# AN EFFICIENT MATRIX METHOD FOR COUPLED SYSTEMS OF VARIABLE FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

by

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*We establish a powerful numerical algorithm to compute numerical solutions of coupled system of variable fractional order differential equations. Our numerical procedure is based on Bernstein polynomials. The mentioned polynomials are non-orthogonal and have the ability to produce good numerical results as compared to some other numerical method like wavelet. By variable fractional order differentiation and integration, some operational matrices are formed. On using the obtained matrices, the proposed coupled system is reduced to a system of algebraic equations. Using MATLAB, we solve the given equation for required results. Graphical presentations and maximum absolute errors are given to illustrate the results. Some useful features of our sachem are those that we need no discretization or collocation technique prior to develop operational matrices. Due to these features the computational complexity is much more reduced. Further, the efficacy of the procedure is enhanced by increasing the scale level. We also compare our results with that of Haar wavelet method to justify the usefulness of our adopted method.*

Key words: *variable fractional order differential equations, coupled system, Bernstein polynomials*

## Introduction

Fractional calculus is the generalized class of classical calculus whose foundation was given by Newton and Leibnitz. Different characteristics of fractional calculus have been sparked in last two decades. Due to these researchers have taken keen interest in the said area. Also the applications with particular attention the simulation of physical problems have got much popularity. To model many real life process, biological and engineering phenomenons have been rigourously investigated through fractional order models. For diverse application in the area bio medical engineering as well as in other scientific and engineering disciplines like mechanics, chemistry, visco-elasticity, signal and image handling phenomenon, financial mathematics, optimization theory, control theory, aerodynamics, electrodynamics of complex medium and polymer rheology, we refer few books as [1-3]. One feature that makes fractional

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order operators more papular is the preserving of memory and hereditary property. Further fractional differential operators are more flexible and hold greater degree of freedom. Therefore, in most cases the results of fractional order models are more accurate and efficient as compared to the integer order [4-6]. The most important property of fractional order differential is non-local property of the fractional operator where as integer order operators are local operators. Geometrically fractional order derivative of a function provides an accumulation which include the corresponding integer counter part as a special case. Inspired from the aforesaid features and advantages of fractional order operators, researchers have given much importance to the mentioned filed [7].

On the other hand, researchers continually investigating various aspects of fractional oeder differential equations (FODE) including existence results, numerical analysis as well stability and control theory by utilizing different tools of non-linear functional analysis and fixed point theory. One of the prominent area of research in fractional calculus is devoted to investigate uniqueness and existence of solutions for fractional order coupled systems. The said area has been extensively well investigated in last two decades [8-10]. On the other hand numerical aspects have also been very well investigated. As the analytical solution of every system is not possible due to the complex nature of fractional order derivative. Therefore, researchers have given attention compute approximate solutions by using various tools from numerical analysis [11]. For instance to compute numerical results, difference methods [12], perturbationols [13], eigen function procedure [14], transform methods [15], decomposition schemes [16], iteration techniques [17], collocation techniques [18], Tau methods [19], wavelet analysis [20] *etc*. have been applied very well. All these techniques have their own merits and demerits. However, for the better results about solutions to FODE, operational matrices have been constructed in order to obtain numerical solutions [21]. In all mentioned work the order of derivative has been considered constant fractional order.

Here we remark that variable order calculus has also got more popularity during last few years. Although the said area has been initiated by Samko and Ross [22]. Variable order differentiations and integrations are the natural extension of their real order counter part. Here the order can vary continuously as a function of either dependent or independent variables of differentiation or integration. The said extension in order is natural and it posses more flexibility than the traditional fractional order. In this regards recently various results including existence theory, numerical analysis and stability analysis have been established, we refer some as [23-26]. Also some authors have worked for variable FODE from numerical point of view like [27].

Inspired from the aforementioned vast applications of fractional calculus and numerical solutions for many phenomenons, in this article, operational matrices of variable order integration and differentiation have been constructed by using Bernstein polynomials (BP) [18]. The important features of the said polynomials are that they are non-orthogonal and hence produce very good numerical results as compared to other orthogonal polynomials. Further, BP have been introduced by a Russian mathematician Bernstein. Also BP have been used in the proof of famous Weierstrass approximation theorem. Also it is interesting that every continuous real valued function over closed interval $[a, b]$ can be approximated uniformly with the help of Bernstein basis. Also BP over $[0, 1]$ play important roles in the description of parametric curve increasingly used in computer graphics. To the best of our knowledge the operational matrix method that omit discretization and collocation via BP for variable FODE are very rarely used in paste. Therefore, we will establish an adequate algorithm for computing numerical solutions to coupled system of variable order FDE:

$$\begin{gathered}{}_0^c\mathrm{D}_t^{\theta(t)}v(t)=g_1(t,v(t),w(t)),\ t\in[0,1]\\ {}_0^c\mathrm{D}_t^{\theta(t)}w(t)=g_2(t,v(t),w(t)),\ t\in[0,1]\\ v(0)=v_0,\ w(0)=w_0\end{gathered}\tag{1}$$

where $0<\theta(t)\leq 1$ and:

$$\begin{gathered}{}_0^c\mathrm{D}_t^{\theta(t)}v(t)=g_1(t,v(t),w(t)),\ t\in[0,1]\\ {}_0^c\mathrm{D}_t^{\theta(t)}w(t)=g_2(t,v(t),w(t)),\ t\in[0,1]\\ v(0)=v_0,\ v'(0)=v_1\ w(0)=w_0,\ w'(0)=w_1\end{gathered}\tag{2}$$

with $1<\theta(t)\leq 2$ and $g_i(i=1,2)$: $[0, 1]\times R\times R\rightarrow R$ are linear continuous functions.

By using BP operational matrices of fractional integration and differentiation, the considered system is reduced to some algebraic system of equations. By computational software, we solve the systems and hence receive the corresponding numerical solution the proposed problems. We treat different version of function $g_i(i=1,2)$ under different conditions for numerical solutions. Our numerical algorithm needs no discretization and collocation for constructing the required operational matrices. Further we treat various coupled systems for numerical solutions graphically. Our procedure accuracy can be increased by enlarging scale level.

## Preliminaries

Here we recollect some basic definitions [23, 28].

*Definition 1*. Let $\theta(t)>0$ and if $v\in L[0,T]$, then variable order integral of order $\theta$ is given:

$${}_0\mathrm{I}_t^{\theta(t)}[v(t)]=\frac{1}{\Gamma(\theta(t))}\int_0^t(t-\omega)^{\theta(\omega)-1}v(\omega)\mathrm{d}\omega\tag{3}$$

Let $v(t)=(t-2)^\mu$ be a function, then:

$${}_0\mathrm{I}_t^{\theta(t)}[(t-2)^\mu]=\frac{\Gamma(\mu+1)}{\Gamma(\mu+\theta(t)+1)}(t-p)^{\mu+\theta(t)}$$

*Definition 2*. If $v\in C[0,T]$, then variable order Caputo derivative for $0<\theta\leq 1$ is defined:

$${}_0^c\mathrm{D}_t^{\theta(t)}[v(t)]=\begin{cases}\dfrac{1}{\Gamma(1-\theta(t))}\displaystyle\int_0^t(t-\omega)^{-\theta(\omega)}v'(\omega)\mathrm{d}\omega\\ \dfrac{\mathrm{d}^n\mathrm{v}}{\mathrm{d}t^n},\quad\theta(t)=1\end{cases}\tag{4}$$

Further if $1<\theta\leq 2$, then:

$${}_0^c\mathrm{D}_t^{\theta(t)}[v(t)]=\begin{cases}\dfrac{1}{\Gamma(2-\theta(t))}\displaystyle\int_0^t(t-\omega)^{1-\theta(\omega)}v''(\omega)\mathrm{d}\omega\\ \dfrac{\mathrm{d}^2v}{\mathrm{d}t^2},\ \theta(t)=2\end{cases}\tag{5}$$

Let for instance:

$${}_0^c\mathrm{D}_t^{\theta(t)}[(t-p)^\gamma]=\frac{\Gamma(\theta(t)+1)}{\Gamma(\gamma-\theta(t)+1)}(t-p)^{\gamma-\theta(t)}$$

*Definition 3*. [29] The BP are defined:

$$\mathrm{P}_{j,m}(t)=\binom{m}{j}t^{j}(1-t)^{m-j},\ \ j=0,1,2,\cdots,k$$

which further in closed form can be expressed:

$$\mathrm{P}_{j,m}(t)=\sum_{i=0}^{m}\sum_{l=0}^{m-j}(-1)^{m}\binom{m}{j}\binom{m-j}{l}t^{l+j},\ \ j=0,1,2,\cdots,m$$

where

$$\mathrm{P}_{i,l,m}=(-1)^{m}\binom{m}{j}\binom{m-j}{j} \tag{6}$$

*Remark 1*. Since, BP are non-orthogonal, in this regard the inner product of two Bernstein basis [29] is given:

$$\int_{0}^{1}P_{i,m}(t)P_{j,m}(t)\mathrm{d}t=\begin{bmatrix}\Bbbk_{0,0} & \Bbbk_{0,1} & \cdots & \Bbbk_{0,r} & \cdots & \Bbbk_{0,m}\\ \Bbbk_{1,0} & \Bbbk_{1,1} & \cdots & \Bbbk_{1,r} & \cdots & \Bbbk_{1,m}\\ \Bbbk_{2,0} & \Bbbk_{2,1} & \cdots & \Bbbk_{2,r} & \cdots & \Bbbk_{2,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Bbbk_{r,0} & \Bbbk_{r,1} & \cdots & \Bbbk_{r,r} & \cdots & \Bbbk_{r,m}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Bbbk_{m,0} & \Bbbk_{m,1} & \cdots & \Bbbk_{m,r} & \cdots & \Bbbk_{m,m}\end{bmatrix}=\mathbf{G}_{\mathrm{K\times K}} \tag{7}$$

where $K = m + 1$.

*Remark 2*. [29] Any square integrable function $v \in L^2[0, 1]$, can be approximated in term of the said basis:

$$v(t)\approx\sum_{i=0}^{m}\mho_i\mathrm{P}_{i,m}(t) \tag{8}$$

Further we have:

$$\mathbf{R}_{1\times K}=\int_{0}^{1}\varpi(t)P_{j,m}(t)\mathrm{d}t=\int_{0}^{1}\sum_{i=0}^{m}\mho_i\mathrm{P}_{i,m}(t)\mathrm{P}_{j,m}(t)\mathrm{d}t=\sum_{i=0}^{m}\mho_i\int_{0}^{1}\mathrm{P}_{i,m}(t)\mathrm{P}_{j,m}(t)\mathrm{d}t \tag{9}$$

where $j = 0, 1, 2,...$ Therefore, we may write eq. (9):

$$\mathbf{R}_{1\times K}=\mathbf{S}_{1\times K}\odot\mathbf{G}_{K\times K} \tag{10}$$

where $\mathbf{S}_{1xK} = [\mho_0, \mho_1,..., \mho_m]$ is the coefficient matrix. Further one has:

$$\mathbf{S}_{1\times\mathrm{K}}=\mathbf{R}_{1\times K}\odot\mathbf{G}_{K\times K}^{-1} \tag{11}$$

*Lemma 1*. Let $P_K^T(t)$ be function vector defined in eq. (6), then the variable fractional order integration is given:

$${}_{0}I_{t}^{\theta(t)}\left[P_{K}^{T}(t)\right]=\bigcup_{K\times K}^{\theta(t)}\odot P_{K}^{T}(t) \tag{12}$$

where $\cup_{K\times K}^{\theta(t)}$ stands for operational matrix of variable order integration. Further:

$$\bigcup_{K\times K}^{\theta(t)} = \underset{K\times K}{\vee} \odot \mathbf{G}_{K\times K}^{-1}$$

where

$$\underset{K\times K}{\overset{\theta(t)}{\vee}} = \begin{bmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,r} & \cdots & \alpha_{0,k} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,r} & \cdots & \alpha_{1,k} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,r} & \cdots & \alpha_{2,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{r,0} & \alpha_{r,1} & \cdots & \alpha_{r,r} & \cdots & \alpha_{r,k} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \alpha_{k,0} & \alpha_{k,1} & \cdots & \alpha_{k,r} & \cdots & \alpha_{k,k} \end{bmatrix} \tag{13}$$

where by (6):

$$\alpha_{i,j} = \sum_{l=0}^{m-i}\sum_{c=0}^{m-j} \mathrm{P}_{i,l,m}\mathrm{P}_{j,c,m} \frac{\Gamma(m+i+1)}{(i+j+m+l+\nu+1)\Gamma(m+i+\theta(t)+1)} \tag{14}$$

*Proof 1*. Following the same fashion of [24, 29], we can easily proved the result.

*Lemma 2*. Let $P_K^T(t)$ be the vector function of BP, then the variable fractional order derivative of $P_K^T(t)$ is given by:

$${}_0^c\mathrm{D}_t^{\theta(t)}[\mathrm{P}_K^T(t)] = \mathbf{W}_{K\times K}^{\theta(t)} \odot \mathrm{P}_K^T(t) \tag{15}$$

where

$$\mathbf{W}_{K\times K}^{\theta(t)} = \mathbf{P}_{K\times K}^{\theta(t)} \odot \mathbf{G}_{K\times K}^{-1} \tag{16}$$

where

$$\mathbf{P}_{K\times K}^{\theta(t)} = \begin{bmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,r} & \cdots & \gamma_{0,m} \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,r} & \cdots & \gamma_{1,m} \\ \gamma_{2,0} & \gamma_{2,1} & \cdots & \gamma_{2,r} & \cdots & \gamma_{2,m} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{r,0} & \gamma_{r,1} & \cdots & \gamma_{r,r} & \cdots & \gamma_{r,m} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \gamma_{m,0} & \gamma_{m,1} & \cdots & \gamma_{m,r} & \cdots & \gamma_{m,m} \end{bmatrix} \tag{17}$$

where

$$\gamma_{i,j} = \begin{cases} \displaystyle\sum_{w=0}^{m-i}\sum_{z=0}^{m-j} P_{i,j,m}P_{j,i,m} \frac{\Gamma(j+i-\theta(t))}{\Gamma(w+i-\theta(t)+1)\Gamma(w+m+i+j-\theta(t)+1)}, & i < \max[\theta(t)] \\ \displaystyle\sum_{w=0}^{m-i}\sum_{z=0}^{m-j} P_{i,j,m}P_{j,i,k} \frac{\Gamma(j+i-\theta(t))}{\Gamma(z+i-\theta(t)+1)\Gamma(z+m+i+j-\theta(t)+1)}, & i >= \max[\theta(t)] \end{cases}$$

*Proof 2*. By same fashion of [24, 29] we can prove this results.

*Remark 3*. [24] Let $v \in L^2[0, 1]$ be $K + 1$times differentiable function and

$$P_K^T(t)=\{P_0(t),P_1(t),P_3(t),\cdots,P_m(t)\}$$

be the Bernstein Basis. Let

$$\sum_{i=0}^{m}\mho_i \mathrm{P}_{i,m}(t)$$

be the best approximation for $v(t)$:

$$\left\| v(t)-\sum_{i=0}^{m}\mho_i P_{i,m}(t)\right\| \le \sup_{t\in[0,1]}\frac{\sqrt{2}\,|\,v^{K+1}(t)\,|\,\mathcal{H}^{\frac{2K+1}{\mathcal{H}}}}{\sqrt{(2K+3)}\Gamma(K+2)} \tag{18}$$

and the given inequality present the maximum bound of error, where $\mathcal{H}=\max\{1-t_0,\ t_0\}$.

**General algorithms for coupled system of differential equations**

Here we describe a general algorithm for our considered system. Here we discuss two cases:

*Case I*: When $0<\theta\le 1$, then let from eq. (1), we consider:

$$ {}_0^c\mathrm{D}_t^{\theta(t)}v(t)=\mathbf{A}_\mathrm{K}\odot\mathbf{P}_K^T(t)$$
$$ {}_0^c\mathrm{D}_t^{\theta(t)}w(t)=\mathbf{B}_\mathrm{K}\odot\mathbf{P}_K^T(t)$$

Then inview of variable order integral ${}_0I_t^{\theta}(t)$ and using initial conditions, we have:

$$\begin{aligned} v(t)&=v_0+\mathbf{A}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\mathbf{P}_K^T(t)\\ w(t)&=\omega_0+\mathbf{B}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\mathbf{P}_K^T(t)\end{aligned} \tag{19}$$

Using approximation for initial values:

$$v_0\approx\mathbf{F}_K^{(1)}\odot P_K^T(t),\ \ w_0\approx\mathbf{F}_K^{(2)}\odot P_K^T(t) \tag{20}$$

Thank to approximation (20), eq. (19) becomes:

$$\begin{aligned} \mathrm{v}(t)&=\mathbf{F}_\mathrm{K}^{(1)}\odot\mathrm{P}_\mathrm{K}^T(t)+\mathbf{A}_\mathrm{K}\odot\bigcup\nolimits_{\mathrm{K\times K}}^{\theta(t)}\odot\mathrm{P}_\mathrm{K}^T(t)\\ w(t)&=\mathbf{F}_K^{(2)}\odot P_K^T(t)+\mathbf{B}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\odot P_K^T(t)\end{aligned} \tag{21}$$

Therefore, in terms of eq. (21), our considered system (1) yields:

$$\mathbf{A}_K\odot\mathbf{P}_K^T(t)=g_1\left(t,\mathbf{F}_K^{(1)}\odot P_K^T(t)+\mathbf{A}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\odot P_K^T(t),\mathbf{F}_K^{(2)}\odot\mathrm{P}_K^T(t)+\mathbf{B}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\odot P_K^T(t)\right)$$

$$\mathbf{B}_K\odot\mathbf{P}_K^T(t)=g_2\left(t,\mathbf{F}_K^{(1)}\odot\mathrm{P}_K^T(t)+\mathbf{A}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\odot P_K^T(t),\mathbf{F}_K^{(2)}\odot\mathrm{P}_K^T(t)+\mathbf{B}_K\odot\bigcup\nolimits_{K\times K}^{\theta(t)}\odot P_K^T(t)\right)$$

The concerned algebraic system can be solved to get the required solution.

*Case II*: If the order $1<\theta\le 2$, by using the following approximations:

$$v_0+v_1t\approx\mathbf{H}_K^{(1)}\odot\mathrm{P}_K^T(t),w_0+w_1t\approx\mathbf{H}_K^{(2)}\odot\mathrm{P}_K^T(t) \tag{22}$$

second system (2) yields:

$$\mathbf{A}_K \odot \mathbf{P}_K^T(t) = g_1\left(t, \mathbf{H}_\mathrm{K}^{(1)} \odot P_K^T(t) + \mathbf{A}_K \odot \bigcup_{K\times K}^{\theta(t)} \odot P_K^T(t), \mathbf{H}_K^{(2)} \odot P_K^T(t) + \mathbf{B}_K \odot \bigcup_{K\times K}^{\theta(t)} \odot P_K^T(t)\right)$$

$$\mathbf{B}_K \odot \mathbf{P}_K^T(t) = g_2\left(t, \mathbf{H}_K^{(1)} \odot \mathrm{P}_K^T(t) + \mathbf{A}_K \odot \bigcup_{K\times K}^{\theta(t)} \odot P_K^T(t), \mathbf{H}_K^{(2)} \odot \mathrm{P}_K^T(t) + \mathbf{B}_K \odot \bigcup_{K\times K}^{\theta(t)} \odot P_K^T(t)\right)$$

By using computational software we will solve these algebraic equations to get the required numerical solutions.

## Illustrative examples

To support our numerical findings, we here present some examples and their numerical interpretation.

*Example 1*. Consider the coupled system which is describing drug therapy in human body [30]:

$$\begin{aligned} {}_0^c\mathrm{D}_t^{\theta(t)} v(t) &= -0.09\mathrm{v} + 0.038w,\ 0 < \theta(t) \le 1 \\ {}_0^c\mathrm{D}_t^{\theta(t)} w(t) &= 0.066\mathrm{v} - 0.038w,\ 0 < \theta(t) \le 1 \\ v(0) &= v_0,\ \ w(0) = w_0 \end{aligned} \tag{23}$$

where amount of lidocaine in the bloodstream is denoted by $v$ and amount of lidocaine in body tissue by $w$. From physically significance point of view initial data is taken zero drug in the bloodstream and injection dosage = $w_0$. Analytical solution of the problem at $\theta = 1$ can be derived:

$$v(t) = -0.2367 w_0 \exp(-0.1204t) + 0.2696 w_0 \exp(-0.0076t)$$

$$w(t) = 0.2696 w_0 \exp(-0.1204t) + 0.7304 w_0 \exp(-0.0076t)$$

Here we present the comparison between exact and numerical solutions of the coupled system (23) in fig. 1. While in fig. 2, we give their absolute errors by taking $w_0 = 250$.

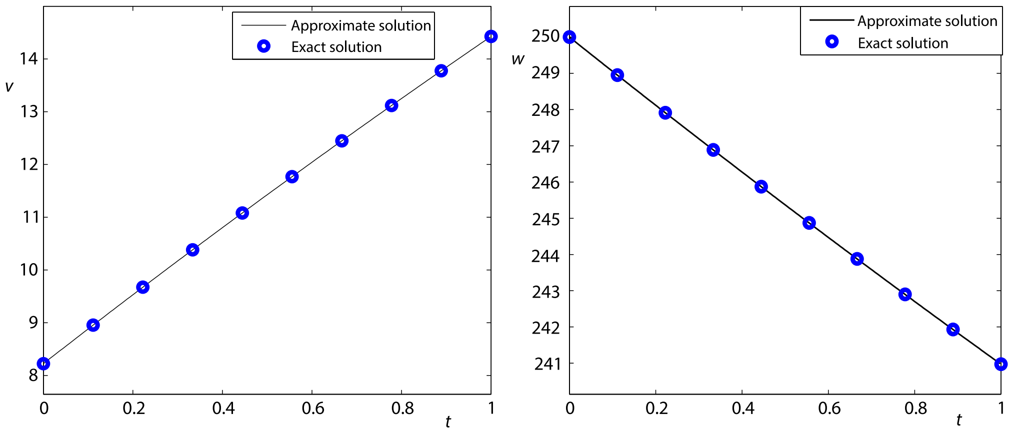


**Figure 1. Analytical and numerical solutions comparison of *v* and *w*, respectively at scale Level 6 of *Example 1***

Here we present the numerical solutions of the coupled system (23) in fig. 3 at some variable order. While in fig. 4, we give the absolute errors at different values of $t$ aking variable order $\theta = 1 - \exp(-t)$.

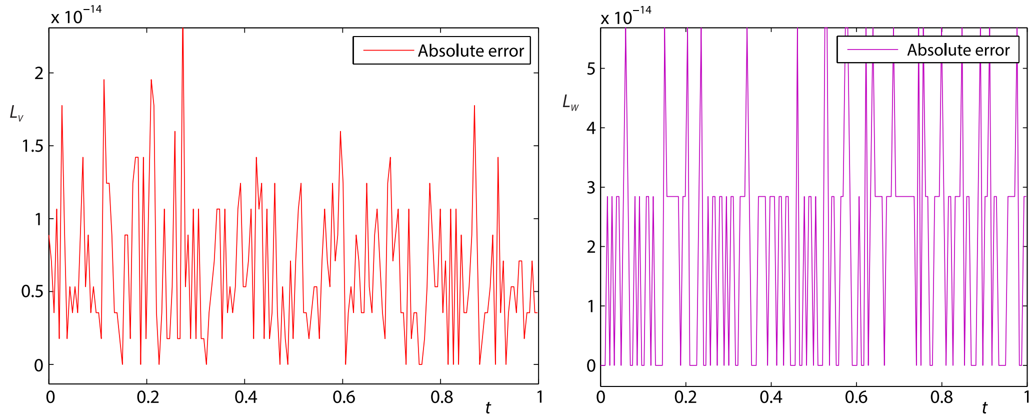


**Figure 2. Absolute errors in *v* and *w* at given scale level of *Example 1***

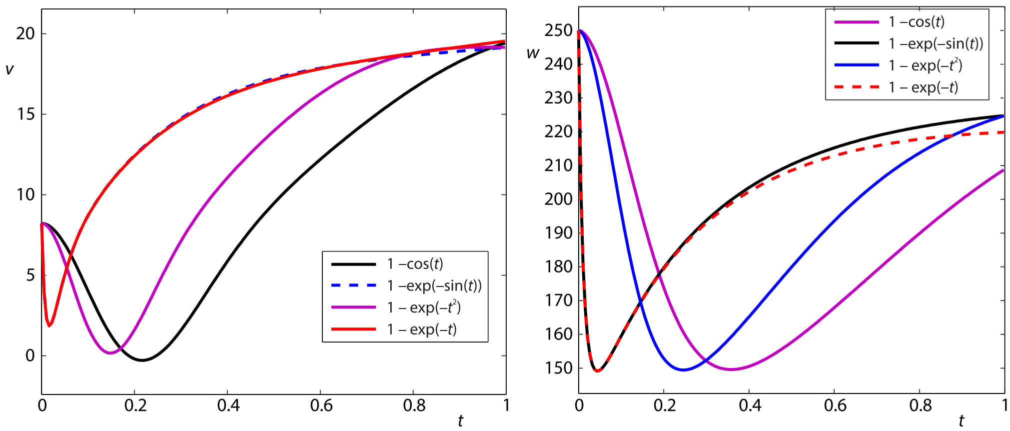


**Figure 3. Graphical presentation of numerical solutions of *v* and *w* at scale Level 6 and different variable orders of *Example 1*** (*for color image see journal web site*)

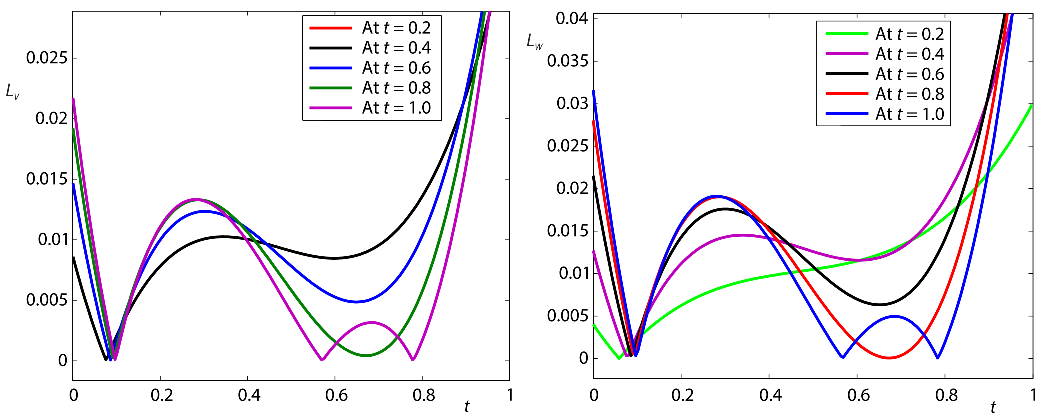


**Figure 4. Absolute errors in *v* and *w* at different values of *t* for variable order $\theta = 1 - \exp(-t)$ and scale Level 6 of *Example 1*** (*for color image see journal web site*)

*Example* 2. Consider the coupled system which is describing growth and decay model of radioactivity when some amount of radium $v$ decays at rate $a$, then as a results Radon creates with amount $w$. But Radon is unstable and decay at rate $b$, thus the phenomenon is modeled as [31]:

$$\begin{gathered}{}_{0}^{c}\mathrm{D}_{t}^{\theta(t)}v(t)=-av(t),\ \ 0<\theta\le 1\\ {}_{0}^{c}\mathrm{D}_{t}^{\theta(t)}w(t)=av(t)-bw(t),\ \ 0<\theta\le 1\\ v(0)=v_0,\ \ w(0)=0\end{gathered}\tag{24}$$

The exact solution of the problem at $\theta$ is given:

$$v(t)=v_0t^{\theta-1}E_{\theta,\theta}\left(-at^{\theta}\right),\ \ w(t)=\frac{av_0}{a-b}\left[E_{\theta,\theta}\left(-at^{\theta}\right)-E_{\theta,\theta}\left(-bt^{\theta}\right)\right]$$

Here we present the comparison between exact and numerical solutions of the coupled system (24) in fig. 5. Using $a=0.00043$, $b=66.14$, and $v_0=10000$ we check the approximation. While in fig. 6, we give their absolute errors at two different scale levels.

Here we present the comparison between exact and numerical solutions of the coupled system (24) in fig. 7. While in fig. 8, we give their absolute errors at different fractional orders.

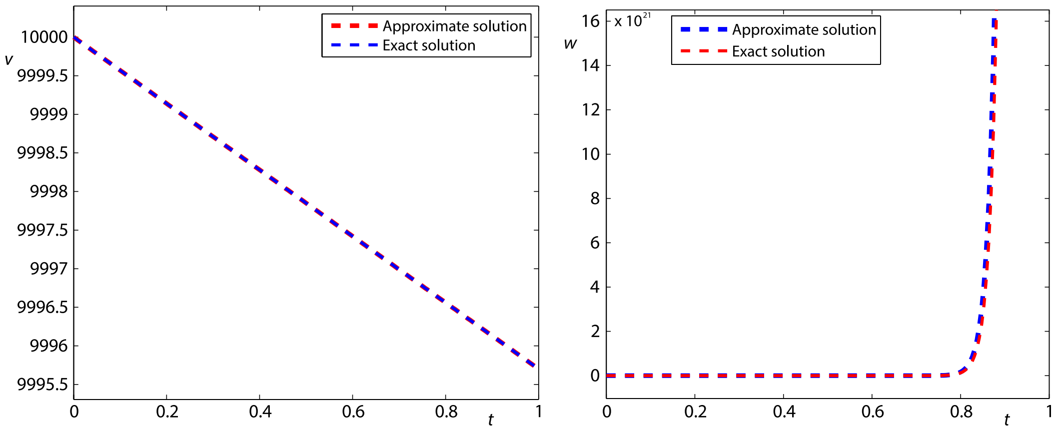


**Figure 5. Analytical and numerical solutions comparison for *v* and *w*, respectively at scale Level 6 of *Example 2***

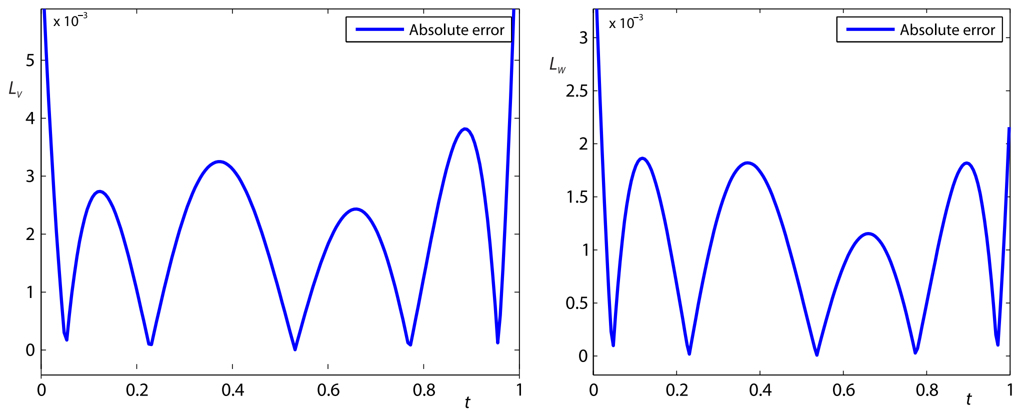


**Figure 6. Absolute errors in *v* and *w* at given scale Level 6 of *Example 2***

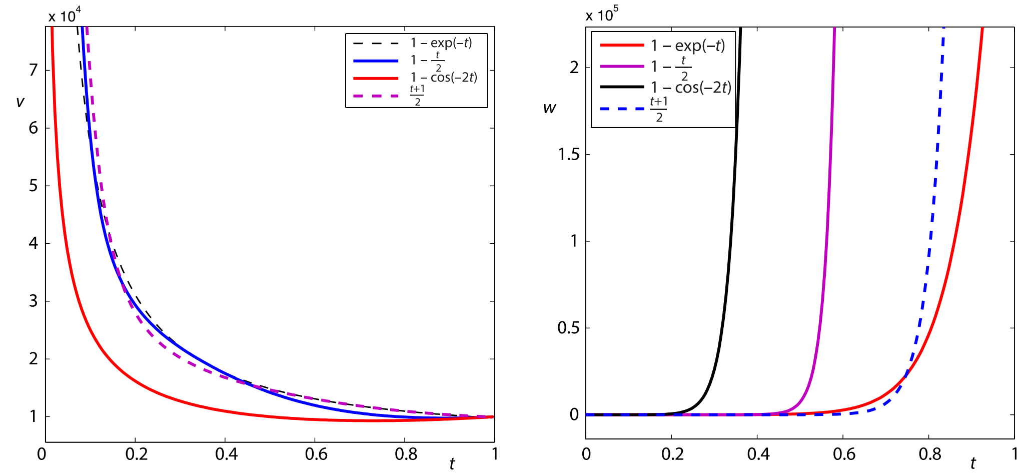


**Figure 7. Graphical presentation of numerical solutions of *v* and *w* at scale Level 6 and different variable orders of *Example 2*** (*for color image see journal web site*)

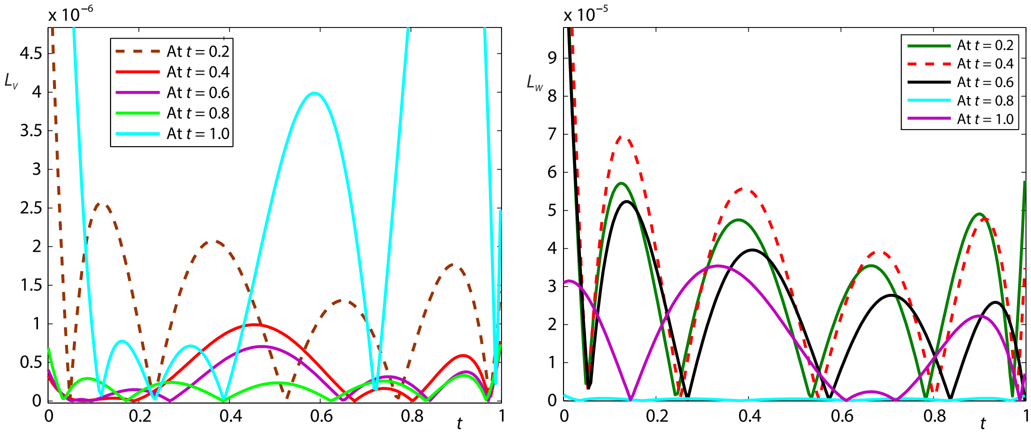


**Figure 8. Absolute errors in *v* and *w* at different values of *t* for variable order $\theta = 1 - t/2$ and scale Level 6 of *Example 2*** (*for color image see journal web site*)

*Example 3*. Consider the coupled system:

$$
\begin{gathered}
{}_0^c\mathrm{D}_t^{\theta(t)}v(t) = v + w + \frac{120t^{5-\theta(t)}}{\Gamma(6-\theta(t))} - t^5 - t^4, \ 1 < \theta \le 2 \\
{}_0^c\mathrm{D}_t^{\theta(t)}w(t) = v + w + \frac{24t^{4-\theta(t)}}{\Gamma(5-\theta(t))} - t^5 - t^4, \ 1 < \theta \le 2 \\
v(0) = 0, \ \ w(0) = 0, \ \ v'(0) = 0, \ \ w'(0) = 0
\end{gathered}
\tag{25}
$$

At $\theta(t) = 2$, the analytical solution can be deduced:

$$v(t) = t^5, \ \ w(t) = t^4$$

Here we present the comparison between exact and numerical solutions of the coupled system (25) in fig. 9. While in fig. 10, we give their absolute errors at two different scale levels.

Here we present the comparison between exact and numerical solutions of the coupled system (25) in figure 11. While in fig. 12, we give their absolute errors at different fractional orders.

Here we compared our numerical scheme with Haar wavelet method [32] in the case of given problem.

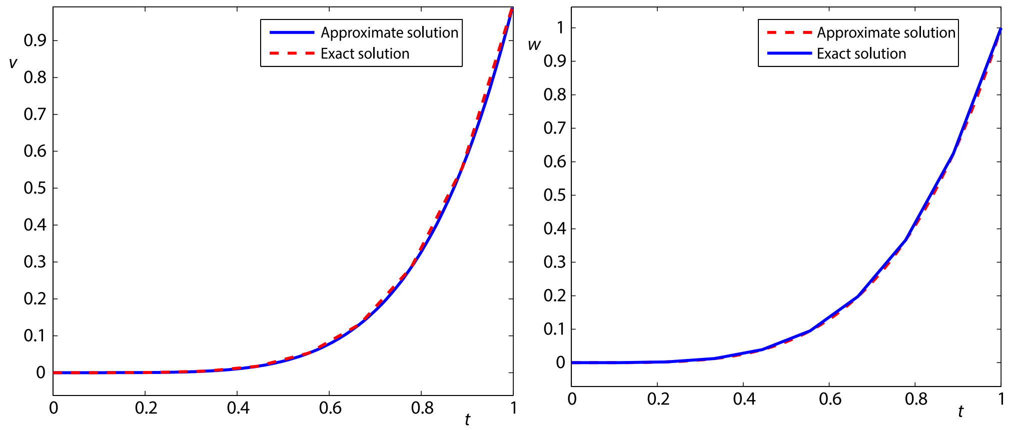


**Figure 9. Analytical and numerical solutions comparison of *v* and *w*, respectively at scale Level 6 of *Example 3***

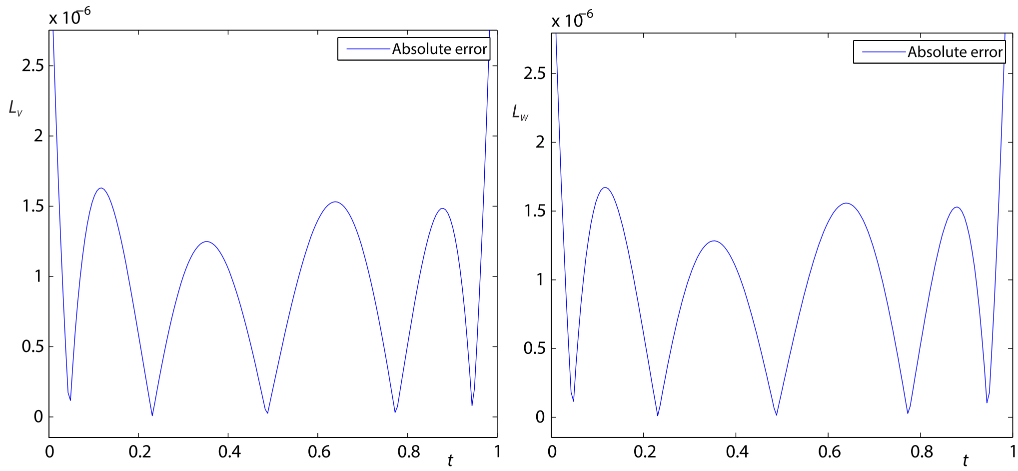


**Figure 10. Absolute errors in *v* and *w* at given scale Level 6 of *Example 3***

*Example 4*. Consider the coupled system:

$$\begin{gathered}{}_0^c\mathrm{D}_t^{\theta(t)}v(t)=v+w,\ \ 0<\theta(t)\le 1\\ {}_0^c\mathrm{D}_t^{\theta(t)}w(t)=-v+w,\ \ 0<\theta(t)\le 1\\ v(0)=0,\ \ w(0)=0\end{gathered}\tag{26}$$

At $\theta(t)=1$, the analytical solution can be deduced:

$$v(t)=\exp(t)\sin t,\ \ w(t)=\exp(t)\cos t$$

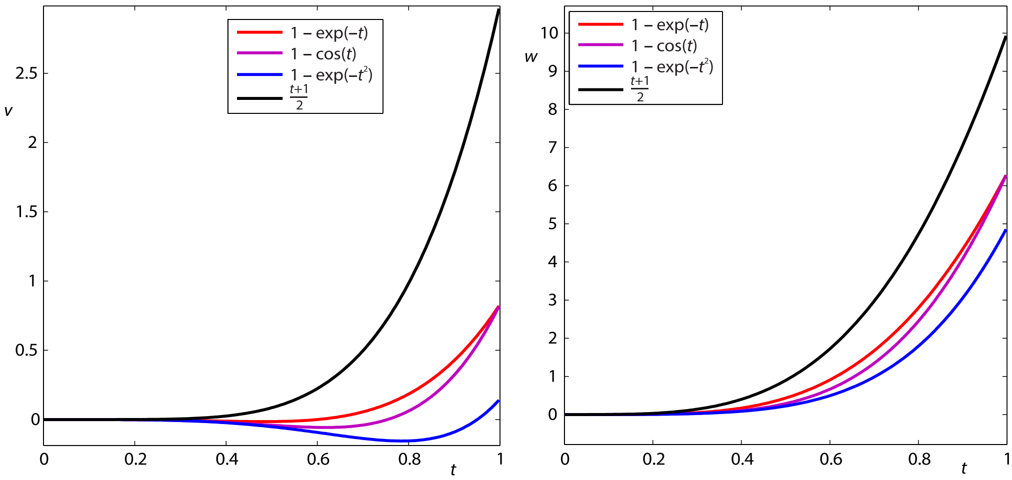


**Figure 11. Graphical presentation of numerical solutions of *v* and *w* at scale Level 6 and different variable orders of *Example 3*** (*for color image see journal web site*)

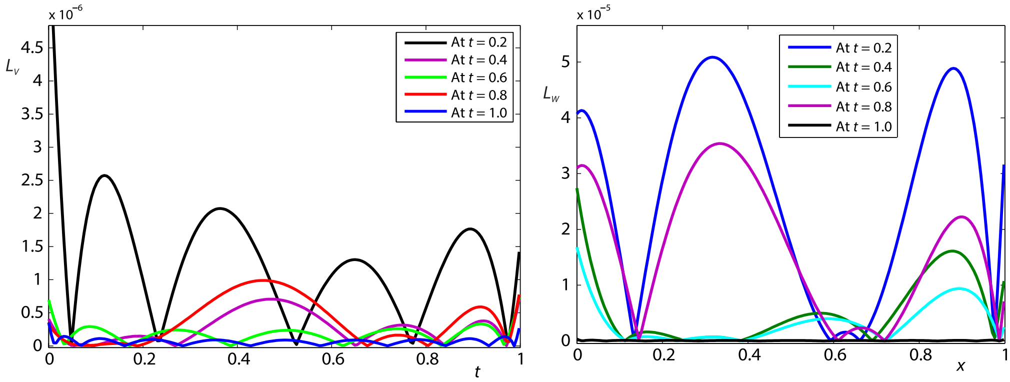


**Figure 12. Absolute errors in *v* and *w* at different values of *t* for $\theta = (t + 1)/2$ and scale Level 6 for *Example 3*** (*for color image see journal web site*)

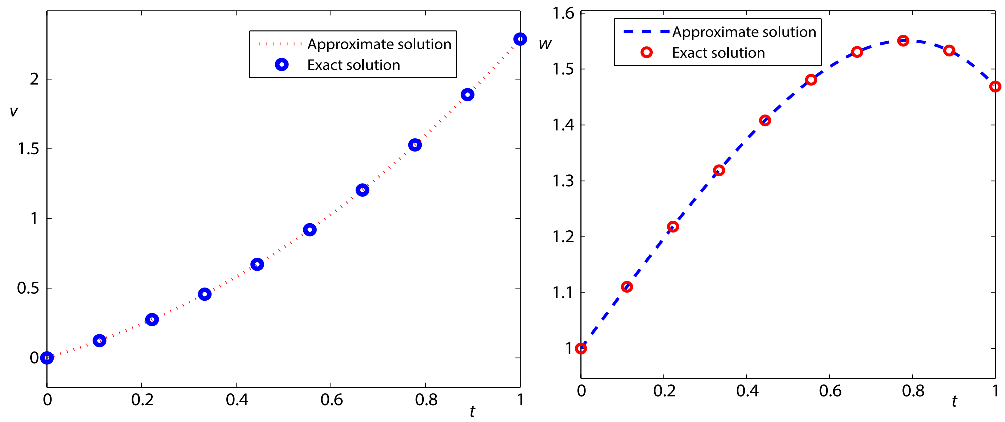


**Figure 13. Analytical and numerical solutions comparison for *v* and *w*, respectively at scale Level 6 of *Example 4***

Here we present the comparison between exact and numerical solutions of the coupled system (26) in fig. 13. While in fig. 14, we give their absolute errors at two different scale levels.

Here in fig. 15, we present the numerical solution for different variable order.

Here in fig. 16, we give absolute errors at different values of $t$ for taking fractional variable order $\theta = 1 - (t/2)$.

Here in tab. 1, we compared our numerical results with the computational results of Haar wavelet method for the given Problem 4. From the present table we see that our spectral method produces much more better results than the mentioned wavelet method. Further as compared to wavelet method the adopted numerical scheme is simple and easy to implement. Also the proposed spectral method apply to any linear problem has been proved stable and convergent, see [33].

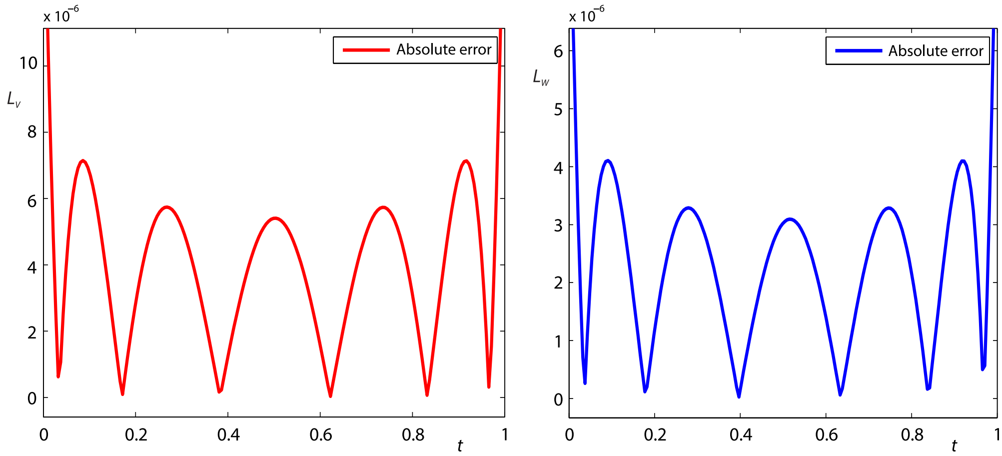


**Figure 14. Absolute errors in $v$ and $w$ at given scale Level 6 of *Example 4***

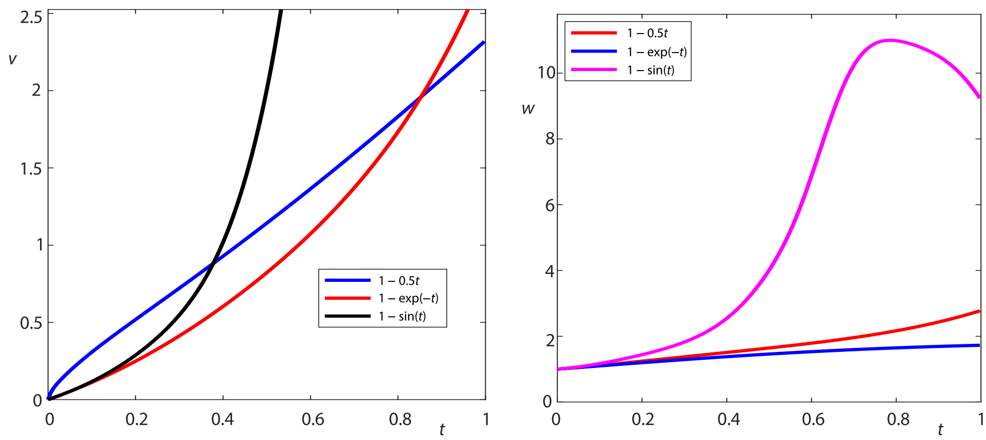


**Figure 15. Graphical presentation of numerical results for $v$ and $w$ at different variable fractional order and scale Level 6 for *Example 4*** (*for color image see journal web site*)

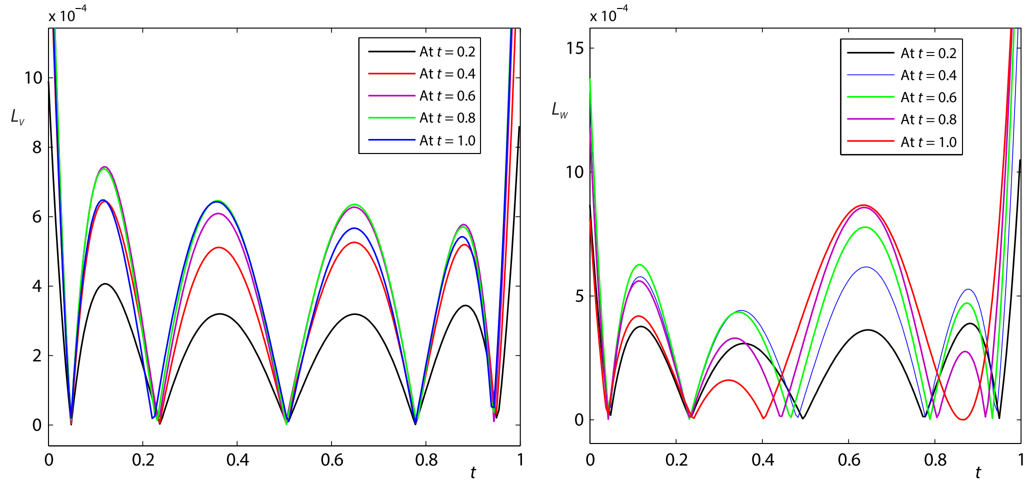


**Figure 16. Absolute errors in $v$ and $w$ at different values of $t$ for $\theta = 1 - (t/2)$ and scale Level 6 for *Example 4*** (*for color image see journal web site*)

**Table 1. Comparison between the absolute errors at proposed method for scale Level 10 and Haar wavelet method at collocation Points 32 for fixed order $\theta = 1$**

| Time | Proposed method | | Haar wavelet method [32] | |
|---|---|---|---|---|
| $t$ | $L_v$ | $L_w$ | $L_v$ | $L_w$ |
| 0.1 | $3.7995 \cdot 10^{-5}$ | $3.08750 \cdot 10^{-5}$ | $4.13150 \cdot 10^{-4}$ | $1.13860 \cdot 10^{-3}$ |
| 0.2 | $3.9876 \cdot 10^{-6}$ | $6.00987 \cdot 10^{-6}$ | $3.90876 \cdot 10^{-4}$ | $.8353 \cdot 10^{-3}$ |
| 0.3 | $6.78650 \cdot 10^{-6}$ | $2.49109 \cdot 10^{-6}$ | $7.00050 \cdot 10^{-4}$ | $2.4860 \cdot 10^{-3}$ |
| 0.4 | $1.98008 \cdot 10^{-6}$ | $7.33214 \cdot 10^{-6}$ | $2.03367 \cdot 10^{-4}$ | $3.12345 \cdot 10^{-3}$ |
| 0.5 | $8.34567 \cdot 10^{-7}$ | $4.48763 \cdot 10^{-7}$ | $3.5678 \cdot 10^{-4}$ | $3.13789 \cdot 10^{-3}$ |
| 0.6 | $6.92340 \cdot 10^{-7}$ | $6.7703 \cdot 10^{-7}$ | $2.34521 \cdot 10^{-4}$ | $4.00123 \cdot 10^{-3}$ |
| 0.7 | $7.98123 \cdot 10^{-7}$ | $4.90903 \cdot 10^{-7}$ | $1.61546 \cdot 10^{-4}$ | $1.03342 \cdot 10^{-3}$ |
| 0.8 | $1.92350 \cdot 10^{-7}$ | $8.12345 \cdot 10^{-7}$ | $3.09871 \cdot 10^{-4}$ | $2.12345 \cdot 10^{-3}$ |
| 0.9 | $9.00431 \cdot 10^{-8}$ | $7.07650 \cdot 10^{-8}$ | $3.3456 \cdot 10^{-5}$ | $8.45678 \cdot 10^{-3}$ |
| 1.0 | $4.98750 \cdot 10^{-8}$ | $2.4130 \cdot 10^{-8}$ | $1.03367 \cdot 10^{-5}$ | $4.05860 \cdot 10^{-4}$ |

## Conclusion and discussion

The over investigation and discussion take us to the conclusion that the modern strategy is exceptionally proficient for the numerical solution of coupled system of fractional order differential equations. Operational matrices in relation with the Bernstein type polynomials gives the best approximation of any system. The approximate solution of any coupled system with initial and boundary conditions can easily be calculated by using the developed technique. Also the developed operational matrices can be extended to higher dimension. The accuracy rate of the developed algorithm is higher for the numerical solutions. Further we have compared our numerical techniques with that of Haar wavelet method solutions in tab. 1, we see that our procedure is much more efficient than the said one. Also the we have compared the numerical solutions with the exact results of the corresponding problem, we see that they have close agreement at small scale number. Also we have recorded some values of absolute errors at different values of $t$ in the case of variable order by using fixed scale level. Briefly the procedure is good to apply various systems or scaler problems of variable order differential equations.

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