m$ and $Q = Q_n + Q_m$,and let $C = \dfrac{1+n}{1+m}$, then

$$\begin{aligned}
P(V_{nx} + CV_{mx}) + Q(V_{ny} + CV_{my}) &= P_nV_{nx} + CP_nV_{mx} + P_mV_{nx} + CP_mV_{mx} \\
&+ Q_nV_{ny} + CQ_nV_{my} + Q_mV_{ny} + CQ_mV_{my} \\
&= (P_nV_{nx} + Q_nV_{ny}) + C(P_mV_{mx} + Q_mV_{my}) \\
&+ CP_nV_{mx} + P_mV_{nx} + CQ_nV_{my} + Q_mV_{ny}
\end{aligned}$$

Since P_n, P_m, Q_n and Q_m are homogeneous polynomials , the functions V_n and V_m satisfy

$$\begin{aligned} P_n V_{nx} + Q_n V_{ny} &= (P_{nx} + Q_{ny}) V_n \\ P_m V_{mx} + Q_m V_{my} &= (P_{mx} + Q_{my}) V_m \end{aligned}$$

and for $k = n, m$

$$\begin{aligned} V_{kx} &= Q_k + x Q_{kx} - y P_{kx} \\ V_{ky} &= x Q_{ky} - P_k - y P_{ky} \end{aligned}$$

Therefore,

$$\begin{aligned} P(V_{nx} + CV_{mx}) + Q(V_{ny} + CV_{my}) &= (P_{nx} + Q_{ny})V_n + C(P_{mx} + Q_{my})V_m \\ &+ \; CP_n V_{mx} + P_m V_{nx} + CQ_n V_{my} + Q_m V_{ny} \\ &= (P_{nx} + Q_{ny})V_n + (P_{mx} + Q_{my})V_m \qquad (3.7) \\ &+ \; CP_n(Q_m + xQ_{mx} - yP_{mx}) + P_m(Q_n + xQ_{nx} - yP_{nx}) \\ &+ \; Q_m(xQ_{ny} - P_n - yP_{ny}) + CQ_n(xQ_{my} - P_m - yP_{my}) \end{aligned}$$

On the other hand,

$$\begin{aligned} (P_x + Q_y)(V_n + CV_m) &= P_{nx}V_n + CV_m P_{nx} + P_{mx}V_n + CP_{mx}V_m \\ &+ \; Q_{ny}V_n + CQ_{ny}V_m + Q_{my}V_n + CV_m Q_{my} \\ &= (P_{nx} + Q_{ny})V_n + C(P_{mx} + Q_{my})V_m \\ &+ \; CV_m P_{nx} + P_{mx}V_n + CQ_{ny}V_m + Q_{my}V_n \\ &= (P_{nx} + Q_{ny})V_n + C(P_{mx} + Q_{my})V_m \\ &+ \; (P_{mx} + Q_{my})V_n + C(P_{nx} + Q_{ny})V_m \end{aligned}$$

From the definition of V_n and V_m we have

$$\begin{aligned}(P_x + Q_y)(V_n + CV_m) &= (P_{nx} + Q_{ny})V_n + C(P_{mx} + Q_{my})V_m \\ &+ (P_{nx} + Q_{ny})(xQ_n - yP_n) + C(P_{mx} + Q_{my})(xQ_m - yP_m)\end{aligned} \tag{3.8}$$

For simplicity, let $G(x, y) = P(V_{nx} + CV_{mx}) + Q(V_{ny} + CV_{my}) - (P_x + Q_y)(V_n + CV_m)$. Then from equations (3.8) and (3.7) we have:

$$\begin{aligned}G(x, y) &= CP_nQ_m - cyP_nP_{mx} + P_mQ_n - yP_mP_{nx} + xQ_mQ_{ny} - Q_mP_n + CxQ_nQ_{my} \\ &- CQ_nP_m - xQ_nP_{mx} + yP_nP_{mx} - xQ_nQ_{my} + yP_nQ_{my} - CxQ_mP_{nx} + CyP_mP_{nx} \\ &- cxQ_mQ_{ny} + CyP_mQ_{ny} + CxP_nQ_{mx} + xP_mQ_{nx} - yQ_mP_{ny} - CyQ_nP_{my}\end{aligned}$$

The terms $CxP_nQ_{mx}, xP_mQ_{nx}, yQ_mP_{ny}$ and CyQ_nP_{my} can be written as

$$\begin{aligned}CxP_nQ_{mx} &= C(xP_nQ_{mx} + yP_nQ_{my} - yP_nQ_{my}) = C(mP_nQ_m - yP_nQ_{my}) \\ CyQ_nP_{my} &= C(yQ_nP_{my} + xQ_nP_{mx} - xQ_nP_{mx}) = C(mP_mQ_n - xQ_nP_{mx}) \\ xP_mQ_{nx} &= xP_mQ_{nx} + yP_mQ_{ny} - yP_mQ_{ny} = nP_mQ_n - yP_mQ_{ny} \\ yQ_mP_{ny} &= yQ_mP_{ny} + xQ_mP_{nx} - xQ_mP_{nx} = nP_nQ_m - xQ_mP_{nx}\end{aligned}$$

Thus, the function $G(x,y)$ becomes

$$\begin{aligned}
G(x,y) &= CP_nQ_m - cyP_nP_{mx} + P_mQ_n - yP_mP_{nx} + xQ_mQ_{ny} - Q_mP_n + CxQ_nQ_{my} \\
&- CQ_nP_m - xQ_nP_{nx} + yP_nP_{nx} - xQ_nQ_{ny} + yP_nQ_{ny} - CxQ_mP_{mx} + CyP_mP_{mx} \\
&- cxQ_mQ_{my} + CyP_mQ_{my} + C(mP_nQ_m - yP_nQ_{my}) + (nP_mQ_n - yP_mQ_{ny}) \\
&- (nP_nQ_m - xQ_mP_{nx}) - C(mP_mQ_n - xQ_nP_{mx}) \\
&= CP_nQ_m + P_mQ_n - Q_mP_n - CQ_nP_m + nP_mQ_n + CmP_nQ_m - nP_nQ_m - CmP_mQ_n \\
&+ (xQ_mP_{nx} + xQ_mQ_{ny} - yP_mP_{nx} - yP_mQ_{ny} - CxQ_mP_{nx} - CxQ_mQ_{ny} + CyP_mP_{nx} \\
&+ CyP_mQ_{ny}) + (CxQ_nP_{mx} + CxQ_nQ_{my} - CyP_nP_{mx} - CyP_nQ_{my} - xQ_nP_{mx} \\
&- xQ_nQ_{my} + yP_nP_{mx} + yP_nQ_{my})
\end{aligned}$$

After rearranging the polynomial in the last equation we get:

$$\begin{aligned}
G(x,y) &= ((1+n) - (1+m)C)(P_mQ_n - P_nQ_m) - (C-1)(xQ_m - yP_m)(P_{nx} + Q_{ny}) \\
&+ (C-1)(xQ_n - yP_n)(P_{mx} + Q_{my}) \\
&= (C-1)[(xQ_n - yP_n)(P_{mx} + Q_{my}) - (xQ_m - yP_m)(P_{nx} + Q_{ny})] = 0.
\end{aligned}$$

□

Example 3.3 *Let* $P_1 = 2x + y, P_2 = x^2 + 2xy + 2y^2, Q_1 = -\dfrac{11}{2}x + y,$ *and* $Q_2 = 11x^2 - 8xy + 2y^2$. *Then*

$$\begin{aligned}
(P_{2x} + Q_{2y})(xQ_1 - yP_1) &= (-\frac{11}{2}x^2 + xy - 2xy - y^2)(6y - 6x) \\
&= 33x^3 - 27x^2y - 6y^3
\end{aligned}$$

and

$$(P_{1x}+Q_{1y})(xQ_2-yP_2) \quad = \quad (11x^3-8x^2y+2xy^2-x^2y-2xy^2-2y^3)(3).$$

The given polynomials satisfy the condition in corollary (3.1), which implies $2C_1 = 3C_2$*. Choose* $C_1 = 1$ *then* $C_2 = \frac{2}{3}$*. Therefore,*

$$\begin{aligned} V(x,y) &= (-y,x)\cdot(C_1X_1+C_2X_2) \\ &= (-y,x)(2x+y,-\frac{11}{2}x+y)+\frac{2}{3}((x^2+2xy+2y^2,11x^2-8xy+2y^2) \\ &= -xy-y^2-\frac{11}{2}x^2-\frac{4}{3}y^3+\frac{22}{3}x^3-6x^2y \end{aligned}$$

Proposition 3.2 *The following homogeneous system*

$$\begin{aligned} x' &= a_1x+a_2y+a_3x^3+_4x^2y+a_5xy^2+a_6y^3 \\ y' &= b_1x+b_2y+b_3x^3+b_4x^2y+b_5xy^2+b_6y^3. \end{aligned} \tag{3.9}$$

where a_i *and* b_i *are assumed to be real and satisfying the following conditions:*

$$\begin{aligned} (a_1+b_2)(b_5-a_4) &= (2a_4+2b_5)(b_2-a_1)-a_2(3a_3+b_4)+b_1(a_5+3b_6) \\ (a_1+b_2)(b_6-a_5) &= (a_5+3b_6)(b_2-a_1)-2a_2(a_4+b_5) \\ b_1(3a_3+b_{4)} &= b_3(a_1+b_2) \\ (3a_3+b_4)(b_2-a_1)+b_1(2a_4+2b_5) &= (a_1+b_2)(b_4-a_3) \\ a_2(a_5+3b_6) &= a_6(a_1+b_2), \end{aligned}$$

Then the function $V(x,y) = (-y,x)\cdot(X_1+\frac{1}{2}X_3)$ *is an inverse integrating factor of system (3.9).*

Proof: Consider $P_1 = a_1x + a_2y,\ Q_1 = b_1x + b_2y,\ P_3 = a_3x^3 +_4 x^2y + a_5xy^2 + a_6y^3$ and $Q_3 = b_3x^3 + b_4x^2y + b_5xy^2 + b_6y^3$. Choose $C_1 = 1$ and $C_3 = \dfrac{1}{2}$. So, it is enough to show $(P_{3x} + Q_{3y})(xQ_1 - yP_1) - (P_{1x} + Q_{1y})(xQ_3 - yP_3) = 0$.

$$\begin{aligned} P_{3x} + Q_{3y} &= (3a_3 + +b_4)x^2 + 2(a_4 + b_5)xy + (a_5 + 3b_6)y^2 \\ xQ_1 - yP_1 &= b_1x^2 + (b_2 - a_1)xy - a_2y^2 \end{aligned}$$

The product of the above equations gives

$$\begin{aligned} (P_{3x} + Q_{3y})(xQ_1 - yP_1) &= b_1(3a_3 + +b_4)x^4 + (2b_1(a_4 + b_5) + (3a_3 + b_4)(b_2 - a_1))x^3y \\ &+ (b_1(a_5 + 3b_6) + 2(a_4 + b_5)(b_2 - a_1) - a_2(3a_3 + b_4))\, x^2y^2 \\ &+ ((a_5 + 3b_6)(b_2 - a_1) - a_2(3a_3 + b_4))xy^3 - a_2(a_5 + 3b_6)y^4. \end{aligned}$$

On the other hand,

$$\begin{aligned} (P_{1_x} + Q_{1_y})(xQ_3 - yP_3) &= (a_1 + b_2)b_3x^4 + (a_1 + b_2)(b_4 - a_3)x^3y - (a_1 + b_2)a_6y^4 \\ &+ (a_1 + b_2)(b_5 - a_4)x^2y^2 + (a_1 + b_2)(b_6 - a_5)xy^3 \end{aligned}$$

then the function $V(x, y) = (-y, x) \cdot (X_1 + \frac{1}{2}X_3)$ is an inverse integrating factor. $\square$

Example 3.4 *Consider system(3.9) where* $P_1 = x + y,\ P_3 = x^3 + \dfrac{-17}{4}x^2y - \dfrac{9}{8}xy^2 + \frac{33}{8}y^3$, $Q_1 = 2x + 2y$ *and* $Q_3 = x^3 + \dfrac{-3}{2}x^2y + 2xy^2 + \frac{9}{2}y^3$.

The given polynomials satisfy the condition in corollary (3.1), so the inverse integrating factor is

$$\begin{aligned} V(x, y) &= (-y, x) \cdot (X_1 + \frac{1}{2}X_2) \\ &= (-y, x) \cdot [(x + y, 2x + 2y) + \frac{1}{2}(x^3 - \frac{17}{4}x^2y - \frac{9}{8}xy^2 + \frac{33}{8}y^3, x^3 - \frac{3}{2}x^2y + 2xy^2 + \frac{9}{2}y^3)] \\ &= xy + 2x^2 - y^2 - \frac{5}{4}x^3y + \frac{25}{8}x^2y^2 + \frac{1}{2}x^4 - \frac{33}{16}y^4 + \frac{45}{16}xy^3 \end{aligned}$$

Case 2: Now, Consider the following system

$$
\begin{aligned}
x' &= P_n(x,y) + P_m(x,y) + P_k(x,y) \\
y' &= Q_n(x,y) + Q_m(x,y) \ + Q_k(x,y)
\end{aligned} \tag{3.10}
$$

To simplify the expressions in the following results, we use the notation $[P_n, Q_m]$ as follows:

$$
\begin{aligned}
[P_n, Q_m] &= (\frac{(1+n)C_n - (1+m)C_m}{C_m - C_n})(P_m Q_n - P_n Q_m) + (xQ_n - yP_n)(P_{m_x} + Q_{m_y}) \\
&- (xQ_m - yP_m)(P_{n_x} + Q_{n_y})
\end{aligned}
$$

Corollary 3.3 *In system (3.10), assume P_i and Q_i are homogeneous polynomials of degree i and satisfying the following condition: $[P_n, Q_m] = 0, [P_m, Q_k] = 0, [P_n, Q_k] = 0$, where C_n, C_m and C_k are distinct constants. Then*

$$
V(x,y) = (-y, x) \cdot (C_n X_n + C_m X_m + C_k X_k) \tag{3.11}
$$

is an inverse integrating factor for the system.

Proof: Let $V_n(x,y) = xQ_n - yP_n$, $V_m(x,y) = xQ_m - yP_m$ and $V_k(x,y) = xQ_k - yP_k$. Then from (3.2) we have

$$
\begin{aligned}
P_n V n_x + Q_n V_{n_y} &= (P_{n_x} + Q_{n_y}) V_n \\
P_m V_{m_x} + Q_m V_{m_y} &= (P_{m_x} + Q_{m_y}) V_m \\
P_k V_{k_x} + Q_k V_{k_y} &= (P_{k_x} + Q_{k_y}) V_k
\end{aligned} \tag{3.12}
$$

Also, equation (3.11) can be written as $V(x,y) = C_nV_n + C_mV_m + C_kV_k$. To show that $V(x,y)$ is an inverse integrating factor for the system, it's enough to show that V satisfies equation (1.4), i.e $PV_x + QV_y = (P_x + Q_y)V$ where $P = P_n + P_m + P_k$ and $Q = Q_n + Q_m + Q_k$.

$$\begin{aligned}
PV_x + QV_y &= C_nP_nV_{n_x} + C_mP_nV_{m_x} + C_kP_nV_{k_x} + C_nP_mV_{n_x} + C_mP_mV_{m_x} + C_kP_mV_{k_x} \\
&+ C_nP_kV_{n_x} + C_mP_kV_{m_x} + C_kP_kV_{k_x} + C_nQ_nV_{n_y} + C_mQ_nV_{m_y} + C_kQ_nV_{k_y} \\
&+ C_nQ_mV_{n_y} + C_mQ_mV_{m_y} + C_kQ_mV_{k_y} + C_nQ_kV_{n_y} + C_mQ_kV_{m_y} + C_kQ_kV_{k_y} \\
&= C_n(P_nV_{n_x} + Q_nV_{n_y}) + C_m(P_mV_{m_x} + Q_mV_{m_y}) + C_k(P_kV_{k_x} + Q_kV_{k_y}) \\
&+ C_n[V_{n_x}(P_m + P_k) + V_{n_y}(Q_m + Q_k)] + C_m[V_{m_x}(P_n + P_k) + V_{m_y}(Q_n + Q_k)] \\
&+ C_k[V_{k_x}(P_n + P_m) + V_{k_y}(Q_n + Q_m)]
\end{aligned}$$

From (3.12) we have:

$$\begin{aligned}
PV_x + QV_y &= C_n(P_{n_x} + Q_{n_y})V_n + C_m(P_{m_x} + Q_{m_y})V_m + C_k(P_{k_x} + Q_{k_y})V_k \\
&+ C_n[V_{n_x}(P_m + P_k) + V_{n_y}(Q_m + Q_k)] + C_m[V_{m_x}(P_n + P_k) + V_{m_y}(Q_n + Q_k)] \\
&+ C_k[V_{k_x}(P_n + P_m) + V_{k_y}(Q_n + Q_m)] \qquad (3.13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(P_x + Q_y)V &= [(P_{n_x} + Q_{n_y}) + (P_{m_x} + Q_{m_y}) + (P_{k_x} + Q_{k_y})](C_nV_n + C_mV_m + C_kV_k) \\
&= C_n(P_{n_x} + Q_{n_y})V_n + C_n(P_{m_x} + Q_{m_y})V_n + C_n(P_{k_x} + Q_{k_y})V_n \\
&+ C_m(P_{n_x} + Q_{n_y})V_m + C_m(P_{m_x} + Q_{m_y})V_m + C_m(P_{k_x} + Q_{k_y})V_m \\
&+ C_k(P_{n_x} + Q_{n_y})V_k + C_k(P_{m_x} + Q_{m_y})V_k + C_k(P_{k_x} + Q_{k_y})V_k. \qquad (3.14)
\end{aligned}$$

From equation(3.13) and(3.14)

$$\begin{aligned}
(P_x + Q_y)V - (PV_x + QV_y) &= C_m(P_{n_x} + Q_{n_y})V_m + C_m(P_{m_x} + Q_{m_y})V_m + C_m(P_{k_x} + Q_{k_y})V_m \\
&+ C_k(P_{n_x} + Q_{n_y})V_k + C_k(P_{m_x} + Q_{m_y})V_k + C_k(P_{k_x} + Q_{k_y})V_k \\
&- C_n[V_{n_x}(P_m + P_k) + V_{n_y}(Q_m + Q_k)] \\
&- C_m[V_{m_x}(P_n + P_k) + V_{m_y}(Q_n + Q_k)] \\
&- C_k[V_{k_x}(P_n + P_m) + V_{k_y}(Q_n + Q_m)] \qquad (3.15)
\end{aligned}$$

The last three terms in equation (3.15) are computed as follows

$$\begin{aligned}
C_n V_{n_x}(P_m + P_k) &= C_n(Q_n + xQ_{n_x} - yP_{n_x})(P_m + P_k) \\
&= C_n[Q_nP_m + Q_nP_k + xQ_{n_x}P_m + xQ_{n_x}P_k - yP_{n_x}P_m - yP_{n_x}P_k] \\
C_n V_{n_y}(Q_m + Q_k) &= C_n(-P_n + xQ_{n_y} - yP_{n_y})(Q_m + Q_k) \\
& \quad C_n[-P_nQ_m - P_nQ_k + xQ_{n_y}Q_m + Q_{n_y}Q_k - P_{n_y}Q_m - P_{n_y}Q_k] \\
C_m V_{m_x}(P_n + P_k) &= C_m(Q_m + xQ_{m_x} - yP_{m_x})(P_n + P_k) \\
&= C_m[Q_mP_n + Q_mP_k + xQ_{m_x}P_n + xQ_{m_x}P_k - yP_{m_x}P_n - yP_{m_x}P_k]
\end{aligned}$$

$$\begin{aligned}
C_m V_{m_y}(Q_n + Q_k) &= C_m(-P_m + xQ_{m_y} - yP_{m_y})(Q_n + Q_k) \\
& \quad C_m[-Q_nP_m - Q_kP_m + xQ_{m_y}Q_n + xQ_{m_y}Q_k - yP_{m_y}Q_n - yP_{m_y}Q_k] \\
C_k V_{k_x}(P_n + P_m) &= C_k(Q_k + xQ_{k_x} - yP_{k_x})(P_n + P_m) \\
&= C_k[Q_kP_n + Q_kP_m + xQ_{k_x}P_n + xQ_{k_x}P_m - yP_{k_x}P_n - yP_{k_x}P_m] \\
C_k V_{k_y}(Q_n + Q_m) &= C_k(-P_k + xQ_{k_y} - yP_{k_y})(Q_n + Q_m) \\
&= C_k[-Q_nP_k - Q_mP_k + xQ_{k_y}Q_n + xQ_{k_y}Q_m - yP_{k_y}Q_n - yP_{k_y}Q_m]
\end{aligned}$$

Substituting the above equations into equation (3.15) and applying the Euler's Theorem

$(C_n - C_m + nC_n - mC_m)P_mQ_n + (C_m - C_n - nC_n + mC_m)P_nQ_m + (C_k - C_n - nC_n + kC_k)P_nQ_k + (C_m - C_k - kC_k + mC_m)P_kQ_m + (C_k - C_m - mC_m + kC_k)P_mQ_k + (C_n - C_k + nC_n - kC_k)P_kQ_n + C_n(xQ_m - yP_m)(P_{n_x} + Q_{n_y}) + C_n(xQ_k - yP_k)(P_{n_x} + Q_{n_y}) + C_m(xQ_n - yP_n)(P_{m_x} + Q_{m_y}) + C_m(xQ_k - yP_k)(P_{m_x} + Q_{m_y}) + C_k(xQ_n - yP_n)(P_{k_x} + Q_{k_y}) + C_k(xQ_m - yP_m)(P_{k_x} + Q_{k_y}) = C_n(xQ_n - yP_n)(P_{m_x} + Q_{m_y}) + C_n(xQ_n - yP_n)(P_{k_x} + Q_{k_y}) + C_m(xQ_m - yP_m)(P_{n_x} + Q_{n_y}) + C_m(xQ_m - yP_m)(P_{k_x} + Q_{k_y}) + C_k(xQ_k - yP_k)(P_{n_x} + Q_{n_y}) + C_k(xQ_k - yP_k)(P_{m_x} + Q_{m_y})$

Rearranging the terms of the above expression and from the condition we get:
$\frac{(1+n)C_n - (1+m)C_m}{C_m - C_n}(P_mQ_n - P_nQ_m) + (xQ_n - yP_n)(P_{m_x} + Q_{m_y}) - (xQ_m - yP_m)(P_{n_x} + Q_{n_y}) + \frac{(1+m)C_m - (1+k)C_k}{C_k - C_m}(P_kQ_m - P_mQ_k) + (xQ_m - yP_m)(P_{k_x} + Q_{k_y}) - (xQ_k - yP_k)(P_{m_x} + Q_{m_y}) + \frac{(1+k)C_k - (1+n)C_n}{C_k - C_n}(P_kQ_n - P_nQ_k) + (xQ_n - yP_n)(P_{k_x} + Q_{k_y}) - (xQ_k - yP_k)(P_{n_x} + Q_{n_y}) = 0.$

Therefor,the function

$$\begin{aligned} V(x,y) &= C_nV_n + C_mV_m + C_kV_k \\ &= (-y,x)\cdot(C_nX_n + C_mX_m + C_kX_k) \end{aligned}$$

is the inverse integrating factor.

□

In system(3.10), consider the following subsystems $x' = P_n(x,y);\ y' = Q_n(x,y)$, $x' = P_m(x,y);\ y' = Q_m(x,y)$ and $x' = P_k(x,y);\ y' = Q_k(x,y)$. Then from remark(3.2), the functions $V_{nm}(x,y) = \frac{1}{2}(C_nV_n + C_mV_m)$, $V_{nk}(x,y) = \frac{1}{2}(C_nV_n + C_kV_k)$ and $V_{mk}(x,y) = \frac{1}{2}(C_kV_k + C_mV_m)$ are inverse integrating factors of the above subsystems respectively. Taking in account that the condition of corollary (3.3) is fulfilled, equation(3.11) can be written as

$$V(x,y) = C_nV_n + C_mV_m + C_kV_k. \tag{3.16}$$

Example 3.5 *Consider the polynomial differential system (3.10) with* $P_1 = -x + y$, $P_2 = x^2 - 3xy$, $P_3 = -x^3 - x^2y - 4xy^2$, $Q_1 = -x + y$, $Q_2 = -3xy + y^2$ *, and* $Q_3 = -5x^3 - x^2y - xy^2 + y^3$.

Choose $C_1 = 1$, then $C_2 = \frac{3}{4}$ and $C_3 = \frac{3}{5}$, easy calculations show that the condition of corollary (3.3) satisfied, then the inverse integrating factor is

$$\begin{aligned} V(x,y) &= (-y,x)\cdot(X_1 + \frac{3}{4}X_2 + \frac{3}{5}X_3) \\ &= (-y,x)\cdot(-x+y,-x+y) + \frac{3}{4}(x^2-3xy,-3xy+y^2) + \frac{3}{5}(-x^3-x^2y-4xy^2,-5x^3-x^2y \\ &= 2xy - x^2 - y^2 - 3x^2y + 3xy^2 + 3xy^3 - 3x^4 \end{aligned}$$

Remark 3.4 *In corollary(3.3), if* $(xQ_n - yP_n)(P_{m_x} + Q_{m_y}) - (xQ_m - yP_m)(P_{n_x} + Q_{n_y}) = 0$ *,*$(xQ_n - yP_n)(P_{k_x} + Q_{k_y}) - (xQ_k - yP_k)(P_{n_x} + Q_{n_y}) = 0$ *,and* $C_n = 1$*, Then the function* $V(x,y) = (-y,x)\cdot(X_n + \frac{1+n}{1+m}X_m + \frac{1+n}{1+k}X_k)$ *is an inverse integrating factor for system (3.10)*

Example 3.6 *Consider the polynomial differential system (3.10) with* $P_1 = 2x$, $P_2 = -\frac{18}{5}x^2 + xy$, $P_3 = \frac{1}{3}x^3$, $Q_1 = 2x + y$, $Q_2 = -\frac{24}{5}x^2 + \frac{2}{5}y^2$ *, and* $Q_3 = \frac{2}{3}x^3$.

Choose values for a set of parameters,Substitute these values in the conditions obtain through corollary(3.3), Choose $C_1 = 1$, then $C_2 = \frac{2}{3}$ and $C_3 = \frac{1}{2}$, long calculations show that the condition of corollary (3.3) satisfied, then the inverse integrating factor is

$$\begin{aligned} V(x,y) &= (-y,x)\cdot(X_1 + \frac{2}{3}X_2 + \frac{1}{2}X_3) \\ &= (-y,x)\cdot(2x, 2x+y) + \frac{2}{3}(-\frac{18}{5}x^2 + xy, -\frac{24}{5}x^2 + \frac{2}{5}y^2) + \frac{1}{2})(\frac{1}{3}x^3, \frac{2}{3}x^3) \\ &= -xy + 2x^2 + \frac{12}{5}x^2y - \frac{2}{5}xy^2 - \frac{16}{5}x^3 - \frac{1}{6}x^3y + \frac{1}{3}x^4 \end{aligned}$$

3.2 Inverse Integrating Factor for Planar Differential System of Class C^1

Consider the planar autonomous differential system of the form (2.1) with vector field of the form (2.2) and divergence $div\tilde{X} = P_x(x, y) + Q_y(x, y)$, where $P(x, y)$ and $Q(x, y)$ are functions of class C^1. Also, consider the following two hypotheses:

Hypothesis 3.1 *The expression* $\frac{1}{P(x, y)Q(x, y)} [P_x(x, y)Q(x, y) - P(x, y)Q_x(x, y)]$ *depends only on* x.

Hypothesis 3.2 *The expression* $\frac{1}{P(x, y)Q(x, y)} [P(x, y)Q_y(x, y) - P_y(x, y)Q(x, y)]$ *depends only on* y.

Theorem 3.4 *[1] (i) If hypothesis (3.1) holds, then the inverse integrating factor of the system (2.2 is*

$$V(x, y) = Q(x, y) \exp \int \frac{1}{P(x, y)Q(x, y)} [P_x(x, y)Q(x, y) - P(x, y)Q_x(x, y)] \, dx. \quad (3.17)$$

(ii) If hypothesis (3.2) holds , then the inverse integrating factor of the system (2.2 is

$$V(x, y) = P(x, y) \exp \int \frac{1}{P(x, y)Q(x, y)} [P(x, y)Q_y(x, y) - P_y(x, y)Q(x, y)] \, dy. \quad (3.18)$$

proof of part (i) is in [1] Below we give the proof of part (ii),

Proof: First let $R(x, y) = \frac{1}{P(x, y)Q(x, y)} [P(x, y)Q_y(x, y) - P_y(x, y)Q(x, y)]$, then $V(x, y) = P(x, y) \exp(\int R(x, y)dy)$. Hence, $V_x = P_x \exp(\int R(x, y)dy)$ and, $V_y = PR(x, y) \exp(\int R(x, y)dy) + P_y \exp(\int R(x, y)dy)$. Now, the function V is an inverse integrating factor if it satisfies equation(1.4),

$$\begin{aligned}
PV_x + QV_y &= Q\Big[PR\exp\Big(\int Rdy\Big) + P_y \exp\Big(\int Rdy\Big)\Big] + P\Big[P_x \exp\Big(\int Rdx\Big)\Big] \\
&= \exp\Big(\int Rdy\Big)[P_xP + P_yQ + PQ_y + QP_y] \\
&= \exp\Big(\int Rdy\Big)[P_xP + PQ_y] \\
&= Q\exp\Big(\int Rdy\Big)[P_x + Q_y] \\
&= (P_x + Q_y)V(x,y).
\end{aligned}$$

One can follow the same line to prove part (i) □

Example 3.7 *Consider the differential system*

$$\begin{aligned}
x' &= y^2x + x, \\
y' &= \frac{1}{2y^2x}.
\end{aligned}$$

Elementary calculations yield $R(x,y) = \frac{2y^2}{y^2+1}(\frac{-2y^2-1}{y^3}) = \frac{-4y^2-2}{y(y^2+1)}$ which depends on y only. Then the inverse integrating factor is

$$\begin{aligned}
V(x,y) &= P(x,y)\exp\int\Big(\frac{1}{P(x,y)Q(x,y)}\Big)\Big[\frac{\partial Q(x,y)}{\partial y}P(x,y) - Q(x,y)\frac{\partial P(x,y)}{\partial y}\Big]\Big]dy \\
&= (y^2x + x)\exp(-\ln(y^4 + y^2)) \\
&= \frac{x}{y^2}.
\end{aligned}$$

Corollary 3.4 *[1] for the system*

$$x' = H(x,y)F_1(x)G_1(y)$$

$$y' = H(x,y)F_2(x)G_2(y)$$

the inverse integrating factor is $V(x,y) = \; H(x,y)F_1(x)G_2(y)$

Proof: Firstly,

$$\begin{aligned}
R(x,y) &= \frac{1}{H^2F_1G_1F_2G_2}[(HF_2G_{2y} + H_yF_2G_2)HF_1G_1 - (H_yF_1G_1 + HF_1G_{1y})HF_2G_2] \\
&= \frac{1}{HG_1G_2}[(HG_{2y} + H_yG_2)G_1 - (H_yG_1 + HG_{1y})G_2] \\
&= \frac{1}{HC_1G_2}[HG_{2y}G_1 - HG_{1y}G_2] \\
&= \frac{G_{2y}}{G_2} - \frac{G_{1y}}{G_1},
\end{aligned}$$

which depends on y only, so by theorem(3.4) the inverse integrating factor is

$$\begin{aligned}
V(x,y) &= P(x,y)\exp\int(\frac{1}{P(x,y)Q(x,y)})[\frac{\partial Q(x,y)}{\partial y}P(x,y) - Q(x,y)\frac{\partial P(x,y)}{\partial y}]]dy \\
&= H(x,y)F_1(x)G_1(y)\exp\int[\frac{G_{2y}}{G_2} - \frac{G_{1y}}{G_1}]dy \\
&= H(x,y)F_1(x)G_1(y)(\frac{G_2}{G_1}) \\
&= H(x,y)F_1(x)G_2(y).
\end{aligned}$$

□

Corollary 3.5 *[1] for the system $x' = F(x)G(y)$, $y' = H(x,y)$.*

(i) If $\frac{H_x}{H}$ depends only on x, then $V(x,y) = H(x,y)\exp\Big(\int \frac{F_xH - H_xF}{FH}dx\Big)$.

(ii) If $\frac{H_x}{H}$ depends only on y, then $V(x,y) = F(x)G(y)\exp\Big(\int \frac{H_yG - HG_y}{GH}dy\Big)$.

Proof: (ii) First compute the function $R(x,y)$in the proof of theorem (3.4)

$$\begin{aligned} R(x,y) &= \frac{1}{H(x,y)F(x)G(y)}[FH_yG - HG_yF] \\ &= \frac{H_y}{H} - \frac{G_y}{G} \end{aligned}$$

if $\frac{H_y}{H}$ depends only on y , then $R(x,y)$ also depends on y only. By theorem(3.4), $V(x,y) = P(x,y)\exp\int R(x,y)dy = F(x)G(y)\exp\left(\int \frac{H_yG - HG_y}{GH}dy\right)$. □

Example 3.8 *Consider the differential system*

$$\begin{aligned} x' &= (5x+1)(y^2-2y) \\ y' &= x^2y + y \end{aligned}$$

$F(x) = 5x+1,\ G(y) = y^2-2y$,and $H(x,y) = x^2y+y$, then $\frac{H_y}{H} = \frac{1}{y}$ then the function $V(x,y) = 5xy + y$ is an inverse integrating factor.

Theorem 3.5 *[1] for system (2.1)) if there are functions $F(x,y),\ G(x,y)$ and $h(y)$ such that*

$P(x,y) = \frac{F(x,y)}{G(x,y)}$

$Q(x,y) = [F(x,y) + \frac{h(y)F(x,y)}{G(x,y)} \int \frac{\partial G(x,y)/\partial x}{(G(x,y)+h(y))^2}\ dy$

then the function $V(x,y) = F(x,y) + \frac{h(y)F(x,y)}{G(x,y)}$ is an inverse integrating factor.

Application on this theorem may involve some guessing and inspection for choosing suitable function $F,\ G$ and$h$ which may be not easy. The function $h(y)$ can be chosen in away that simplifies the integral and integration operation. One of suitable choices could be $h(y) = -G(0,y)$.

Example 3.9 *In theorem (3.5), let $F(x,y) = x^3+2y, G(x,y) = y^2+xy$ and $Q(x,y) = \frac{x^5y^2 + x^6y + 2x^2y^3 + 2x^3y^2 - ln(x^3y^2+2y^3)}{x^2y^2+x^3y}$. Then $P(x,y) = \frac{x^3+2y}{y^2+xy}$ and*

$h(y) = -G(0, y) = -y^2$. *Therefore, the function* $V(x, y) = \dfrac{x^4y + 2xy}{x + y}$ *is an inverse integrating factor.*

Corollary 3.6 *for the system* $x' = H(x, y)F(x)$, $y' = G(y)$.

(i) If $\dfrac{H_x}{H}$ *depends only on* x*, then* $V(x, y) = G(y)\exp\left(\int \dfrac{F_xH + H_xF}{FH}dx\right)$.

(ii) If $\dfrac{H_y}{H}$ *depends only on* y*, then* $V(x, y) = H(x, y)F(x)\exp\left(\int \dfrac{HG_y - H_yG}{GH}dy\right)$.

Proof: (i) First compute the function $R(x, y)$in the proof of theorem (3.4)

$$\begin{aligned} R(x, y) &= \frac{1}{H(x, y)F(x)G(y)}[(HF_x + FH_x)G] \\ &= \frac{F_x}{F} + \frac{H_x}{H} \end{aligned}$$

if $\dfrac{H_x}{H}$ depends only on x , then $R(x, y)$ also depends on x only. By theorem(3.4), $V(x, y) = Q(x, y)\exp\int R(x, y)dx = G(y)\exp\left(\int \dfrac{F_xH + H_xF}{FH}dx\right)$

(ii) First compute the function $R(x, y)$in the proof of theorem (3.4)

$$\begin{aligned} R(x, y) &= \frac{1}{H(x, y)F(x)G(y)}[(HF_xG_y - FH_y)G] \\ &= \frac{G_y}{G} - \frac{H_y}{H} \end{aligned}$$

if $\dfrac{H_y}{H}$ depends only on y , then $R(x, y)$ also depends on y only. By theorem(3.4), $V(x, y) = P(x, y)\exp\int R(x, y)dy = H(x, y)F(x)\exp\left(\int \dfrac{HG_y - H_yG}{GH}dy\right)$

□

Example 3.10 *Consider the differential system*

$$
\begin{aligned}
x' &= 2x^3(xy^2+x) \\
y' &= 2y^2+1
\end{aligned}
$$

$F(x) = 2x^3$, $G(y) = y^2+1$,and $H(x,y) = xy^2+x$, then $\dfrac{H_x}{H} = \dfrac{1}{x}$ depends only on x then the function $V(x,y) = 2x^4y^2 + x^4$ is an inverse integrating factor.

Example 3.11 *Consider the differential system*

$$
\begin{aligned}
x' &= xy^2(x^2+1) \\
y' &= y^3
\end{aligned}
$$

$F(x) = x^2+1$, $G(y) = y^3$,and $H(x,y) = xy^2$, then $\dfrac{H_y}{H} = \dfrac{2}{y}$ depends only on y then the function $V(x,y) = 2\ln(y)xy^2(x^2+1)$ is an inverse integrating factor.

Example 3.12 *consider the polynomial differential system*

$\acute{x} = x^2(y+1)$ $\acute{y} = 2xy$ $F(x) = x^2$, $G(y) = y+1$and $H(x,y) = 2xy$ by corollary (2) , $\dfrac{\partial H/\partial x}{H} = \dfrac{1}{x}$ depend on x

$$
\begin{aligned}
V(x,y) &= H(x,y)\exp\int \frac{(\frac{\partial F}{\partial x})H - (\frac{\partial H}{\partial x})F}{FH}dx \\
&= 2xy\exp\int (\frac{2}{x} - \frac{1}{x})dx \\
&= 2xy\exp\int \frac{1}{x}\, dx \\
&= 2x^2y
\end{aligned}
$$

$\mu(x,y) = \dfrac{1}{2x^2y}$ is an integrating factor so, the system $\acute{x} = \dfrac{x^2(y+1)}{2x^2y}$, $\acute{y} = \dfrac{2xy}{2x^2y}$ has zero divergence $div(\frac{x^2(y+1)}{2x^2y}\ ,\frac{2xy}{2x^2y}) = \frac{\partial \frac{x^2(y+1)}{2x^2y}}{\partial x} + \frac{\frac{2xy}{2x^2y}}{\partial y}\ = 0$

Example 3.13 *consider the system*

$(3xy + y^2)\partial x + (x^2 + xy)\partial y = 0$

$V(x, y) = y(P(x, y) - x(Q(x, y)\ \ = y(3xy + y^2) - x(x^2 + xy)\ = 3xy^2\ + y^3\ - x^3 - x^2y$

is an inverse integrating factor

$\frac{1}{V(x,y)} = \frac{1}{3xy^2 + y^3 - x^3 - x^2y}$is an integrating factor then

$div(\frac{P}{v}, \frac{Q}{V}) = \frac{6x^2y^2 + 6x^3y + 2xy^3 - 6x^2y^2 - 6x^3y - 2xy^3}{(3xy^2 + y^3 - x^3 - x^2y)^2} = 0.$

3.3 Limit Cycle and Inverse Integrating Factor

Inverse integrating factors are of the useful methodologies in the study of differential systems. Indeed, investigating the behavior of trajectories of that system in a neighborhood of each stationary point is important in analyzing differential systems which is connected to the so-called limit cycle. Therefore, many authors have been studied the relationship between inverse integrating factor and limit cycle [7, 15, 17, 19, 23, 29, 40]

In this section, we present a brief survey on the existence and nonexistence of limit cycle of autonomous differential system using Bendexison theorem, linearization method and finally through the existence of inverse integrating factor. Consider the following differential system

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y). \end{aligned} \tag{3.19}$$

Suppose (x, y) is a solution of system(3.19), its parametric representation defines a smooth curve in $xy-$plane. As t changes, the position $(x(t), y(t))$ will change a long that curve. Such a curve with direction is called orbit or trajectory of system(3.19). If there is a postive number k such that $x(t) = x(t+k)$ and $y(t) = y(t+k)$ then the orbit is called periodic.

Definition 3.3 *[28] A point $(x_o y_o)$ is a critical point of system(3.19) if and only if $P(x_o y_o) = 0$ and $Q(x_o y_o) = 0$.*

Definition 3.4 *[28] Let U be an open subset in $\mathbb{R}^2$. A limit cycle of system (3.19) is a periodic orbit $\gamma \in U$ for which at least one other solution tends toward γ as $t \to -\infty$ or $\gamma \to \infty$.*

Definition 3.5 *A domain D is called simply connected if every simple closed in D encloses only point of D.*

Theorem 3.6 (Bendixson) *[28] Suppose that U is a simply connected open subset of $\mathbb{R}^2$. If $P_x + Q_y$ is of one sign then system(3.19) has no limit cycle.*

Example 3.14

$$x' = x + y - x(x^2 + y^2)$$
$$y' = -x + y - y(x^2 + y^2)$$

Since $P_x + Q_y = y^2 + x^2 + 1 > 0$ for all x and y, the differential system has no limit cycle.

Now, assume that system (3.19) has $(0,0)$ as a critical point, if not it can be moved to the origin by a suitable transformation. So, by Taylor expansion, the functions $P(x,y)$ and $Q(x,y)$ can be expanded as follows

$$\begin{aligned} P(x,y) &= ax + by + p(x,y) \\ Q(x,y) &= cx + dy + q(x,y) \end{aligned}$$

where $a = P_x(0,0),\ b + P_y(0,0),\ c = Q_x(0,0)$ and $d = Q_y(0,0)$. Thus, the linearization of a non-linear system is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} P_x(x_o, y_o) & P_y(x_o, y_o) \\ Q_x(x_o, y_o) & Q_y(x_o, y_o) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

To understand the behavior of trajectories of the nonlinear system we convert the system to polar coordinates by letting $x = r\cos\theta$,$y = r\sin\theta$, $\frac{y}{x} = tan\theta$ and $r^2 = x^2 + y^2$. Then $r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = rcos\theta\ P + rsin\theta\ Q$, $sec^2\theta\ \frac{d\theta}{dt} = -\frac{rsin\theta}{r^2cos^2\theta}\ P + \frac{1}{rcos\theta}\ Q$. Thus,

$$\begin{aligned} \frac{dr}{dt} &= cos\theta\ P + rsin\theta\ Q \\ \frac{d\theta}{dt} &= -\frac{sin\theta}{r}\ P + \frac{cos\theta}{r}\ Q. \end{aligned} \tag{3.20}$$

where $P = P(r,\theta)$ and $Q = Q(r,\theta)$

Example 3.15 *Consider the polynomial differential system*

$$x' = x + y - x(x^2 + y^2)$$

$$y' = -x + y - y(x^2 + y^2)$$

We notice here that $P_x + Q_y$ has a negative sign in the region $\left\{(x,y) : x^2 + y^2 < \frac{1}{2}\right\}$ and a positive sign in $\left\{(x,y) : x^2 + y^2 > \frac{1}{2}\right\}$. So, Bendixson theorem says nothing about the existence of the limit cycle for the given system. So, in order to see whether the system has a limit cycle or not we use the method of linearization of nonlinear systems together with the polar transformation as follows:

Since the critical point of this system is $(0,0)$,the linearization near the critical point, is given by

$$\begin{bmatrix} x\prime \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The polar form of the system using (3.20) is
$\dfrac{dr}{dt} = r(1-r^2)$, $\dfrac{d\theta}{dt} = -1$.
From the polar differential equations, $\acute{r} = 0$ when$r = 0$ or $r = 1.r = 0$ corresponds to the critical point(0,0) ,but $r = 1$ corresponds to a periodic cycle$x^2 + y^2 = 1$

note that θ is decreasing and if $r > 1$,then $\acute{r} < 0$ so the direction of trajectory spiral in(goes toward the origin). If $r < 1$, then $r' > 0$,and the direction of the trajectory spiral in. All trajectories converge to the periodic trajectory $r = 1$.so, $r = 1$ is an stable limit cycle of the system.

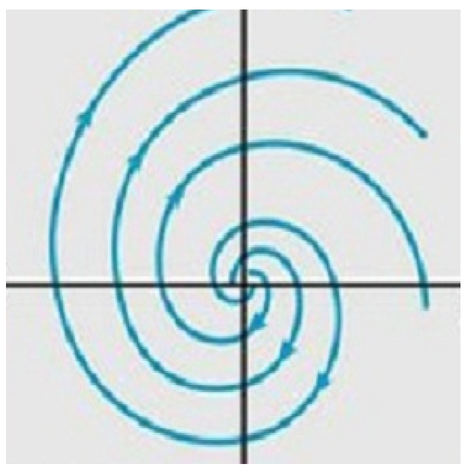

Fig. 3.1: Stable limit cycle

consider a planar autonomous differential system of the form (2.1) defined in an open subset U of $\mathbb{R}^2$. Recall that a non-constant continuously differentiable function $V : U \to \mathbb{R}$ an inverse integrating factor vector field $\tilde{X} = (P, Q)$ if it satisfies $\tilde{X}V = V div \tilde{X}$ on U.

Theorem 3.7 *[14,29]. Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable functions* *the vector field $\tilde{X} = (P, Q)$ defined in open subset U of R^2 admits an inverse integrating factor , if γ is alimit cycle of the vector field (P, Q)in the simply connected domain of definition V then is contained $\sum = \{(x, y) \in U : V\ (x, y) = 0\}$*

Theorem 3.8 *[19] the vector field $\tilde{X}$= (P,Q) defined in open subset U of R^2 admits an inverse integrating factor of the form V= V(Z) , where Z= f(x,y) if and only if $\dfrac{divX}{(P\ f_x\ + Q\ f_y)} = \ell(Z)$, where $\ell(Z)$ is a function exclusively of Z and in such case the inverse integrating factor is of the form*

$$V = \exp(\int^{Z} \ell(s)\partial s). \tag{3.21}$$

moreover if γ is a limit cycle of the vector field (P, Q)in the simply connected domain of the form of (1.1) then γ is contained in $\sum = \left\{(x, y) \in u : \exp(\int^{Z} \ell(s)\partial s = 0\right\}$.

Example 3.16 *Consider the differential system*

$P = x$

$Q\ = x^2 - 2y$

show has no limit cycles in the domain $R^2 / \{(x, y) : x = 0\}$?

Assume that system has an inverse integrating factor of the form V= V(Z) , where Z= f(x,y) = x

$$\frac{\frac{\partial V}{\partial z}}{V} = \frac{(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})}{P} = \frac{-1}{x} = \ell(x)$$

hence $v = \exp(\int^{x} \frac{-1}{s}\partial s) = \frac{1}{x}$

applying theorem (1.1) system has a limit cycle in the domain V if a limit cycle contained in $\sum = \left\{(x, y) \in u : \frac{1}{x} = 0\right\}$,but $V^{-1}(0) = \phi$ therefore system has no limit cycles in the domain of V $R^2/\{(x,y) : x = 0\}$

Note :-the correct conclusion the system does not have limit cycles in the simply connected domain of deinitin of V **,** this domain is not all R^2

For example suppose $\ell(s) = \dfrac{1}{(s \ln s)}$ which implies that $v = \exp(\int^x \ell(s)\partial s = \ln x$, therefore applying theorem (1) if γ is a limit cycle of the vector field (P, Q) in the domain of V ,

which is in this case $x > 0$, then γ is contained in $\sum = \{(x, y) \in u : V$ (x,y) $= 0\}$,therefore the vector field (P, Q) has no limit cycle for $x > 0$,but it has limit cycles in the domain $x \leq 0$.

Corollary 3.7 *[29] . If P, Q are homogeneous polynomials of the same degree then the differential equation (1) has no limit cycle*

Example 3.17 *consider the differential system show it,s has no limit cycle*

$P = x^2y + y^3$

$Q = x^3 + y^3$

P, Q are homogeneous polynomials of the degree 3,then $V = yP - xQ$ is an inverse integrating factor

$V = x^2y^2 + y^4 - x^4 - xy^3$ is homogeneous polynomials of the degree 4 ,by corollary (3.7) the system has no limit cycle.

Chapter 4

Integrability Problem and Inverse Integrating Factor

The integrability problem is mainly related to planar polynomial differential systems of the form (2.1). The problem of finding such a first integral and the functional class it must belong to is what we call the integrability problem. To find an integrating factor or an inverse integrating factor is closely related to finding a first integral for it. An important fact of their results is that invariant algebraic curves play a distinguished role in this characterization. Moreover, this characterization is expressed in terms of the inverse integrating factor.

In this chapter, we study some aspects of the integrability problem .This chapter consists of three sections. In the first section, the definition of first integral and examples are given. Invariant algebraic curves for planar polynomials system is presented in section two. Besides, two methods for finding the invariant algebraic curves are described. In Section three we discussed the Darboux Theorem.

4.1 First Integral

Consider the planar autonomous differential system (2.1) ,where $P(x, y)$ and $Q(x, y)$ are functions of class C^1, or elements in the ring of real polynomials $\mathbb{R}[x, y]$, with associated vector field of the form (2.2).

Definition 4.1 *[43] Let U be an open subset of $\mathbb{R}^2$. A continuously differentiable function $H : U \to \mathbb{R}$ is said to be first integral of system (2.1) if $H(x(t), y(t)) =$ constant for any t such that $(x(t), y(t))$ is contained in U.*

Remark 4.1 *H is the first integral of system(2.1) if and only if* $\frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial H}{\partial y} \cdot \frac{\partial y}{\partial t} = 0.$

We say system (2.1) is integrable on an open subset $U \subset \mathbb{R}^2$ if the system has a first integral function in U, see [43]. In fact there is no general method for finding the first integral for system (2.1). Writing the two differential equations in terms of $\frac{\partial y}{\partial x}$ is one of the tricks for finding the first integral, which we will see in the examples below.

Example 4.1 *[43][Hamilton system]*

$$\begin{aligned} x' &= -\frac{\partial H(x,y)}{\partial y}, \\ y' &= \frac{\partial H(x,y)}{\partial x}. \end{aligned}$$

Since $\frac{\partial H}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial H}{\partial y} \cdot \frac{\partial y}{\partial t} = \frac{\partial H}{\partial x} \cdot (-\frac{\partial H}{\partial y}) + \frac{\partial H}{\partial y} \cdot (\frac{\partial H}{\partial x}) = 0$, the function $H(x, y)$ is a first integral of the system.

Example 4.2 *Consider the system*

$$\begin{aligned} x' &= y, \\ y' &= -x + x^3. \end{aligned}$$

divide the second equation by the first one to get $\frac{dy}{dx} = \frac{-x + x^3}{y}$. Then the first integral followed immediately which is $H(x, y) = y^2 + x^2 - \frac{1}{2}x^4$.

Example 4.3 *Consider the system*

$$x' = x - xy,$$
$$y' = y^2 - y.$$

divide the second equation by the first one to get $\frac{dy}{dx} = -\frac{y}{x}$. Then, the first integral followed immediately which is $H(x,y) = xy$.

Example 4.4 *Consider the system*

$$x' = x\cos y + xy,$$
$$y' = x\sin x.$$

divide the second equation by the first one to get $\frac{dy}{dx} = \frac{\sin x}{y+\cos y}$. Then, the first integral followed immediately which is $H(x,y) = -\cos x - \sin y - \frac{1}{2}y^2$.

4.2 Invariant Algebraic Curves for Planar Polynomials Systems

Consider the planar autonomous differential system (2.1) where $P(x,y)$,$Q(x,y)$ are polynomials. Let $m = max\{degP, degQ\}$ and $f(x,y) \in \mathbb{C}[x,y]$.

Definition 4.2 *[8, 37] The algebraic curve $f(x,y) = 0$ is called invariant Algebraic Curves if there exists a polynomial $K(x,y)$ of degree at most $m-1$ such that*

$$P(x,y)\,\frac{\partial f(x,y)}{\partial x} + Q(x,y)\,\frac{\partial f(x,y)}{\partial y} = K(x,y)\,f(x,y). \tag{4.1}$$

The polynomial $K(x,y)$ is called the cofactor of $f(x,y) = 0$.

If f (x, y) = 0 is an invariant algebraic curve with nonzero imaginary part and a certain cofactor k(x, y), then its conjugate curve $\bar{f}(x,y) = 0$ also satisfies (4.1) with cofactor $\bar{\text{k}}$(x, y). Hence, the product of these complex polynomials $f(x,y)\bar{f}(x,y) \in R$[x,y] [14]. Usually Invariant algebraic curves can be computed by the method of undetermined coefficients. In other words, we write:$P\frac{\partial f}{\partial x}+Q\frac{\partial f}{\partial y} = Kf$. This gives a polynomial system in the unknown coefficients of K and f. In the following two subsections, the algorithm of two methods given in [36] will be desribed.

4.2.1 The First Method

Definition 4.3 *A monic polynomial is a polynomial in which the coefficient of the highest order is 1.*

The leading term of a monic polynomial is the term of highest order, denoted by $l(f)$. We summarize the algorithm of the first method in the following steps:

1. Construct all monic polynomials f_i such that $degf_i \leq m$
2. For each f_i do the following
   (a) Calculate $F_i = P(x,y)\frac{\partial f_i}{\partial x} + Q(x,y)\frac{\partial f_i}{\partial y}$.
   (b) If $l(f_i)$ divides $l(F_i)$, then construct plynomial k_i with undetemined coefficients of equal to $degree(F_i) - degree(f_i)$
   (c) If the system is consistenct then all unkown cofficient f_i and K_i can be determine, otherwise jump to the next f_i .

Example 4.5 *Consider the polynomial differential system(2.1) with the polynomials* $P(x,y) = y$*, and* $Q = -y^2 - 4x^2 - 4x$*.*

Apply the algorithm when the degree of the monic polynomial is 1, so we construct the polynomials $f_1 = a_1 + x$ and, $f_2 = a_1 + a_2x + y$. For f_1, we have $F_1 = y$, and it's clear that $l(F_1)$ is not divisible by $l(f_1)$, the method fails. For f_2, we have $F_2 = -4x+a^2y-y^2-4x^2$. since $l(F_2) = -y^2$ is divisible by $l(f_2) = y$, we choose $K_2 = b_1-b_2y$, then all unkown cofficient f_2 and, K_2 can be determined with undetemined coefficients, we have $b_1a_2 = -4, a_2 = 0$, the method fails. Therefore, Apply algorithm when the degree of the monic polynomial is 2, so we construct the polynomials$f_1 = a_1 + x^2$ and, $f_2 = a_1+a_2x^2+y^2$. For f_1, we have $F(1) = 2xy$ and, it's clear that $l(F_1)$ is not divisible by $l(f_1)$. For f_2, we have $F_2 = 2a_2xy-2y^3-8x^2y-8xy$. since $l(F_2) = -2y^3$ is divisible by $l(f_2) = y^2$, Therefore, we choose $K_2 = b_1 - 2b_2y$, then all unkown cofficient f_2 and K_2 can be determined with undetemined coefficients, we have $f = 4x^2 + y^2, K = -2y$ Similarly, we can do this for other cases.

4.2.2 The Second Method

The algorithm of the second method can be summarized as follows:

1. Construct all monic polynomials with undetermined coefficients f_i
2. For each f_i :
   (a) calculate $F_i = P(x,y)\frac{\partial f_i}{\partial x} + Q(x,y)\frac{\partial f_i}{\partial y}$.

   (b) if $l(f_i)$ divides $l(F_i)$ then

$$\begin{aligned} k_i(new) &= k_i(old) + \frac{l(F(new))}{l(f(i))} \\ F_i(new) &= F_i(old) - f_i \times \frac{l(F(old))}{l(f(i))} \end{aligned}$$

   (c) Do this process until $F_i = 0$

   (d) If the system consistenct, then all unkown cofficient f_i and K_i can be determined , otherwise jump to the next f_i .

Example 4.6 *Consider the polynomial differential system (2.1) where*

$P(x,y) = 2x^2y - x$ and $Q(x,y) = 2xy^2 + y$

Apply the algorithm when the degree of the monic polynomial is 1. First, let $f_1 = a_1 + x$, then $F_1 = 2x^2y - x$. it's clear that $l(F_1)$ is divisible by $l(f_1)$then, $k = 2xy$.Also, we have $F_{1new} = -2a_1xy - x$, and it's clear that $l(F_1)$ is divisible by $l(f_1)$then, $k_{1new} = 2xy - 2a1y$. next we have $F_{1new} = 2a_1^2y - x$, Then, $l(F_1) = 2a_1^2y$ is divisible by $l(f_1 = x)$ only if $a_1 = 0$ so, $k_{1new} = 2xy$ we have $F_{1new} = -x$, and it's clear that $l(F_1)$ is divisible by $l(f_1)$therefore we obtain $f = x$ and $k = 2xy - 1$ in this case.

For $f_2 = a_1 + a_2x + y$ f_2, we have $F_2 = 2x^2ya_2 - xa_2 + 2xy^2 + y$ and it's clear that $l(F_2)$ is divisible by $l(f_2)$then, $k = 2xy$. Also, we have $F_2 = -2a_1xy + y - a_2x$, and it's clear that $l(F_2)$ is divisible by$l(f_2)$then, $k = 2xy - 2a_1x$. next we have $F_2 = 2a_1a_2x^2 + y + (2a_1^2 - a_2)x$, then, $l(F_2) = 2a_1a_2x^2$ is divisible by $l(f_2 = y)$ only if $a_1a_2 = 0$ so, $k = 2xy - 2a_1x$. Also, we have $F_2 = y + 2a_1^2 - a_2)x$, that, $l(F_2)$ is divisible by$l(f_2)$then, $k = 2xy - 2a_1x + 1$. Next, we have $(F_2) = (2a_1^2 - 2a_2)x - a_1$, so, $l(F_2) = (2a_1^2 - 2a_2)x$ is divisible by $l(f_2 = y)$ only if $(2a_1^2 - 2a_2) = 0$, solving $a_1a_2 = 0$ and, $2a_1^2 - 2a_2 = 0$, we get $a_1 = 0, a_2 = 0$. so we obtain $f = y$ and, $k = 2xy + 1$.

4.3 Darboux Theorem

Draboux in 1878 showed that if a planar polynomial system(2.1) of degree m has at least $1 + \frac{m(m+1)}{2}$ invariant algebraic curves, then it has a first integral. The original Draboux theory of integrability of planar polynomial systems has extended to many versions considering not only the invariant algebraic curves but also the multiplicity of algebraic curves, the exponential factors and some other, for aspects. for details, see [38]. The idea was to construct an integrating factor, first integral and inverse integrating factor are summarized in the following theorem

Theorem 4.1 (Darboux) *[36] Suppose that a polynomial system (2.1) of degree m*

admits q invariant algebraic curves f_i =0 with cofactor k_i for $i = 1, \ldots, q$.

(a) If $q \geq 1 + \dfrac{m(m+1)}{2}$ then the function $f_1^{\ell_1} \cdots f_q^{\ell_q}$ is a first integral of system (2.1), such that $\sum_i^q \ell_i k_i = 0$ where ℓ_i's not all zero.

(b) If $q = \dfrac{m(m+1)}{2}$ then the function $f_1^{\ell_1} \cdots f_q^{\ell_q}$ for suitable ℓ_i not all zero is either a first integral where $\sum_i^q \ell_i k_i = 0$, or an integrating factor where $\sum_i^q \ell_i k_i = -div(P, Q)$.

(c) If $q < \dfrac{m(m+1)}{2}$ then the function $f_1^{\ell_1} \cdots f_q^{\ell_q}$ and there exist $\ell_i \in F$ not all zero such that $\sum_i^q \ell_i k_i = 0$, $H = f_1^{\ell_1} \cdots f_q^{\ell_q}$ then is a first integral.

(d) If $q < \dfrac{m(m+1)}{2}$ then the function $f_1^{\ell_1} \cdots f_q^{\ell_q}$ and there exist $\ell_i \in F$ not all zero such that $\sum_i^q \ell_i k_i = -div(P, Q)$. then $R = f_1^{\ell_1} \cdots f_q^{\ell_q}$ is an integrating factor.

The function $f_1^{\ell_1} \cdots \cdots f_q^{\ell_q}$ is called either a Draboux first integral or a Draboux integrating factor for system (2.1).

Remark 4.2 *The function $V = f_1^{\ell_1} \cdots \cdots f_q^{\ell_q}$ for suitable $\ell_i \in F$ not all zero and $\sum_i^q \ell_i k_i = div(P, Q)$ is an inverse integrating factor for system (2.1).*

Proof: To show $V = f_1^{\ell_1} \cdots \cdots f_q^{\ell_q}$ is an inverse integrating factor for system (2.1), it's enough to show that equation (3.1) is satisfied. The partial derivatives of V with respect to x and y are $V_x = \sum_{i=1}^q \ell_i f_i^{l_i - 1} \frac{\partial f_i}{\partial x} \prod_{k \neq i}^q f_k^{\ell k}$ and $V_y = \sum_{i=1}^q \ell_i f_i^{l_i - 1} \frac{\partial f_i}{\partial y} \prod_{k \neq i}^q f_k^{\ell k}$ respectively. Hence

$$\begin{aligned} PV_x + QV_y &= P \sum_{i=1}^q \ell_i f_i^{l_i - 1} \frac{\partial f_i}{\partial x} \prod_{k \neq i}^q f_k^{\ell k} + Q \sum_{i=1}^q \ell_i f_i^{l_i - 1} \frac{\partial f_i}{\partial y} \prod_{k \neq i}^q f_k^{\ell k} \\ &= \sum_{i=1}^q \ell_i f_i^{\ell_i - 1} (\prod_{k \neq i}^q f_k^{\ell_k})(P \frac{\partial f_i}{\partial x} + Q \frac{\partial f_i}{\partial y}). \end{aligned}$$

but f_i satisfies $P\frac{\partial f_i}{\partial x} + Q\frac{\partial f_i}{\partial y} = K_i f_i$ since it's an invariant algebraic curve with cofactor

k_i for $i = 1, \ldots, q$. So,

$$\begin{aligned} PV_x + QV_y &= \sum_{i=1}^{q} \ell_i f_i^{\ell_i - 1} (\prod_{k \neq i}^{q} f_k^{\ell_k})(K_i f_i). \\ &= \sum_{i=1}^{q} \ell_i K_i f_i^{\ell_i - 1} (\prod_{k=1}^{q} f_k^{\ell_k}) \\ &= V \sum_{i=1}^{q} \ell_i K_i = (P_x + Q_y) V \end{aligned}$$

□

As an application on this theorem, we solve the following two examples.

Example 4.7 *Consider the polynomial differential system (2.1) with* $P = x^2 + 1$ *and* $Q = x^3 + x - xy$.

The degree of the system is $m = 3$, from example 4.5. the invariant algebraic curves are $f_1 = x + i$, $f_2 = x - i$ with cofactor $k_1 = x - i, k_2 = x + i$. Since the number of invariant algebraic curves is $< \dfrac{m(m+1)}{2}$, the function $f_1^{\ell_1} \cdots f_q^{\ell_q}$ for suitable $\ell_i \in F$ not all zero is either a first integral and $\sum_i^q \ell_i k_i = 0$, or an integrating factor and $\sum_i^q \ell_i k_i = -div(P, Q) = -x$. So, $\ell_1(x - i) + \ell_2(x + i) = -x$ and hence $\ell_1 = \ell_2 = \dfrac{-1}{2}$.

Then, $R = f_1^{\ell_1} f_2^{\ell_2} = (x + i)^{\frac{-1}{2}} (x - i)^{\frac{-1}{2}} = (x^2 + 1)^{\frac{-1}{2}}$.

On the other hand, V is an inverse integrating factor if $\sum_i^q \ell_i k_i = div(P, Q)$. Thus, $V = f_1^{\ell_1} f_2^{\ell_2} = (x + i)^{\frac{1}{2}} (x - i)^{\frac{1}{2}} = (x^2 + 1)^{\frac{1}{2}}$.

Example 4.8 *Consider the polynomial differential system (2.1) with* $P = 2x^2y - x$ *and* $Q = 2xy^2 + y$.

The degree of system $m = 3$, Invariant algebraic curves are $f_1 = x$ $f_2 = y$ with cofactor $k_1 = 2xy - 1$, $k_2 = 2xy + 1$. Since the number of invariant algebraic curves is less than 6, the function $f_1^{\ell_1} \cdot \cdots \cdot f_q^{\ell_q}$ for suitable $\ell_i \in F$ not all zero is either a first integral and $\sum_i^q \ell_i k_i = 0$, or an integrating factor and $\sum_i^q \ell_i k_i = -div(P, Q)$. So, $-div(P, Q) =$

$-8xy = \ell_1(2xy-1) + \ell_2(2xy+1) - 2(2xy-1) + -2(2xy+1)$. Then, $R = f_1^{\ell_1} f_2^{\ell_2} = (xy)^{-2}$. Also, V is an inverse integrating factor if $\sum_i^q \ell_i k_i = div(P,Q)$, which implies $V = f_1^{\ell_1} f_2^{\ell_2} = (xy)^2$

Remark 4.3 *the relationship between a first integral H and its associated inverse integrating factor V is given by*

$$V = \frac{-P(x,y)}{\frac{\partial H}{\partial y}} = \frac{Q(x,y)}{\frac{\partial H}{\partial x}} \tag{4.2}$$

Proof: If the differential system (1) has μ integrating factor then

$\acute{x} = P(x,y)\mu(x,y)$

$\acute{y} = Q(x,y)\mu(x,y)$

is Hamilton system , there exists a function H such that

$\acute{x} = P\mu \ = -H_y$, $\acute{y} = Q\mu = H_x$

We say that H is the first integral associated to the integrating factor μ

then $H = \int -P \ \mu \ dy + h(x)$, satisfying $\dfrac{\partial H}{\ell x} = QR$

V is inverse integrating facor then $\mu = \dfrac{1}{V}$ is integrating factor

hence $\dfrac{-\partial H}{\partial y} = \mu P = \dfrac{P}{V}$

then, $V = \dfrac{-P(x,y)}{\frac{\partial H}{\partial y}}$

and, $\dfrac{\partial H}{\partial x} = \mu Q = \dfrac{Q}{V}$ then ,$V = \dfrac{Q(x,y)}{\frac{\partial H}{\partial x}}$

so $,V = \dfrac{-P(x,y)}{\dfrac{\partial H}{\partial y}} = \dfrac{Q(x,y)}{\dfrac{\partial H}{\partial x}}$ □

Example 4.9 *Consider the polynomial differential system*

$$x' = -y(x^2 + y^2)$$
$$y' = x(x^2 + y^2)$$

Divide the second equation by the first one, $\dfrac{\partial y}{\partial x} = \dfrac{x(x^2+y^2)}{-y(x^2+y^2)} = -\dfrac{x}{y}$. Hence, $H(x,y) = x^2 + y^2$ is a first integral. Therefore, the inverse integrating factor is

$$\begin{aligned} V &= \frac{-P(x,y)}{\dfrac{\partial H}{\partial y}} = \frac{Q(x,y)}{\dfrac{\partial H}{\partial x}} \\ &= \frac{y(x^2+y^2)}{2y} = \frac{x(x^2+y^2)}{2x} \\ &= \frac{1}{2}(x^2+y^2) \end{aligned}$$

Corollary 4.1 *if system (2.1) has inverse integrating factor* V *and a first integral* H *on open subset* U *of* R^2*, then* VH *is another inverse integrating factor on open subset* U

Proof: VH is an inverse integrating factor if it satiating the partial differential equation (3.1). Therefore,

$$\begin{aligned} P(VH)_x + Q(VH)_y &= P(VH_x + HV_x) + Q(VH_y + HV_y) \\ &= PVH_x + QVH_y + PHV_x + QHV_y \\ &= V(PH_x + QH_y) + H(PV_x + QV_y). \end{aligned}$$

But V is an inverse integrating factor and H is a first integral so they satisfy the following relations $PV_x+QV_y = (P_x+Q_y)V$ and $PH_x+QH_y = 0$ respectively. Therefore,

$$\begin{aligned} P(VH)_x + Q(VH)_y &= V(0) + H(P_x + Q\partial y)V \\ &= (P_x + Q_y)VH, \end{aligned}$$

Consequently, VH, is an inverse integrating factor. □

Example 4.10 $V = \frac{1}{2}(x^2 + y^2), H = x^2 + y^2$, *in the last example*

then $VH = \frac{1}{2}(x^2 + y^2)^2$ is another inverse integrating factor

Example 4.11 *Consider the polynomial differential system*

$$x' = -2x^2y - x$$

$$y' = 2xy^2 + y$$

For N=1, we find two darboux polynomial $f_1 = x$ and, $f_2 = y$, with cofactor $k_1 = 2xy - 1$and, $k_2 = 2xy + 1$, $\ell_1(2xy - 1) + \ell_2(2xy + 1) = -4xy$.

Hence, $\mu(x, y) = x^{-2}y^{-2}$is integrating factor Also, $V(x, y) = x^2y^2$ is an inverse integrating factor. We can find the first integral through this $P\mu = -H_y$ $Q\mu = H_x$. so, $H(x, y) = 2ln\frac{x}{y} - \frac{1}{xy}$. then VH= $2x^2y^2lnxy - xy$, is another inverse integrating factor .

Conclusions

1- In this work, We studied the inverse integrating Factor Method for Homogeneous poynomial system, we give formulas of inverse integrating factors of class C^1.

2- We investigated the relation between the inverse integrating factor and the integrating factor for autonomous differential system.

3- We presented a brief survey on the existence and nonexistence of limit cycle of autonomous differential using inverse integrating Factor.

4- We offered some aspects concerning of the integrability problem , and the goal of that is to construct the integrating factor, first integral and inverse integrating factor for autonomous differential system and showing some results concerning them.

5-In the future work, we will classify all the quadratic system in to ten normal forms to find a polynomial inverse integrating factor .

References

[1] K. I. Al-Dosary, *Inverse integrating factor for classes of planar differential systems*, International Journal of Mathematical Analysis, 4 (2010), pp. 1433–1446.

[2] J. C. Artés, J. Llibre, and N. Vulpe, *Quadratic systems with a polynomial first integral: a complete classification in the coefficient space r12*, Journal of Differential Equations, 246 (2009), pp. 3535–3558.

[3] L. R. Berrone and H. J. Giacomini, *On the vanishing set of inverse integrating factors*, Qualitative Theory of Dynamical Systems, 1 (2000), pp. 211–230.

[4] L. Cairó, M. R. Feix, and J. Llibre, *Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems*, Resenhas do Instituto de Matemática e Estatística da Universidade de São Paulo, 4 (1999), pp. 127–161.

[5] J. Chavarriga, H. Giacomini, J. Giné, and J. Llibre, *On the integrability of two-dimensional flows*, Journal of Differential Equations, 157 (1999), pp. 163–182.

[6] J. Chavarriga, H. Giacomini, and M. Grau, *Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems*, Bulletin des sciences mathematiques, 129 (2005), pp. 99–126.

[7] G. Chen and Y. Wu, *On the shape of limit cycles that bifurcate from isochronous center*, The Scientific World Journal, 2014 (2014).

[8] G. Cheze, *Computation of darboux polynomials and rational first integrals with bounded degree in polynomial time*, Journal of Complexity, 27 (2011), pp. 246–262.

[9] C. Christopher, J. Llibre, C. Pantazi, and S. Walcher, *Inverse problems in darboux'theory of integrability*, Acta applicandae mathematicae, 120 (2012), pp. 101–126.

[10] C. Christopher, J. Llibre, C. Pantazi, and X. Zhang, *Darboux integrability and invariant algebraic curves for planar polynomial systems*, Journal of Physics A: Mathematical, Nuclear and General, 35 (2002), pp. 2457–2476.

[11] A. Enciso and D. Peralta-Salas, *Existence and vanishing set of inverse integrating factors for analytic vector fields*, Bulletin of the London Mathematical Society, (2009), p. bdp090.

[12] A. Ferragut, J. Llibre, and B. Coll Vicens, *Polynomial inverse integrating factors of quadratic differential systems and other results*, Universitat Autònoma de Barcelona,, 2006.

[13] I. García, H. Giacomini, and M. Grau, *The inverse integrating factor and the poincaré map*, Transactions of the American Mathematical Society, 362 (2010), pp. 3591–3612.

[14] I. A. García and M. Grau, *A survey on the inverse integrating factor*, Qualitative Theory of Dynamical Systems, 9 (2010), pp. 115–166.

[15] A. Gasull, H. Giacomini, and J. Torregrosa, *Explicit non-algebraic limit cycles for polynomial systems*, Journal of computational and applied mathematics, 200 (2007), pp. 448–457.

[16] H. Giacomini, J. Giné, and M. Grau, *The role of algebraic solutions in planar polynomial differential systems*, in Mathematical Proceedings of the Cambridge Philosophical Society, vol. 143, Cambridge Univ Press, 2007, p. 487.

[17] H. Giacomini, J. Llibre, and M. Viano, *On the nonexistence, existence and uniqueness of limit cycles*, Nonlinearity, 9 (1996), p. 501.

[18] J. Giné, *The nondegenerate center problem and the inverse integrating factor*, Bulletin des sciences mathematiques, 130 (2006), pp. 152–161.

[19] ——, *Non-existence of limit cycles for planar vector fields*, Electronic Journal of Differential Equations, 2014, vol. 2014, núm. 75, p. 1-8, (2014).

[20] J. Giné and J. Chavarriaga, *Integrable systems via inverse integrating factor*, Extracta mathematicae, 13 (1998), pp. 41–60.

[21] J. Giné and H. Giacomini, *An algorithmic method to determine integrability for polynomial planar vector fields*, Eur J Appl Math, (2006).

[22] J. Giné and J. Llibre, *Integrability, degenerate centers, and limit cycles for a class of polynomial differential systems*, Computers & Mathematics with Applications, 51 (2006), pp. 1453–1462.

[23] L. R. González-Ramirez, O. Osuna, and R. Santaella-Forero, *On the limit cycles of quasihomogeneous polynomial systems*, Revista Colombiana de Matemáticas, 49 (2015), pp. 261–268.

[24] A. Goriely, *Integrability and nonintegrability of dynamical systems*, vol. 19, World Scientific, 2001.

[25] M. Grau, J. Llibre, et al., *On the extensions of the darboux theory of integrability*, Nonlinearity, 26 (2013), p. 2221.

[26] M. Hayashi, *On polynomial liénard systems which have invariant algebraic curves*, Funkcialaj Ekvacioj, 39 (1996), pp. 403–408.

[27] A. P. Hiwarekar, *Extension of euler's theorem on homogeneous function to*

higher derivatives, Bulletin of the Marathwada Mathematical Society, 10 (2009), p. 16–19.

[28] D. W. Jordan and P. Smith, *Nonlinear ordinary differential equations: an introduction to dynamical systems*, vol. 2, Oxford University Press, USA, 1999.

[29] L. Laura-Guarachi, O. Osuna, and G. Villasenor-Aguilar, *Non-existence of limit cycles via inverse integrating factors*, Electronic Journal of Differential Equations, 2011 (2011), pp. 1–6.

[30] J. Llibre, M. Messias, and A. C. Reinol, *Darboux invariants for planar polynomial differential systems having an invariant conic*, Zeitschrift für angewandte Mathematik und Physik, 65 (2014), pp. 1127–1136.

[31] J. Llibre and C. Pantazi, *Polynomial differential systems having a given darbouxian first integral*, Bulletin des sciences mathematiques, 128 (2004), pp. 775–788.

[32] J. Llibre and C. Pantazi, *Darboux theory of integrability for a class of nonautonomous vector fields*, Journal of mathematical physics, 50 (2009), p. 102705.

[33] J. Llibre and J. V. Pereira, *Multiplicity of invariant algebraic curves and darboux integrability*, arXiv preprint math/0009020, (2000).

[34] J. Llibre and C. Valls, *Weierstrass integrability for a class of differential systems*, Journal of Nonlinear Mathematical Physics, 21 (2014), pp. 294–307.

[35] Y.-K. Man, *On the liouvillian solution of second-order linear differential equations and algebraic invariant curves*, Journal of Physics A: Mathematical and Theoretical, 43 (2010), p. 412001.

[36] Y.-K. Man and M. A. MacCallum, *A rational approach to the prelle–singer algorithm*, Journal of Symbolic Computation, 24 (1997), pp. 31–43.

[37] L. C. Ngô, *Finding rational solutions of rational systems of autonomous odes*, RISC Report Series, 10, 2 (2010).

[38] C. Pantazi and J. Llibre, *Inverse problems of the Darboux theory of integrability for planar polynomial differential systems*, Universitat Autònoma de Barcelona,, 2004.

[39] M. J. Prelle and M. F. Singer, *Elementary first integrals of differential equations*, Transactions of the American Mathematical Society, 279 (1983), pp. 215–229.

[40] J. Qiao and S. Shui, *Limit cycles for a class of polynomial differential systems*, Electronic Journal of Qualitative Theory of Differential Equations, 2016 (2016), pp. 1–9.

[41] D. Schlomiuk, N. Vulpe, et al., *Integrals and phase portraits of planar quadratic differential systems with invariant lines of at least five total multiplicity*, Rocky Mountain J. Math, 38 (2008), pp. 2015–2075.

[42] V. Shah and J. Sharma, *Extension of euler's theorem on homogeneous functions for finite variables and higher derivatives*, International Journal of Engineering and Innovative Technology, 4 (2014), pp. 84–87.

[43] M. Tenenbaum and H. Pollard, *Ordinary differential equations: an elementary textbook for students of mathematics, engineering, and the sciences*, Dover Publications, 1985.

[44] Y. you Pan and X. Zhang, *Algebraic aspects of integrability for polynomial differential systems*, Journal of Applied Analysis and Computation, 3 (2013), pp. 51–69.

ملخص

طريقة معامل التكامل العكسي لأنظمة مستوى كثيرات الحدود التفاضلية المستقلة

براء مجلي جميل ابو خضره

في هذه الأطروحة، تم دراسة طريقة معامل التكامل العكسي لأنظمة مستوى كثيرات الحدود التفاضلية المستقلة وعرض بعض الأمثلة التوضيحية،وكذلك علاقته بمعامل التكامل للمعادلات التفاضلية من الدرجة، ولمحة سريعة بين معامل التكامل العكسي ودورة الحد ،علاوه على ذلك، تم معالجه بعض جوانب مسألة التكامل المنحنيات الجبرية الثابتة وعلاقاتها مع معامل التكامل العكسي للانظمه. .