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Methods for solving some types of Fully Fuzzy nonlinear programming problems

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This thesis was submitted

in partial fulfillment of the requirements for the Master's degree

in

Applied Mathematics

July/ 2024

Thesis Approval

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Declaration

I declare that the content of this thesis are the product of my own effort, except what has been indicated and that this thesis or any part of it has not been previously submitted to obtain any other scientific or research degree.

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Dedication

This thesis is dedicated to my dear daughter Rafeef whose presence in my life helps me to success. I dedicate this dissertation to my wife, Fatima for her unwavering patience, compassion, and most importantly her love. I will never be able to thank her enough for the sacrifices she made throughout this process. Your encouragement and support are what made this dissertation possible. To my parents who encouraged me, without your guidance I wouldn't have made it this far. To my supervisors who have challenged me, I am so grateful for their mentorship and expertise.

Acknowledgments

First of all, great thanks to Allah who gave me the ability to study and to do this work.
I would like to express my special thanks to my supervisors Dr.Abdelhalim Ziqan and Dr. Ammar Qarariyah for their great help and support .
My great thanks to the examination committee for their careful reading to my thesis and valuable feedback.

Abstract

In this thesis, we used two methods to solve two types of fully fuzzy nonlinear programming problem. The first method was the ranking function method, which is used to solve some types of fully fuzzy nonlinear programming problem that satisfies Karuch-Kuhn-Tucker (KKT) conditions and separable nonlinear problems. In the first method, the fully fuzzy nonlinear programming problem is converted to a crisp programming problem by the ranking function.

In the second method, the separable programming problem was converted to an interval programming problem. We show how the procedure works in case of having triangular fuzzy number and we generalized the method in case of having trapezoidal and hexagonal fuzzy numbers . each method was explained in details and we have shown numerical examples of each method in case of having triangular, trapezoidal and hexagonal fuzzy numbers. Last we have shown a comparison between the two methods in terms of the number of equations in each method and the complexity level results of the examples that we have got from the two methods and showed how they are close.

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List of Abbreviations

Abbreviation	Definition
FFNLPP	Fully fuzzy nonlinear programming problem
INPP	Interval nonlinear programming problem
KKT	Karush-Kuhn-Tucker
NLPP	Nonlinear programming problem

Introduction

The nonlinear programming is the process of solving an optimization problem where the objective function is nonlinear or some of the constraints are nonlinear . The nonlinear programming is widely applicable in real life , as an example it is used in the problem of packing items within bounded region in the euclidean space which has many applications in physics, chemistry and engineering Birgin [2016]. However, in real life problems such as engineering problems cannot be determined by crisp real values. Therefore it is important to deal with such problems by using the concept of fuzzy set which can handle this uncertainty. it means that instead of the crisp real numbers we will consider the coefficients as fuzzy numbers to be more expressive of the real system in our life. In the fully fuzzy nonlinear programming problem , the coefficient are all fuzzy numbers , we can show the general form for the fully fuzzy nonlinear programming problem that we used as follows : Loganathan and Lalitha [2017]

$$
\begin{aligned}
&\text{maximize (or minimize)}\tilde{z} = \tilde{c}^T \otimes \tilde{x}^{\alpha_j} \\
&\text{subject to} \\
&\tilde{A} \otimes \tilde{X} (\geq, \leq, =) \tilde{b} \\
&\tilde{X} \geq 0 \\
&\text{where } \tilde{c}^T = [\tilde{c}_j]^T, \tilde{A} = [\tilde{a}_{ij}]_{m\times n} \\
&\tilde{b} = [\tilde{b}_i]_{m\times 1}, \tilde{X} = [\tilde{x}_j]_{n\times 1}
\end{aligned}
\tag{0.1}
$$

Several methods have been applied to solve the fully fuzzy nonlinear programming problem . In Loganathan and Lalitha [2017], M.Lalitha [2018] ranking function method was used in case the parameters are triangular fuzzy numbers . In Akrami et al. [2016a] the interval method was used to solve fully nonlinear programming problem in case of having triangular fuzzy numbers. In this thesis we will study two methods for solving two types of fully fuzzy non linear programming problem . In chapter 1 we introduced some basic definitions about fuzzy numbers. In chapter 2 crisp non linear programming problem (NLPP) was considered, we consider first the non linear programming problems with KKT necessary and sufficient conditions and then we considered the separable programming, we showed a numerical example for each of them.

In chapter 3 we have explained the ranking function method that converts the fully fuzzy non linear programming problem to a crisp non linear programming problem, We generalized the problem in Loganathan and Lalitha [2017], M.Lalitha [2018] to solve fully fuzzy nonlinear programming problem in case of dealing with trapezoidal and hexagonal fuzzy numbers . We have shown also a numerical examples for the fully fuzzy nonlinear programming problem (FFNLPP) that satisfies KKT conditions and fully fuzzy separable nonlinear programming problems. In Chapter 4, interval programming procedure was considered to solve separable FFNLPP, by converting the FFNLPP, to an interval programming problem using the alpha cut of each fuzzy number and then solving it by the existing method. We generalized the method in Akrami et al. [2016a] to solve the problems in case of dealing with trapezoidal and hexagonal fuzzy numbers, numerical examples was shown also for each case .

Chapter 1

Fuzzy sets and fuzzy numbers

This chapter shows the basic definitions of the fuzzy set and fuzzy number, and also the membership function for the triangular and trapezoidal and hexagonal fuzzy numbers that we will deal with in our thesis.

1.1 Fuzzy set.

In this section we will introduce some basic definitions of fuzzy set that are needed in our thesis .

Definition 1.1.1 *A S Karthick [2022]*

Let X be an universal set and $x \in X$, then the fuzzy set $\widetilde{A}$ is defined as a collection of ordered pairs as follows :

$$\widetilde{A} = \{(x, \mu_{\tilde{A}}(x), x \in X\}$$

where $\mu_{\tilde{A}}(x)$ is a membership function,

$$\mu_{\tilde{A}} : X \longrightarrow [0,1]$$

Definition 1.1.2 *C.Loganathan [2017]*

The α cut for a fuzzy set is defined by $\widetilde{A_\alpha} = \{x \in x \mid \mu_{\tilde{A}}(x) \geq \alpha\}$ which is a crisp set

Definition 1.1.3 *Taghi-Nezhad and Taleshian [2018]*

A fuzzy set $\tilde{A}$ on X is convex if $\forall x, y \in X$ we have $\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$ where $\lambda \in [0,1]$

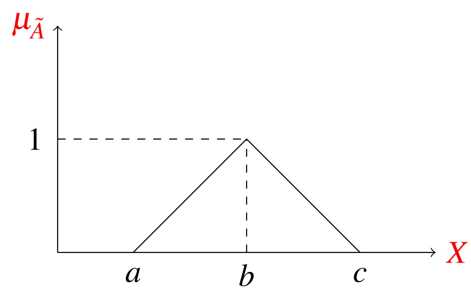

Fig. 1.1: Triangular Fuzzy Number

Definition 1.1.4 *Taghi-Nezhad and Taleshian [2018]*

A fuzzy set $\tilde{A}$ is called normal if there exist at least one point $x \in X$ with $\mu_{\tilde{A}}(x) = 1$

1.2 fuzzy numbers

In this section we will show the definition of the fuzzy number and each type that we will need in our thesis.

Definition 1.2.1 *Nasseri [2008] $\tilde{A}$ fuzzy set $\tilde{A}$ is called a fuzzy number if its defined on the real line and its convex and normal set .*

1.2.1 Triangular fuzzy number

In this subsection the definition of the triangular fuzzy number and its membership function and some operators will be shown.

Definition 1.2.2 *Purnima Raj [2021]*

A fuzzy number $\tilde{A} = (a, b, c)$ that is shown in fig 1.1 is called a triangular fuzzy number if it has the following membership function :

$$\mu_{\widetilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a, b] \\ \dfrac{x-c}{b-c}, & x \in [b, c] \\ 0, & otherwise \end{cases}$$

for a triangular fuzzy numbers $\widetilde{A} = (a, b, c), \quad \widetilde{B} = (e, f, g)$ we will show the following arithmetic operations : C.Loganathan [2017]

(i) $\widetilde{A} \oplus \widetilde{B} = (a+e, b+f, c+g)$

(ii) $\widetilde{A} \ominus \widetilde{B} = (a-g, b-f, c-e)$

(iii) $\ominus\widetilde{A} = (-c, -b, -a)$

Definition 1.2.3 *Chandrasekaran [2015]*

for a triangular fuzzy number $\widetilde{A} = (a,b,c)$ *the* α *cut of* $\widetilde{A}$ *is:*

$$\tilde{A}_\alpha = [a+\alpha(b-a), c-\alpha(c-b)]$$

1.2.2 Trapezoidal fuzzy number

In this subsection we will show the definition of the trapezoidal fuzzy number and some operations on it.

Definition 1.2.4 *Allahdadi and Mishmast Nehi [2011]*

A fuzzy number $\tilde{A} = (a,b,c,d)$ *that is shown in fig 1.2is called a trapezoidal fuzzy number if it has the following membership function.*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x \leq a, x \geq d \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & b < x < c \\ \dfrac{x-d}{c-d} & c < x < d \end{cases}$$

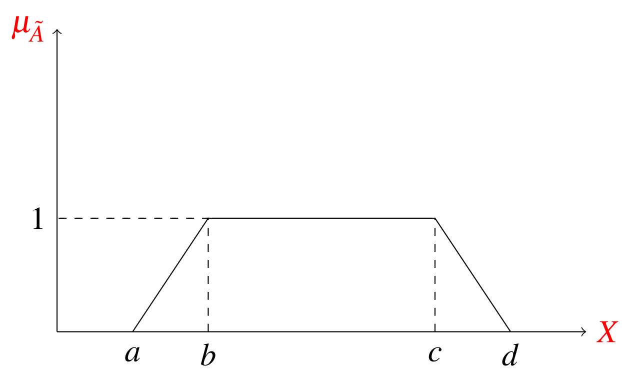

Fig. 1.2: Trapezoidal fuzzy number

for a trapezoidal fuzzy numbers $\widetilde{A} = (a,b,c,d)$, $\widetilde{B} = (e,f,g,h)$ we will show the following arithmetic operations : Hasan et al. [2023]

(i) $\widetilde{A} \oplus \widetilde{B} = (a+e, b+f, c+g, d+h)$

(ii) $\widetilde{A} \ominus \widetilde{B} = (a-h, b-g, c-f, d-e)$

Definition 1.2.5 *Kumar and Dhiman [2021] if $\tilde{A} = (a,b,c,d)$ is a trapezoidal fuzzy number then the α-cut of $\tilde{A}$ is :*

$$\tilde{A}_\alpha = [a + \alpha(b-a), d - \alpha(d-c)]$$

1.2.3 Hexagonal fuzzy number

A fuzzy set $\tilde{A} = (a,b,c,d,e,f)$ that is shown in fig 1.3 is called hexagonal fuzzy number if its membership function is given by P et al. [2013]:

$$\mu_{\tilde{A}} = \begin{cases} 0 & ,x < a \\ \frac{1}{2}\left(\frac{x-a}{b-a}\right), & a \le x \le b \\ \frac{1}{2} + \frac{1}{2}\left(\frac{x-b}{c-b}\right), & b \le x \le c \\ 1 & ,c \le x \le d \\ 1 - \frac{1}{2}\left(\frac{x-d}{e-d}\right), & d \le x \le e \\ \frac{1}{2}\left(\frac{f-x}{f-e}\right), & e \le x \le f \\ 0 & ,x \ge f \end{cases}$$

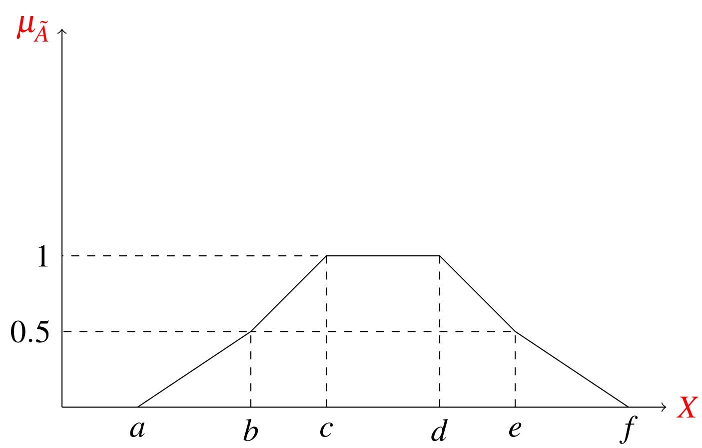

Fig. 1.3: Hexagonal Fuzzy Number

Definition 1.2.6 ?

If $\widetilde{A} = (a,b,c,d,e,f)$ is hexagonal fuzzy number , $\widetilde{A}_\alpha$ is given by:

$$\widetilde{A}_\alpha = \begin{cases} 2\alpha(b-a)+a, -2\alpha(f-e)+f & \text{for } \alpha \in [0, 0.5] \\ 2\alpha(c-b)-c+2b, -2\alpha(e-d)+2e-d & \text{for } \alpha \in [0.5, 1] \end{cases}$$

Chapter 2

Nonlinear Programming Problem

Nonlinear programming problem has many application including financial modeling , optimal control and image processing. In this chapter we will show the definition of the nonlinear programming problem and two types of the nonlinear programming problem , we will show the procedure of solving each type with a numerical examples .

Definition 2.0.1 *The problem is called nonlinear programming problem (NLPP) if the objective function is nonlinear and/or the feasible region is determined by nonlinear constraints.*

The general form for the nonlinear programming problem is given by :

$$\begin{aligned} \text{Maximize} \quad & Z = \sum_{j=1}^{n} c_j x_j \\ \text{Subject to} \quad & g_i(x) \leq 0, \quad i = 1, ..., m \\ X \geq 0 & \end{aligned} \tag{2.1}$$

where $x = (x_1, x_2, \ldots, x_n)$, Z is nonlinear or one or more of the constraints are nonlinear or both. B.Cobacho [2012].

2.1 Karuch-Kuhn-Tucker (KKT) conditions.

In this section we will show how to solve the nonlinear programming problems that satisfy the KKT sufficient and necessary conditions.

The KKT necessary conditions for 2.1 can be summarized as follows :

$$\begin{aligned} \lambda_i g_i(x) &= 0 \\ \frac{\partial L(X,\lambda)}{\partial X} &= 0 \quad \text{where} \quad X = \{x_1, x_2, ..., x_n\} \\ g_i(X) &\leq 0 \\ X &\geq 0 \\ \lambda_i &\geq 0 \end{aligned} \tag{2.2}$$

note that $L(X,\lambda) = Z - \lambda_i g_i(X)$ and its called the Lagrangian Function.

The KKT necessary conditions (2.2) will be also sufficient to have an exteremum point when Z is concave in the case of maximization (i.e. when the hessian matrix is negative semi definite) and Z is convex in case of minimization (i.e when the hessian matrix is positive semi definite). B.Cobacho [2012], Taha and Taha [2003]

Note that the hessian matrix is positive semi definite if its symmetric and its eigen values are nonnegative. and it, negative semi definite when its symmetric and its eigenvalues are positive semi definite. The following table summurizes the sufficient conditions

optimization	Required conditions			
	$f(X)$	$g_i(X)$	λ_i	
maximization	concave	*Convex*	≥ 0	$g(X) \leq 0$
		Concave	≤ 0	$g(X) \geq 0$
		Linear	*Unrestricted*	$g(X) = 0$
minimization	convex	*Convex*	≤ 0	$g(X) \leq 0$
		Concave	≥ 0	$g(X) \geq 0$
		Linear	*Unrestricted*	$g(X) = 0$

Example 2.1.1 *consider the following nonlinear programming problem*

$$\begin{aligned} &\max f(x) = 4x_1 + 6x_2 - x_1^2 - x_2^2 - x_3^2 \\ &\textit{Subject to} \\ &x_1 + x_2 \leq 2 \\ &2x_1 + 3x_2 \leq 12 \\ &x_1, x_2 \geq 0 \end{aligned}$$

First we have to show that the KKT conditions are sufficient and necessary for this example , since the problem is to maximize , $f(x)$ should be concave and the constraints must be convex . for $f(x)$ to be concave the symmetric hessian matrix must be negative semi definite (i.e the eigen values must be nonpositve) . To show that , we will first construct the hessian matrix as follows :

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

we can see that the eigen values are $-1,-1,-1$ (all negative) hence the symmetric hessian matrix is negative semi definite and hence $f(x)$ is concave , the constraints are convex since they are linear.

Thus, the KKT conditions will be necessary and sufficient for a maximum to be exist .

Now the Lagrangian function $L(x,\lambda)$

$$\begin{aligned} &L(x,\lambda) = \left(4x_1+6x_2-x_1^2-x_2^2-x_3^2\right) - \lambda_1(x_1+x_2-2) \\ &-\lambda_2(2x_1+3x_2-12) \end{aligned}$$

The necessary conditions are:

$$\begin{aligned} &4-2x_1-\lambda_1-2\lambda_2=0 \\ &6-2x_2-\lambda_1-3\lambda_2=0 \\ &-2x_3=0 \\ &\lambda_1(x_1+x_2-2)=0 \\ &\lambda_2(2x_1+3x_2-12)=0 \\ &\lambda_1\ ,\lambda_2 \geq 0 \\ &x_1+x_2 \leq 2 \\ &2x_1+3x_2 \leq 12 \\ &x_1,x_2 \geq 0 \end{aligned}$$

Solving the above problem then the values of x_1,x_2,x_3 and λ_1,λ_2, that satisfy all the constraints are:

$$x_1 = \frac{1}{2} \quad x_2 = \frac{3}{2} \quad x_3 = 0 \quad \lambda_1 = 3 \quad \lambda_2 = 0$$

so the optimal solution is:

$$f(x) = 4\left(\frac{1}{2}\right) + 6\left(\frac{3}{2}\right) - \left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - (0)^2 = \frac{17}{2}$$

2.2 Separable Programming

In this section we will introduce some definitions and the useful procedures that are needed for solving nonlinear programming problems.

Definition 2.2.1 *PATE [2014]*

The nonlinear programming problem in which the nonlinear functions can be expressed as a sum of single variable functions is called a separable programming problem.

An approximated solution of a separable nonlinear programming problem can be deduced easily by approximating each nonlinear function in a problem to a linear function, which can be solved by any well known method . The procedure can be explained as follows:

suppose that $h(X)$ is a non linear function it will be approximated over the interval $[a,b]$ as follows:

$$h(X) \approx \sum_{k=1}^{K} h(a_k)\omega_k$$
$$X \approx \sum_{k=1}^{K} a_k\omega_k$$

where a_k is the $k-th$ break point on x-axis $\omega_k \geq 0$ and $\sum_{k=1}^{K} w_k = 1$

note that at most two w_k are positive. The following example will illustrate the procedure:

Example 2.2.1 *suppose we have the following nonlinear programming problem*

$$\begin{aligned} &\max f(x) = 3x_1 + 2x_2 \\ &\textit{subject to} \\ &4x_1^2 + x_2^2 \leq 16 \\ &x_1, x_2 \geq 0 \end{aligned} \tag{2.3}$$

$$\text{let } f_1(x_1) = 3x_1, f_2(x_2) = 2x_2$$
$$g_1(x_1) = 4x_1^2, g_2(x_2) = x_2^2$$

the problem (2.3) will be :

$$\max f(x) = f_1(x_1) + f_2(x_2)$$
subject to
$$g_1(x_1) + g_2(x_2) \le 16$$
$$x_1, x_2 \ge 0$$

Now from the constraint $\left(4{x_1}^2 + {x_2}^2 \le 16\right)$ we can observe that the upper limit for x_1 is 2 and for x_2 is 4 (explanation: $4x_1^2 + 0 \le 16 \Rightarrow x_1^2 \le 4 \Rightarrow x_1 \le 2$ since x_1 is nonnegative, $0 + x_2^2 \le 16 \Rightarrow x_2 \le 4$ since x_2 is nonnegative)

we will divide the intervals [0,2] and [0,4]into equal subintervals.

k	a_k	$f_1(a_k) = 3a_k$	$g_1(a_k) = 4a_k^2$
1	0	0	0
2	1	3	4
3	2	6	16
4	3	9	36
5	4	12	64

k	a_k	$f_2(a_k) = 2a_k$	$g_2(a_k) = a_k^2$
1	0	0	0
2	1	2	1
3	2	4	4

$$f_1(x_1) \approx 0w_{11} + 3w_{12} + 6w_{13} + 9w_{14} + 12w_{15}$$
$$g_1(x_1) \approx 0w_{11} + 4w_{12} + 16w_{13} + 36w_{14} + 64w_{15}$$
$$f_1(x_2) \approx 0w_{21} + 2w_{22} + 4w_{23}$$
$$g_2(x_2) \approx 0w_{21} + w_{22} + 4w_{23}$$

Now by using the piecewise linear approximation obtained we will get the given problem:

$$max\, f(X) = (0w_{11} + 3w_{12} + 6\omega_{13} + 9\omega_{14} + 12\omega_{15}) +$$
$$(0\omega_{21} + 2\omega_{22} + 4\omega_{23})$$

subject to

$$(0\omega_{11}+4\omega_{12}+16\omega_{13}+36\omega_{14}+64\omega_{15})+\quad(0\omega_{21}+\omega_{22}+4\omega_{23})\leq 16$$

$$\omega_{11}+\omega_{12}+\omega_{13}+\omega_{14}+\omega_{15}=1$$

$$\omega_{21}+\omega_{22}+\omega_{23}=1$$

$$\omega_{jk}\geq 0,\ j=1,2,\ k=1,2,3,4,5$$

solving the problem by any well known method we get:

$$\omega_{12}=0.33 \quad \omega_{13}=0.667 \quad \omega_{23}=1$$

$$\omega_{14}=\omega_{15}=\omega_{22}=\omega 11=\omega_{21}=0$$

hence the approximated optimal solution is:

$$\begin{aligned} x_1 &\approx a_{11}\omega_{11}+a_{12}\omega_{12}+a_{12}\omega_{13}+a_{14}\omega_{14}+a_{15}\omega_{15} \\ &= 0\times 0+1\times 0.33+2\times 0.667+3\times 0+4\times 0 \\ &= 1.664 \end{aligned}$$

$$\begin{aligned} x_2 &\approx a_{21}\omega_{21}+a_{22}\omega_{22}+a_{23}\omega_{23} \\ &= 1\times 0+2\times 0+3\times 1=3 \\ z=3x_1+2x_2 &= 3(1.664)+2\times 3 \\ &= 4.992+6 \\ &= 10.992 \end{aligned}$$

Chapter 3

Ranking function method for solving some types of non linear programming problems

In this chapter we will introduce the ranking function method to solve some types of non linear programming problems

3.1 Ranking function method for solving non linear programming problems that satisfy KKT conditions.

In this section we will show the definition of the ranking function and how it can be used for solving the nonlinear programming problems that satisfy the KKT conditions.

Definition 3.1.1 *Alkanani and Adnan [2014], Thaher [2018]*

The ranking function is a function $R : F(R) \to R$ where $F(R)$ is the set of fuzzy numbers which maps each fuzzy number into the real live where a nature order exist

If $\widetilde{A} = (a,b,c)$ is a triangular fuzzy number then the pascal triangular ranking function $\tilde{A}$ is defined as:

$$R(\tilde{A}) = \frac{a+2b+c}{4} \quad \text{Akrami et al. [2016b]}$$

If $\tilde{A} = (a,b,c,d)$ is a trapezoidal fuzzy number then the ranking function of $\tilde{A}$ is defined as:

$$R(\tilde{A}) = \frac{a+b+c+d}{4} \quad \text{Temelcan et al. [2022]}$$

If $\tilde{A} = (a,b,c,d,e,f)$ is a hexagonal fuzzy number then the ranking function of $\tilde{A}$ is defined as:

$$R(\tilde{A}) = \frac{a+b+c+d+e+f}{6} \quad \text{Vafaei et al. [2018]}$$

for solving the FFNLPP with KKT conditions by using the ranking function we can follow the following procedure : M.Lalitha [2018]

1. we will express any variable in the problem as fuzzy variable.
2. we will convert the problem to a crisp one by using the ranking function.
3. we will check the KKT sufficient conditions.
4. obtain the Lagrangian function
5. Write the KKT necessary conditions and then solve the equations to find the values of the variables.
6. check the optimality conditions at x_j

Example 3.1.1 *Take the following FFNLPP that satisfies KKT conditions with triangular fuzzy number using ranking function.*

$$\min \tilde{z} = (0.5,1,1.5) \otimes \tilde{x}^2 \oplus (0.5,1,1.5) \otimes \tilde{y}^2 \oplus (0.5,1,1.5) \otimes \tilde{v}^2$$

Subject to

$$(1,2,3) \otimes \tilde{x} \oplus (0.5,1,1.5) \otimes \tilde{y} \leq (3,5,7)$$
$$(0.5,1,1.5) \otimes \tilde{x} \oplus (0.5,1,1.5) \otimes \tilde{v} \leq (1,2,3)$$
$$(0.5,1,1.5) \ominus (0.5,1,1.5)\tilde{x} \leq 0$$
$$(1,2,3) \ominus (1,3,5) \otimes \tilde{y} \leq 0$$
$$\ominus (0.5,1,1.5) \otimes \tilde{v} \leq 0$$
$$\tilde{x}, \tilde{y}, \tilde{v} \geq 0$$

let $\tilde{x} = (x_1, x_2, x_3)\,, \tilde{y} = (y_1, y_2, y_3)\,, \tilde{v} = (v_1, v_2, v_3)$, hence the problem will be :

$$\min \tilde{z} = (0.5,1,1.5) \otimes \left(x_1{}^2, x_2{}^2, x_3^2\right) \oplus (0.5,1,1.5) \otimes \left(y_1{}^2, y_2^2, y_3^2\right) \oplus (0.5,1,1.5) \otimes \left(v_1{}^2, v_2^2, v_3^2\right)$$

Subject to

$$(1,2,3) \otimes (x_1, x_2, x_3) \oplus (0.5,1,1.5) \otimes (y_1, y_2, y_3) \leq (3,5,7)$$
$$(0.5,1,1.5) \otimes (x_1, x_2, x_3) \oplus (0.5,1,1.5) \otimes (v_1, v_2, v_3) \leq (1,2,3)$$
$$(0.5,1,1.5) \ominus (0.5,1,1.5) \otimes (x_1, x_2, x_3) \leq 0$$
$$(1,2,3) \ominus (1,3,5) \otimes (y_1, y_2, y_3) \leq 0$$
$$\ominus (0.5,1,1.5) \otimes (v_1, v_2, v_3) \leq 0.$$

$$x_1, x_2, x_3, y_1, y_2, y_3, v_1, v_2, v_3 \geq 0$$

using multiplication of triangular fuzzy numbers we get:

$$
\begin{aligned}
&\min z = \left(0.5x_1^2, x_2^2, 1.5x_3^2\right) \oplus \left(0.5y_1^2, y_2^2, 1.5y_3^2\right) \oplus \left(0.5v_1^2, v_2^2, 1.5v_3^2\right) \\
&\text{Subject to} \\
&(x_1, 2x_2, 3x_3) \oplus (0.5y_1, y_2, 1.5y_3) \ominus (3,5,7) \leq 0 \\
&(0.5x_1, x_2, 5x_3) \oplus (0.5v_1, v_2, 1.5v_3) \ominus (1,2,3) \leq 0 \\
&(0.5,1,1,5) \ominus (0.5x_1, x_2, 1.5x_3) \leq 0 \\
&(1,2,3) \ominus (y_1, 3y_2, 5y_3) \leq 0 \\
&\ominus (0.5v_1, v_2, 1.5v_3) \leq 0.
\end{aligned}
$$

converting the problem to a crisp one by using the pascal triangular ranking function, $R(\tilde{A}) = \frac{a+2b+c}{4}$
we get:

$$
\begin{aligned}
&\min z = \frac{1}{4}\left(0.5x_1^2 + 2x_2^2 + 1.5x_3^2 + 0.5y_1^2 + 2y_2^2 + 1.5y_3^2 + 0.5v_1^2 + 2v_2^2 + 1.5v_3^2\right) \\
&\text{Subject to} \\
&\frac{1}{4}(x_1 + 4x_2 + 3x_3 + 0.5y_1 + 2y_2 + 1.5y_3 - 3 - 10 - 7) \leq 0 \\
&\frac{1}{4}(0.5x_1 + 2x_2 + 1.5x_3 + 0.5v_1 + 2v_2 + 1.5v_3 - 1 - 4 - 3) \leq 0 \\
&\frac{1}{4}(0.5 + 2 + 1.5 - 0.5x_1 - 2x_2 - 1.5x_3) \leq 0 \\
&\frac{1}{4}(1 + 4 + 3 - y_1 - 6y_2 - 5y_3) \leq 0 \\
&\frac{1}{4}(-0.5v_1 - 2v_2 - 1.5v_3) \leq 0. \\
&x_1, x_2, x_3, y_1, y_2, y_3, v_1, v_2, v_3 \geq 0
\end{aligned}
$$

Now, since the problem is minimization type , z should he convex to satisfy the KKT sufficient condition. i.e the hessian matrix must be symmetric with non negative eigen values.

the eigen values for the symmetric hessian matrix of Z are all positive $\left(11, 1, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$.

and hence z is convex

the constraints are all linear and hence they are convex and concave at the same time.

The Lagrangian function will be :

$$
\begin{aligned}
L(\tilde{x},\tilde{y},\tilde{v},\lambda) &= z-\lambda_1 g_1(x)-\lambda_2 g_2(x)-\lambda_3 g_3(x)\\
&-\lambda_4 g_4(x)-\lambda_5 g_5(x)\\
&=\frac{1}{4}\begin{pmatrix}0.5x_1^2+2x_2^2+1.5x_3^2+0.5y_1^2+y_2^2+\\ 1.5y_3^2+0.5v_1^2+2v_2^2+1.5v_3^2\end{pmatrix}\\
&-\frac{\lambda_1}{4}\left(x_1+4x_2+3x_3+0.5y_1+2y_2+1.5y_3-20\right)\\
&-\frac{\lambda_2}{4}\left(0.5x_1+2x_2+1.5x_3+0.5v_1+2v_2+1.5v_3-8\right)\\
&-\frac{\lambda_3}{4}\left(4-0.5x_1-2x_2-1.5x_3\right)-\frac{\lambda_4}{4}\left(8-y_1-6y_2-5y_3\right)\\
&-\frac{\lambda_5}{4}\left(-0.5v_1-2v_2-1.5v_3\right)
\end{aligned}
$$

The necessary conditions are:

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= 0.25x_1-\frac{\lambda_1}{4}-0.125\lambda_2+0.375\lambda_3=0\\
\frac{\partial L}{\partial x_2} &= x_2-\lambda_1-\frac{\lambda_2}{2}+\frac{\lambda_3}{2}=0\\
\frac{\partial L}{\partial x_3} &= 0.75x_3-\frac{3}{4}\lambda_1-0.375\lambda_2+0.125\lambda_3=0\\
\frac{\partial L}{\partial y_1} &= 0.25y_1-0.125\lambda_1+\frac{5}{4}\lambda_4=0\\
\frac{\partial L}{\partial y_2} &= y_2-\frac{\lambda_1}{2}+\frac{3}{2}\lambda_4=0\\
\frac{\partial L}{\partial y_3} &= 0.75y_3-0.375\lambda_1+\frac{\lambda_4}{4}=0\\
\frac{\partial L}{\partial v_1} &= 0.25v_1-0.125\lambda_2+0.375\lambda_5=0\\
\frac{\partial L}{\partial v_2} &= v_2-\frac{\lambda_2}{2}+\frac{\lambda_5}{2}=0\\
\frac{\partial L}{\partial v_3} &= 0.75v_3-0.375\lambda_2+0.125\lambda_5=0
\end{aligned}
$$

$$\lambda_1 g_1 = \frac{\lambda_1}{4}(x_1 + 4x_2 + 3x_3 + 0.5y_1 + 2y_2 + 1.5y_3 - 20) = 0$$
$$\lambda_2 g_2 = \frac{\lambda_2}{4}(0.5x_1 + 2x_2 + 1.5x_3 + 0.5v_1 + 1.5v_3 + 2v_2 - 8) = 0$$
$$\lambda_3 g_3 = \frac{\lambda_3}{4}(4 - 0.5x_1 - 2x_2 - 1.5x_3) = 0$$
$$\lambda_4 g_4 = \frac{\lambda_4}{4}(8 - y_1 - 6y_2 - 9y_3) = 0$$
$$\lambda_5 g_5 = \frac{\lambda_5}{4}(0.5v_1 - 2v_2 - 1.5v_3) = 0$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \leq 0$$

solving by any known method we get:

$x_1 = 1.8 \quad x_2 = 0.6 \quad x_3 = 0.2$

$y_1 = 1.16505 \quad y_2 = 0.349515 \quad y_3 = 0.0776694$

$v_1 = 0 \quad v_2 = 0 \quad v_3 = 0$

$\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = -1.2 \quad \lambda_4 = -0.23301 \quad \lambda_5 = 0$

Now we will check whether the solution satisfies the constraints or no , substituting the values in g_1, g_2, g_3, g_4 and g_5, we insure that they satisfy the constraints, hence the optimal solution is:

$\tilde{x} = (0.2, 0.6, 1.8) \quad \tilde{y} = (1.16505, 0.0776699, 0.3495) \quad \tilde{v} = (0,0,0) \quad \tilde{z} = (0.6904, 0.482, 2.298)$

Example 3.1.2 *Take the following FFNLPP that satisfies KKT condition with trapezoidal fuzzy number using ranking function.*

$$\begin{aligned} &min\,\tilde{z} = (4,9,19,24) \otimes \tilde{x}^4 \oplus (1,3,7,9) \otimes \tilde{y}^2 \\ &subject\ to \\ &(1,3,5,7) \otimes \tilde{x} \oplus (3,4,8,9) \otimes \tilde{y} \ominus (2,4,6,8) \geq 0 \\ &(1,1.5,2.5,3) \otimes \tilde{x} \ominus (2,3,5,6) \otimes \tilde{y} \ominus (1,2,4,5) \geq 0 \\ &\tilde{x}, \tilde{y} \geq 0 \end{aligned} \tag{3.1}$$

$$let\ \tilde{x} = (x_1, x_2, x_3, x_4), \tilde{y} = (y_1, y_2, y_3, y_4)$$

Now by substituting $\tilde{x}$ and $\tilde{y}$ and using multiplication we get:

$$\begin{aligned}
&\min\widetilde{z} = (4x_1^4, 9x_2^4, 19x_3^4, 24x_4^4) \oplus (y_1^2, 3y_2^2, 7y_3^2, 9y_4^2) \\
&\quad \text{subject to} \\
&\quad (x_1, 3x_2, 5x_3, 7x_4) \oplus (3y_1, 4y_2, 8y_3, 9y_4) \ominus (2,4,6,8) \geq 0 \\
&\quad (x_1, 1.5x_2, 2.5x_3, 3x_4) \ominus (2y_1, 3y_2, 5y_3, 6y_4) \ominus (1,2,4,5) \geq 0 \\
&\quad x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \geq 0
\end{aligned}$$

converting the problem to a crisp one by using the ranking function $R(\tilde{A}) = \frac{a+b+c+d}{4}$ we get:

$$\begin{aligned}
&\min\tilde{z} = \frac{1}{4}\left(4x_1^4 + 9x_2^4 + 19x_3^4 + 24x_4^4 + y_1^2 + 3y_2^2 + 7y_3^2 + 9y_4^2\right) \\
&\quad \text{subject to} \\
&\quad \frac{1}{4}(x_1 + 3x_2 + 5x_3 + 7x_4 + 3y_1 + 4y_2 + 8y_3 + 9y_4 - 20) \geq 0 \\
&\quad \frac{1}{4}(x_1 + 1.5x_2 + 2.5x_3 + 3x_4 - 2y_1 - 3y_2 - 5y_3 - 6y_4 - 12) \geq 0 \\
&\quad x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \geq 0
\end{aligned}$$

Now since the type of the problem is minimization , z must be convex to satisfy the KKT sufficient condition i.e. the hessian matrix must be symmetric positive definite (with positive eigenvalues)

The eigen values of the symmetric hessian matrix of z are $\left(\frac{9}{2}, \frac{7}{2}, \frac{3}{2}, \frac{1}{2}, 12x_1^2, 27x_2^2, 57x_3^2, 72x_4^2\right)$. Since the eigen values are all positive, z is convex.

Note that since $\tilde{z}$ is convex and the constraints are (greater than or equal) so the values of λ have to be non-negative.

Now we will obtain the Lagrangian function:

$$
\begin{aligned}
L(\tilde{x};\tilde{y},\lambda) &= z - \lambda_1 g_1 - \lambda_2 g_2 \\
&= \frac{1}{4}\left(4x_1^4 + 9x_2^4 + 19x_3^4 + 24x_4^4 + y_1^2 + 3y_2^2 + 7y_3^2 + 9y_4^2\right) \\
&- \lambda_1 \left(\frac{x_1 + 3x_2 + 5x_3 + 7x_4 + 3y_1 + 4y_2 + 8y_3 + 9y_4 - 20}{4}\right) \\
&- \lambda_2 \left(\frac{x_1 + 1.5x_2 + 2.5x_3 + 3x_4 - 2y_1 - 3y_2 - 5y_3 - 6y_4 - 12}{4}\right)
\end{aligned}
$$

The necessary conditions are:

$$
\begin{aligned}
&\frac{\partial L}{\partial x_1} = 4x_1^3 - \frac{\lambda_1}{4} - \frac{\lambda_2}{4} = 0 \\
&\frac{\partial L}{\partial x_2} = 9x_2^3 - \frac{3\lambda_1}{4} - 0.375\lambda_2 = 0 \\
&\frac{\partial L}{\partial x_3} = 19x_3^3 - \frac{5\lambda_1}{4} - 0.625\lambda_2 = 0 \\
&\frac{\partial L}{\partial x_4} = 24x_4^3 - \frac{7\lambda_1}{4} - \frac{3\lambda_2}{4} = 0 \\
&\frac{\partial L}{\partial y_1} = \frac{y_1}{2} - \frac{3\lambda_1}{4} - \frac{\lambda_2}{2} = 0 \\
&\frac{\partial L}{\partial y_2} = \frac{3y_2}{2} - \lambda_1 - \frac{3\lambda_2}{4} = 0 \\
&\frac{\partial L}{\partial y_3} = \frac{7y_3}{2} - 2\lambda_1 - \frac{5\lambda_2}{4} = 0 \\
&\frac{\partial L}{\partial y_4} = \frac{9y_4}{2} - \frac{9\lambda_1}{4} - \frac{3\lambda_2}{2} = 0 \\
&\lambda_1 g_1 = \lambda_1 \left(\frac{x_1 + 3x_2 + 5x_3 + 7x_4 + 3y_1 + 4y_2 + 8y_3 + 9y_4 - 20}{4}\right) = 0 \\
&\lambda_2 g_2 = \lambda_2 \left(\frac{x_1 + 1.5x_2 + 2.5x_3 + 3x_4 - 2y_1 - 3y_2 - 5y_3 - 6y_4}{4}\right) = 0 \\
&\lambda_1, \lambda_2 \geq 0 \\
&g_1(x) \geq 0 \\
&g_2(x) \geq 0 \\
&x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \geq 0
\end{aligned}
$$

solving the system of equation we get:

$x_1 = 1.282303$, $x_2 = 1.379273$, $x_3 = 1.475643$, $x_4 = 1.572213$

$y_1 = 2.974041$, $y_2 = 1.321796$, $y_3 = 1.132968$, $y_4 = 0.9913470$

$\lambda_1 = 1.982694$, $\lambda_2 = 0$

substituting these values in the constraints g_1 and g_2 we insure that they satisfy them. hence the optimal solution is:

$$\tilde{x} = (1.282803, 1.379273, 1.475743, 1.572213)$$
$$\tilde{y} = (2.974041, 1.321796, 1.132968, 0.9913470)$$
$$\tilde{z} = (4,9,19,24) \otimes \tilde{x}^4 \oplus (1,3,7,9) \otimes \tilde{y}^2$$
$$= (4,9,19,24) \otimes (1.282803^4, 1.379273^4, 1.475743^4, 1.572213^4)$$
$$\oplus (1,3,7,9) \otimes (2.974041^2, 1.13296^2, 1.132968^2, 0.9913470^2)$$
$$= (19.655, 37.813, 99.09, 155.486)$$

Example 3.1.3 *consider the following FFNLPP that satisfies KKT condition with hexagonal fuzzy number using ranking function.*

$$max\, \tilde{z} = \ominus(6,8,10,14,16,18) \otimes \tilde{x}^4 \ominus (5,6,7,9,10,11) \otimes \tilde{y}^4$$
$$subject\ to$$
$$(1,3,5,9,11,13) \otimes \tilde{x}^3 \ominus (25,28,31,37,40,43) \otimes \tilde{y} \oplus (10,14,18,26,30,34) \leq 0$$
$$\ominus (3,8,13,23,28,33) \otimes \tilde{x} \ominus (4,6,8,12,14,16) \otimes \tilde{y} \ominus (20,24,28,36,40,44) \leq 0$$
$$\tilde{x}, \tilde{y} \geq 0$$

$$let\, \tilde{x} = (x_1, x_2, x_3, x_4, x_5, x_6), \tilde{y} = (y_1, y_2, y_3, y_4, y_5, y_6)$$

using the ranking function $R(\tilde{A}) = \frac{a+b+c+d+e+f}{6}$ to convert the problem to a crisp one we get:

$$\max\, \tilde{z} = \frac{1}{6}\left(-6x_1^4 - 8x_2^4 - 10x_3^4 - 14x_4^4 - 16x_5^4 - 18x_6^4 - 5y_1^4 - 6y_2^4 - 7y_3^4 - 9y_4^4 - 10y_5^4 - 11y_6^4\right)$$

subject to

$$\frac{1}{6}\left(x_1^3 + 3x_2^3 + 5x_3^3 + 9x_4^3 + 11x_5^3 + 13x_6^3 - 25y_1 - 28y_2 - 31y_3 - 37y_4 - 40y_5 - 43y_6 + 132\right)$$
$$\frac{1}{6}\left(-3x_1 - 8x_2 - 13x_3 - 23x_4 - 28x_5 - 33x_6 - 4y_1 - 6y_2 - 8y_3 - 12y_4 - 14y_5 - 16y_6 - 192\right)$$
$$x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$$

Now since the type of the problem is maximization z must be concave (i.e. the hessian matrix have to be symmetric with negative eigen values). The eigen Values for the symmetric hessian matrix of z are all negative and hence z is concave. The constraints have to be convex since they are (less than or equal) i.e. the eigen values must be non-negative for the symmetric values hessian matrix. For the first constraint the eigen values are non-negative and hence it is convex. The second constraint is linear and hence it is convex. Now the Lagrangian function will be :

$$
\begin{aligned}
&L(\tilde{x},\tilde{y},\lambda) = z - \lambda_1 g_1 - \lambda_2 g_2 \\
&= \frac{1}{6}\left(-6x_1^4 - 8x_2^4 - 10x_3^4 - 14x_4^4 - 16x_5^4 - 18x_6^4 - 5y_1^4 - 6y_2^4 - 7y_3^4 - 9y_4^4 - 10y_5^4 - 11y_6^4\right) \\
&-\lambda_1\left(\frac{x_1^3 + 3x_2^3 + 5x_3^3 + 9x_4^3 + 11x_5^3 + 13x_6^3 - 25y_1 - 28y_2 - 31y_3 - 37y_4 - 40y_5 - 43y_6 + 132}{6}\right) \\
&-\lambda_2\left(-3x_1 - 8x_2 - 13x_3 - 23x_4 - 28x_5 - 33x_6 - 4y_1 - 6y_2 - 8y_3 - 12y_4 - 14y_5 - 16y_6 - 192\right)
\end{aligned}
$$

The necessary conditions are:

$$
\begin{aligned}
\frac{\partial L}{\partial x_1} &= -4x_1^3 + 2x_2^2\lambda_1 - \frac{\lambda_2}{2} = 0 \\
\frac{\partial L}{\partial x_2} &= \frac{-16x_2^3}{3} + \frac{7x_2^2\lambda_1}{2} - \frac{4\lambda_2}{3} = 0 \\
\frac{\partial L}{\partial x_3} &= \frac{-20x_3^3}{3} + 5x_3^2\lambda_1 + \frac{13\lambda_2}{6} = 0 \\
\frac{\partial L}{\partial x_4} &= \frac{-28x_4^3}{3} + 8x_4^2\lambda_1 + \frac{23\lambda_2}{6} = 0 \\
\frac{\partial L}{\partial x_5} &= \frac{-32x_5^3}{3} + \frac{19x_5^2\lambda_1}{2} + \frac{14\lambda_2}{3} = 0 \\
\frac{\partial L}{\partial x_6} &= -12x_6^3 + 11x_6^2\lambda_1 + \frac{11\lambda_2}{2} = 0
\end{aligned}
$$

$$\frac{\partial L}{\partial y_1} = \frac{-10y_1^3}{3} - \frac{7\lambda_1}{6} + \frac{2\lambda_2}{3} = 0$$
$$\frac{\partial L}{\partial y_2} = -4y_2^3 - \frac{3\lambda_1}{2} + \lambda_2 = 0$$
$$\frac{\partial L}{\partial y_3} = \frac{-14y_3^3}{3} - \frac{-11\lambda_1}{6} + \frac{4\lambda_2}{3} = 0$$
$$\frac{\partial L}{\partial y_4} = -6y_4^3 - \frac{5\lambda_1}{2} + 2\lambda_2 = 0$$
$$\frac{\partial L}{\partial y_5} = \frac{-20y_5^3}{3} - \frac{17\lambda_1}{6} + \frac{7\lambda_2}{3} = 0$$
$$\frac{\partial L}{\partial y_6} = -\frac{19\lambda_1}{6} + \frac{8\lambda_2}{3} = 0$$
$$\lambda_1 g_1 = 0$$
$$\lambda_2 g_2 = 0$$
$$\lambda_1, \lambda_2 \geq 0$$
$$g_1, g_2 \leq 0$$
$$x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0$$

Solving the system of equations we get:

$x_1 = 1.242773$, $x_2 = 1.256449$, $x_3 = 1.270121$, $x_4 = 1.297475$, $x_5 = 1.311151$, $x_6 = 1.324827$

$y_1 = 1.204242$, $y_2 = 1.193234$, $y_3 = 1.182225$, $y_4 = 1.160207$, $y_5 = 1.149199$, $y_6 = 1.138190$

$\lambda_1 = \lambda_2 = 0$

substituting these values in the constraints g_1 and g_2 we insure that they satisfy them and hence the optimal solution is:

$\tilde{x} = (1.242773, 1.256449, 1.270121, 1.297475, 1.311151, 1.324827)$

$\tilde{y} = (1.204242, 1.193234, 1.182225, 1.160207, 1.149199, 1.138190)$

$\tilde{z} = (-23.995, -32.101, -39.698, -55.983, -64.727, -73.912)$

3.2 Solving fully fuzzy nonlinear separable programming problem by ranking function method

In this section we will show the procedure of the ranking function that we will use to solve separable fully fuzzy nonlinear programming problem.

Assume that $\tilde{c} = [\tilde{c_j}]_{1\times n}$, $\tilde{b} = [\tilde{b_i}]_{m\times 1}$ are fuzzy numbers, $A = [\tilde{a_{ij}}]_{m\times n}$ is a matrix of fuzzy numbers, $X = [\tilde{x_j}]_{1\times n}$ is the fuzzy solution vector. The problem0.1 will be

$$\max z = \sum_{j=1}^{n} (q_j, r_j, s_j, ...) \otimes (x_j, y_j, z_j)$$

subject to

$$\sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}, ...) \otimes (x_j, y_j, z_j, ...) \leq (b_i, g_i, h_i, ...)$$

$$i = 1, ..., m \quad j = 1, ..., n \quad \tilde{x_j} \geq 0$$

by using the definition 3.1.1 the objective function will be:

$\max \tilde{z} = R\left(\sum_{j=1}^{n} (q_j, r_j, s_j, ...) \otimes (x_j, y_j, z_j, ...)\right)$

constraints will be :

$\sum_{j=1}^{n} (a_{ij}, b_{ij}, c_{ij}, ...) \otimes (x_j, y_j, z_j, ...) \leq (b_i, g_i, h_i, ...)$

$\left(\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} b_{ij}y_j, \sum_{j=1}^{n} c_{ij}z_j, ...\right) \leq (b_i, g_i, h_i, ...)$

by comparison of the components we will the following crisp non linear programming problem

$$\max z = R\left(\sum_{j=1}^{n} (q_j, r_j, s_j, ...) \otimes (x_j, y_j, z_j, ...)\right)$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

$$\sum_{j=1}^{n} b_{ij}y_j \leq g_i$$

$$\sum_{j=1}^{n} c_{ij}z_j \leq h_i$$

.

.

.

$x_j \geq 0 \quad y_j \geq 0 \quad z_j \geq 0 \quad$ j= 1,...,n

Example 3.2.1 *Take the following separable fully fuzzy non linear programming problem with triangular fuzzy number using ranking function method*

$$\begin{aligned}
&\max \tilde{z} = (0.5,1,1.5)\otimes\tilde{x}\oplus(0.5,1,1.5)\otimes\tilde{y}^5\\
&\textit{subject to}\\
&(2,3,4)\otimes\tilde{x}\oplus(1,2,3)\otimes\tilde{y}^2 \leq (7,9,11)\\
&\tilde{x},\tilde{y}\geq 0
\end{aligned}$$

$\tilde{x} = (x_1,x_2,x_3)$ $\quad \tilde{y} = (y_1,y_2,y_3)$ rewriting the problem we get:

$$\begin{aligned}
&\max \tilde{z} = (0.5,1,1.5)\otimes(x_1,x_2,x_3)\oplus(0.5,1,1.5)\otimes(y_1^5,y_2^5,y_3^5)\\
&\text{subject to}\\
&(2,3,4)\otimes(x_1,x_2,x_3)\oplus(1,2,3)\otimes(y_1^2,y_2^2,y_3^3)\leq(7,9,11)\\
&x_1,x_2,x_3,y_1,y_2,y_3\geq 0
\end{aligned}$$

using the addition and multiplication of triangular fuzzy numbers we get:

$$\begin{aligned}
&\max \tilde{z} = (0.5x_1+0.5y_1^5, x_2+y_2^5, 1.5x_3+1.5y_3^5)\\
&\text{subject to}\\
&(2x_1+y_1^2, 3x_2+2y_2^2, 4x_3+3y_3^2)\leq(7,9,1)\\
&x_1,x_2,x_3,y_1,y_2,y_3\geq 0
\end{aligned}$$

Applying the pascal triangular ranking function $R(\tilde{A}) = \dfrac{a+2b+c}{4}$ on the objective function $\tilde{z}$ we get:

$$\begin{aligned}
&\max \tilde{z} = \frac{0.5x_1+0.5y_1^5+2x_2+2y_2^5+1.5x_3+1.5y_3^5}{4}\\
&\text{subject to}\\
&2x_1+y_1^2\leq 7\\
&3x_2+2y_2^2\leq 9\\
&4x_3+3y_3^2\leq 11\\
&x_1,x_2,x_3,y_1,y_2,y_3\geq 0
\end{aligned}$$

This is a crisp separable non linear programming problem that can be solved easily using the separable programming method , consider the separable non linear functions:

$f_1(y_1) = 0.125y_1^5 \quad f_2(y_2) = 0.5y_2^5 \quad f_3(y_3) = 0.375y_3^5$
$g_1(y_1) = y_1^2 \quad g_2(y_2) = 2y_2^2 \quad g_3(y_3) = 3y_3^2$

we can observe from the constraints that

$0 \leq y_1 \leq 3$ (explanation: $2x_1 + y_1^2 \leq 7$ let $x_1 = 0$ we get $y_1^2 \leq 7$ by taking square root we get $y_1 \leq 2.64575$)

$0 \leq y_2 \leq 3$ (explanation: $3x_2 + 2y_2^2 \leq 9$ let $x_2 = 0$ we get $y_2^2 \leq 4.5$ by taking square root we get $y_2 \leq 2.121$)

$0 \leq y_3 \leq 2$ (explanation: $4x_3 + 3y_3^2 \leq 11$ let $x_3 = 0$ we get $y_3^2 \leq 3.67$ by taking square root we get $y_3 \leq 1.915$)

k	a_{1k}	$f_1(a_{1k}) = 0.125a_{1k}^5$	$g_1(a_{1k} = a_{1k}^2)$
1	0	0	0
2	1	0.125	1
3	2	4	4
4	3	30.375	9

k	a_{2k}	$f_2(a_{2k}) = 0.5a_{2k}^5$	$g_2(a_{2k}) = 2a_{2k}^2$
1	0	0	0
2	1	0.5	2
3	2	16	8
4	3	121.5	18

k	a_{3k}	$f_3(a_{3k}) = 0.375y_3^5$	$g_3(a_{3k} = 3a_{3k}^2)$
1	0	0	0
2	1	0.375	3
3	2	12	12

This yields :

$$
\begin{aligned}
f_1(y_1) &= w_{11}f_1(a_{11}) + w_{12}f_1(a_{12}) + w_{13}f_1(a_{13}) + w_{14}f_1(a_{14}) \\
&= 0w_{11} + 0.125w_{12} + 4w_{13} + 30.375w_{14} \\
f_2(y_2) &= w_{21}f_2(a_{21}) + w_{22}f_2(a_{22}) + w_{23}f_2(a_{23}) + w_{24}f_2(a_{24}) \\
&= 0w_{21} + 0.5w_{22} + 16w_{23} + 121.5w_{24} \\
f_3(y_3) &= w_{31}f_3(a_{31}) + w_{32}f_3(a_{32}) + w_{33}f_3(a_{33}) \\
&= 0w_{31} + 0.375w_{32} + 12w_{33} \\
g_1(y_1) &= w_{11}g_1(a_{11}) + w_{12}g_2(a_{12}) + w_{13}g_3(a_{13}) + w_{14}g_4(a_{14}) \\
&= 0w_{11} + w_{12} + 4w_{13} + 9w_{14} \\
g_2(y_2) &= w_{21}g_2(a_{21}) + w_{22}g_2(a_{22}) + w_{23}g_3(a_{23}) + w_{24}g_4(a_{24}) \\
&= 0w_{21} + 2w_{22} + 8w_{23} + 18w_{24} \\
g_3(y_3) &= w_{31}g_3(a_{31}) + w_{32}g_3(a_{32}) + w_{33}g_3(a_{33}) \\
&= 0w_{31} + 3w_{32} + 12w_{33}
\end{aligned}
\tag{3.2}
$$

now substituting $f_1(y_1), f_2(y_2), f_3(y_3), g_1(y_1), g_2(y_2), g_3(y_3)$ we get :

$$
\begin{aligned}
&\max \tilde{z} \approx 0.125x_1 + 0.125w_{12} + 4w_{13} + 30.375w_{14} + 0.5x_2 + 0.5w_{22} + \\
&\quad 16w_{23} + 121.5w_{24} + 0.375x_3 + 0.375w_{32} + 12w_{33} \\
&\text{subject to} \\
&2x_1 + w_{12} + 4w_{13} + 9w_{14} \leq 7 \\
&3x_2 + 2w_{22} + 8w_{23} + 18w_{24} \leq 9 \\
&4x_3 + 3w_{23} + 12w_{34} \leq 11 \\
&w_{11} + w_{12} + w_{13} + w_{14} = 1 \\
&w_{21} + w_{22} + w_{23} + w_{24} = 1 \\
&w_{31} + w_{32} + w_{33} = 1 \\
&x_1, x_2, x_3 \geq 0 \quad w_{1i} \geq 0, \quad i = 1,2,3,4 \\
&w_{2i} \geq 0, \quad i = 1,2,3,4 \\
&w_{3i} \geq 0, \quad i = 1,2,3,4
\end{aligned}
$$

This is a linear programming problem that can be solved easily . solving the previous linear programming problem we get :

$$\begin{aligned}
&w_{12} = w_{13} = w_{22} = w_{23} = w_{31} = w_{32} = 0 \\
&w_{11} = 0.23, w_{14} = 0.77, w_{21} = 0.5, w_{24} = 0.5 \\
&w_{33} = 1, x_1 = 0, x_2 = 0, x_3 = 1.25 \\
&y_1 \approx 0w_{11} + w_{12} + 2w_{13} + 3w_{14} \\
&\quad = 0 + 0 + 2 \times (0) + 3 \times 0.77 = 2.31 \\
&y_2 \approx 0w_{21} + w_{22} + 2w_{23} + 3w_{24} \\
&\quad = 0 + 0 + 2 \times 0 + 3 \times 0.5 = 1.5 \\
&y_3 \approx 0w_{31} + w_{32} + 2w_{33} \\
&\quad = 0 + 0 + 2 \times 1 = 2
\end{aligned}$$

hence the approximated optimal solution is

$\tilde{x} = (0, 0, 1.25) \quad \tilde{y} = (2.31, 1.5, 2) \quad \tilde{z} = (32.89, 7.6, 49.9)$

Example 3.2.2 *consider the following separable fully fuzzy non linear programming problem with trapezoidal fuzzy number using ranking function method*

$$\max \widetilde{z} = (0.5, 0.75, 1.25, 1.5) \otimes \tilde{x} \oplus (0.5, 0.75, 1.25, 1.5) \otimes \tilde{y}^4 \tag{1}$$

$$\textit{subject to} \tag{3.3}$$

$$(1, 2, 4, 5) \otimes \widetilde{x} \oplus (1, 1.5, 3, 3.5) \otimes \tilde{y}^3 \leq (11, 13, 15, 17) \tag{3.4}$$

$$(0.5, 0.75, 1.25, 1.5) \otimes \tilde{x} \ominus (0.5, 0.75, 1.25, 1.5) \otimes \tilde{y} \leq (1.5, 2.5, 3.5, 4.5) \tag{3.5}$$

$$\widetilde{x}, \tilde{y} \geq 0 \tag{3.6}$$

$$\text{Let } \widetilde{x} = (x_1, x_2, x_3, x_4), \tilde{y} = (y_1, y_2, y_3, y_4)$$

rewriting the problem we get:

$$\max \widetilde{z} = (0.5x_1, 0.75x_2, 1.25x_3, 1.5x_4) \oplus \left(0.5y_1^4, 0.75y_2^4, 1.25y_3^4, 1.5y_4^4\right)$$

subject to

$\left(x_1+y_1^3, 2x_2+1.5y_2^3, 4x_3+3y_3^3, 5x_4+3.5y_4^3\right) \leq (11,13,15,17)$

$(0.5x_1-1.5y_4, 0.75x_2-1.25y_3, 1.25x_3-0.75y_2, 1.5x4-0.5y_1) \leq (1.5,2.5,3.5,4.5)$

$x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4 \geq 0$

now we will use the ranking function method that we discussed before by applying the ranking function $R(\widetilde{A}) = \frac{a+b+c+d}{4}$ on the objective function and using the comparison of components, the problem will be:

$$max\,\tilde{z} = \frac{0.5x_1+0.75x_2+1.25x_3+1.5x_4+0.5y_1^4+0.75y_2^4+1.25y_3^4+1.5y_4 4}{4}$$

subject to

$x_1+y_1^3 \leq 11$

$2x_2+1.5y_2^3 \leq 13$

$4x_3+3y_3^3 \leq 15$

$5x_4+3.5y_4^3 \leq 17$

$0.5x_1-1.5y_4 \leq 1.5$

$0.75x_2-1.25y_3 \leq 2.5$

$1.25x_3-0.75y_2 \leq 3.5$

$1.5x_4-0.5y_1 \leq 4.5$

$x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4 \geq 0$

new using the separable programming procedure:

from the constraints we observe that $0 \leq y_1 \leq 3 \quad 0 \leq y_2 \leq 3 \quad 0 \leq y_3 \leq 2 \quad 0 \leq y_4 \leq 2$

k	a_k	$a_k^4 = y^4$	$a_k^3 = y^3$
1	0	0	0
2	1	1	1
3	2	16	8
4	3	81	27

hence

$y_i^4 \approx 0w_{i1} + w_{i2} + 16w_{i3} + 81w_{i4} \quad \text{for } i = 1,2$

$y_i^{;4} \approx 0w_{i1} + w_{i2} + 16w_{i3} \quad \text{for } i = 3,4$

$y_i^3 \approx 0w_{i1} + w_{i2} + 8w_{i3} + 27w_{i4} \quad \text{for } i = 1,2$

$y_i^3 \simeq 0w_{i1} + w_{i2} + 8w_{i3} \quad \text{for } i = 1,2$

now after substituting these functions in the problem 1:

$$max\,\tilde{z} = \frac{1}{4}(0.5x_1 + 0.75x_2 + 1.25x_3 + 1.5x_4 + 0.5\,(0w_{11} + w_{12} + 16w_{13} + 81w_{14})$$
$$+0.75\,(0w_{21} + w_{22} + 16w_{33} + 81w_{24}) + 1.25\,(0w_{31} + w_{32} + 16w_{33}) + 1.5\,(0w_{41} + w_{42} + 16w_{43}))$$

subject to

$$x_1 + (0w_{11} + w_{12} + 8w_{13} + 27w_{14}) \leq 11$$
$$2x_2 + 1.5\,(0w_{21} + w_{22} + 8w_{23} + 27w_{24}) \leq 13$$
$$4x_3 + 3\,(0w_{31} + w_{32} + 8w_{33}) \leq 15$$
$$5x_4 + 3.5\,(0w_{41} + w_{42} + 8w_{43}) \leq 17$$
$$0.5x_1 - 1.5\,(0w_{41} + w_{42} + 2w_{43}) \leq 1.5$$
$$0.75x_2 - 1.25\,(0w_{31} + w_{32} + 2w_{33}) \leq 2.5$$
$$1.25x_3 - 0.75\,(0w_{21} + w_{22} + 2w_{23} + 3w_{24}) \leq 3.5$$
$$1.5x_4 - 0.5\,(0w_{11} + w_{12} + 2w_{13+3}w_{14}) \leq 4.5$$
$$w_{11} + w_{12} + w_{13} + w_{14} = 1$$
$$w_{21} + w_{22} + w_{23} + w_{24} = 1$$
$$w_{31} + w_{32} + w_{33} = 1$$
$$w_{41} + w_{42} + w_{43} = 1$$
$$x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \geq 0$$
$$w_{ij} \geq 0 \ for \quad i = 1,2,\, j = 1,2,3,4$$
$$w_{ij} \geq 0 \ for \quad i = 3,4,\, j = 1,2,3$$

Solving the system by any method we get

$$x_1 = x_2 = x_3 = x_4 = 0$$
$$w_{11} = 0.5925926 \quad w_{14} = 0.4074074$$
$$w_{21} = 0.6790123 \quad w_{24} = 0.3209877$$
$$w_{31} = 0.375 \quad w_{33} = 0.6250$$
$$w_{41} = 0.3928571 \quad w_{43} = 0.6071429$$
$$w_{12} = w_{13} = w_{22} = w_{23} = w_{32} = w_{42} = 0$$

$$\begin{aligned} y_1 &\approx 0w_{11} + w_{12} + 2w_3 + 3w_{14} \\ &= 0+0+0+3\times(0.4074074) \\ &= 1.\overline{2} \end{aligned}$$

$$\begin{aligned} y_2 &\approx 0w_{21} + w_{22} + 2w_{23} + 3w_{24} \\ &= 0+0+0+3\times(0.3209877) \\ &= 0.98963 \end{aligned}$$

$$\begin{aligned} y_3 &\approx 0w_{31} + w_{32} + 2w_{33} \\ &= 0+0+2\times(0.6250) = 1.25 \end{aligned}$$

$$\begin{aligned} y_4 &\approx 0w_{41} + w_{42} + 2w_{43} \\ &= 0+0+2\times(0.6071429) = 1.2143 \end{aligned}$$

hence the approximated optimal solution is:

$$\tilde{x} = (0,0,0,0) \quad \tilde{y} = \left(1.\overline{2}, 0.989631, 1.25, 1.2143\right)$$
$$\tilde{z} = (1.115, 0.7194, 3.06, 3.261)$$

Example 3.2.3 *consider the following separable fully fuzzy non linear programming problem with hexagonal fuzzy number using ranking function method.*

$$max\,(\tilde{z}) = (0.25,0.5,0.8,1.2,1.5,1.75)\otimes\tilde{x}^3 \oplus (0.2,0.5,0.8,1.2,1.5,1.8)\otimes\widetilde{y}$$

subject to

$$(1,2,3,4,6,8)\otimes\tilde{x}^2\oplus(0.5,1,1.5,2.5,3,3.5)\otimes\tilde{y}\leq(15,17,19,21,23,25)$$
$$(0.25,0.5,0.8,1.2,1.5,1.75)\otimes\tilde{y}\ominus(0.2,0.5,0.8,1.2,1.5,1.8)\otimes\tilde{x}\leq(5,6,7,8,10,12)$$
$$\tilde{x},\tilde{y}\geq 0$$

$$\text{let } \tilde{x}=(x_1,x_2,x_3,x_4,x_5,x_6)\,,\tilde{y}=(y_1,y_2,y_3,y_4,y_5,y_6)$$

applying the ranking function $R(\widetilde{A})=\frac{a+b+c+d+e+f}{6}$
on the objective function and using the comprising comparison of components we get:

$$max\tilde{z}=\frac{1}{6}(0.25x_1^3+0.5x_2^3+0.8x_3^3+1.2x_4^3+1.5x_5^3$$
$$+1.75x_6^3+0.2y_1+0.5y_2+0.8y_3+1.2y_4+1.5y_5+1.8y_6)$$

Subject to

$$x_1^2+0.5y_1\leq 15$$
$$2x_2^2+y_2\leq 17$$
$$3x_3^2+1.5y_3\leq 19$$
$$4x_4^2+2.5y_4\leq 21$$
$$6x_5^2+3y_5\leq 23$$
$$8x_6^2+3.5y_6\leq 25$$
$$0.25y_1-1.8x_6\leq 5$$
$$0.5y_2-1.5x_5\leq 6$$
$$0.8y_3-1.2x_4\leq 7$$
$$1.2y_4-0.8x_3\leq 8$$
$$1.5y_5-0.5x_2\leq 10$$
$$1.75y_6-0.2x_1\leq 12$$
$$x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2,y_3,y_4,y_5,y_6\geq 0$$

now using the separable programming procedure from the constraints we observe that:

$0\leq x_1\leq 4,\quad 0\leq x_2\leq 3,\quad 0\leq x_3\leq 3,\quad 0\leq x_4\leq 3,\quad 0\leq x_4\leq 3,$

$0\leq x_5\leq 3,\quad 0\leq x_6\leq 2$

k	a_k	$a_k^3 = x^3$	$a_k^2 = x^2$
1	0	0	0
2	1	1	1
3	2	8	4
4	3	27	9
5	4	64	16

after substituting these functions the problem will be :

$$max\,\tilde{z} \approx \frac{1}{6}(0.25\,(0w_{11}+w_{12}+8w_{13}+27w_{14}+64w_{15})+0.5\,(0w_{21}+w_{22}+8w_{23}+27w_{24})$$
$$+0.8\,(0w_{31}+w_{32}+8w_{33}+27w_{34})+1.2\,(0w_{41}+w_{42}+8w_{43})+1.5\,(0w_{51}+w_{52}+8w_{53}+27w_{54})$$
$$+1.75\,(0w_{61}+w_{62}+8w_{63})+0.2y_1+0.5y_2+0.8y_3+1.2y_4+1.5y_5+1.8y_6)$$

subject to

$$0w_{11}+w_{12}+4w_{13}+9w_{14}+16w_{15}+0.5y_1 \leq 15$$
$$2\,(0w_{21}+w_{22}+4w_{23}+9w_{24})+y_2 \leq 17$$
$$3\,(0w_{31}+w_{32}+4w_{33}+9w_{34})+1.5y_3 \leq 19$$
$$4\,(0w_{41}+w_{42}+4w_{43}+9w_{44})+2.5y_4 \leq 21$$
$$6\,(0w_{51}+w_{52}+4w_{53}+9w_{54})+3y_5 \leq 23$$
$$8\,(0w_{61}+w_{62}+4w_{63})+3\cdot 5y_6 \leq 25$$
$$0.25y_1-1.8\,(0w_{61}+w_{62}+2w_{63}) \leq 5$$
$$0.5y_2-1.5\,(0w_{51}+w_{52}+2w_{53}+3w_{54}) \leq 6$$
$$0.8y_3-1.2\,(0w_{41}+w_{42}+2w_{43}+3w_{44}) \leq 7$$
$$1.2y_4-0.8\,(0w_{31}+w_{32}+2w_{33}+3w_{34}) \leq 8$$
$$1.5y_5-0.5\,(0w_{21}+w_{22}+2w_{23}+3w_{24}) \leq 10$$
$$1.75y_6-0.2\,(0w_{11}+w_{12}+2w_{13}+3w_{14}+4w_{15}) \leq 12$$
$$w_{11}+w_{12}+w_{13}+w_{14}+w_{15} = 1$$
$$w_{21}+w_{22}+w_{23}+w_{24} = 1$$
$$w_{31}+w_{32}+w_{33}+w_{34} = 1$$

$$w_{41}+w_{42}+w_{43}+w_{44}=1$$
$$w_{51}+w_{52}+w_{53}+w_{54}=1$$
$$w_{61}+w_{62}+w_{63}=1$$
$$x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2,y_3,y_4,y_5,y_6\geq 0$$
$$w_{1j}\geq 0 \quad \text{for } j=1,2,3,4,5$$
$$w_{ij}\geq 0 \quad \text{for } i=2,3,4,5 \quad , \quad j=1,2,3,4$$
$$w_{6j}\geq 0 \quad \text{for } j=1,2,3$$

Solving by any well known method we get:

$$y_1=y_2=y_3=y_4=y_3=0, \quad y_6=7.142857$$
$$w_{11}=0.6\times 10^{-1}, \quad w_1=0.937$$
$$w_{21}=0.556\times 10^{-1}, \quad w_{24}=0.944$$
$$w_{31}=0.2962, \quad w_{34}=0.7037$$
$$w_{41}=0.4167, \quad w_{44}=0.583$$
$$w_{51}=0.5740, w_{54}=0.4259259$$
$$w_{61}=1$$
$$w_{12}=w_{13}=w_{14}=w_{22}=w_{23}=w_{32}=w_{33}=w_{42}=w_{43}=w_{52}=w_{53}=w_{62}=w_{63}=0$$
$$x_1\approx 4\times w_{15}=4\times 0.937=3.748$$
$$x_2\approx 3\times w_{24}=3\times 0.944=2.832$$
$$x_3\approx 3\times w_{34}=3\times 0.7037=2.111$$
$$x_4\approx 3\times w_{44}=3\times 0.583=1.749$$
$$x_5\approx 3\times w_{54}=3\times 0.4259259=1.277$$
$$x_6=0$$

hence the approximated optimal solution is:

$$\tilde{x}=(3.748,2.832,2.111,1.749,1.277,0)$$
$$\tilde{y}=(0,0,0,0,0,7.142857)$$
$$\tilde{z}=(13.1625,11.356,7.526,6.42,3.123,12.858)$$

Chapter 4

Interval programming approach for solving fully fuzzy nonlinear programming problems

In this chapter we will convert the FFNLPP to an interval programming problem (IPP) and then we will use interval programming approach to find the optimal solution for the FFNLPP with triangular fuzzy number we also extended the procedure for solving FFNLPP with trapezoidal and hexagonal fuzzy number

4.1 Interval programming problem

In this section we will show the procedure for solving interval programming problem .

Definition 4.1.1 *An interval number X in general is represented as* $[\underline{x}, \bar{x}]$.

Definition 4.1.2 *Akrami et al. [2016b] An interval programming problem is defined as :*

$$\begin{aligned}
&\max f(x) = \sum_{j=1}^{n} \left[\underline{c_j}, \overline{c_j}\right] f_j(x) \\
&\text{Subject to} \\
&\sum_{j=1}^{n} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] g_{jj}(x) \leq \left[\underline{b_i}, \overline{b_i}\right] \quad , \quad i = 1, 2, \ldots m \\
&x \geq 0
\end{aligned} \tag{4.1}$$

where $f_j(x)$ or $g_j(x)$ be nonlinear real valued functions.

Theorem 1: Akrami et al. [2016b] For the interval nonlinear programming problem (4.1), the best and worst optimum values can be obtained by solving the following problems respectively.

$$\max \overline{z} = \sum_{j=1}^{n} c''_j f_j(x) \tag{2}$$

Subject to (4.2)

$$\sum_{j=1}^{n} a''_{ij} g_j(x) \leq \overline{b_i} \quad , \quad i = 1,2,\ldots,m \tag{4.3}$$

$$x \geq 0 \tag{4.4}$$

$$\max \underline{z} = \sum_{j=1}^{n} c'_j f_j(x) \tag{3}$$

Subject to (4.5)

$$\sum_{j=1}^{n} a'_{ij} g_j(x) \leq \underline{b_i} \quad , \quad i = 1,2,\ldots,m \tag{4.6}$$

$$x \geq 0 \tag{4.7}$$

$$c'_j = \begin{cases} \underline{c_j} & , f_j(x) \geq 0 \\ \overline{c_j} & , f_j(x) \leq 0 \end{cases} \qquad a'_{ij} = \begin{cases} \overline{a_{ij}} & , g_j(x) \geq 0 \\ \underline{a_{ij}} & , g_j(x) \leq 0 \end{cases}$$

$$c''_j = \begin{cases} \overline{c_j} & , f_j(x) \geq 0 \\ \underline{c_j} & , f_j(x) \leq 0 \end{cases} \qquad a''_{ij} = \begin{cases} \underline{a_{ij}} & , g_j(x) \geq 0 \\ \overline{a_{ij}} & , g_j(x) \leq 0 \end{cases}$$

Theorem 2 :Akrami et al. [2016b] If the objective function for problem (4.1) is changed to min then the best and worst solution will be obtained by solving the following problems respectively

$$\min \underline{z} = \sum_{j=1}^{n} c'_j f_j(x)$$

Subject to

$$\sum_{j=1}^{n} a''_{ij} g_j(x) \leq \overline{b_i} \quad , \quad i = 1,2,\ldots m$$

$$x \geq 0$$

$$\min \bar{z} = \sum_{j=1}^{n} c''_j f_j(x)$$

Subject to

$$\sum_{j=1}^{n} a'_{ij} g_j(x) \le \underline{b_i} \quad , \quad i = 1,2\ldots m$$

$$x \ge 0$$

where $a'_{ij}, a''_{ij}, c'_j, c''_j$ are the same as defined in theorem (1).

4.2 Solving FFNLPP with triangular fuzzy number using interval programming approach.

In this section we will show how the interval programming will be used to solve the FFNLPP with triangular fuzzy number and we will shew a numerical example.

First we will convert the FFNLPP (0.1) using the alpha cut of the triangular fuzzy number as in definition (1.2.3) to the following interval nonlinear programming problem (INLPP) : Akrami et al. [2016b]

$$\begin{aligned} &\max z = \sum_{j=1}^{n} \left[\underline{c_j}, \overline{c_j}\right] f_j(x) \\ &\text{Subject to} \\ &\sum_{j=1}^{n} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] g_j(x) \le \left[\underline{b_i}, \overline{b_i}\right] \quad , \quad i = 1,2,\ldots m \\ &x \ge 0 \end{aligned} \tag{4.8}$$

where $\underline{c_j}, \underline{a_{ij}}, \underline{b_i}$ are called a lower bounds, $\overline{c_j}, \overline{a_{ij}}, \overline{b_i}$ are called a upper bounds

for $j = 1, \ldots, n$ and i= $1, \ldots m$:

$$\underline{c_j} = \left(c_j^2 - c_j^1\right)\alpha + c_j^1, \overline{c_j} = c_j^3 - \left(c_j^3 - c_j^2\right)\alpha$$
$$\underline{a_{ij}} = \left(a_{ij}^2 - a_{ij}^1\right)\alpha + a_{ij}^1, \overline{a_{ij}} = a_{ij}^3 - \left(a_{ij}^3 - a_{ij}^2\right)\alpha$$
$$\underline{b_i} = \left(b_j^2 - b_j^1\right)\alpha + b_j^1, \overline{b_{ij}} = b_j^3 - \left(b_j^3 - b_j^2\right)\alpha$$

where $\alpha \in [0,1]$.

Set $\alpha = 0$ in problem (4.8) we get:

$$maxz = \sum_{j=1}^{n} \left[c_j^1, c_j^3\right] f_j(x)$$

Subject to

$$\sum_{j=1}^{n} \left[a_{ij}^1, a_{ij}^3\right] g_j(x) \leq \left[b_j^1, b_j^3\right] \quad , \quad i = 1, 2, \ldots m$$

from this problem we will get two problems as in the problems (2) and (3) which will be solved to get two solutions

$$\underline{x}^* = \left(\underline{x}_1^*, \underline{x}_2^*, \ldots \underline{x}_n^*\right)^T, \overline{x}^* = \left(\overline{x}_1^*, \overline{x}_2^*, \ldots \overline{x}_n^*\right)^T$$

and the optimal values are $\underline{z}^*$ and $\overline{z}^*$ respectively

setting $\alpha = 1$ in problem (4.8) we will get the following problem :

$$\max z' = \sum_{j=1}^{n} c_j^2 f_j\left(x'\right) \quad , \quad i = 1, \ldots m$$

Subject to

$$\sum_{j=1}^{n} a_{ij}^2 g_j\left(x'\right) \leq b_j^2$$

$$x' \geq 0$$

solving this problem we get :

$$x' = \left(x_1', x_2', \ldots, x_n'\right) \text{ and the optimal value is } z'$$

hence the optimal solution for the problem (4.8)

$$x^* = \begin{cases} \left(\underline{x}_1^*, x_1', \overline{x}_1^*\right), \left(\underline{x}_2^*, x_2', \overline{x}_2^*\right), \ldots \\ \left(\underline{x}_n^*, x_n', \overline{x}_n^*\right) \end{cases}$$

$$z^* = \left(\underline{z}^*, z', \overline{z}^*\right)$$

Example 4.2.1 *consider the following FFNLPP with triangular fuzzy number*

$$
\begin{aligned}
&\max \widetilde{z} = (0.5,1,1.5) \otimes \widetilde{x} \oplus (0.5,1,1.5) \otimes \tilde{y}^5 \\
&\textit{Subject to} \\
&(2,3,4) \otimes \widetilde{x} \oplus (1,2,3) \otimes \widetilde{y}^2 \leq (7,9,11) \\
&\widetilde{x}, \widetilde{y} \geq 0
\end{aligned}
$$

converting the problem to an interval problem programming problem using alpha cut for triangular fuzzy number we get:

$$
\begin{aligned}
&\max z = [0.5+\alpha(0.5), 1.5-\alpha(0.5)]x + [0.5+\alpha(0.5), 1.5-\alpha(0.5)]y^5 \\
&\text{Subject to} \\
&[2+\alpha, 4-\alpha]x + [1+\alpha, 3-\alpha] \leq [7+2\alpha, 11-2\alpha]
\end{aligned}
$$

now set $\alpha = 1$ we get:

$$
\begin{aligned}
&\max z^* = x^* + y^{*5} \\
&\text{subject to} \\
&3x^* + 2y^{*2} \leq 9 \\
&x^*, y^* \geq 0
\end{aligned}
\tag{4.9}
$$

This is a crisp nonlinear programming problem that can be solved by separable programming method that was discussed in chapter 3
from the constraints we observe that $0 \leq y^* \leq 3$ let $f_1(y^*) = y^{*5}, g_1(y^*) = y^{*2}$

k	a_k	$f_1(a_k) = a_k^5$	$g_1(a_k) = a_k^2$
1	0	0	0
2	1	1	1
3	2	32	4
4	3	243	9

$$
\begin{aligned}
&f_1(y^*) = 0\omega_{11} + \omega_{12} + 32\omega_{13} + 243\omega_{14} \\
&g_1(y^*) = 0\omega_{11} + \omega_{12} + 4\omega_{13} + 9\omega_{14}
\end{aligned}
$$

substituting in the problem(4.9) we get:

$$\begin{aligned}
&\max z^* = x^* + \omega_{12} + 32\omega_{13} + 243\omega_{14} \\
&\text{ subject to} \\
&3x^* + 2(\omega_{12} + 4\omega_{13} + 9\omega_{14}) \leq 9 \\
&w_{11} + \omega_{12} + \omega_{13} + \omega_{14} = 1 \\
&x^*, \omega_{11}, \omega_{12}, \omega_{13}, \omega_{14} \geq 0
\end{aligned} \tag{4.10}$$

This is a linear programming problem, we solved it by lingo 11 and we got the following solution :

$$\begin{aligned}
&x^* = 0 \quad \omega_{11} = 0.5 \quad \omega_{12} = \omega_{13} = 0 \quad \omega_{14} = 0.5 \\
&y^* \approx \omega_{12} + 2\omega_{13} + 3\omega_{14} \\
&\quad = 0 + 0 + 3(0.5) = 1.5
\end{aligned}$$

hence, the optimal solution is:

$$x^* = 0, \quad y^* = 1.5 \quad z^* = 0 + 1.5^3 = 7.59$$

Now setting $\alpha = 0$ we get the following model:

$$\begin{aligned}
&\max z = [0.5, 1.5]x + [0.5, 1.5]y^5 \\
&\text{ Subject to} \\
&[2, 4]x + [1, 3]y^2 \leq [7, 11] \\
&x, y \geq 0
\end{aligned}$$

Now by considering theorem1 we have two problems, the first problem is

$$\begin{aligned}
&\max \bar{z} = 1.5\bar{x} + 1.5\bar{y}^5 \\
&\text{ subject to} \\
&2\bar{x} + \bar{y}^2 \leq 11 \\
&\bar{x}, \bar{y} \geq 0
\end{aligned} \tag{4.11}$$

this is a crisp nonlinear programming problem that can be solved by separable programming procedure as follows:

$$f_1(\bar{y}) = \bar{y}^5 \quad g_1(\bar{y}) = \bar{y}^2$$

from the constraint we observe that $0 \leq y \leq 4$
(explanation: $2x + y^2 \leq 11 \Rightarrow 0 + y^2 \leq 11 \Rightarrow y \leq 3.3$)

$$f_1(\bar{y}) = 0\omega_{11} + \omega_{12} + 32\omega_{13} + 243\omega_{14} + 1024\omega_{15}$$
$$g_1(\bar{y}) = 0\omega_{11} + \omega_{12} + 4\omega_{13} + 9\omega_{14} + 16\omega_{15}$$

substituting in the problem (4.11) we get :

$$\begin{aligned} &\max \bar{z} = 1.5\bar{x} + 1.5\left(0\omega_{11} + \omega_{12} + 32\omega_{13} + 243\omega_{14} + 1024\omega_{15}\right) \\ &\text{ subject to} \\ &2\bar{x} + (ow_{11} + w_{12} + 4w_{13} + 9w_{14} + 16w_{15}) \leq 11 \\ &w_{11} + w_{12} + w_{13} + w_{14} + \omega_{15} = 1 \\ &\bar{x}, w_{11}, w_{12}, w_{13}, w_{14}, \omega_{15} \geq 0 \end{aligned} \tag{4.12}$$

solving by big M method we get:

$$\begin{aligned} \bar{x} \quad &= 0 \quad \omega_{11} = 0.31 \quad \omega_{12} = \omega_{13} = \omega_{14} = 0 \quad \omega_{15} = 0.6875 \\ \bar{y} \quad &\approx 0\omega_{11} + \omega_{12} + 2\omega_{13} + 3\omega_{14} + 4\omega_{15} \\ &= 4 \times 0.6875 = 2.75 \end{aligned}$$

hence the optimal solution for the problem(4.12)

$$\bar{x} = 0 \quad \bar{y} = 2.75 \quad \bar{z} = 235.9$$

The second problem is:

$$\begin{gathered} \max \underline{z} = 0.5\underline{x} + 0.5\underline{y}^5 \\ \text{subject to} \\ 4\underline{x} + 3\underline{y}^2 \leq 7 \\ \underline{x}, \underline{y} \geq 0 \end{gathered}$$

solving the problem by separable programming procedure we get:

$$
\begin{aligned}
&\max \underline{z} = 0.5\underline{x} + 0.5\left(0w_{11} + w_{12} + 32\omega_{13}\right)\\
&\text{subject to}\\
&4\underline{x} + 3\left(0\omega_{11} + w_{12} + 4\omega_{19}\right) \leq 7\\
&w_{11} + w_{12} + \omega_{13} = 1\\
&\underline{x}, \omega_{11}, w_{12}, w_{13} \geq, 0
\end{aligned}
$$

$$
\begin{aligned}
&\underline{x} = 0 \quad \omega_{11} = 0.4167 \quad \omega_{12} = 0 \quad \omega_{13} = 0.5833\\
&\underline{y} = 2 \times 0.583 = 1.166\\
&\underline{z} = 0.5 \times (1.166)^5 = 1.0776
\end{aligned}
$$

hence the optimal solution for the original problem is:

$$
\begin{aligned}
&\widetilde{x} = (\underline{x}, x^*, \bar{x}) = (0,0,0)\\
&\widetilde{y} = \left(\underline{y}, y^*, \bar{y}\right) = (1.1665, 1.5, 2.75)\\
&\widetilde{z} = (\underline{z}, z^*, \bar{z}) = (1.0776, 7.59375, 235.9)
\end{aligned}
$$

4.3 Solving FFNLPP with trapezoidal fuzzy number using interval programming approach

In this section we will show how the interval programming will be used to Solve the FFNLPP and with trapezoidal fuzzy number .

first we will convert the FFNLPP (0.1) using alpha cut for trapezoidal fuzzy number in definition (0.1) to the INPP :

$$
\begin{aligned}
&\max f(x) = \sum_{j=1}^{n} \left[\underline{c_j}, \overline{c_j}\right] f_j(x)\\
&\text{Subject to}\\
&\sum_{j=1}^{n} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] g_{jj}(x) \leq \left[\underline{b_i}, \overline{b_i}\right] \quad , \quad i = 1, 2, \ldots m\\
&x \geq 0
\end{aligned} \tag{4.13}
$$

in which

$$\underline{c_j} = \left(c_j^1 + \alpha\left(c_j^2 - c_j^1\right)\right.$$
$$\overline{c_j} = \left(c_j^4 - \alpha\left(c_j^4 - c_j^3\right)\right.$$
$$\underline{a_{ij}} = {a_{ij}}^1 + \alpha\left(a_{ij}^2 - a_{ij}^1\right)$$
$$\overline{a_{ij}} = \left(a_{ij}^4 - \alpha\left(a_{ij}^4 - a_{ij}^3\right)\right)$$
$$\underline{b_i} = b_j^1 + \alpha\left(b_j^2 - b_j^1\right)$$
$$\overline{b_i} = b_j^4 - \alpha\left(b_j^4 - b_j^3\right)$$

Set $\alpha = 0$ in problem (4.13) we get :

$$\max \quad z = \sum_{j=1}^{n} \left[c_j^1, {c_j}^4\right] f_j(x)$$

Subject to

$$\sum_{j=1}^{n} \left[a_{ij}^1, a_{ij}^4\right] g_j(x) \leq \left[b_j^1, b_j^4\right]$$
$$x \geq 0$$

from this problem we will get two problems as problems (2) and (3) which will be Solved to get two solutions which are :

$$\underline{x}^* = (\underline{x}_1^*, \underline{x}_2^*, \ldots . \underline{x}_n^*), \overline{x}^* = (\overline{x}_1^*, \overline{x}_2^*, \ldots \overline{x}_n^*)$$

and the optimal values are $\bar{z}^*$ and $\underline{z}^*$ respectively

Setting $\alpha = 1$ in problem (4.13) we get the following problem :

$$\max z = \sum_{j=1}^{n} \left[c_j^2, c_j^3\right] f_j(x)$$

subject to

$$\sum_{j=1}^{n} \left[a_{ij}^2, a_{ij}^3\right] g_j(x) \leq \left[g_j^2, b_j^3\right] \quad i = 1, \ldots, m$$
$$x \geq 0$$

from this problem we will get two problems as problems (2) and (3) which will be solved to get two solutions which are :

$$\underline{x} = (\underline{x}_1, \underline{x}_2, \dots \underline{x}_n) \quad , \quad \overline{x} = (\overline{x}_1, \overline{x}_2, \dots \overline{x}_n)$$

and the optimal values are $\underline{z}$ and $\overline{z}$· respective.

hence the optimal solution for problem (4.13) is

$$\widetilde{x} = \left\{ \left(\underline{x_1^*}, \underline{x_1}, \overline{x_1}, \overline{x_1^*}\right), \left(\underline{x_2^*}, \underline{x_2}, \overline{x_2}, \overline{x_2^*}\right), \dots \left(\underline{x_n^*}, \underline{x_n}, \overline{x_n}, \overline{x_n^*}\right) \right\}$$
$$\widetilde{z} = \left(\underline{z^*}, \underline{z}, \overline{z}, \overline{z^*}\right)$$

Example 4.3.1 *Consider the following FFNLPP with trapezoidal fuzzy number.*

$$\begin{aligned}
&max\ \widetilde{z} = (0.5, 0.75, 1.25, 1.5) \otimes \widetilde{x} \oplus (0.5, 0.75, 1.25, 1.5) \otimes \widetilde{y}^4 \\
&Subject\ to \\
&(1, 2, 4, 5) \otimes \widetilde{x} \oplus (1, 1.5, 3, 3.5) \otimes \widetilde{y}^3 \leq (11, 13, 15, 17) \\
&(0.5, 0.75, 1.25, 1.5) \otimes \widetilde{x} \oplus (05, 0.75, 1.25, 1.5) \otimes \widetilde{y} \leq (1.5, 25, 3.5, 45) \\
&\widetilde{x}, \widetilde{y} \geq 0
\end{aligned}$$

converting the problem to an interval programming problem using alpha cut we get:

$$\begin{aligned}
&\max z = [0.5 + \alpha(0.25), 1.5 - \alpha(0.25)]x + [0.5 + \alpha(0.25), 1.5 - \alpha(0.25)]y^4 \\
&\text{Subject to} \\
&[1 + \alpha(1), 5 - \alpha]x + [1 + 0.5\alpha, 3.5 - 0.5\alpha]y^3 \leq [11 + 2\alpha, 17 - 2\alpha] \\
&[0.5 + 0.25\alpha, 1.5 - 0.25\alpha]x - [0.5 + 0.25\alpha, 1.5 - 0.25\alpha]y \leq [1.5 + \alpha, 4.5 - \alpha] \\
&x, y \geq 0
\end{aligned}$$

Set $\alpha = 1$ we get:

$$\begin{aligned}
&\max z = [0.75, 1.25]x + [0.75, 1.25]y^4 \\
&\text{subject to} \\
&[2.4]x + [1.5, 3]y^3 \leq [13.15] \\
&[0.75, 1.25]x - [0.75, 1.25]y \leq [2.5, 3.5] \\
&x, y \geq 0
\end{aligned}$$

from this problem we will get two problems The first problem is :

$$\max \bar{z} = 1.25\bar{x} + 1.25\bar{y}^4$$

subject to

$$2\bar{x} + 1.5\bar{y}^3 \leq 15$$

$$0.75\bar{x} - 1.25\bar{y} \leq 3.5$$

$$\bar{x}, \bar{y} \geq 0$$

This is a crisp nonlinear programming that can be solved by separable programming procedure as follows:

$$f_1(\bar{y}) = \bar{y}^4 g_1(\bar{y}) = \bar{y}^3$$

from the first constraint we observe that $0 \leq \bar{y} \leq 3$.

k	a_k	$f_1(a_k) = a_k^4$	$g_1(y_1) = a_k^3$
1	0	0	0
2	1	1	1
3	2	16	8
4	3	81	27

$$f_1(\bar{y}) = 0w_{11} + w_{12} + 16w_{13} + 81w_{14}$$

$$g_1(\bar{y}) = 0w_{11} + w_{12} + 8w_{13} + 27w_{14}$$

substituting $f_1(\bar{y})$ and $g_1(\bar{y})$ in the problem we get:

$$max\bar{z} = 1.25\bar{x} + 1.25(\omega_{12} + 16\omega_{19} + 81\omega_{14})$$

subject to

$$2\bar{x} + 1.5(w_{12} + 8w_{13} + 27w_{14}) \leq 15$$

$$0.75\bar{x} - 1.25(w_{12} + 2w_{13} + 3w_{14}) \leq 3.5$$

$$w_{11} + w_{12} + w_{13} + w_{14} = 1$$

$$\bar{x}, w_{11}, w_{12}, w_{13}, w_{14} \geq 0$$

solving the problem we get :

$$\bar{x} = 0, \quad w_{11} = 0.63, \quad w_{12} = w_{19} = 0, \quad w_{14} = 0.37$$
$$\bar{y} = 3 \times w_{14} = 3 \times 0.37 = 1.11$$
$$\bar{z} = 1.25(0) + 1.25(1.11)^4 = 1.8976$$

The second problem is :

$$\begin{aligned} &\max \underline{z} = 0.75\underline{x} + 0.75\underline{y}^4 \\ &\text{subject to} \\ &4\underline{x} + 3\underline{y}^3 \leq 13 \\ &1.25\underline{x} - 0.75\underline{y} \leq 2.5 \\ &\underline{x}, \underline{y} \geq 0 \end{aligned}$$

This is also a crisp nonlinear programming problem that can be solved by separable programming.

substituting the values of $f_1(\underline{y}) = \underline{y}^4$ and $g_1(\underline{y}) = \underline{y}^3$ noting that $0 \leq \underline{y} \leq 2$ we get:

$$\begin{aligned} &\max \underline{z} = 0.75\underline{x} + 0.75\,(w_{12} + 16w_{13}) \\ &\text{subject to} \\ &4\underline{x} + 3\,(\omega_{12} + 8\omega_{13}) \leq 13 \\ &1.25\underline{x} - 0.75\,(\omega_{12} + 2\omega_{13}) \leq 2.5 \\ &w_{11} + w_{12} + w_{13} = 1 \\ &\underline{x}, w_{11}, w_{12}, w_{13} \geq 0 \end{aligned}$$

solving the problem we get :

$$\underline{x} = 0 \quad w_{11} = 0.458 \quad \omega_{12} = 0 \quad w_{13} = 0.542$$
$$\underline{y} \approx 2 \times 0.542 = 1.08$$
$$\underline{z} = 0.75(0) + 0.75(1.08)^4 = 1.02$$

now set $\alpha = 0$ we will get:

$$\begin{aligned}
&\max z = [0.5,1.5]x + [0.5,1.5]y^4 \\
&\text{ subject to} \\
&[1,5]x + [1,3.5]y^3 \leq [11,17] \\
&[0.5,1,5]x - [0.5,1.5]y \leq [1.5,4.5] \\
&x, y \geq 0
\end{aligned}$$

from this interval programming problem we will get two problems, the first problem is

$$\begin{aligned}
&\max \overline{z}^* = 1.5\overline{x}^* + 1.5\overline{y}^{*4} \\
&\text{ subject to} \\
&\overline{x}^* + \overline{y}^{*3} \leq 17 \\
&\ 0.5\overline{x}^* - 1.5\overline{y}^* \leq 4.5 \\
&\overline{x}^*, \overline{y}^* \geq 0
\end{aligned} \tag{4.14}$$

this nonlinear programming problem will be solved by separable programming procedure:

we observe from the first constraint that $0 \leq \overline{y}^* \leq 3$

$$\begin{aligned}
&f_1(\overline{y}^*) = \overline{y}^{*4}, g_1(\overline{y}^*) = \overline{y}^{*3} \\
&f_1(\overline{y}^*) = 0w_{11} + w_{12} + 16w_{13} + 81w_{14} \\
&g_1(\overline{y}^*) = 0w_{11} + w_{12} + 8w_{13} + 27w_{14}
\end{aligned}$$

Substituting these values in the problem (4.14) we get :

$$\begin{aligned}
&max\ \overline{z}^* = 1.5\overline{x}^* + 1.5(w_{12} + 16w_{13} + 81w_{14}) \\
&\text{ subject to} \\
&\overline{x}^* + (w_{12} + 8w_{13} + 27w_{14}) \leq 17 \\
&0.5\overline{x}^* - 1.5(w_{12} + 2w_{13} + 3w_{14}) \leq 4.5 \\
&w_{11} + w_{12} + w_{13} + w_{14} = 1 \\
&\overline{x}^*, w_{11}, w_{12}, w_{13}, w_{14} \geq 0
\end{aligned}$$

solving this problem we get :

$$\bar{x}^* = 0 \quad w_{11} = 0.37, \quad w_{12} = w_{13} = 0, \quad w_{14} = 0.63$$
$$\bar{x}^* \approx 3 \times 0.63 = 1.89$$
$$\bar{x}^* = 1.5(0) + 1.5(1.89)^4 = 19.7599$$

The second problem is :

$$\begin{aligned}
&max\ \underline{z}^* = 0.5\underline{x}^* + 0.5\underline{y}^{*^4} \\
&\text{subject to} \\
&5\underline{x}^* + 3.5\underline{y}^{*^3} \leq 11 \\
&1.5\underline{x}^* - 0.5\underline{y}^* \leq 1.5 \\
&\underline{x}^*, \underline{y}^* \geq 0
\end{aligned}$$

Solving by separable programming procedure. noting that $0 \leq \underline{y}^* \leq 2$ (from the second constraint) ,the problem will be:

$$\begin{aligned}
&max\ \underline{z}^* = 0.5\underline{x}^* + 0.5\left(w_{12} + 16w_{13}\right) \\
&\text{subject to} \\
&5\underline{x}^* + 3.5\left(w_{12} + 8w_{13}\right) \leq 11 \\
&1.5\underline{x}^* - 0.5\left(w_{12} + 2w_{13}\right) \leq 1.5 \\
&w_{11} + w_{12} + w_{13} = 1 \\
&\underline{x}^*, w_{11}, w_{12}, w_{13} \geq 0
\end{aligned}$$

solving the problem we get :

$$\underline{x}^* = 0, \quad w_{11} = 0.61, \quad w_{13} = 0.39, \quad w_{12} = 0$$
$$\underline{y}^* = 2 \times 0.39 = 0.78$$
$$\underline{z}^* = 0.01$$

hence , the optimal solution for the original problem is :

$$\widetilde{x} = (0, 0, 0, 0), \quad \widetilde{y} = (0.78, 1.02, 1.11, 1.89)$$
$$\widetilde{z} = (0.01, 1.02, 1.89, 19.7599)$$

4.4 Solving FFNLPP with hexagonal fuzzy number using interval programming approach

In this section we will show how the interval programming will be used to solve the FFNLPP with hexagonal fuzzy number and we will show a numerical example.

First we will convert the FFNLPP using the alpha cut of the hexagonal fuzzy number in definition(1.2.5) to an INPP as follows :

$$\begin{aligned} &\max f(x) = \sum_{j=1}^{n} \left[\underline{c_j}, \overline{c_j}\right] f_j(x) \\ &\text{subject to} \\ &\sum_{j=1}^{n} \left[\underline{a_{ij}}, \overline{a_{ij}}\right] g_j(x) \leq \left[\underline{b_i}, \overline{b_i}\right] \\ &x \geq 0 \end{aligned} \tag{4.15}$$

for $j = 1, \ldots, n$ and $i = 1, \ldots m$

$$\underline{c_j} = \begin{cases} 2\alpha\left(c_j^2 - c_j^1\right) + c_j^1 & \alpha \in [0, 0.5) \\ 2\alpha\left(c_j^3 - c_j^2\right) - c_j^3 + 2c_j^2 & \alpha \in [0, 5, 1] \end{cases}$$

$$\overline{c_j} = \begin{cases} -2\alpha\left(c_j^6 - c_j^5\right) + c_j^6 & \alpha \in [0, 0.5) \\ -2\alpha\left(c_j^5 - c_j^4\right) + 2c_j^5 - c_j^4 & \alpha \in [0.5, 1] \end{cases}$$

$$\underline{a_{ij}} = \begin{cases} 2\alpha\left(a_{ij}^2 - a_{ij}^1\right) + a_{ij}^1 & \alpha \in [0, 0.5) \\ 2\alpha\left(a_{ij}^3 - {a_{ij}}^2\right) - a_{ij}^3 + 2a_{ij}^2 & \alpha \in [0.5, 1] \end{cases}$$

$$\overline{a_{ij}} = \begin{cases} -2\alpha\left(a_{ij}^6 - a_{ij}^5\right) + a_{ij}^6 & \alpha \in [0, 0.5) \\ -2\alpha\left(a_{ij}^5 - {a_{ij}}^4\right) + 2a_{ij}^5 - {a_{ij}}^4 & \alpha \in [0, 5, 1] \end{cases}$$

$$\underline{b_i} = \begin{cases} 2\alpha\left(b_j^2 - b_j^1\right) + b_j^1 & \alpha \in [0, 0.5) \\ 2\alpha\left(b_j^3 - b_j^2\right) - b_j^3 & \alpha \in [0.5, 1] \end{cases}$$

$$\overline{b_i} = \begin{cases} -2\alpha\left(b_j^6 - b_j^5\right) + b_j^6 & \alpha \in [0, 0.5) \\ -2\alpha\left(b_j^5 - b_j^4\right) + 2b_j^5 - b_j^4 & \alpha \in [0.5, 1] \end{cases}$$

set $\alpha = 0$ in problem (4.15) we get

$$\max z = \sum_{j=1}^{n} \left[c_j^1, c_j^6\right] f_j(x)$$

subject to

$$\sum_{j=1}^{n} \left[a_{ij}^1, a_{ij}^6\right] g_j(x) \leq \left[b_i^1, b_i^6\right]$$

$$i = 1, \ldots, m$$

$$x \geq 0$$

from this problem we will get two problems which will be solved to get two solutions which are :

$$\underline{x^*} = \left(\underline{x_1^*}, \underline{x_2^*}, \ldots \underline{x_n^*}\right) \quad , \quad \overline{x^*} = \left(\overline{x_1^*}, \overline{x_2^*}, \ldots \overline{x_n^*}\right)$$

and the optimal values are $\underline{z^*}$ and $\overline{z^*}$ respectively .

setting $\alpha = 0.5$ in problem (4.15) wo get the following problems:

$$\max z = \sum_{j=1}^{n} \left[c_j^2, c_j^5\right] f_j(x)$$

subject to

$$\sum_{j=1}^{n} \left[a_{ij}^2, a_{ij}^5\right] g_j(x) \leq \left[b_i^2, b_i^5\right]$$

$$i = 1, \ldots, m$$

$$x \geq 0$$

from this problem we will get two problems as problems (2) and (3) which will be solved to get two solutions which are :

$$\underline{x'} = \left(\underline{x_1'}, \underline{x_2'}, \ldots \underline{x_n'}\right) \quad , \quad \overline{x'} = \left(\overline{x_1'}, \overline{x_2'}, \ldots \overline{x_n'}\right)$$

and the optimal values are $\underline{z'}$ and $\overline{z'}$ respectively.

setting $\alpha = 1$ in problem (4.15) we will get the following problem

$$\max z = \sum_{j=1} \left[c_j^3, c_j^4\right] f_j(x)$$

subject to

$$\sum_{j=1}^{n} \left[a_{ij}^3, a_{ij}^4\right] g_j(x) \leq \left[b_j^3, b_j^4\right]$$

$$i = 1, \ldots, m$$

$$x \geq 0$$

from this problem we will get two problems a problem (2) and (3) that can be solved, to get two solutions which are:

$$\underline{x''} = \left(\underline{x_1''}, \underline{x_2''}, \ldots \underline{x_n''}\right) \quad , \quad \overline{x''} = \left(\overline{x_1''}, \overline{x_2''}, \ldots \overline{x_n''}\right)$$

and the optimal values are $\underline{z''}$ and $\overline{z''}$ respectively.

hence the optimal Solution for problem (4.15) is :

$$x = \left\{ \ \left(\underline{x_1^*}, \underline{x_1'}, \underline{x_1''}, \overline{x_1''}, \overline{x_1'}, \overline{x_1^*}\right), \left(\underline{x_2^*}, \underline{x_2'}, \underline{x_2''}, \overline{x_2''}, \overline{x_2'}, \overline{x_2^*}\right), \ldots \left(\underline{x_n^*}, \underline{x_n'}, \underline{x_n''}, \overline{x_n''}, \overline{x_n'}, \overline{x_n^*}\right) \ \right\}$$

The objective value is

$$z = \left(\underline{z^*}, \underline{z'}, \underline{z''}, \overline{z''}, \overline{z'}, \overline{z^*}\right)$$

Example 4.4.1 *Consider the following FFNLPP with hexagonal fuzzy number*

$$max\widetilde{z} = (0.25, 0.5, 0.8, 1.2, 1.5, 1.75) \otimes \widetilde{x}^3 \oplus (0.2, 0.5, 0.8, 1.2, 1.5, 1.8) \otimes \widetilde{y}$$

subject to

$$(1, 2, 3, 4, 6, 8) \otimes \widetilde{x}^2 \oplus (0.5, 1, 1.5, 2.5, 3, 3.5) \otimes \widetilde{y} \leq (15, 17, 19, 21, 23, 25)$$

$$(0.25, 0.5, 0.8, 1.2, 1.5, 1.75) \otimes \widetilde{y} \ominus (0.2, 0.5, 0.8, 1.2, 1.5, 1.8) \otimes \widetilde{x} \leq (5, 6, 7, 8, 10, 12)$$

$$\widetilde{x}, \widetilde{y} \geq 0$$

first , converting the problem to an interval programming using alpha cut for hexagonal fuzzy number we get :

(i) for $\alpha \in [0,0,5)$:

$$\max z = [0.5\alpha+0.25, -0.5\alpha+1.75]x^3 + [0.6\alpha+0.2, -0.6\alpha+1.8]y$$
subject to
$$[2\alpha+1, -4\alpha+8]x^2 + [2\alpha+0.5, -2\alpha+3,5]y \leq [4\alpha+15) - 4\alpha+25]$$
$$[0.5\alpha+0.25, -0.5\alpha+1.75]y - [0.6\alpha+0.2, -0.6\alpha+1.8]x \leq [2\alpha+5, -4\alpha+12]$$
$$x, y \geq 0$$

(ii) for $\alpha \in [0.5, 1]$:

$$\max z = [0.6\alpha+0.2, -0.6\alpha+1.8]x^2 + [0.6\alpha+0.2, -0.6\alpha+1.8]y$$
subject to
$$[2\alpha+1, -4\alpha+8]x^2 + [\alpha+0.5, -\alpha+3.5]y \leq [4\alpha+15, -4\alpha+25]$$
$$[0.6\alpha+0.2, -0.6\alpha+1.8]y - [0.6\alpha+0.2, -0.6\alpha+1.8]x \leq [2\alpha+5, -4\alpha+12]$$
$$x, y \geq 0$$

when $\alpha = 0$ we will get the following interval programming problem

$$\max z = [0.25, 1.75]x^3 + [0.2, 1.8]y$$
subject to
$$[1,8]x^2 + [0.5, 3.5]y \leq [15, 25]$$
$$[0.25, 1.75]y - [0.21.8]x \leq [5, 12]$$
$$x, y \geq 0$$

Thus, we will get two problems The first problem is :

$$\max \bar{z}^* = 1.75\bar{x}^{*^3} + 1.8\bar{y}^*$$
subject to
$$\bar{x}^{*^2} + 0.5\bar{y}^* \leq 25$$
$$0.25\bar{y}^* - 1.8\bar{x}^* \leq 12$$
$$\bar{x}^*, \bar{y}^* \geq 0$$

solving by separable programming procedure and assuming that

$\overline{x}^* = 0w_{11} + w_{12} + 2w_{13} + 3w_{14} + 4w_{15} + 5w_{16}$, we get :

$$\overline{y}^* = 0$$
$$w_{11} = w_{12} = w_{13} = w_{15} = 0, w_{16} = 1$$
$$\overline{x}^* \approx 5$$
$$\overline{z}^* = 218.75$$

The second problem is :

$$\max \underline{z}^* = 0.25\underline{x}^{*^3} + 0.2\underline{y}^*$$
$$\text{subject to}$$
$$8\underline{x}^{*^2} + 3.5\underline{y}^* \leq 15$$
$$1.75\underline{y}^* - 0.2\underline{x}^* \leq 5$$
$$\underline{x}^*, \underline{y}^* \geq 0$$

Solving by separable programming procedure assuming that $\underline{x}^* = 0w_{11} + w_{12} + 2w_{13}$, we get :

$$\underline{y}^* = 0$$
$$w_{11} = 0.53, \quad w_{12} = 0, \quad w_{13} = 0.47$$
$$\underline{x}^* \approx 2 \times 0.47 = 0.94$$

when $\alpha = 0.5$ we will get the following interval programming :

$$\max z = [0.5, 1.5]x^3 + [0.5, 1.5]y$$
$$\text{subject to}$$
$$[2, 6]x^2 + [1, 3]y \leq [17, 23]$$
$$[0.5, 1.5]y - [0.5, 1.2]x \leq [6, 10]$$
$$x, y \geq 0$$

from this interval programming problem we will get two problems :

$$\begin{aligned}
&\max \overline{z}' = 1.5\overline{x}'^3 + 1.5\overline{y}' \\
&\text{subject to} \\
&2\overline{x}'^2 + \overline{y}' \leq 23 \\
&0.5\overline{y}' - 1.2\overline{x}' \leq 10 \\
&\overline{x}', \overline{y}' \geq 0
\end{aligned}$$

solving by separable programming procedure and assuming that
$\overline{x}' = 0w_{11} + w_{12} + 2w_{13} + 3w_{14} + 4w_{15}$, we get :

$$\begin{aligned}
&\overline{y}' = 0 \\
&w_{11} = 0.28, \quad w_{12} = w_{13} = w_{14} = 0, \quad w_{15} = 0.72 \\
&\overline{x}' = 2.88
\end{aligned}$$

The second problem is:

$$\begin{aligned}
&\max \underline{z}' = 0.5\underline{x}'^3 + 0.5\underline{y}' \\
&\text{subject to} \\
&6\underline{x}'^2 + 3\underline{y}' \leq 17 \\
&1.5\underline{y}' - 0.5\underline{x}' \leq 6 \\
&\underline{x}', \underline{y}' \geq 0
\end{aligned}$$

solving by separable programming procedure and assuming that
$\underline{x}' = 0w_{11} + w_{12} + 2w_{13}$, we get :

$$\begin{aligned}
&\underline{y}' = 4.13 \\
&w_{11} = 0.81, \quad w_{12} = 0, \quad w_{13} = 0.19 \\
&\overline{x}' = 0.38
\end{aligned}$$

When $\alpha = 1$

we will get the following interval programing problem :

$$\begin{aligned}
&\max z = [0.8, 1.2]x^3 + [0.8, 1.2]y \\
&\text{subject to} \\
&[3, 4]x^2 + [1.5, 2.5]y \leq [19, 21] \\
&[0.8, 1.2]y - [0.8, 1.2]x \leq [7, 8] \\
&x, y \geq 0
\end{aligned}$$

hence we will get two problems , the first problem is :

$$\begin{aligned}
&\max \overline{z}'' = 1.2\overline{x}''^3 + 1.2\overline{y}'' \\
&\text{subject to} \\
&3\overline{x}''^2 + 1.2\overline{y}'' \\
&0.8\overline{y}'' - 1.2\overline{x}'' \leq 8 \\
&\overline{x}'', \overline{y}'' \geq 0
\end{aligned}$$

solving by separable programming procedure and assuming that $\overline{x}'' = 0w_{11} + w_{12} + 2w_{13} + 3w_{14}$, we get :

$$\begin{aligned}
&\overline{y}'' = 0 \\
&w_{11} = 0.22, \quad w_{12} = w_{13} = 0, \quad w_{14} = 0.78 \\
&\overline{x}'' = 2.34
\end{aligned}$$

The second problem is

$$\begin{aligned}
&\max \underline{z}'' = 0.8\underline{x}''^3 + 0.8\underline{y}'' \\
&4\underline{x}''^2 + 2.5\underline{y}'' \leq 19 \\
&1.2\underline{y}'' - 0.8\underline{x}'' \leq 7 \\
&\underline{x}'', \underline{y}'' \geq 0
\end{aligned}$$

solving by separable programming procedure and assuming that $\underline{x}'' = 0w_{11} + w_{12} + 2w_{13} + 3w_{14}$, we get :

$$\begin{aligned}
&\underline{y}'' = 0 \\
&w_{11} = 0.47, \quad w_{12} = w_{13} = 0, \quad w_{14} = 0.53 \\
&\underline{x}'' = 1.59
\end{aligned}$$

hence , the optimal solution for the original problem is :

$$\widetilde{x} = (0.94, 0.38, 1.59, 2.34, 2.88, 5), \quad \widetilde{y} = (0, 4.13, 0, 0, 0, 0)$$
$$\widetilde{z} = (0.201, 2.1, 3.2, 5.4, 35.8, 218.75)$$

4.5 comparison between the ranking function method and the interval programming procedure

We observed that the interval programming method needs a large number of problems to be solved individually while that the ranking function method consists only of one problem to solve. In ranking function method the problem consists of large number of variables which may affect on each other while that each problem in the interval programming method consists of less number of variables which make each problem easier to solve.

comparing the results of the fuzzy variables between the ranking function method and the interval programming method (by calculating the ranking function value of each variable) we can see that the results are close to each other. the below tables show the comparison between the results .

In table (4.1) we have shown a comparison between the ranking function method and the interval programming procedure in terms of number of problems, number of variables and the level of complexity of the problems

Tab. 4.1: comparison between the ranking function method and the interval programming procedure

	Ranking Function method	Interval programming Procedure
number of problems	one problem	large number of problems
number of variables	large number of variables	less number of variables
the level of complexity of the problems	the problem is complex and needs much time to solve	each problems is very simple and easier to solve

In table (4.2) we compared the results of example(4.2.1) and example(5.2.1)

Tab. 4.2: triangular fuzzy number example

	Ranking Function method	Interval programming Procedure
$\widetilde{x}$	$(0, 0, 1.25)$	$(0, 0, 0)$
$\widetilde{y}$	$(2.31, 1.5, 2)$	$(1.16, 1.5, 2.75)$
$R(\widetilde{x})$	0.31	0
$R(\widetilde{y})$	1.83	1.73

In table (4.3) we compared the results of example(4.2.2) and example(5.3.1)

Tab. 4.3: trapezoidal fuzzy number example

	Ranking Function method	Interval programming Procedure
$\widetilde{x}$	$(0,0,0,0)$	$(0,0,0,0)$
$\widetilde{y}$	$(1.2,0.9896,1.25,1.2143)$	$(0.78,1.02,1.11,1.89)$
$R(\widetilde{x})$	0	0
$R(\widetilde{y})$	1.16	1.2

In table (4.4) we compared the results of example(4.2.3) and example(5.4.1)

Tab. 4.4: hexagonal fuzzy number example

	Ranking Function method	Interval programming Procedure
$\widetilde{x}$	$(3.748,2.832,2.111,1.749,1.27799)$	$(0.94,0.38,1.59,2.34,2.88,5)$
$\widetilde{y}$	$(0,0,0,0,0,7.142857)$	$(0,4.13,0,0,0,0)$
$R(\widetilde{x})$	1.95	2.19
$R(\widetilde{y})$	1.19	0.69

Conclusion

In this thesis we have discussed two methods for solving some types of fully fuzzy nonlinear programming problem, the first method is ranking function method that converts the FFNLPP to a crisp non linear programming problem that can be solved by existing methods.
The second method is the interval programming procedure which converts the FFNLPP to an interval programming problem using the alpha cut of each type of the fuzzy numbers and then solving it by the existing method for interval programming problems.

As a future work we may consider other types of fully fuzzy non linear programming problems, and we may use other methods to solve the fully fuzzy nonlinear programming problem and make a comparison between the new methods .

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الملخص

في هذه الأطروحة، استخدمنا طريقتين لحل نوعين من مشاكل البرمجة غير الخطية الضبابية بالكامل. الطريقة الأولى كانت طريقة اقتران الترتيب، والتي تستخدم لحل بعض أنواع مشاكل البرمجة غير الخطية الضبابية بالكامل التي تلبي شروط كاروش كون توكر والمشاكل غير الخطية القابلة للفصل. في الطريقة الأولى، يتم تحويل مشكلة البرمجة غير الخطية الضبابية بالكامل إلى مشكلة برمجة واضحة بواسطة اقتران الترتيب.

في الطريقة الثانية، تم تحويل مشكلة البرمجة القابلة للفصل إلى مشكلة برمجة فاصلة. نوضح كيف تعمل الإجراءات في حالة وجود رقم ضبابي مثلثي وقمنا بتعميم الطريقة في حالة وجود أرقام ضبابية شبه منحرفة وسداسية، وتم شرح كل طريقة بالتفصيل وقد أظهرنا أمثلة عددية لكل طريقة في حالة وجود أرقام ضبابية مثلثية وشبه منحرفة وسداسية. أخيرًا، أظهرنا مقارنة بين الطريقتين من حيث عدد المعادلات في كل طريقة ونتائج مستوى التعقيد للأمثلة التي حصلنا عليها من الطريقتين وأظهرنا مدى قربهما.