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Faculty of Graduate Studies

Methods for solving Fully Fuzzy Multi-objective Linear programming problem

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Declaration

I declare that the content of this thesis are the product of my own effort, except what has been indicated and that this thesis or any part of it has not been previously submitted to obtain any other scientific or research degree.

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Dedication

This thesis is dedicated to my parents who encouraged me and supported me a lot, without your guidance I wouldn't have made it this far.

To my husband khaled, who has stood by me through thick and thin and provided unwavering support and encouragement when I needed it most. This accomplishment is as much theirs as it is mine.

To my dear daughter Rafeef whose presence in my life was the biggest motivation for success.

To my supervisors who have challenged me and pushed me to grow as a scholar, I am grateful for their mentorship and expertise, which have been invaluable throughout my studies.

To my sister and brothers, who have been my source of motivation.

To my husband family I am grateful for their support, and I hope this dedication serves as a small token of my appreciation.

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Abstract

Methods for solving Fully Fuzzy Multi-objective Linear programming problem

In this thesis fully fuzzy multi objective linear programming problem will be considered, we will study two methods which are ranking function method and game theory approach, in these methods the fully fuzzy multi-objective Linear programming problem (FFMOLPP) will be converted to a crisp multi objective linear programming problem (MOLPP) and then the problem will be solved by weighted method or preemptive method. And alpha cut method will be used for solving fuzzy multi objective linear programming (FMOLPP).

Contents

Chapter 1

Introduction

Multi objective linear programming is an efficient operational research technique in decision making problems. It is usually used in cases which the decision maker needs to take more than one aspect simultaneously instead of one. However, since the problems in our real world are complex, the decision maker cannot set a deterministic values for the coefficients so it is more convenient to deal with this uncertainty by means of fuzzy concepts and replace the crisp number by fuzzy one, this will lead to a fully fuzzy multi objective linear programming problem (FFMOLPP) in which all the coefficients are fuzzy numbers.

Several methods have been used to solve the fuzzy multi objective linear programming problems such as ranking function method in [3], max-min method [7]. Hossein Zadeh [11] discussed the fuzzy multi objective linear programming problems in which the variables are symmetric trapezoidal fuzzy numbers, the authors in [10, 20] used fuzzy simplex algorithm to solve such problems. In [9] revised simplex method was used in case that the constraints are crisp.

The general form of the FFMOLPP is given as : [5]

$$Maximize \quad \tilde{Z}^1, \tilde{Z}^2, ..., \tilde{Z}^K$$

$$where \quad \tilde{Z}^k = \sum_{j=1}^{n} \tilde{c}_j^k \otimes \tilde{x}_j, \qquad k = 1, 2, ..., K$$

$$such\ that$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \leq \tilde{b}_i \qquad i = 1, 2, ..., m$$

$$\tilde{x}_j \geq 0 \qquad j = 1, 2, ..., n$$

where $\tilde{c}_j^k, \tilde{a}_{ij}, \tilde{b}_i$ are fuzzy numbers and $\tilde{x}_j$ are fuzzy variables.

In this work we will study two methods for solving FFMOLPP. In chapter 2 we introduced some basic definitions about triangular, trapezoidal and hexagonal fuzzy numbers. In chapter 3 crisp multi objective linear programming problem (MOLPP) was considered, we have shown two methods for solving a MOLPP which are preemptive method and weighted sum method, numerical examples were also presented. In chapter 4 we used the ranking function method to solve fully fuzzy multi objective linear programming problem with triangular, trapezoidal and hexagonal fuzzy numbers by converting the FFMOLPP to a crisp MOLPP and then solving it by preemptive and weighted sum method, the method was illustrated by some numerical examples. In chapter 5, game theory approach was used to solve the FF-MOLPP. In chapter 6, alpha cut method was used for solving FMOLPP.

Chapter 2

Fuzzy sets and fuzzy numbers

To solve FFMOLPP and FMOLPP , we need first to understand the basic definitions about fuzzy set and its basic definitions and the types of fuzzy numbers that we will use in our thesis.
In this chapter we will introduce the definitions of fuzzy set and fuzzy number and we will show the membership function for the fuzzy numbers that we will deal with in our thesis which are triangular, trapezoidal and hexagonal fuzzy number.

2.1 Fuzzy set

Definition 2.1.1 *Fuzzy set*

For a set X, a subset A is called fuzzy set if $\mu_A : X \longrightarrow [0,1]$, where μ denote the degree of belongingness of A in X [22].

Its obvious that the fuzzy set is a generalization for the classical crisp set that has a characteristic function which maps each element in X to 0 or 1 i.e $(X_A : X \longrightarrow \{0,1\})$.

Definition 2.1.2 *Convex Set*

A fuzzy Set A on X is said to be convex if for any $x, y \in X$ and $\lambda \in [0,1]$, we have $\mu_A(\lambda x + (1-\lambda)y) \geqslant min\{\mu_A(x), \mu_A(y)\} \forall x, y \in X$ [16] .

Definition 2.1.3 *Normal set*

We called a fuzzy set normal set if there exist at least one point $x \in X$ *with* $\mu_A(x) = 1$, *i,e* $(\exists x \in X ,s.t\ \mu_A(x) = 1)$ *[16].*

Definition 2.1.4 *For a fuzzy set A,* $\alpha -$ *cut is a crisp set, defined by* $A_\alpha = \{x \in X|\mu_A(x \geqslant \alpha)\}$ *[16].*

2.2 Fuzzy numbers

In this section we will show the definitions of fuzzy number and each type.

Definition 2.2.1 *Fuzzy number*

A fuzzy set A with membership function defined on the set of the real numbers is called fuzzy number if it is normal , convex and $\mu_A(x))\}$ *is piecewise continuous [18].*

Definition 2.2.2 *A fuzzy number A is called positive fuzzy number* $(A > 0)$ *if* $\mu_A(x) = 0, \forall x \leqslant 0$ *and it is called negative fuzzy number* $(A < 0)$ *if* $\mu_A(x) = 0\ \forall x \geqslant 0$ *[15].*

2.3 Triangular fuzzy number

In this subsection we will show the definition of the triangular fuzzy number and some other basic definitions and operators on them.

Definition 2.3.1 *Triangular fuzzy number*

Let a, b, c *be real numbers with* $a < b < c$ *,Then the triangular fuzzy number (TFN)* $A = (a, b, c)$ *is the fuzzy number with the following membership function [28]:*

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a}, & x \in [a,b] \\ \\ \dfrac{c-x}{c-b}, & x \in [b,c] \\ \\ 0, & x < a \quad and \quad x > c \end{cases}$$

The graph of the triangular fuzzy number is shown in Fig 2.1.

Definition 2.3.2 *Two triangular fuzzy numbers* $A=(a_1, a_2, a_3)$ *and* $B = (b_1, b_2, b_3)$ *are said to be equal i.e* $A = B$ *if* $a_1 = b_1$ *and* $a_2 = b_2$ *and* $a_3 = b_3$ *[12]*

Arithmetic operations on the triangular fuzzy numbers.

$(i) A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$(ii) A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$

$(iii) - A = (-a_3, -a_2, -a_1)$

(iv)

$$k(a_1, a_2, a_3) = \begin{cases} (ka_1, ka_2, ka_3) \ \ for \quad k \geq 0 \\ \\ (ka_1, ka_2, ka_3) \ \ for \quad k < 0 \end{cases} \tag{2.1}$$

Definition 2.3.3 *A fuzzy triangular number* $A = (a, b, c)$ *is called non negative triangular fuzzy number if* $a \geq 0$ *[16].*

Definition 2.3.4 $\alpha - cut$ *for triangular number [6]*

If $A = (a, b, c)$ is a triangular fuzzy number ,then $\alpha - cut$ of A

$$A_\alpha = [(b-a)\alpha + a, c - (c-b)\alpha] \tag{2.2}$$

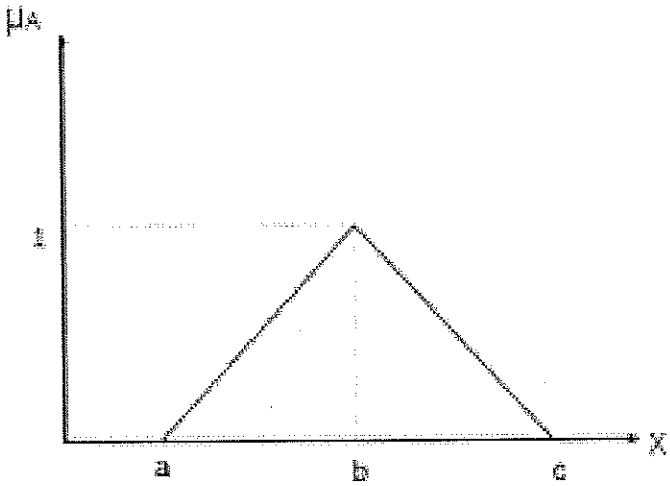

Fig. 2.1: Triangular fuzzy number.

2.3.1 Trapezoidal fuzzy number

In this subsection we will introduce the definition of the trapezoidal fuzzy number.

Definition 2.3.5 *Trapezoidal fuzzy number [2]*

A fuzzy number $A = (a, b, c, d)$is a trapezoidal fuzzy number if

$$\mu_A(x) = \begin{cases} 0 & x \leq a, x \geq d \\ \frac{x-a}{b-a}, & a < x < b \\ 1 & b \leq x \leq c \\ \frac{d-x}{d-c}, & c < x < d \end{cases} \tag{2.3}$$

The graph of the trapezoidal fuzzy number is shown in 2.2 , we can see that when $b = c$ we will get triangular fuzzy number.

Definition 2.3.6 *$\alpha - cut$ of trapezoidal fuzzy number*

The $\alpha - cut$ of trapezoidal fuzzy number(a, b, c, d) is given by [22] :

$$A_\alpha = [a + \alpha(b - a), d - \alpha(d - c)] \tag{2.4}$$

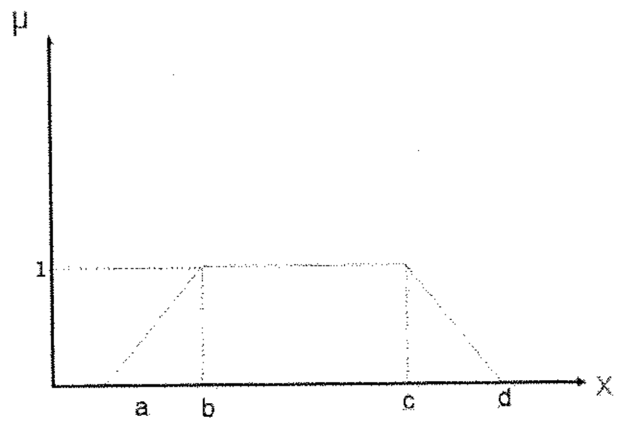

Fig. 2.2: Trapezoidal fuzzy number.

2.3.2 Hexagonal fuzzy number

In this subsection we will show the definition of the hexagonal fuzzy number.

Definition 2.3.7 *Hexagonal fuzzy number [8]*

A fuzzy number A_H is a hexagonal fuzzy number denoted by
$A_H = (a_1, a_2, a_3, a_4, a_5, a_6)$where $a_1, a_2, a_3, a_4, a_5, a_6$ are real numbers and its membership function μ_{AH} is given by:

$$\mu_{A_H} = \begin{cases} 0 & x < a_1 \\ \frac{1}{2}\left(\frac{x-a_1}{a_2-a_1}\right) & a_1 \leq x \leq a_2 \\ \frac{1}{2}+\frac{1}{2}\left(\frac{x-a_2}{a_3-a_2}\right) & a_2 \leq x \leq a_3 \\ 1 & a_3 \leq x \leq a_4 \\ 1-\frac{1}{2}\left(\frac{x-a_4}{a_5-a_4}\right) & a_4 \leq x \leq a_5 \\ \frac{1}{2}\left(\frac{a_6-x}{a_6-a_5}\right) & a_5 \leq x \leq a_6 \\ 0 & x > a_6. \end{cases} \tag{2.5}$$

The graph of the hexagonal fuzzy number is shown in 2.3.

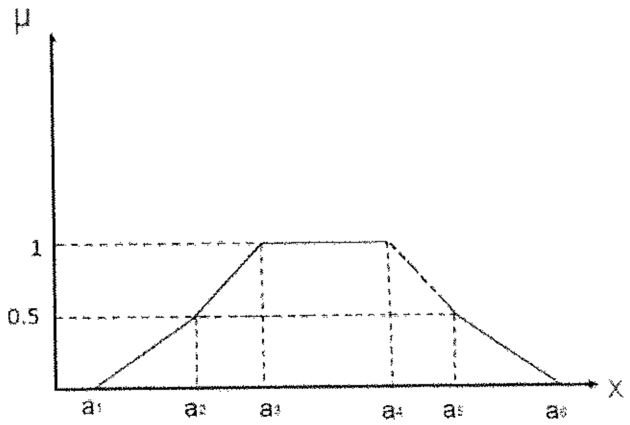

Fig. 2.3: Hexagonal fuzzy number.

Definition 2.3.8 *α-cut for hexagonal fuzzy number [21]*

if $A = (a_1, a_2, a_3, a_4, a_5, a_6)$ is hexagonal fuzzy number then

$$A_\alpha = \begin{cases} [2\alpha(a_2 - a_1) + a_1, -2\alpha(a_6 - a_5) + a_6] \quad for \;\; \alpha \in [0, 0.5) \\ [2\alpha(a_3 - a_2) - a_3 + 2a_2, -2\alpha(a_5 - a_4) + 2a_5 - a_4] \quad for \;\; \alpha \in [0.5, 1] \end{cases} \tag{2.6}$$

Chapter 3

Multi objective linear programming problem

In solving FFMOLPP we need to convert the problem into a crisp multi objective linear programming problem (MOLPP).
In this chapter we will introduce the definition of the multi objective linear programming and some methods for solving it with numerical examples.

Definition 3.0.1 *The process of optimizing systemically and simultaneously a collection of objective functions is called multi objective optimization (MOO), when all the objective functions and constraints are linear this will lead to multi objective linear programming which we want to study [22].*

Multi objective linear programming has applications in real world such as economic, industrial, etc [4].
The crisp multi objective linear programming model usually presented as [27]:

$$
\begin{aligned}
&\text{max or min } \; Z_k = C_k x \\
&\text{subject to} \\
&\qquad\qquad Ax\{\leq, \geq, =\}b \\
&\qquad\qquad x \geq 0
\end{aligned}
$$

where x is an n-dimensional vector of decision variables.

$Z_1(x)$, $Z_2(x)$,..., $Z_k(x)$ are k distinct linear objective function of the decision vector x.

$C_1, C_2, ..., C_k$ are n-dimensional cost factor vector.

A is $m * n$ constraints matrix.

b is n-dimensional constant vector.

Many methods have been used for solving MOLPP such as preemptive method and weighted sum method [26, 27].

3.1 Preemptive method

In preemptive method, the objective functions are ranked according to their priority or importance for decision maker and at each step we will optimize only one objective function (starting with the highest priority) subject to the constraints and neglecting all other objective functions, at each step we will add an additional constraint from the answer of the previous optimization step, we will show an example to illustrate the method [23].

Example 3.1.1

$$\min Z_1 = 40x_1 + 32x_2$$

$$\min Z_2 = 800x_1 + 1250x_2$$

$$\min Z_3 = 0.2x_1 + 0.45x_2$$

subject to

$$\begin{aligned} &12x_1 + 4x_2 \geq 48 \\ &4x_1 + 4x_2 \geq 28 \\ &10x_1 + 20x_2 \geq 100 \\ &x_1, x_2 \geq 0 \end{aligned} \tag{3.1}$$

solution:

step 1: we will solve LPP1

$$min \;\; Z_1 = 40x_1 + 32x_2$$

subject to

$$12x_1 + 4x_2 \geq 48$$

$$4x_1 + 4x_2 \geq 28$$

$$10x_1 + 20x_2 \geq 100$$

$$x_1, x_2 \geq 0$$

by using simplex method we will get :

$$x_1 = 2.5 \qquad x_2 = 4.5 \qquad Z_1 = 244$$

step 2: we will solve LPP2

$$min \;\; Z_2 = 800x_1 + 1250x_2$$

subject to the set of constraints (3.1)

$40x_1 + 32x_2 \leq 244$ (additional constraint)

by using Big M-method or linear programming solver we will get :

$$x_1 = 2.5 \qquad x_2 = 4.5 \qquad Z_2 = 7625$$

step 3: solve LPP3

$$min \;\; Z_3 = 0.2x_1 + 0.45x_2$$

subject to the set of constraints (3.1)

$$800x_1 + 1250x_2 \leq 7625$$

$40x_1 + 32x_2 \leq 244$ (additional constraint)

by simplex method we will get :

$$x_1 = 2.5 \qquad x_2 = 4.5 \qquad Z_3 = 2.525$$

hence the optimal solution is :

$$x_1 = 2.5 \qquad x_2 = 4.5 \qquad Z_1 = 244 \qquad Z_2 = 7625$$

$$Z_3 = 2.525$$

3.2 Weighted sum method

Weighted sum method in multi objective linear programming is used to combine two or more objective functions together by multiplying each objective by a weight which will be determined by the decision maker according to the importance of each objective function to him/her, combining the objective functions will lead to a single objective function which can be solve easily by simplex method or other methods.

Since objective functions have different units and magnitude, normalization is needed to make them comparable before combining it together.

Here we will show steps fore solving MOLPP by weighted sum method:

Step 1:

Solve each objective function individually subject to all constraints to find the optimal value of the objective function, denote the optimal value for the objective function Z_k by Z_k^*.

step 2:

we will combine the objective functions together by using weighted sum, and we will normalize each objective function to make the objectives comparable, hence the single objective function is written as follows:

$$Z = \sum_{k=1}^{n} w_k \frac{Z_k}{Z_k^*}$$

step 3:

solve the following LPP by any method or by a linear programming solver to get the optimal solution:

$$\text{max/min } Z = \sum_{k=1}^{n} w_k \frac{Z_k}{Z_k^*}$$

s.t

$$Ax\{\leq, \geq, =\}b$$

$$x \geq 0$$

* Note that the weights w_k will be chosen by the decision maker.

Example 3.2.1

We will solve the previous example by weighted method.

Step 1: we have to solve each objective subject to constraints individually to find the optimal solution.

For each objective, denote the optimal value for Z_k objective by Z_k^*, $k = 1, 2, ...$ where k is the number of objective functions.

solving Z_1 subject to the constraints (3.1) we get :

$$x_1 = 4 \qquad x_2 = 3 \qquad Z_1^* = 256$$

solving Z_2 subject to the constraints (3.1) we get :

$$x_1 = 4 \qquad x_2 = 3 \qquad Z_2^* = 6950$$

solving Z_3 subject to the constraints (3.1) we get :

$$x_1 = 4 \qquad x_2 = 3 \qquad Z_3^* = 2.15$$

Step 2:

construct a normalized weighted Linear programming problem :

$$\min Z = w_1 * (\tfrac{Z_1}{Z_1^*}) + w_2 * (\tfrac{Z_2}{Z_2^*}) + w_3 * (\tfrac{Z_3}{Z_3^*})$$

subject to the set of constraints (3.1)

w_1,w_2,w_3 are chosen by decision maker, assume that $w_1 = 0.6$ $\qquad w_2 = 0.2$ $w_3 = 0.2$

solving by big-M method we get :

$$x_1 = 4 \qquad x_2 = 3$$

$$Z_1 = 256 \qquad Z_2 = 6950 \qquad Z_3 = 2.15$$

which is the optimal solution.

Chapter 4

Ranking function method for solving fully fuzzy multi objective linear programming problem

In this chapter, we will use the ranking functions for the fuzzy numbers to convert the fully fuzzy multi objective linear programming problem to a crisp one, and then solve it by the previous methods, this method will be applied to the triangular, trapezoidal and hexagonal fuzzy numbers.

4.1 Ranking function

Definition 4.1.1 *ranking function*

The ranking function is a function$R : F(R) \rightarrow R$, where $F(R)$ is the set of fuzzy numbers defined one the real numbers wich maps each fuzzy number into the real line where a nature order exist [1, 25].

The ranking function for a triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is defined as:

$$R(\tilde{A}) = \frac{a_1+8a_2+a_3}{10} \text{ [13]} .$$

The ranking function for trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$ is defined as:

$$R(\tilde{A}) = \frac{a_1+a_2+a_3+a_4}{4} \text{ [24]} .$$

The ranking function for hexagonal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4, a_5, a_6)$ is defined as:

$$R(\tilde{A}) = \frac{a_1+a_2+a_3+a_4+a_5+a_6}{6} \text{ [26] }.$$

To use the ranking function wa have to know the following properties:

For a two fuzzy numbers $\tilde{A}$ and $\tilde{B}$:

1. $\tilde{A} > \tilde{B}$ iff $R(\tilde{A}) > R(\tilde{B})$
2. $\tilde{A} < \tilde{B}$ iff $R(\tilde{A}) < R(\tilde{B})$
3. $\tilde{A} = \tilde{B}$ iff $R(\tilde{A}) = R(\tilde{B})$

4.2 Ranking function method

In this section we will study the ranking function method and give some numerical examples [25].

let $c = [\widehat{c_j}]_{1xm}, b = [\widehat{b_i}]_{mx1}$ be fuzzy numbers and $A = [\widehat{a_{ij}}]_{mxn}$ be a matrix whose entries are fuzzy numbers, $x = [\widehat{x_j}]_{1xn}$ is the solution which is fuzzy vector, then the FFMOLPP (1) can be written as follows:

maximize or minimize $\tilde{Z}^k = \sum_{j=1}^{n}(d_j, e_j, f_j, ...) \otimes (x_j, y_j, z_j, ...)$

subject to

$$\sum_{j=1}^{n}(a_{ij}, b_{ij}, c_{ij}, ...) \otimes (x_j, y_j, z_j, ...) \leq (b_i, g_i, h_i, ...),$$

$$i = 1, ..., m$$

$$\tilde{x}_j \geq 0, j = 1, ..., n$$

by using the definition (4.1.1) the objective function will be

max or min $\tilde{Z}^k = R(\sum_{j=1}^{n}(d_j, e_j, f_j, ...) \otimes (x_j, y_j, z_j, ...))$

and constraints will be as follows:

$$\sum_{j=1}^{n}(a_{ij}, b_{ij}, c_{ij}, ...) \otimes (x_j, y_j, z_j, ...) \leq (b_i, g_i, h_i, ...)$$

$$(\sum_{j=i}^{n} a_{ij}x_j, \sum_{j=1}^{n} b_{ij}y_j, \sum_{j=1}^{n} c_{ij}z_j, ...) \leq (b_i, g_i, h_i, ...)$$

by the comparison of components we will get the following crisp MOLPP :

max/min $Z^k = R(\sum_{j=1}^{n}(d_j, e_j, f_j, ...) \otimes (x_j, y_j, z_j, ...))$, $k = 1, 2, ..., K$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$
$$\sum_{j=1}^{n} b_{ij}y_j \leq g_i$$
$$\sum_{j=1}^{n} c_{ij}z_j \leq h_i$$

.

.

.

$$x_j \geq 0, y_j \geq 0, z_j \geq 0, ... \qquad j = 1, ...n$$

and then this model can be solved by any method for solving MOLPP such as preemptive method and weighted sum method that we discussed before.

Numerical examples:

Example 4.2.1 *Solving FFMOLPP With Triangular Fuzzy Number Using Ranking Function Method [19].*

$$\max \tilde{z}_1 = (7, 10, 14)\tilde{x} \oplus (20, 25, 35)\tilde{y}$$
$$\max \tilde{z}_2 = (10, 14, 25)\tilde{x} \oplus (25, 35, 40)\tilde{y}$$

subject to

$$(1, 2, 3)x \oplus (1, 4, 6)y \leq (2, 5, 13)$$
$$(1, 2, 3)x \oplus (4, 5, 6)y \leq (2, 4, 7)$$

$\tilde{x}, \tilde{y} \geq 0$ solution:

let $\tilde{x} = (x_1, x_2, x_3)$, $\tilde{y} = (y_1, y_2, y_3)$

$$\max \tilde{z}_1 = (7, 10, 14) \otimes (x_1, x_2, x_3) \oplus (20, 25, 35) \otimes (y_1, y_2, y_3)$$

$$\max \tilde{z}_2 = (10, 14, 25) \otimes (x_1, x_2, x_3) \oplus (25, 35, 40) \otimes (y_1, y_2, y_3)$$

subject to

$$(1,2,3) \otimes (x_1, x_2, x_3) \oplus (1,4,6) \otimes (y_1, y_2, y_3) \leq (2,5,13)$$

$$(1,2,4) \otimes (x_1, x_2, x_3) \oplus (4,5,6) \otimes (y_1, y_2, y_3) \leq (2,4,7)$$

$$(x_1, x_2, x_3), (y_1, y_2, y_3) \geq 0$$

using multiplication of triangular fuzzy numbers we will get:

$$\max \tilde{z}_1 = (7x_1, 10x_2, 14x_3) \oplus (20y_1, 25y_2, 35y_3)$$

$$\max \tilde{z}_2 = (10x_1, 14x_2, 25x_3) \oplus (25y_1, 35y_2, 40y_3)$$

subject to

$$(x_1, 2x_2, 3x_3) \oplus (y_1, 4y_2, 6y_3) \leq (2,5,13)$$

$$(x_1, 2x_2, 4x_3) \oplus (4y_1, 5y_2, 6y_3) \leq (2,4,7)$$

$$(x_2, x_2, x_3), (y_1, y_2, y_3) \geq 0$$

by using addition of triangular fuzzy numbers

$$\max \tilde{z}_1 = (7x_1 + 20y_1, 10x_2 + 25y_2, 14x_3 + 35y_3)$$

$$\max \tilde{z}_2 = (10x_1 + 25y_1, 14x_2 + 35y_2, 25x_3 + 40y_3)$$

subject to

$$(x_1 + y_1, 2x_2 + 4y_2, 3x_3 + 6y_3) \leq (2,5,13)$$

$$(x_1 + 4y_1, 2x_2 + 5y_2, 4x_3 + 6y_3) \leq (2,4,7)$$

$$(x_2, x_2, x_3), (y_1, y_2, y_3) \geq 0$$

Now by applying the ranking function $R(\tilde{A}) = \frac{a_1+8a_2+a_3}{10}$ on the objective functions we will get :

$$\max z_1 = 0.7x_1 + 2y_1 + 8x_2 + 20y_2 + 1.4x_3 + 3.5y_3$$

$$\max z_2 = x_1 + 2.5y_1 + 11.2x_2 + 28y_2 + 2.5x_3 + 4y_3$$

subject to

$$x_1 + y_1 \leq 2$$

$$2x_2 + 4y_2 \leq 5$$

$$3x_3 + 6y_3 \leq 13$$

$$x_1 + 4y_1 \leq 2$$
$$2x_2 + 5y_2 \leq 4$$
$$4x_3 + 6y_3 \leq 7$$
$$x_1, y_1, x_2, y_2, x_3, y_3 \geq 0$$

This is a crisp MOLPP that we will solve by preemptive method and weighted sum method.

(i) Using preemptive method

We will solve LPP1 which is :

$$\max z_1 = 0.7x_1 + 2y_1 + 8x_2 + 20y_2 + 1.4x_3 + 3.5y_3$$

subject to

$$\begin{aligned} &x_1 + y_1 \leq 2 \\ &2x_2 + 4y_2 \leq 5 \\ &3x_3 + 6y_3 \leq 13 \\ &x_1 + 4y_1 \leq 2 \\ &2x_2 + 5y_2 \leq 4 \\ &4x_3 + 6y_3 \leq 7 \\ &x_1, y_1, x_2, y_2, x_3, y_3 \geq 0 \end{aligned} \tag{4.1}$$

using simplex method :

$x_1 = 2 \quad x_2 = 0 \quad x_3 = 0 \quad y_1 = 0 \quad y_2 = 0.8 \quad y_3 = 1.167$

$z_1 = 21.483$

Now we will solve LPP2 which is :

$$\max z_2 = x_1 + 2.5y_1 + 11.2x_2 + 28y_2 + 2.5x_3 + 4y_3$$

subject to the set of constraints (4.1)

$$0.7x_1 + 2y_1 + 8x_2 + 20y_2 + 1.4x_3 + 3.5y_3 \geq 21.483$$

by a linear programming solver we get :

$x_1 = 2 \quad x_2 = 2 \quad x_3 = 0 \quad y_1 = 0 \quad y_2 = 0 \quad y_3 = 1.167$

$z_2 = 29$

hence the optimal solution is:

$$\tilde{x} = (x_1, x_2, x_3) = (2, 2, 0)$$

$$\tilde{y} = (y_1, y_2, y_3) = (0, 0, 1.167)$$

$$\tilde{z}_1 = (7x_1, 10x_2, 14x_3) \oplus (20y_1, 25y_2, 35y_3) = (7x_1+20y_1, 10x_2+25y_2, 14x_3+35y_3)$$
$$= (14, 20, 40.85)$$

$$\tilde{z}_2 = (10x_1 + 25y_1, 14x_2 + 35y_2, 25x_3 + 40y_3)$$
$$= (20, 28, 46.68)$$

(ii) Now we will solve the problem by using weighted sum method.

First we have to solve LPP1 which is :

max z_1 subject to the set of constraints(4.1)

by simplex method we get :

$$x_1 = 2 \qquad x_2 = 0 \qquad x_3 = 0 \qquad y_1 = 0 \qquad y_2 = 0.8 \qquad y_3 = 1.167$$

$$z_1^* = 21.483$$

then we will solve LPP2 which is

max z_2 subject to the set of constraints (4.1)

by simplex method we get :

$$x_1 = 2 \qquad x_2 = 0 \qquad x_3 = 0 \qquad y_1 = 0 \qquad y_2 = 0.8 \qquad y_3 = 1.167$$

$$z_2^* = 29.13$$

Now we will form a single objective function by using weighted sum method and normalization for the objective function.

assuming that $w_1 = 0.8$, $w_2 = 0.2$, the LPP will be written as follows:

$$\max z = 0.8 * \frac{(0.7x_1+2y_1+8x_2+20y_2+1.4x_3+3.5y_3)}{21.4833} + 0.2 * \frac{x_1+2.5y_1+11.2x_2+28y_2+2.5x_3+4y_3}{29.1}$$

subject to the set of constraints (4.1)

by a linear programming solver, the optimal solution is :

$$\tilde{x} = (2, 0, 0)$$

$$\tilde{y} = (0, 0.8, 1.167)$$

$\tilde{z}_1 = (14, 20, 40.85)$

$\tilde{z}_2 = (20, 28, 46.68)$

Example 4.2.2 *Solving FFMOLPP With Trapezoidal Fuzzy Numbers By Using Ranking Function Method.*

$$\begin{aligned}
&\text{maximize } \tilde{z}_1 = (1, 3, 5, 7)\tilde{x} \oplus (4, 6, 10, 12)\tilde{y} \\
&\text{maximize } \tilde{z}_2 = (4, 6, 10, 12)\tilde{x} \oplus (20, 22, 26, 28)\tilde{y} \\
&\text{subject to} \\
&\qquad (0.25, 0.5, 1.25, 2)\tilde{x} \oplus (1, 1.5, 2.5, 3)\tilde{y} \leq (10, 15, 25, 30) \\
&\qquad (0.25, 0.5, 1.25, 2)\tilde{x} \leq (2, 4, 8, 10) \\
&\qquad \tilde{x}, \tilde{y} \geq 0
\end{aligned}$$

Solution:

Let $\tilde{x} = (x_1, x_2, x_3, x_4)$, $\tilde{y} = (y_1, y_2, y_3, y_4)$

$$\begin{aligned}
&\max \tilde{z}_1 = (1, 3, 5, 7) \otimes (x_1, x_2, x_3, x_4) \oplus (4, 6, 10, 12) \otimes (y_1, y_2, y_3, y_4) \\
&\max \tilde{z}_2 = (4, 6, 10, 12) \otimes (x_1, x_2, x_3, x_4) \oplus (20, 22, 26, 28) \otimes (y_1, y_2, y_3, y_4) \\
&\text{subject to} \\
&(0.25, 0.5, 1.25, 2) \otimes (x_1, x_2, x_3, x_4) \oplus (1, 1.5, 2.5, 3) \otimes (y_1, y_2, y_3, y_4) \leq (10, 15, 25, 30) \\
&(0.25, 0.5, 1.25, 2) \otimes (x_1, x_2, x_3, x_4) \leq (2, 4, 8, 10) \\
&(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \geq 0
\end{aligned}$$

using multiplication of trapezoidal fuzzy numbers we will get

$$\begin{aligned}
&\max \tilde{z}_1 = (x_1, 3x_2, 5x_3, 7x_4) \oplus (4y_1, 6y_2, 10y_3, 12y_4) \\
&\max \tilde{z}_2 = (4x_1, 6x_2, 10x_3, 12x_4) \oplus (20y_1, 22y_2, 26y_3, 28y_4) \\
&\text{subject to} \\
&(0.25x_1, 0.5x_2, 1.25x_3, 2x_4) \oplus (y_1, 1.5y_2, 2.5y_3, 3y_4) \leq (10, 15, 25, 30) \\
&(0.25x_1, 0.5x_2, 1.25x_3, 2x_4) \leq (2, 4, 8, 10)
\end{aligned}$$

$(x_1, x_2, x_3, x_4) \geq 0$, $(y_1, y_2, y_3, y_4) \geq 0$

using addition of trapezoidal fuzzy numbers we get:

$$\max \tilde{z}_1 = (x_1 + 4y_1, 3x_2 + 6y_2, 5x_3 + 10y_3, 7x_4 + 12y_4)$$

$$\max \tilde{z}_1 = (4x_1 + 20y_1, 6x_2 + 22y_2, 10x_3 + 26y_3, 12x_4 + 28y_4)$$

subject to

$$(0.25x_1 + y_1, 0.5x_2 + 1.5y_2, 1.25x_3 + 2.5y_3, 2x_4 + 3y_4) \leq (10, 15, 25, 30)$$

$$(0.25x_1, 0.5x_2, 1.25x_3, 2x_4) \leq (2, 4, 8, 10)$$

$$(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \geq 0$$

Using the ranking function $R(\tilde{A}) = \frac{a_1+a_2+a_3+a_4}{4}$ for the objective functions we get: max $z_1 = 0.25x_1 + 0.75x_2 + 1.25x_3 + 1.75x_4 + y_1 + 1.5y_2 + 2.5y_3 + 3y_4$

$$\max z_2 = x_1 + 1.5x_2 + 2.5x_3 + 3x_4 + 5y_1 + 5.5y_2 + 6.5y_3 + 7y_4$$

subject to

$$\begin{aligned} &0.5x_1 + y_1 \leq 10 \\ &0.5x_2 + 1.5y_2 \leq 15 \\ &1.25x_3 + 2.5y3 \leq 25 \\ &2x_4 + 3y_4 \leq 30 \\ &0.25x_1 \leq 2 \\ &0.5x_2 \leq 4 \\ &1.25x_3 \leq 8 \\ &2x_4 \leq 10 \\ &x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \geq 0 \end{aligned} \tag{4.2}$$

This is a crisp MOLPP that we will solve first by preemptive method and then by weighted sum method.

(i) Using preemptive method

we will solve LPP1 which is

$$\max z_1 = 0.25x_1 + 0.75x_2 + 1.25x_3 + 1.75x_4 + y_1 + 1.5y_2 + 2.5y_3 + 3y_4$$

subject to the set of constraints (4.2)

using simplex method we get :

$x_1 = 0 \qquad x_2 = 8 \qquad x_3 = 0 \qquad x_4 = 0$

$y_1 = 10 \qquad y_2 = 7.33 \qquad y_3 = 10 \qquad y_4 = 10 \qquad z_1 = 82$

then we will solve LPP2 which is

$$\max z_2 = x_1 + 1.5x_2 + 2.5x_3 + 3x_4 + 5y_1 + 5.5y_2 + 6.5y_3 + 7y_4$$

subject to the set of constraints (4.2)

$$0.25x_1 + 0.75x_2 + 1.25x_3 + 1.75x_4 + y_1 + 1.5y_2 + 2.5y_3 + 3y_4 \geq 82$$

solving by a linear programming solver we get :

$x_1 = 0 \qquad x_2 = 8 \qquad x_3 = 0 \qquad x_4 = 0$

$y_1 = 10 \qquad y_2 = 7.33 \qquad y_3 = 10 \qquad y_4 = 10 \qquad z_2 = 237.315$

(ii) weighted sum method

now we will use weighted sum method and normalization to form a single objective function.

first we have to find z_1^* and z_2^*

$z_1^* = 82 \qquad z_2^* = 237.3$

The LPP is constructed as follows:

$$\max z = w_1 \times \frac{z_1}{z_1^*} + w_2 \times \frac{z_2}{z_2^*}$$

subject to the set of constraints (4.2)

assume that $w_1 = 0.2$ and $w_2 = 0.8$

hence

$\tilde{x} = (0, 0, 0, 0)$

$\tilde{y} = (10, 10, 10, 10)$

$\tilde{z}_1 = (40, 60, 100, 120)$

$\tilde{z}_2 = (200, 220, 100, 280)$

assume that $w_1 = 0.7$, $w_2 = 0.3$,

$\tilde{x} = (0, 8, 0, 0)$

$\tilde{y} = (10, 7.33, 10, 10)$

we can see how changing of weights may change the optimal solution.

Example 4.2.3 *Solving FFMOLPP With Hexagonal Fuzzy Number Using The Ranking Function Method.*

$$\max \tilde{z}_1 = (50, 60, 70, 80, 90, 100)\tilde{x} \oplus (65, 75, 85, 95, 105, 115)\tilde{y}$$

$$\max \tilde{z}_2 = (35, 45, 55, 65, 75, 85)\tilde{x} \oplus (50, 60, 70, 80, 90, 100)\tilde{y}$$

subject to

$$(180, 190, 200, 200, 210, 220)\tilde{x} \oplus (230, 240, 250, 250, 260, 270)\tilde{y}$$

$$\leq (12000, 13000, 14000, 14000, 15000, 16000)$$

$$(280, 290, 300, 300, 310, 320)\tilde{x} \oplus (180, 190, 200, 200, 210, 220)\tilde{y}$$

$$\leq (11500, 12000, 12500, 12500, 13000, 13500)$$

$$\tilde{x}, \tilde{y} \geq 0$$

solution :

let $\tilde{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$, $\tilde{y} = (y_1, y_2, y_3, y_4, y_5, y_6)$

$\max \tilde{z}_1 = (50, 60, 70, 80, 90, 100) \otimes (x_1, x_2, x_3, x_4, x_5, x_6) \oplus (65, 75, 85, 95, 105, 115) \otimes (y_1, y_2, y_3, y_4, y_5, y_6)$

$\max \tilde{z}_2 = (35, 45, 55, 65, 75, 85) \otimes (x_1, x_2, x_3, x_4, x_5, x_6)$

$\oplus (50, 60, 70, 80, 90, 100) \otimes (y_1, y_2, y_3, y_4, y_5, y_6)$

subject to

$(180, 190, 200, 200, 210, 220) \otimes (x_1, x_2, x_3, x_4, x_5, x_6)$

$\oplus (230, 240, 250, 250, 260, 270) \otimes (y_1, y_2, y_3, y_4, y_5, y_6) \leq (12000, 13000, 14000, 14000, 15000, 16000)$

$(280, 290, 300, 300, 310, 320) \otimes (x_1, x_2, x_3, x_4, x_5, x_6)$

$\oplus(180, 190, 200, 200, 210, 220)\otimes(y_1, y_2, y_3, y_4, y_5, y_6) \leq (11500, 12000, 12500, 12500, 13000, 13500)$

$(x_1, x_2, x_3, x_4, x_5, x_6), (y_1, y_2, y_3, y_4, y_5, y_6) \geq 0$

using multiplication we get :

max $\tilde{z}_1 = (50x_1, 60x_2, 70x_3, 80x_4, 90x_5, 100x_6) \oplus (50y_1, 60y_2, 70y_3, 80y_4, 90y_5, 1000y_6)$

subject to

$(280x_1, 290x_2, 300x_3, 300x_4, 310x_5, 320x_6) \oplus (180y_1, 190y_2, 200y_3, 200y_4, 210y_5, 220y_6)$

$\leq (12000, 13000, 14000, 14000, 15000, 16000)$

$(180x_1, 190x_2, 200x_3, 200x_4, 210x_5, 220x_6) \oplus (230y_1, 240y_2, 250y_3, 250y_4, 260y_5, 270y_6)$

$\leq (11500, 12000, 12500, 12500, 13000, 13500)$

$(x_1, x_2, x_3, x_4, x_5, x_6), (y_1, y_2, y_3, y_4, y_5, y_6) \geq 0$

using addition of hexagonal fuzzy numbers we get:

max $\tilde{z}_1 = (50x_1+65y_1, 65x_2+75y_2, 70x_3+85y_3, 80x_4+95y_4, 90x_5+105y_5, 100x_6+115y_6)$

max $\tilde{z}_2 = (35x_1+50y_1, 45x_2+60y_2, 55x_3+70y_3, 65x_4+80y_4, 75x_5+90y_5, 85x_6+100y_6)$

subject to

$(180x_1 + 230y_1, 190x_1 + 240y_2, 200x_3 + 250y_3, 200x_4 + 250y_4, 210x_5 + 260y_5, 220x_6 + 270x_6) \leq (12000, 13000, 14000, 14000, 15000, 16000)$

$(280x_1 + 180y_1, 290x_2 + 190y_2, 300x_3 + 200y_3, 300x_4 + 200y_4, 310x_5 + 210y_5, 320x_6 + 220y_6) \leq (11500, 12000, 12500, 12500, 13000, 13500)$

Applying the ranking function $R(\tilde{A}) = \frac{a_1+a_2+a_3+a_4+a_5+a_6}{6}$ on the objective functions we get:

max $z_1 = \frac{50}{6}x_1 + \frac{65}{6}y_1 + x_2 + 12.5y_2 + \frac{70}{6}x_3 + \frac{85}{6}y_3 + \frac{80}{6}x_4 + \frac{95}{6}y_4 + 15x_5 + 17.5y_5 + \frac{100}{6}x_6 + \frac{115}{6}y_6$

max $z_2 = \frac{35}{6}x_1 + \frac{50}{6}y_1 + 7.5x_2 + y_2 + \frac{55}{6}x_3 + \frac{70}{6}y_3 + \frac{65}{6}x_4 + \frac{80}{6}y_4 + 12.5x_5 + 15y_5 + \frac{85}{6}x_6 + \frac{100}{6}y_6$

subject to

$$\begin{aligned}
180x_1 + 230y_1 &\le 12000\\
190x_2 + 240y_2 &\le 13000\\
200x_3 + 250y_3 &\le 14000\\
200x_4 + 250y_4 &\le 14000\\
210x_5 + 260y_5 &\le 15000\\
220x_6 + 270y_6 &\le 16000\\
280x_1 + 180y_2 &\le 11500\\
290x_2 + 190y_2 &\le 12000\\
300x_3 + 200y_3 &\le 12500\\
300x_4 + 200y_4 &\le 12500\\
310x_5 + 210y_5 &\le 13000\\
320x_6 + 220y_6 &\le 13500\\
x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 &\ge 0
\end{aligned} \tag{4.3}$$

(i) Solving by using preemptive method

solve LPP1 which is :

max $z_1 = \frac{50}{6}x_1 + \frac{65}{6}y_1 + x_2 + 12.5y_2 + \frac{70}{6}x_3 + \frac{85}{6}y_3 + \frac{80}{6}x_4 + \frac{95}{6}y_4 + 15x_5 + 17.5y_5 + \frac{100}{6}x_6 + \frac{115}{6}y_6$

subject to the constraints (4.3)

using linear programming solver we get:

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 0 \quad x_5 = 6.3x_6 = 3.29$

$y_1 = 52 \quad y_2 = 54 \quad y_3 = 56 \quad y_4 = 62.5 \quad y_5 = 52.6 \quad y_6 = 56.57$

hence the optimal solution is:

$$\tilde{x} = (0, 0, 0, 0, 6.3, 3.29)$$

$$\tilde{y} = (52, 54, 56, 62.5, 52.6, 56.57)$$

$z_1 = 5972.9$

second we will solve LPP2 which is :

$\max z_2 = \frac{35}{6}x_1 + \frac{50}{6}y_1 + 7.5x_2 + y_2 + \frac{55}{6}x_3 + \frac{70}{6}y_3 + \frac{65}{6}x_4 + \frac{80}{6}y_4 + 12.5x_5 + 15y_5 + \frac{85}{6}x_6 + \frac{100}{6}y_6$

subject to the set of constraints (4.3)

$\frac{50}{6}x_1 + \frac{65}{6}y_1 + x_2 + 12.5y_2 + \frac{70}{6}x_3 + \frac{85}{6}y_3 + \frac{80}{6}x_4 + \frac{95}{6}y_4 + 15x_5 + 17.5y_5 + \frac{100}{6}x_6 + \frac{115}{6}y_6 \geq 5972.9$

using a linear programming solver we get :

$x_1 = 0 \quad x_2 = 0.02 \quad x_3 = 0 \quad x_4 = 0 \quad x_5 = 6.3 \quad x_6 = 3.3$

$y_1 = 52.17 \quad y_2 = 54.15 \quad y_3 = 56 \quad y_4 = 62.5 \quad y_5 = 52.6 \quad y_6 = 56.6$

$z_2 = 4486.47$

hence the optimal solution is :

$$\tilde{x} = (0, 0.02, 0, 0, 6.3, 3.3)$$

$$\tilde{y} = (52.17, 54.15, 56, 62.5, 52.6, 56.6)$$

$$\tilde{z}_1 = (3519, 4058.7, 4760, 5937.5, 6090, 6839)$$

$$\tilde{z}_2 = (2608.5, 3250, 3920, 5000, 5206.5, 5940.5)$$

(ii) Using weighted sum method:

$$z_1^* = 5972 \qquad z_2^* = 4742.5$$

The linear programming problem is constructed as follows:

$$\max z = w_1 \times \frac{z_1}{5972} + w_2 \times \frac{z_2}{4742.5}$$

subject to the set of constraints (4.3)

assume that $w_1 = 0.4 \qquad w_2 = 0.6$

The optimal solution is:

$$\tilde{x} = (0, 12.2, 0, 0, 6.3, 3.28)$$

$$\tilde{y} = (52.2, 44.5, 56, 62.5, 52.6, 56.6)$$

$$\tilde{z}_1 = (3393, 4069.5, 4760, 5937.5, 7129, 6837)$$

$$\tilde{z}_2 = (2610, 2670, 39200, 5000, 5206.5, 5938.8)$$

Chapter 5

Solving fully fuzzy multi objective linear programming problem by using game theory approach

In this chapter we will study a method for solving FFMOLPP by using game theory approach and we will show some numerical examples.

In [23] game theory approach and ranking function were used to solve FFMOLPP with triangular fuzzy numbers , in our work we will consider also trapezoidal and hexagonal fuzzy numbers.

Consider the general form for the FFMOLPP in (1) the method for solving it is given in the following steps:

Step 1: separate the FFMOLPP into k FFLP problems where k is the number of objective functions.

Step 2: solve individually each FFLPP using any method to find the optimal solution $\tilde{P}_q = (\tilde{x}_1, ..., \tilde{x}_n) \forall q \in 1, ..., k$ we will solve by using the method in [17].

Step 3: find the value of each objective function at all fuzzy optimal solutions and then use the ranking function to defuzzify these values.

Step 4: use the values that were obtained in the previous step to construct a payoff matrix as shown in the table 5.1

The payoff matrix

$\tilde{\mathbf{P}}_{\mathbf{q}}$	$\mathbf{R}(\tilde{\mathbf{z}}_1(\tilde{\mathbf{P}}_{\mathbf{q}}))$	$\mathbf{R}(\tilde{\mathbf{z}}_2(\tilde{\mathbf{P}}_{\mathbf{q}}))$	$\cdots$	$\mathbf{R}(\tilde{\mathbf{z}}_k(\tilde{\mathbf{P}}_{\mathbf{q}}))$
$\tilde{P}_1$	$R(\tilde{z}_1(\tilde{P}_1))$	$R(\tilde{z}_2(\tilde{P}_1))$	$\cdots$	$R(\tilde{z}_k(\tilde{P}_1))$
$\tilde{P}_2$	$R(\tilde{z}_1(\tilde{P}_2))$	$R(\tilde{z}_2(\tilde{P}_2))$	$\cdots$	$R(\tilde{z}_k(\tilde{P}_2))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\tilde{P}_k$	$R(\tilde{z}_1(\tilde{P}_k))$	$R(\tilde{z}_2(\tilde{P}_k))$	$\cdots$	$R(\tilde{z}_k(\tilde{P}_k))$

Tab. 5.1: The payoff matrix.

Note if we get any negative entry in the payoff matrix then we have to convert it to a matrix which all entries are positive by adding consecutive integer of the absolute values of the least negative entry.

step 5: take the ratios of the values of thr rows of the payoff matrix to form a ratio matrix , the ratio matrix is shown in the table 5.2.

A representative ratio matrix

	z_1	z_2	$\cdots$	z_k
$\tilde{P}_1/\tilde{P}_2$	$\frac{R(\tilde{z}_1(\tilde{P}_1))}{R(\tilde{z}_1(\tilde{P}_2))}$	$\frac{R(\tilde{z}_2(\tilde{P}_1))}{R(\tilde{z}_2(\tilde{P}_2))}$	$\cdots$	$\frac{R(\tilde{z}_k(P_1))}{R(\tilde{z}_k(P_2))}$
$\tilde{P}_2/\tilde{P}_3$	$\frac{R(\tilde{z}_1(\tilde{P}_2))}{R(\tilde{z}_1(\tilde{P}_3))}$	$\frac{R(\tilde{z}_2(\tilde{P}_2))}{R(\tilde{z}_1(\tilde{P}_3))}$	$\cdots$	$\frac{R(\tilde{z}_k(\tilde{P}_2))}{R(\tilde{z}_k(\tilde{P}_3))}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\tilde{P}_{k-1}/\tilde{P}_k$	$\frac{R(\tilde{z}_1(\tilde{P}_{k-1}))}{R(\tilde{z}_1(\tilde{P}_k))}$	$\frac{R(\tilde{z}_2(\tilde{P}_{k-1}))}{R(\tilde{z}_2(\tilde{P}_k))}$	$\cdots$	$\frac{R(\tilde{z}_k(\tilde{P}_{k-1}))}{R(\tilde{z}_k(\tilde{P}_k))}$

Tab. 5.2: A representative ratio matrix.

Note that if the FFMOLPP has two objectives only then we have to add an extra row to the ratio matrix. This will be by solving a Lp problem with crisp parameters which was obtained in step 2 and with objective function which equals zero.

Step 6: solve the ratio matrix as a two player zero sum game to get the weights for each objective function

Step 7: multiply each objective function with its weight and then add them up and use normalization to obtain a normalized weighted linear programming problem and then solve it under the constraints that was obtained in step 2.

Step 8: express the fuzzy compromise solution for the FFMOLPP problem.

Note that if we change the order if the row in the payoff matrix we will get an alternative ratio matrices and hence another fuzzy compromise solutions can be generated and hence the decision maker(s) can choose the most preferred one.

Definition 5.0.1 *Fuzzy compromise solution*

In case of minimization, a fuzzy feasible solution $\tilde{x}^*$ of the fully fuzzy multi objective linear programming problem is called compromise solution iff there is not exist another feasible solution $\tilde{x}^1$ s.t $R(\tilde{z}_s(\tilde{x^*})) \leq R(\tilde{z}_s(\tilde{x^1}))\ \forall s$, and $R(\tilde{z}_r(\tilde{x^*})) < R(\tilde{z}_r(\tilde{x^1}))$ for at least one [24].

Numerical examples:

Example 5.0.1 *Solving FFMOLPP With Triangular Fuzzy Numbers Using Game Theory Approach* .

$$\text{maximize } \tilde{z}_1 = (7, 10, 14)\tilde{x} \oplus (20, 25, 35)\tilde{y}$$
$$\text{maximize } \tilde{z}_2 = (10, 14, 25)\tilde{x} \oplus (25, 35, 40)\tilde{y}$$
$$\text{subject to}$$
$$(1, 2, 3)\tilde{x} \oplus (1, 4, 6)\tilde{y} \leq (2, 5, 13)$$
$$(1, 2, 4)\tilde{x} \oplus (4, 5, 6)\tilde{y} \leq (2, 4, 7)$$
$$\tilde{x}, \tilde{y} \geq 0$$

solution :

First we will separate the problem into two FFLP problems and solve each one individually by the method in [17]

The first FFLP problem is written as:

$$\max \tilde{z}_1 = (7,10,14)\tilde{x} \oplus (20,25,35)\tilde{y}$$

subject to

$$(1,2,3)\tilde{x} \oplus (1,4,6)\tilde{y} \le (2,5,13)$$

$$(1,2,4)\tilde{x} \oplus (4,5,6)\tilde{y} \le (2,4,7)$$

$$\tilde{x},\tilde{y} \ge 0$$

let $\tilde{x} = (x_1,x_2,x_3)$ $\quad \tilde{y} = (y_1,y_2,y_3)$

now we will convert the inequalities to equations by adding arbitrary fuzzy variables $\tilde{L}_1 = (L_{11},L_{12},L_{13})$, $\tilde{L}_2 = (L_{21},L_{22},L_{23})$ and $\tilde{r}_1 = (r_{11},r_{12},r_{13})$, $\tilde{r}_2 = (r_{21},r_{22},r_{23})$ to the left and right sides respectively as follows:

$$\max \tilde{z}_1 = (7,10,14) \otimes (x_1,x_2,x_3) \oplus (20,25,35) \otimes (y_1,y_2,y_3)$$

subject to

$$(1,2,3) \otimes (x_1,x_2,x_3) \oplus (1,4,6) \otimes (y_1,y_2,y_3) \oplus (L_{11},L_{12},L_{13}) = (2,5,13) \oplus (r_{11},r_{12},r_{13})$$

$$(1,2,4) \otimes (x_1,x_2,x_3) \oplus (4,5,6) \otimes (y_1,y_2,y_3) \oplus (L_{21},L_{22},L_{23}) = (2,4,7) \oplus (r_{11},r_{12},r_{13})$$

By multiplication then addition of triangular fuzzy numbers we get

$$\max \tilde{z}_1 = (7x_1 + 20y_1, 10x_2 + 25y_2, 14x_3 + 35y_3)$$

subject to

$$(x_1 + y_1 + L_{11}, 2x_2 + 4y_2 + L_{12}, 3x_3 + 6y_3 + L_{13}) = (2 + r_{11}, 5 + r_{12}, 13 + r_{13})$$

$$(x_1 + 4y_1 + L_{21}, 2x_2 + 5y_2 + L_{22}, 4x_3 + 6y_3 + L_{23}) = (2 + r_{21}, 4 + r_{22}, 7 + r_{23})$$

Now, use the ranking function $R(\tilde{A}) = \frac{a_1+8a_2+a_3}{10}$ on the objective function, we get :

$$\max z_1 = 0.7x_1 + 2y_1 + 8x_2 + 20y_2 + 1.4x_3 + 3.5y_3$$

subject to

$$
\begin{aligned}
&x_1 + y_1 + L_{11} = 2 + r_{11} \\
&2x_2 + 4y_2 + L_{12} = 5 + r_{12} \\
&3x_3 + 6y_3 + L_{13} = 13 + r_{13} \\
&x_1 + 4y_1 + L_{21} = 2 + r_{21} \\
&2x_2 + 5y_2 + L_{22} = 4 + r_{22} \\
&4x_3 + 6y_3 + L_{23} = 7 + r_{23} \\
&x_2 - x_1 \geq 0, x_3 - x_2 \geq 0 \\
&y_2 - y_1 \geq 0, y_3 - y_2 \geq 0 \\
&L_{12} - L_{11} \geq 0, L_{13} - L_{12} \geq 0 \\
&L_{22} - L_{21} \geq 0, L_{23} - L_{22} \geq 0 \\
&r_{12} - r_{11} \geq 0, r_{13} - r_{12} \geq 0 \\
&r_{22} - r_{21} \geq 0, r_{23} - r_{22} \geq 0 \\
&L_{11} + 8L_{12} + L_{13} - r_{11} - 8r_{12} - r_{13} \geq 0 \\
&L_{21} + 8L_{22} + L_{23} - r_{21} - 8r_{22} - r_{23} \geq 0
\end{aligned}
\tag{5.1}
$$

Solving by using a linear programming solver we get :

$$
\begin{aligned}
&x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0 \\
&y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 6.8 \\
&z_1 = 23.9
\end{aligned}
$$

The optimal solution is :

$$\tilde{x} = (0, 0, 0)$$

$$\tilde{y} = (0, 0, 6.8)$$

$$\tilde{z}_1 = (7x_1 + 20y_1, 10x_2 + 25y_2, 14x_3 + 35y_3) = (0, 0, 238)$$

Now we will solve the second FFLP problem

$$\max \tilde{z}_2 = (10, 14, 25)\tilde{x} \oplus (25, 35, 40)\tilde{y}$$

subject to

$$(1, 2, 3)\tilde{x} \oplus (1, 4, 6)\tilde{y} \leq (2, 5, 13)$$

$$(1, 2, 4)\tilde{x} \oplus (4, 5, 6)\tilde{y} \leq (2, 4, 7)$$

$$\tilde{x}, \tilde{y} \geq 0$$

Using ranking function $R(\tilde{A}) = \frac{a_1+8a_2+a_3}{10}$ On the objective function we get :

$$\max z_2 = x_1 + 2.5y_1 + 11.2x_2 + 28y_2 + 20x_3 + 32y_3$$

subject to the set of constraints 5.1

Solving by using simplex method we get:

$$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0$$

$$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 6.83$$

$$z_2 = 218.7$$

The optimal solution is :

$$\tilde{x} = (0, 0, 0)$$

$$\tilde{y} = (0, 0, 6.8)$$

$$\tilde{z}_2 = (10x_1 + 25y_1, 14x_2 + 35y_2, 25x_3 + 40y_3) = (0, 0, 273.2)$$

step 4: construct the payoff matrix(5.3)

Since we have only two objective functions we will get p_3 by solving the problem with the same constraints with objective function equal zero.

$\tilde{P}_q$	$R(\tilde{z}_1, \tilde{P}_q)$	$R(\tilde{z}_2, \tilde{P}_q)$
$\tilde{P}_1$	23.8	27.32
$\tilde{P}_2$	23.8	27.32
$\tilde{P}_3$	21.11	27.96

Tab. 5.3: The payoff matrix

$$\tilde{P}_1 = (\tilde{x}, \tilde{y}),\ \tilde{x} = (0, 0, 0),\ \tilde{y} = (0, 0, 6.83)$$

$$\tilde{P}_2 = (\tilde{x}, \tilde{y}),\ \tilde{x} = (0, 0, 0),\ \tilde{y} = (0, 0, 6.83)$$

$$\tilde{P}_3 = (\tilde{x}, \tilde{y}),\ \tilde{x} = (0.14, 0.14, 0.14),\ \tilde{y} = (0.47, 0.74, 1)$$

$$\tilde{z}_1(\tilde{P}_1) = (0, 0, 238) \Rightarrow R(\tilde{z}_1(\tilde{P}_1) = 23.8$$

$$\tilde{z}_2(\tilde{P}_1) = (0, 0, 273.2) \Rightarrow R(\tilde{z}_2(\tilde{P}_1) = 27.32$$

$$\tilde{z}_1(\tilde{P}_2) = (0, 0, 238) \Rightarrow R(\tilde{z}_1(\tilde{P}_2) = 23.8$$

$$\tilde{z}_2(\tilde{P}_2) = (0, 0, 273.2) \Rightarrow R(\tilde{z}_2(\tilde{P}_2) = 27.32$$

$$\tilde{z}_1(\tilde{P}_3) = (14.98, 19.9, 36.96) \Rightarrow R(\tilde{z}_1(\tilde{P}_2) = 23.8$$

$$\tilde{z}_2(\tilde{P}_3) = (13.15, 27.86, 43.5) \Rightarrow R(\tilde{z}_2(\tilde{P}_3) = 27.6$$

Step 5: construct the ratio matrix (5.4)

	z_1	z_2
P_1/P_2	1	1
P_2/P_3	1.13	0.98

Tab. 5.4: The ratio matrix

Step 6: Solve the following matrix as two-player zero sum game

minimize $z = v$

subject to

$$v - \sum a_{ij} w_j \geq 0$$

$w_1 + w_2 = 1$

$w_i \geq 0$ i= $1, ..., n$

minimize $z = v$

subject to

$v - w_1 - w_2 \geq 0$

$v - 1.13w_1 - 0.98w_2 \geq 0$

$w_1 + w_2 = 1$

v unrestricted

$w_1, w_2 \geq 0$

by using a linear programming solver we get:

$$w_1 = 0 \qquad w_2 = 1 \qquad v = 1$$

hence the weights for z_1 and z_2 are 0 and 1 respectively

Step 7: we will multiply z_1 and z_2 by the normalized weights to form a single linear programming problem as follows:

$z = w_1 * \frac{z_1}{z_1^*} + w_2 * \frac{z_2}{z_2^*}$

$\Rightarrow \max z = \frac{z_2}{218.7}$

subject to the set of constraints 5.1

hence the fuzzy compromise solution of the FFMOLPP is :

$$(\tilde{x}, \tilde{y}) = ((0, 0, 0), (0, 0, 6.8)) \qquad \tilde{z}_1 = (0, 0, 238) \qquad \tilde{z}_2 = (0, 0, 273.2)$$

Example 5.0.2 *Solving FFMOLPP With Trapezoidal Fuzzy Numbers By Using Game Theory Approach.*

maximize $\tilde{z}_1 = (1, 3, 5, 7)\tilde{x} \oplus (4, 6, 10, 12)\tilde{y}$

maximize $\tilde{z}_2 = (4, 6, 10, 12)\tilde{x} \oplus (20, 22, 26, 28)\tilde{y}$

subject to

$$(0.25, 0.5, 1.25, 2)\tilde{x} \oplus (1, 1.5, 2.5, 3)\tilde{y} \leq (10, 15, 25, 30)$$

$$(0.25, 0.5, 1.25, 2)\tilde{x} \leq (2, 4, 8, 10)$$

$$\tilde{x}, \tilde{y} \geq 0$$

solution:

Step 1: separate the problem into two FFLP problem and solve each one individually.

The first FFLP problem is:

$$\max \tilde{z}_1 = (1, 3, 5, 7)\tilde{x} \oplus (4, 6, 10, 12)\tilde{y}$$

subject to

$$(0.25, 0.5, 1.25, 2)\tilde{x} \oplus (1, 1.5, 2.5, 3)\tilde{y} \leq (10, 15, 25, 30)$$

$$(0.25, 0.5, 1.25, 2)\tilde{x} \leq (2, 4, 8, 10)$$

Let $\tilde{x} = (x_1, x_2, x_3, x_4)$, $\tilde{y} = (y_1, y_2, y_3, y_4)$

Converting the inequalities to equalities by adding arbitrary trapezoidal fuzzy variables $\tilde{L}_1 = (L_{11}, L_{12}, L_{13}, L_{14})$, $\tilde{L}_2 = (L_{21}, L_{22}, L_{23}, L_{24})$ and $\tilde{r}_1 = (r_{11}, r_{12}, r_{13}, r_{14})$, $\tilde{r}_2 = (r_{21}, r_{22}, r_{23}, r_{24})$ to the left and right sides respectively

$\max \tilde{z}_1 = (1, 3, 5, 7)\tilde{x} \oplus (4, 6, 10, 12)\tilde{y}$

subject to

$(0.25, 0.5, 1.25, 2)\tilde{x} \oplus (1, 1.5, 2.5, 3)\tilde{y} \oplus (L_{11}, L_{12}, L_{13}, L_{14}) \leq (10, 15, 25, 30) \oplus (r_{11}, r_{12}, r_{13}, r_{14})$

$(0.25, 0.5, 1.25, 2)\tilde{x} \oplus (L_{21}, L_{22}, L_{23}, L_{24}) \leq (2, 4, 6, 8) \oplus (r_{21}, r_{22}, r_{23}, r_{24}))$

$\tilde{x}, \tilde{y} \geq 0$

now, using multiplication and then addition of trapezoidal fuzzy numbers we get

$\max \tilde{z}_1 = (x_1 + 4y_1, 3x_2 + 6y_2, 5x_3 + 10y_3, 7x_4 + 12y_4)$

subject to

$(0.25x_1 + y_1 + L_{11}, 0.5x_2 + 1.5y_2 + L_{12}, 1.25x_3 + L_{13}, 2x_4 + 3y_4 + L_{14}) \leq (10 + r_{11}, 15 + r_{12}, 25 + r_{13}, 30 + r_{14})$

$(0.25x_1 + L_{21}, 0.55x_2 + L_{22}, 1.25x_3 + L_{23}, 2x_4 + L_{24}) \leq (2 + r_{21}, 4 + r_{22}, 8 + r_{23}, 10 + r_{24})$

now applying $R(\tilde{A}) = \frac{a_1+a_2+a_3+a_4}{4}$ on $\tilde{z}_1$ we get

max $z_1 = 0.25x_1 + 0.75x_2 + 1.25x_3 + 1.75x_4 + y_1 + 1.5y_2 + 2.5y_3 + 3y_4$

subject to

$$
\begin{aligned}
&0.5x_1 + y_1 + L_{11} = 10 + r_{11} \\
&0.5x_2 + 1.5y_2 + L_{12} = 15 + r_{12} \\
&1.25x_3 + 2.5y_3 + L_{13} = 25 + r_{13} \\
&2x_4 + 3y_4 + L_{14} = 30 + r_{14} \\
&0.25x_1 + L_{21} = 2 + r_{21} \\
&0.5x_2 + L_{22} = 4 + r_{22} \\
&1.25x_3 + L_{23} = 8 + r_{23} \\
&2x_4 + L_{24} = 10 + r_{24} \\
&L_{12} - L_{11} \geq 0 \quad L_{13} - L_{12} \geq 0 \quad L_{14} - L_{13} \geq 0 \\
&L_{22} - L_{21} \geq 0 \quad L_{23} - L_{22} \geq 0 \quad L_{24} - L_{23} \geq 0 \\
&r_{12} - r_{11} \geq 0 \quad r_{13} - r_{12} \geq 0 \quad r_{14} - r_{13} \geq 0 \\
&r_{22} - r_{21} \geq 0 \quad r_{23} - r_{22} \geq 0 \quad r_{24} - r_{23} \geq 0 \\
&x_2 - x_1 \geq 0 \quad x_3 - x_2 \geq 0 \quad x_4 - x_3 \geq 0 \\
&y_2 - y_1 \geq 0 \quad y_3 - y_2 \geq 0 \quad y_4 - y_3 \geq 0 \\
&L_{11} + L_{12} + L_{13} + L_{14} - r_{11} - r_{12} - r_{13} - r_{14} \geq 0 \\
&L_{21} + L_{22} + L_{23} + L_{24} - r_{21} - r_{22} - r_{23} - r_{24} \geq 0
\end{aligned}
\tag{5.2}
$$

by a linear programming solver we get :

$$
\begin{array}{llll}
x_1 = 0 & x_2 = 13.7 & x_3 = 13.7 & x_4 = 0 \\
y_1 = 5.4 & y_2 = 5.4 & y_3 = 2.6 & y_4 = 2.6 \\
z_1 = 55 & & &
\end{array}
$$

The second FFLPP is:

$\max \tilde{z}_2 = (4, 6, 10, 12)\tilde{x} \oplus (20, 22, 26, 28)\tilde{y}$

subject to

$(0.25, 0.5, 1.25, 2)\tilde{x} \oplus (1, 1.5, 2.5, 3)\tilde{y} \leq (10, 15, 25, 30)$

$(0.25, 0.5, 1.25, 2)\tilde{x} \leq (2, 4, 8, 10)$

$\tilde{x}, \tilde{y} \geq 0$

by using multiplication and then addition of trapezoidal fuzzy numbers as the previous step and then applying $R(\tilde{A})$ on $\tilde{z}_2$ finally we get the following LPP :

maximize $z_2 = x_1 + 1.5x_2 + 2.5x_3 + 3x_4 + 5y_1 + 5.5y_2 + 6.5y_3 + 7y_4$

subject to the set of constraints (5.2)

solving by any LP solver we get :

$$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0 \qquad x_4 = 0$$

$$y_1 = 20.8 \qquad y_2 = 0.8 \qquad y_3 = 0 \qquad y_4 = 0$$

$$z_2 = 218.4$$

construct the payoff matrix (5.5)

$\tilde{P}_q$	$R(\tilde{z}_1(\tilde{P}_q))$	$R(\tilde{z}_2(\tilde{P}_q))$
$\tilde{P}_1$	55.2	146.6
$\tilde{P}_2$	22	108.4
$\tilde{P}_3$	50.8	158

Tab. 5.5: The payoff matrix

$\tilde{P}_1 = (\tilde{x}, \tilde{y}) = ((0, 13.7, 13.7, 0), (5.4, 5.4, 2.6, 2.6))$

$\tilde{P}_2 = (\tilde{x}, \tilde{y}) = ((0, 0, 0, 0), (20.8, 0.8, 0, 0))$

$\tilde{P}_3 = (\tilde{x}, \tilde{y}) = ((10, 0, 6.4, 5), (10, 10, 1.64, 1.64))$

$\tilde{z}_1(\tilde{P}_1) = (21.6, 73.8, 94.5, 31.2) \Rightarrow R(\tilde{z}_1(\tilde{P}_1)) = 55.2$

$\tilde{z}_2(\tilde{P}_1) = (108, 201, 204.6, 72.8) \Rightarrow R(\tilde{z}_2(\tilde{P}_2)) = 146.5$

$\tilde{z}_1(\tilde{P}_2) = (83.2, 4.8, 0, 0) \Rightarrow R(\tilde{z}_1(\tilde{P}_2)) = 22$

$\tilde{z}_2(\tilde{P}_2) = (417, 17.6, 0, 0) \Rightarrow R(\tilde{z}_1(\tilde{P}_2)) = 108.4$

$\tilde{z}_1(\tilde{P}_3) = (40, 60, 48.4, 53.68) \Rightarrow R(\tilde{z}_3(\tilde{P}_3)) = 50.8$

$\tilde{z}_2(\tilde{P}_3) = (200, 220, 106.6, 105.92) = 158.1$

construct the ratio matrix (5.6)

	z_1	z_2
P_1/P_2	2.5	1.35
P_3/P_3	0.43	0.667

Tab. 5.6: The ratio matrix

now, solve the matrix as a two-player zero sum game :

minimize $z = v$

subject to

$v - 2.5y_1 - 1.35y_2 \geq 0$

$v - 0.43 - 0.67y_2 \geq 0$

$w_1 + w_2 = 1$

$w_i \geq 0 \ ,i = 1, ..., n$

v unrestricted

hence

$w_1 = 0 \qquad w_2 = 1 \qquad v = 1.35$

Now we will form a single linear programming problem as follows:

$\max z = w_1 * \frac{z_1}{z_1^*} + w_2 * \frac{z_2}{z_2^*}$

subject to the set of constraints (5.2)

$\Rightarrow \max z = 0 + \frac{z_2}{108.4}$

subject to the set of constraints (5.2)

using any LP solver we get :

$$\tilde{x} = (0, 0, 0, 0)$$

$$\tilde{y} = (20.8, 0.8, 0, 0)$$

$$\tilde{z}_1 = (83.2, 4.8, 0, 0)$$

$$\tilde{z}_2 = (416, 17.6, 0, 0)$$

Example 5.0.3 *solving FFMOLPP with hexagonal fuzzy number using game theory approach.*

$\max \tilde{z}_1 = (50, 60, 70, 80, 90, 100)\tilde{x} \oplus (65, 75, 85, 95, 105, 115)\tilde{y}$

$\max \tilde{z}_2 = (35, 45, 55, 65, 75, 85)\tilde{x} \oplus (50, 60, 70, 80, 90, 100)\tilde{y}$

subject to

$(280, 290, 300, 300, 310, 320)\tilde{x} \oplus (230, 240, 250, 250, 260, 270)\tilde{y}$

$\leq (12000, 13000, 14000, 14000, 15000, 16000)$

$(180, 190, 200, 200, 210, 220)\tilde{x} \oplus (180, 190, 200, 200, 210, 220)\tilde{y}$

$\leq (11500, 12000, 12500, 12500, 13000, 13500)$

$\tilde{x}, \tilde{y} \geq 0$

Solution:

Let $\tilde{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$, $\tilde{y} = (y_1, y_2, y_3, y_4, y_5, y_6)$

Separate the problem into two FFLpp

The first FFLPP is

max $\tilde{z}_1 = (50, 60, 70, 80, 90, 100)\tilde{x} \oplus (65, 75, 85, 95, 105, 115)\tilde{y}$

subject to

$(180, 190, 200, 200, 210, 220)\tilde{x} \oplus (230, 240, 250, 250, 260, 270)\tilde{y}$

$\leq (12000, 13000, 14000, 14000, 15000, 16000)$

$(280, 290, 300, 300, 310, 320)\tilde{x} \oplus (180, 190, 200, 200, 210, 220)\tilde{y}$

$\leq (11500, 12000, 12500, 12500, 13000, 13500)$

Convert the inequalities into equalities by adding arbitrary hexagonal fuzzy variables $\tilde{L}_1 = (L_{11}, L_{12}, L_{13}, L_{14}, L_{15}, L_{16})$, $\tilde{L}_2 = (L_{21}, L_{22}, L_{23}, L_{24}, L_{25}, L_{26})$ to the left side and $\tilde{r}_1 = (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16})$, $\tilde{r}_2 = (r_{21}, r_{22}, r_{23}, r_{24}, r_{25}, r_{26})$ to the right side.

max $\tilde{z}_1 = (50, 60, 70, 80, 90, 100)\tilde{x} \oplus (65, 75, 85, 95, 105, 115)\tilde{y}$

subject to

$(180, 190, 200, 200, 210, 220)\tilde{x} \oplus (230, 240, 250, 250, 260, 270)\tilde{y} \oplus (L_{11}, L_{12}, L_{13}, L_{14}, L_{15}, L_{16}) = (12000, 13000, 14000, 14000, 15000, 16000) \oplus (r_{11}, r_{12}, r_{13}, r_{14}, r_{15}, r_{16})$

$(280, 290, 300, 300, 310, 320)\tilde{x} \oplus (180, 190, 200, 200, 210, 220)\tilde{y} \oplus (L_{21}, L_{22}, L_{23}, L_{24}, L_{25}, L_{26}) = (11500, 12000, 12500, 12500, 13000, 13500)$

$\tilde{x}, \tilde{y} \geq 0$

Using multiplication and then addition of hexagonal fuzzy numbers we get:

max $\tilde{z}_1 = (50x_1 + 65y_1, 60x_2 + 75y_2, 70x_3 + 85y_3, 80x_4 + 95y_4, 90x_5 + 105y_5, 100x_6 + 115y_6)$

subject to$(180x_1 + 230y_1 + L_{11}, 190x_2 + 240y_2 + L_{12}, 200x_3 + 250y_3 + L_{13}, 200x_4 + 250y_4 + L_{14}, 210x_5 + 260y_5 + L_{15}, 220x_6 + 260y_6 + L_{16})$

$= (12000 + r_{11}, 13000 + r_{12}, 14000 + r_{13}, 14000 + r_{14}, 15000 + r_{15}, 16000 + r_{16})$

$(280x_1 + 180y_1 + L_{21}, 290x_2 + 190y_2 + L_{22}, 300x_3 + 200y_3 + L_{23}, 300x_4 + 200y_4 + L_{24}, 310x_5 + 210y_5 + L_{25}, 320x_6 + 220y_6 + L_{26})$

$= (11500 + r_{21}, 12000 + r_{22}, 12500 + r_{23}, 12500 + r_{24}, 13000 + r_{25}, 13500 + r_{26})$

Applying $R(\tilde{A}) = \frac{a_1+a_2+a_3+a_4+a_5+a_6}{6}$ on $\tilde{z}_1$ we get :

max $z_1 = \frac{50}{6}x_1+\frac{65}{6}y_1+10x_2+12.5y_2+\frac{79}{6}x_3+\frac{85}{6}y_3+\frac{80}{6}x_4+\frac{95}{6}y_4+15x_5+17.5y_5+\frac{100}{6}x_6+\frac{65}{6}y_6$

subject to

$$
\begin{aligned}
&180x_1 + 230y_1 + L_{11} = 12000 + r_{11}\\
&190x_2 + 240y_2 + L_{12} = 13000 + r_{12}\\
&200x_3 + 250y_3 + L_{13} = 14000 + r_{13}\\
&200x_4 + 250y_4 + L_{14} = 14000 + r_{14}\\
&210x_5 + 260y_5 + L_{15} = 15000 + r_{15}\\
&220x_6 + 270y_6 + L_{16} = 16000 + r_{16}\\
&280x_1 + 180y_1 + L_{21} = 11500 + r_{21}\\
&290x_1 + 190y_2 + L_{22} = 12000 + r_{22}\\
&300x_1 + 200y_3 + L_{23} = 12500 + r_{23}\\
&300x_1 + 200y_4 + L_{24} = 12500 + r_{24}\\
&310x_1 + 210y_5 + L_{25} = 13000 + r_{25}\\
&320x_1 + 220y_6 + L_{26} = 13500 + r_{26}\\
&x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5, y_6 \geq 0\\
&x_2 - x_1 \geq 0, x_3 - x_2 \geq 0, x_4 - x_3 \geq 0, x_5 - x_4 \geq 0, x_6 - x_5 \geq 0\\
&y_2 - y_1 \geq 0, y_3 - y_2 \geq 0, y_4 - y_3 \geq 0, y_5 - y_4 \geq 0, y_6 - y_5 \geq 0\\
&L_{12} - L_{11} \geq 0, L_{13} - L_{12} \geq 0, L_{14} - L_{13} \geq 0, L_{15} - L_{14} \geq 0, L_{16} - L_{15} \geq 0\\
&L_{22} - L_{21} \geq 0, L_{23} - L_{22} \geq 0, L_{24} - L_{23} \geq 0, L_{25} - L_{24} \geq 0, L_{26} - L_{25} \geq 0\\
&r_{12} - r_{11} \geq 0, r_{13} - r_{12} \geq 0, r_{14} - r_{13} \geq 0, r_{15} - r_{14} \geq 0, r_{16} - r_{15} \geq 0\\
&r_{22} - r_{21 \geq 0}, r_{23} - r_{22} \geq 0, r_{24} - r_{23} \geq 0, r_{25} - r_{24} \geq 0, r_{26} - r_{25} \geq 0\\
&L_{11} + L_{12} + L_{13} + L_{14} + L_{15} + L_{16} - r_{11} - r_{12} - r_{13} - r_{14} - r_{15} - r_{16} \geq 0\\
&L_{21} + L_{22} + L_{23} + L_{24} + L_{25} + L_{26} - r_{21} - r_{22} - r_{23} - r_{24} - r_{25} - r_{26} \geq 0
\end{aligned}
\tag{5.3}
$$

Solution:

$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0 \qquad x_4 = 0 \qquad x_5 = 0 \qquad x_6 = 0$

$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 0 \qquad y_4 = 0 \qquad y_5 = 0 \qquad y_6 = 295.5$

The optimal solution for FFLPP 1 is

$\tilde{x} = (0, 0, 0, 0, 0, 0), \tilde{y} = (0, 0, 0, 0, 0, 295.5)$

$\tilde{z}_1 = (0, 0, 0, 0, 0, 33982.5)$

Now we will solve FFLPP 2

which is:

max $\tilde{z}_2 = (35, 45, 55, 65, 75, 85)\tilde{x} \oplus (50, 60, 70, 80, 90, 100)\tilde{y}$

subject to

$(180, 190, 200, 200, 210, 220)\tilde{x} \oplus (230, 240, 250, 250, 260, 270)\tilde{y}$

$\leq (12000, 13000, 14000, 14000, 15000, 16000)$

$(280, 290, 300, 300, 310, 320)\tilde{x}\oplus, (180, 190, 200, 200, 210, 220)\tilde{y}$

$\leq (11500, 12000, 12500, 12500, 13000, 13500)$

$\tilde{x}, \tilde{y} \geq 0$

Applying $R(\tilde{A}) = \frac{a_1, a_2, a_3, a_4, a_5, a_6}{6}$ on $\tilde{z}_2$ and adding fuzzy variables to the left and right sides of the constraints the final LPP will be:

max $z_2 = \frac{35}{6}x_1 + \frac{50}{6}y_1 + 7.5x_2 + y_2 + \frac{55}{6}x_3 + \frac{70}{6}y_3 + \frac{65}{6}x_4 + \frac{80}{6}y_4 + 12.5y_5 + 15y_5 + \frac{85}{6}x_6 + \frac{100}{6}y_6$

subject to the set of constraints (5.2)

The solution is :

$x_1 = 0 \qquad x_2 = 0 \qquad x_3 = 0 \qquad x_4 = 0 \qquad x_5 = 0 \qquad x_6 = 0$

$y_1 = 0 \qquad y_2 = 0 \qquad y_3 = 0 \qquad y_4 = 0 \qquad y_5 = 0 \qquad y_6 = 295.5$

$z_2 = 4924$

The optimal solution for FFLPP 2 is

$\tilde{x} = (0, 0, 0, 0, 0, 0), \tilde{y} = (0, 0, 0, 0, 0, 295.5)$

$\tilde{z}_2 = (0, 0, 0, 0, 0, 29550)$

Using weighted sum method with normalization:

$z_1^* = 5972 \qquad z_2^* = 4742.5$

The LPP is formed as follows:

$\max z = w_1 \times \frac{\tilde{z}_1}{5972} + w_2 \times \frac{\tilde{z}_2}{4742.5}$

subject to the set of constraints (4.3)

assume that $w_1 = 0.4$ $w_2 = 0.6$

The optimal solution is:

$\tilde{x} = (0, 12.2, 0, 0, 6.3, 3.28)$

$\tilde{y} = (52.2, 44.5, 56, 62.5, 52.6, 56.6)$

$\tilde{z}_1 = (3393, 4069.5, 4760, 5937.5, 7129, 6837)$

$\tilde{z}_2 = (2610, 2670, 3920, 5000, 5206.5, 5938.8)$

Brief comparison between the previous two methods :

first we want to recall that we can get the best or optimal solution for each objective function by solving each objective individually neglecting the other objectives , we assume that the best value for the objective function$z_i is$ denoted by z_i^*.

to compare the results that we got we will convert the fuzzy solution to crisp by using the ranking function and then see how much is it closs to the optimal value . In example 4.2.1 ,$z_1^* = 21.483$ and $z_2^* = 29.13$ we use the ranking function method to get a crisp MOLPP and then we solved it by preemptive method and by weighted sum method . using preemptive method the solution was : $\tilde{z}_1 = (14, 20, 40.85)$, $\tilde{z}_2 = (20, 28, 46.68)$

applying the ranking function we get : $z_1 = 21.485$ and $z_2 = 29$ which are very closed to z_1^* and z_2^* . using the game theory approach for the same example we got $\tilde{z}_1 = (0, 0, 238)$ and $\tilde{z}_2 = (0, 0, 273.2)$ using the ranking function for the triangular fuzzy number we got $z_1 = 23.8$ and $z_2 = 27.32$ these results are also very closed to z_1^* and z_2^* . the first method will be better to use when the decision maker has a priority levels for the objective functions in this case he/she can either choose weights for each objective function or rank the objective functions according to their priorities for the decision maker. the second method will be better to use in case that the decision maker has no priorities for the objective functions so that when using the game theory approach, the ratio matrix will automatically generate the weights without the interference of the decision maker.

Chapter 6

Alpha cut method for solving fuzzy multi objective linear programming problem

In this chapter alpha cut method will be used to solve FMOLPP, the method will be illustrated by numerical examples.

Consider the following FMOLPP

$$
\begin{aligned}
&Max/Min \quad \tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_k \\
&subjectto \\
&\tilde{A}x(\leq \geq =)\tilde{b} \\
&x \geq 0
\end{aligned}
$$

- where $\tilde{z}_i = C_i^T x$, $i = 1, 2, ..., K$ is the i^{th} objective function.
- $\tilde{C}_i = (\tilde{C}_1, \tilde{C}_2, ..., \tilde{C}_n)$ consists of coefficients.
- the vector $x = (x_1, x_2, ..., x_n)$ consists of decision variable vector.
- $\tilde{A}$ is the coefficient $m \times n$ matrix ,where m is the number of constraints.
- $x \in X$ where X is the feasible region.

6.1 Alpha cut method

For a specific α, if the objective function $\tilde{z}_i$ is to minimize then $\tilde{z}_i$, $i = 1, ..., k$ can be replaced by the lower bound using alpha cut i.e $(\tilde{z}_i)^L_\alpha = (\tilde{C}_i^T)^L_\alpha x$, $i = 1, ..., k$ [14].

If the objective function is to maximize then $\tilde{z}_i$, $i = 1, ..., k$ can be replaced with the upper bound of its alpha-cut i.e $(\tilde{z}_i)^U_\alpha = (\tilde{C}_i^T)^U_\alpha x$, $i = 1, ..., k$

The constraints:

$\tilde{A}_j x \geq \tilde{b}_j \qquad j = 1, 2, ..., q$

$\tilde{A}_j x \leq \tilde{b}_j \qquad j = q + 1, ..., m$

will be rewritten as follows:

$(\tilde{A}_j)^U_\alpha x \geq (\tilde{b}_j)^L_\alpha \qquad j = 1, 2, ..., q$

$(\tilde{A}_j)^L_\alpha x \leq (\tilde{b}_j)^U_\alpha \qquad j = q + 1, ..., m$

hence the FMOLPP will be converted to α-multi objective LPP as follows:

min $((\tilde{z}_1(x))^L_\alpha, (\tilde{z}_2(x))^L_\alpha, ..., (\tilde{z}_k(x))^L_\alpha)$ / max $((\tilde{z}_1(x))^U_\alpha, (\tilde{z}_2(x))^U_\alpha, ..., (\tilde{z}_k(x))^U_\alpha)$

subject to

$$\begin{aligned} (\tilde{A}_j)^U_\alpha x \geq (\tilde{b}_j)^L_\alpha \qquad j = 1, 2, ..., q \\ (\tilde{A}_j)^L_\alpha x \leq (\tilde{b}_j)^U_\alpha \qquad j = q + 1, ..., m \end{aligned} \tag{6.1}$$

Let $E_i = min[(\tilde{z}_i(x))^L_\alpha]$, $\quad i = 1, 2, ..., k$ be the minimum solution for the individual $(\tilde{z}_i(x))^L_\alpha$: which will be obtained by solving the following problem:

$\min(\tilde{z}_i(x))^L_\alpha$ subject to the set of constraints (6.1).

Let $F_i = max[(\tilde{z}_i(x))^U_\alpha]$, $\quad i = 1, 2, ..., k$ be the maximum solution for each individual $(\tilde{z}_i(x))^U_\alpha$ which will be obtained by solving the following problem:

$\max(\tilde{z}_i(x))^U_\alpha$ subject to the set of constraints (6.1).

the normalized weights for each objective function will be obtained as follows:

$$w_i = \frac{F_i - E_i}{\sum_{i=1}^{k}(F_i - E_i)}, \quad \sum_{i=1}^{k} w_i = 1, \quad i = 1, 2, ..., k \tag{6.2}$$

Here we will show the solution procedure that we will use for alpha cut method :

step 1: the decision maker have to choose the value of α where $\alpha \in [0,1]$ and then construct the α-multi objective LPP.

step 2: calculate E_i which is the lower band for the objective function and calculate F_i which is the upper bound of the objective function.

step 3: find the corresponding weights for each objective function from (6.2).

step 4: form a single LPP by using the weighted sum and then solve to get the optimal solution denoted by S.

Numerical examples:

Example 6.1.1 *Solving FMOLPP with triangular fuzzy numbers using alpha cut method*

max $\tilde{z}_1 = (7, 10, 14)x \oplus (20, 25, 35)y$

max $\tilde{z}_2 = (10, 14, 25)x \oplus (25, 35, 40)y$

subject to

$(1, 2, 3)x \oplus (1, 4, 6)y \leq (2, 5, 13)$

$(1, 2, 3)x \oplus (4, 5, 6)y \leq (2, 4, 7)$

step 1:

Formulate the α-MOLPP:

Max $(z_1)^U_\alpha = (14(1-\alpha) + 10\alpha)x + (35(1-\alpha) + 25\alpha)y$

Max $(z_2)^U_\alpha = (25(1-\alpha) + 14\alpha)x + (40(1-\alpha) + 35\alpha)y$

Subject to

$((1-\alpha) + 2\alpha)x + ((1-\alpha) + 4\alpha)y \leq 13$

$((1-\alpha) + 2\alpha)x + (4(1-\alpha) + 5\alpha)y \leq 7$

let $\alpha = 0.5$ the problem will be:

$$\text{Max}(z_1)^u_\alpha = 12x + 30y$$

$$\text{Max}(z_2)^u_\alpha = 19.5x + 37.5y$$

subject to

$$
\begin{aligned}
1.5x + 2.5y &\leq 13 \\
1.5x + 4.5y &\leq 7
\end{aligned}
\tag{6.3}
$$

step 2:

Find E_i (the minimum value of each objective function) and F_i (the maximum value of each objective function)

Solving $\max(z_1)^u_{0.5} = 12x + 30y$

subject to the set of constraints (6.3)

we get:

$(x, y) = (4.7, 0)$

$(z_1)^U_{0.5} = 56 = F_1$

Now solving $\max(z_2)^U_{0.5} = 19.5x + 37.5y$

subject to the set of constraints (6.3)

using any linear programming solver we get :

$(x, y) = (4.7, 0)$

$(z_2)^U_{0.5} = 91 = F_2$

$\min(z_1)^L_\alpha = ((1-\alpha) \times 7 + 10\alpha)x + ((1-\alpha) \times 20 + 25\alpha)y$

$\min(z_2)^L_\alpha = (14\alpha + 10(1-\alpha))x + ((1-\alpha) \times 25 + 35\alpha)y$

assume $\alpha = 0.5$ hence ,

$\min(z_1)^L_{0.5} = 8.5x + 22.5y$

$\min(z_2)^L_{0.5} = 12x + 3y$

Solving $\min(z_1)^L_{0.5} = 8.5x + 22.5y$

subject to the set of constraints (6.3)

we get :

$(x, y) = (0.0)$

$(z_1)^L_{0.5} = 0 = E_1$

Solving $\min(z_2)^L_{0.5} = 12x + 30y$

subject to the set of constraints (6.3)

we get :

$(x, y) = (0, 0)$

$(z_2)^L_{0.5} = 0 = E_2$

Summarizing the results in tables 6.1 and 6.2

Objective Function	$(z_1)^L_{0.5}$	$(z_2)^L_{0.5}$
E_i	0	0

Tab. 6.1: The minimum optimum values for the individual objective functions z_1 and z_2

Objective Function	$(z_1)^U_{0.5}$	$(z_2)^U_{0.5}$
F_i	56	91

Tab. 6.2: The maximum optimum values for the individual objective functions z_1 and z_2

step 3:

computing the weights from (6.2)

$w_1 = \frac{F_1 - E_1}{(F_1 - E_1) + (F_2 - E_2)} = \frac{56-0}{147} = 0.38$

$w_2 = \frac{F_2 - E_2}{(F_1 - E_1) + (F_2 - E_2)} = \frac{91-0}{147} = 0.62$

step 4:

form a single LPP by multiplying z_1 by w_1 and z_2 by w_2

$max\ z = w_1 \times (z_1)^U_{0.5} + w_2 \times (z_2)^U_{0.5}$

$= 0.38(12x + 30y) + 0.62(19.5x + 37.5)y$

$\Rightarrow\ max\ z = 16.65x + 34.65y$

subject to the set of constraints (6.3)

solving by a linear programming solver the optimal solution is :

$(x^*, y^*) = (4.7, 0)$

$z_1^* = (32.9, 47, 65.8)$

$z_2^* = (47, 65.8, 117.5)$

Example 6.1.2 *Solving FMOLPP with trapezoidal numbers using alpha cut method.*

maximize $\tilde{z}_1 = (1, 3, 5, 7)x \oplus (4, 6, 10, 12)y$

maximize $\tilde{z}_2 = (4, 6, 10, 12)x \oplus (20, 22, 26, 28)y$

subject to

$(0.25, 0.5, 1.25, 2)x \oplus (1, 1.5, 2.5, 3)y \leq (10, 15, 25, 30)$

$(0.25, 0.5, 1.25, 2)x \leq (2, 4, 8, 10)$

$x, y \geq 0$

Solution:

step 1:

formulate the α-MOLPP, let $\alpha = 0.6$

$$\text{Max}(z_1)_{0.6}^{U} = 5.8x + 10.8y$$

$$\text{Max}(z_2)_{0.6}^{U} = 10.8x + 26.8y$$

subject to

$$\begin{aligned} 3.4x + 1.3y &\leq 27 \\ 0.4x &\leq 8.8 \end{aligned} \qquad (6.4)$$

step 2:

Find E_i and F_i

solving max $(z_1)_{0.6}^{u} = 5.8x + 10.8y$

subject to the set of constraints (6.4) , we get :

$(x, y) = (0, 20.8)$

$(z_1)_{0.6}^{U} = 224.3 = F_1$

solving $\max(z_2)_{0.6}^{U} = 10.8x + 26.8y$

subject to the set of constraints (6.4), we get:

$(x, y) = (0, 20.8)$

$(z_2)^U_{0.6} = 556.6 = F_2$

Solving $\min(z_1)^L_{0.6} = 2.2x + 5y$

subject to the set of constraints (6.4) , we get :

$(x, y) = (0, 0)$

$(z_1)^L_{0.6} = 0 = E_1$

Solving $\min(z_2)^L_{0.6} = 5x + 21.2y$

subject to the set of constraints (6.4) , we get :

$(x, y) = (0, 0)$

$(z_2)^L_{0.6} = 0 = E_2$

step 3:

computing the weights for z_1 and z_2 respectively :

$w_1 = \frac{F_1 - E - 1}{(F_1 - E_1) + (F_2 - E_2)} = \frac{224.3 - 0}{224.3 + 556.6} = 0.29$

$w_2 = \frac{F_2 - E - 2}{(F_1 - E_1) + (F_2 - E_2)} = \frac{556.6 - 0}{224.3 + 556.6} = 0.71$

step 4:

$\max z = 0.29 \times (5.8x + 10.8y) + 0.71 \times (10.8x + 26.8)y$

$= 9.35x + 22.16y$

subject to the set of constraints (6.4)

the optimal solution is :

$(x^*, y^*) = (3, 6) = S$

$z_1^* = (27, 45, 75, 93)$

$z_2^* = (132, 150, 186, 204)$

In case the decision maker has a priority levels for the objective functions we can use pre-emptive method as follows:

Solving $\max(z_1)^U_{0.6} = 5.8x + 10.8y$

subject to the set of constraints (6.4)

we get :

$(x, y) = (0, 20.79) = S$

$(z_1(S))^U_{0.6} = 224.5$

Solving $\max(z_2)^U_{0.6} = 10.8x + 26.8y$

subject to the set of constraint (6.4)

$5.8x + 10.8 \geq 224.5$ (additional constraint)

we get the optimal solution :

$(x^*, y^*) = (0, 20.79)$

Example 6.1.3 *Solving FMOLPP with hexagonal fuzzy numbers using alpha cut method*

$\max \tilde{z}_1 = (50, 60, 70, 80, 90, 100)x_1 \oplus (65, 75, 85, 95, 105, 115)x_2$

$\max \tilde{z}_2 = (35, 45, 55, 65, 75, 85)x_1 \oplus (50, 60, 70, 80, 90, 100)x_2$

subject to

$(180, 190, 200, 200, 210, 220)x_1 \oplus (230, 240, 250, 250, 260, 270)y$

$\leq (12000, 13000, 14000, 15000, 16000)$

$(280, 290, 300, 300, 310, 320)x_1 \oplus (180, 190, 200, 200, 210, 220)x_2$

$\leq (11500, 12000, 12500, 12500, 13000, 13500)$

Solution:

step 1:

formulate the α-cut MOLPP, let $\alpha = 0$

$$\text{Max}(z_1)^U_0 = 100x_1 + 115x_2$$

$$\text{Max}(z_2)^U_0 = 85x_1 + 100x_2$$

subject to

$$180x_1 + 230x_2 \leq 16000$$
$$280x_1 + 180x_2 \leq 13500 \tag{6.5}$$

step 2:

find E_1, E_2, F_1, F_2

solving $\max(z_1(x))_0^U = 100x_1 + 115x_2$

subject to the set of constraints (6.5) , we get:

$(x_1, x_2) = (7, 64.06)$

$(z_1(x))_0^U = 8070.3 = F_1$

Solving $\max(z_1(x))_0^U = 85x_1 + 100x_2$

subject to the set of constraints (6.5) ,we get

$(x_1, x_2) = (7, 64.06)$

$(z_2(x))_0^U = 7003.9 = F_2$

Solving $\min(z_1(x_1))_0^L = 50x_1 + 65x_2$

subject to the set of constraint (6.5), we get :

$(x_1, x_2) = (0, 0)$

$(z_1(x))_0^L = 0 = E_1$

Solving $\min(z_2(x))_0^L = 35x_1 + 50x_2$

we get :

$(x_1, x_2) = (0, 0)$

$(z_2(x))_0^L = 0 = E_2$

step 3:

$w_1 = \frac{8070.3}{7003.9+8070.3} = 0.54$

$w_2 = \frac{7003.9}{7003.9+8070.3} = 0.46$

step 4:

$\max z = 0.54 \times (100x_1 + 115x_2) + 0.46 \times (85x_1 + 100x_2)$

$\Rightarrow \quad z = 93.1x_1 + 108.1x_2$

subject to the set of constraints (6.5)

Solving by a LP solver we get the optimal solution:

$(x_1^*, x_2^*) = (7, 64.06) = S$

$(z_1(S))_0^U = 8070.3$

$(z_2(S))_0^U = 7003.9$

$\tilde{z}_1 = (4510, 5220, 5930, 6640, 7350, 8060)$

$\tilde{z}_2 = (3445, 4155, 4865, 5575, 6285, 6995)$

Conclusion

In this thesis we have discussed two methods for solving the fully fuzzy multi objective linear programming problem , the ranking function method converts the FFMOLPP to a crisp multi objective linear programming problem which can be solved by weights sum method or preemptive method.

The game theory approach also uses the ranking function to convert the problem to a crisp one , but the weights here are generated by the ratio matrix and then using these weights , a linear programming problem will be formed and can be solved easily by using any linear programming solver . we discussed also alpha cut method for solving fuzzy multi objective linear programming problem.

As a future work alpha cut method method may be extended to solve FFMOLPP .

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ملخص

طرق حل مسائل البرمجة الخطية متعددة الأهداف كاملة الضبابية

فاطمة الملّاك

في هذه الأطروحة تناول عدة طرق لحل مسائل البرمجة الخطية متعددة الأهداف الضبابية و كاملة الضبابية ، كل طريقة تم ذكر عدة أمثلة عليها . تعمل هذه الطرق على تحويل مسائل البرمجة الخطية الضبابية متعددة الأهداف إلى مسائل برمجة عادية و التي تحل يمكن أن تحل بسهولة باستخدام طرق عديدة.