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Fuzzy Differential Transform Method for Solving Fuzzy Volterra Integral Equations in Two Dimension

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Declaration

My name is Ghadeer Hussam Rawajbeh, and I am a student of the AAUP university number 201712817. I confirm that I worked on this Master's thesis myself. And I declare that I have complied with all regulations, instructions, Arab American University standards of Academic codes of conduct. I also adhere to the Dean's Council regulations in withdrawing their confirmation of this degree in case of any violations.

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Date: 16\6\2022

Dedication

I dedicate this thesis with my great gratitude
To my partners,
To my husband Mohammad,
To my children Celeen and Zaina,
To all of my family.

Acknowledgment

First, the great thanks and gratitude to God Almighty, who guided me and inspired me with patience and theability to complete my educational career satisfactorily.

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Abstract

In this thesis, we will solve Fuzzy Volterra Integral equations in 2 Dimensions with separable kernels. To this end, we will use the well-known Fuzzy Differential Transform Method (DTM). Without the use of DTM, the problem of solving such equations becomes challenging. DTM simplifies the process by finding an approximate solution for the fuzzy solution. We will implement this computational method using numerical examples to reflect the potential and applicability of such numerical technique.

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Abbreviations

F.D.T.M	Fuzzy Differential Transform Method
F.D.T.M.T.D	Fuzzy Differential Transform Method Two Dimensional
$x(t; r)$	Parameter Form of Fuzzy Function
$x(s, t; r)$	Parameter Form of Fuzzy Function in Two Dimension
$X(k; r)$	Differential Transform of Function
$X(k, h; r)$	Differential Transform of Function in Two Dimension
$\odot$	Dot Product

Introduction

In mathematical modeling, fuzzy set theory is considered a dominating tool when it comes to modeling uncertainty and vague problems [15]. This provided a practical solution for many real-life problems in different scientific fields. For instance, fuzzy logic helps solving many applied problems in computer programming, decision making and communications [17].

The merge of fuzzy sets with integral equations is one of the benefits that different parts engineering and physical fields gained. The importance of considering integral equations with fuzzy properties emerges from the wide range of problems that incorporate fuzzy concepts in many fields within applied mathematics. In practice fields that incorporates mathematics with physics, medicine and communication. In such applications, fuzzy number replaces crisp representatives for some of the parameters. This type of problems motivated researchers to engage in developing different mathematical models and a various number of numerical procedures that would appropriately help solving general fuzzy integral equations [3, 18-20].

The solution of Fuzzy integral equations as an active research topic with new methods that is presented frequently. In [1], H.Seiheii et.al. solved fuzzy Volterra integral equation of the first kind using differential transforms method for one dimension. Salah shourand and collaborators proposed an application of the well-known fuzzy differential transform method in order to solve fuzzy Volterra integral equations in [2].FarshidMirzae et.al. [3] proposed a mathematical method that solves numerically the two-dimensional fuzzy Fredholm integral equations of the second kind that utilize the triangular functions. In [21], a numerical approach that uses a hybrid method is presented for solving fuzzy volterra integral equations with separable kernels. They proposed a method where Laplace transform operation along with A-domain Decomposition Method is utilized to get the numerical results. Authors of [22] presented a numerical solution for the nonlinear fuzzy Volterra integral equations of the second kind.

Many of the present methods for solving fuzzy integral equations concentrated on solving such problems in 1 dimension. While these methods are effective but the need for higher dimensional solutions is required to deal with abroad types or practical applications. One of such equations is fuzzy Volterra integral equations with separable kernels which can be defined as follows:

Let us defined $u(x)$ be a fuzzy function, $f(x)$ is given as any known function, and $k(x,t)$ is an integral kernel.

The resulting general fuzzy Volterra integral equation which is of the second kind is presented in the following form:

$$u(x) + \int_a^x \big(K_1(x,t) \odot u(t)\big)dt = f(x) + \int_a^x \big(K_2(x,t) \odot u(t)\big)dt \qquad ,$$

Where $f(x){:}\mathbb{R} \longrightarrow \mathbb{E}$and $x \in (a,b), b < \infty$.

In this work, we concentrate on Equation (1) and we present a fuzzy differential transform method to approximate the solution in 2 dimensions. We extend the work presented in [1] and develop a 2 dimensional framework for the method by expanding the theoretical work to entertain 2 dimensional problems. Afterword's, we apply our proposed method to solve some benchmark examples and present the results to confirm the theoretical work. This thesis is organized as follows:

Chapter 1 presents basic fuzzy logic definitions along with a brief fuzzy volterra integration equations basics. In Chapter2, We discuss the one dimensional fuzzy differential transform method and we present some of its features. We conclude this chapter by some numerical examples to enhance the understanding of such method. Chapter 3 presents the two dimensional fuzzy differential transform method along with the expanded theoretical framework. We solve the fuzzy volterra integral equation using our proposed method in chapter 4 and we conclude this thesis by presenting a number of numerical examples.

Chapter one

Basic Concepts

This chapter describes fuzzy numbers along with some properties. Moreover, some basic operations and definitions are presented.

1.1 Fuzzy Numbers:

An interval of real numbers is a subset $A$of$\mathbb{R}$, such that for any two real numbers $xandyinA$ and any real number z with the property $x \leq z \leq y$, $z \in A$, and simply we write $A = [a_1, a_2]$. The interval$A$can be expressed by the so-called membership function as follows:

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ 1, & a_1 \leq x \leq a_2 \\ 0, & a_2 \leq x \end{cases} \qquad (1.1)$$

A fuzzy number is defined as an interval in the set $\mathbb{R}$ of the real numbers. While the bounds of the interval are vague, this interval is considered a fuzzy set. In general, the fuzzy interval is represented as $[a_{1,}\ a_{2,}\ a_3]$, where $a_{1,}$ *and* a_3 are two end points incorporated with a peak point a_2. The resulting fuzzy number is denoted by$\tilde{A}$. Let us consider the following definitions for a fuzzy set:

Definition 1.1.1[14]: A fuzzy set$A$defined on a set R_F is represented by a membership function $\mu_A: R_F \to [0,1]$, and is characterized by the ordered pair denoted as$(x, \mu_A(x))$.

Definition 1.1.2[14]: The $r - cut$ of the fuzzy number $\tilde{A}$whose membership function is μ_A, is the set denoted by A_r

$$A_r = \{x: \mu_A(x) \geq r\} \qquad (1.2)$$

Definition 1.1.3 [3]: A fuzzy set denoted by V and defined onR_F. V is called a fuzzy interval if:

- It is normal, i.e. $\exists x_0 \in R_F$ where $V(x_0) = 1$.

- It is convex, If $V(\lambda x+(1-\lambda)\,t) \geq \min\{V(x), V(t)\}$ holds for all numbers $x, t \in R_F$ and $0 \leq \lambda \leq 1$.
- The interval V isupper semi-continuous, such that $V(x_0) \geq \lim_{x\to x_0^{\mp}} V(x)$,for any $x_0 \in R_F$
- The set $[V]^0 = Cl\{x \in R_F | V(x) > 0\}$ is a compact subset of R_F.

For a fuzzy interval V, its $r-cuts$ are closed intervals in R_F, and denoted by $[V]^r = [\underline{v}(r), \overline{v}(r)]$.

The parametric form of a fuzzy number that yields the same set R_F can be presented as follows:

Definition 1.1.4[3]: The parametric form of a fuzzy number denoted by $\tilde{u}$ is organized as an ordered function pair presented as$[\underline{u}(r), \overline{u}(r)]$, if it satisfy the following:

- The lower bound $\underline{u}(r)$is bounded left-continuous function which is non-decreasing on [0,1],
- The upper bound $\overline{u}(r)$is a bounded right-continuous function which is non-increasing on [0,1],
- $\underline{u}(r) \leq \overline{u}(r)$, for all values of r where $0 \leq r \leq 1$.

Definition1.1.5[3]: Let$\tilde{v} = [\underline{v}(r), \overline{v}(r)]$, and $\tilde{w} = [\underline{w}(r), \overline{w}(r)]$be two fuzzy numbers and let λ be a real number, then

- $\tilde{v} = \tilde{w}$if and only if$\underline{v}(r) = \underline{w}(r)$ and $\overline{v}(r) = \overline{w}(r)$,
- $\tilde{v} \oplus \tilde{w} = (\underline{v}(r) + \underline{w}(r), \overline{v}(r) + \overline{w}(r))$,
- $(\lambda \otimes \tilde{v}) = \begin{cases} [\lambda \underline{v}(r), \lambda \overline{v}(r)], & \lambda \geq 0 \\ [\lambda \overline{v}(r), \lambda \underline{v}(r)], & \lambda < 0 \end{cases}$

Each$y \in \mathbb{R}$ can be regarded as a fuzzy number $\tilde{y}$defined by

$$\tilde{y}(t) = \begin{cases} 1, & t = y \\ 0, & t \neq y \end{cases}$$

Definition 1.1.6[12]: Let Ebe the Hausdorff distance that is between fuzzy numbers given by$D: E \times E \to \boldsymbol{R}_+ \cup \{0\}$, where

$$D(\tilde{u}, \tilde{v}) = sup_{r\in[0,1]} max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}.$$

The mapping$D$is a metric in E and has the following properties [8]:

- $D(u \oplus w, v \oplus w) = D(u,v), \forall u,v,w \in E$
- $D(k \odot u, k \odot v) = |k| D(u,v), k \in R, \forall u,v \in E$
- $D(u \oplus v, w \oplus e) \leq D(u,w) + D(v,e), \forall u,v,w,e \in E$
- (D, E)is a complete metric space.

Definition1.1.7[12,9]: Let f: R →E be a fuzzy function. f is continuous if, for an arbitrary fixed pointt_0 ∈R,$\epsilon > 0$, and$\delta > 0$, then

$$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon$$

Definition 1.1.8[12]: If$z \in E$exist where $x = y + z$, and x,y∈ E.z is called the Hukuhara difference of both $x$and$y$, and it is represented by $x \ominus y$.

Definition 1.1.9[12]: Let f : (a,b) → E where $x_0 \in$ (a, b). $f$is called differential at x_0, If there is an element $\acute{f}(x_0) \in$ E, such that

1. f is called 1-diffeentiable if for all positive h with value near to 0, there exist $f(x_0 + h) \ominus f(x_0)$and $f(x_0) \ominus f(x_0 - h)$, then the limit (in the D metric) is presented as

$$\lim_{h \to 0^+} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^+} \frac{f(x_0) \ominus f(x_0 - h)}{h} = \acute{f}(x_0) \quad (1.3)$$

Or

(2) f is called 2-diffeentiable if for all positive h with value near to 0, $\exists f(x_0 + h) \ominus f(x_0), f(x_0) \ominus f(x_0 - h)$and the limit (in the D metric) would be

$$\lim_{h \to 0^-} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \to 0^-} \frac{f(x_0) \ominus f(x_0 - h)}{h} = \acute{f}(x_0) \quad (1.4)$$

When $f$is a fuzzy function, we have the following theorem:

Theorem 1.1.1[12]:
f: R → E is a function and let the r-cut representation if the function be

$$f(t;r) = \left[\underline{f}(t;r), \overline{f}(t;r)\right]$$

for any $r \in (0,1)$. We can conclude the following:

1. If the function$f$ is 1-differentiable, then $\underline{f}(t;r)$and$\overline{f}(t;r)$are two differentiable functions where
$$\acute{f}(t;r) = \left[\underline{\acute{f}}(t;r), \overline{\acute{f}}(t;r)\right].$$
2. If the function is 2-differentiable, then $\underline{f}(t;r)$and$\overline{f}(t;r)$are two differentiable functions with
$$\acute{f}(t;r) = \left[\overline{\acute{f}}(t;r), \underline{\acute{f}}(t;r)\right].$$

We easily can present the second order for the second point in last definition with H-differentiability as in the following theorem:

Theorem 1.1.2 [12]:
Let the function f : R → E be a fuzzy function with the r-cut presentation $f(t;r) = \left[\underline{f}(t;r), \overline{f}(t;r)\right]$for any $r \in (0,1)$. Then we conclude the following:

i. When$f$ and $\acute{f}$ are differentiable as in (1) or when$f$ and $\acute{f}$ are differentiable as in (2), then $\underline{f}(t;r) and \overline{f}(t;r)$are both differentiable functions with$\acute{\acute{f}}(t;r) = \left[\underline{\acute{\acute{f}}}(t;r), \overline{\acute{\acute{f}}}(t;r)\right]$.

ii. When$f$ is differentiable as in (1) and f' is differentiable as in (2) of the same definition or, on the other hand if f is differentiable following (2) and$\acute{f}$ is differentiable following (1), then $\underline{f}(t;r) and \overline{f}(t;r)$ are both differentiable functions with$\acute{\acute{f}}(t;r) = \left[\overline{\acute{\acute{f}}}(t;r), \underline{\acute{\acute{f}}}(t;r)\right]$.

Definition 1.1.10[12]: For the function$f(x)$, the transformation of the nth derivative can be defined as:

$$F(k) = \frac{1}{K!}\left[\frac{d^k}{dx^k} f(x)\right]_{x=x_0} (1.5)$$

While the inverse of this transformation is defined as:

$$f(x) = \sum_{k=0}^{\infty} F(k)\,(x - x_0)^k (1.6)$$

1.2 Fuzzy Volterra Integral Equations

In this section, we define the fuzzy Volterra integral equations that uses separable kernels. Moreover, we define these equations for two- dimensional problems [2]. We start by the one-dimensional case as follows:

Let $u(x)$ be a fuzzy function which is the main solution, $f(x)$ is any given function, and let $k(x, t)$ be a separable function kernel.

We can define the general Volterra integral equation of fuzzy nature and a second kind as in the following form:

$$u(x) + \int_a^x \big(k_1(x,t) \odot u(t)\big)dt = f(x) + \int_a^x \big(k_2(x,t) \odot u(t)\big)dt (1.7)$$

Where $f(x): R \longrightarrow E$and $x \in (a,b), b < \infty$.

f is the fuzzy differential transform method with separable kernels. The fuzzy differential transform method can be defined for separable kernels as follows:

$k_1(x,t) = \sum_{j=0}^{n} Q_{j\,(x)} P_{j(t)}$ and similarly can be defined for k_2.

The equation of interest in this work can be constructed in the same manner that the one dimensional is defined. We can define the fuzzy Volterra integral equation in two dimensions and of the second type as follows:

$$u(x,y) + \int_b^y \int_a^x \big(k_1(x,y,s,t) \odot u(s,t)\big) dt ds$$
$$= f(x,y) + \int_b^y \int_a^x \big(k_2(x,y,s,t) \odot u(s,t)\big) dt ds \quad (1.8)$$

Where K and f are continuous functions and k_1 has the following form:

$$k_1(x,y,s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} Q_{j\,(x)} P_{j(t)} G_{i\,(y)} H_{i(s)}$$

And similarly we can define it for k_2.

Chapter two

One-Dimensional Fuzzy Differential Transform Method:

In this chapter, we briefly discuss the construction of one dimensional fuzzy differential transform method. The chapter is concluded by some numerical examples to better explain the method. The interested reader could turn to [1,2] for more details.

2.3Definitions Of One-Dimensional Fuzzy Differential Transform Method:

Definition 2.1.1: Let$x(t)$be a differentiable function of order k which is defined on the time domain T, then

$$\underline{x}_i(k;r)=\frac{d^k\underline{x}(t;r)}{dt^k}\bigg]_{t=t_j},\quad \overline{x}_i(k;r)=\frac{d^k\overline{x}(t;r)}{dt^k}\bigg]_{t=t_j},\quad \forall k\in K=0,1,...(2.1)$$

when $x(t)$ is (1)- differentiable, and when$k$ is odd

$$\begin{cases}\underline{X}_i(k;r)=\dfrac{d^k\overline{X}(t;r)}{dt^k}\bigg]_{t=t_i}\\ \overline{X}_i(k;r)=\dfrac{d^k\underline{X}(t;r)}{dt^k}\bigg]_{t=t_j}\end{cases}$$

and when k is even

$$\begin{cases}\underline{X}_i(k;r)=\dfrac{d^k\underline{x}(t;r)}{dt^k}\bigg]_{,t=t_i}\\ \overline{x}_i(k;r)=\dfrac{d^k\overline{x}(t;r)}{dt^k}\bigg]_{,t=t_j}\end{cases}$$

when$x(t)$is (2)-differentiable ([1.1.9]). (2.2)

We note that $\underline{X}_i(k;r) and \overline{X}_i(k;r)$ can be considered as the lower and upper lower spectrum and the upper spectrum of$x(t) at t=t_i$,respectively.

Definition 2.1.2: Let $x(t)$ be (1)-differentiable, then for $0 \leq r \leq 1, x(t)$ can be represented in the following manner:

$$\underline{x}(t;r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \underline{X}(k;r)$$

$$\overline{x}(t;r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \overline{X}(k;r) \quad (2.3)$$

and if $x(t)$ be (2)-differentiable, then $x(t)$ can berepresented as follows:

$$\underline{x}(t;r) = \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} \overline{X}(k;r) + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} \underline{X}(k;r)$$

$$\overline{x}(t;r) = \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} \underline{X}(k;r) + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} \overline{X}(k;r) \quad (2.4)$$

Definition 2.1.3: The inverse transformation for $X(k,r)$ can be defined for $x(t)$ when it is (1)-differentiable and $\forall k \in K$, $K = 0,1, \dots$ as follows:

$$\underline{X}(k;r) = \mathrm{M(k)} \frac{d^k \underline{x}(t;r)}{dt^k}\Big]_{t=t_0},$$

$$\overline{X}(k;r) = \mathrm{M(k)} \frac{d^k \overline{x}(t;r)}{dt^k}\Big]_{t=t_0} \quad (2.5)$$

And for $x(t)$ when it is (2)-differentiable. $X(k,r)$ can be defined as

$$\begin{cases} \underline{X}(k;r) = \mathrm{M(k)} \frac{d^k \overline{x}(t;r)}{dt^k}\Big]_{t=t_0} \\ \overline{X}(k;r) = \mathrm{M(k)} \frac{d^k \underline{x}(t;r)}{dt^k}\Big]_{,t=t_0} \end{cases} (k \text{ is odd})$$

$$\begin{cases} \underline{X}(k;r) = \mathrm{M(k)} \frac{d^k \underline{x}(t;r)}{dt^k}\Big]_{t=t_0} \\ \overline{X}(k;r) = \mathrm{M(k)} \frac{d^k \overline{x}(t;r)}{dt^k}\Big]_{,t=t_0} \end{cases} (k \text{is even }) \quad (2.6)$$

Definition 2.1.4: The function $x(t)$ can be represented as follows:

For $0 \leq r \leq 1$,

$$\underline{x}(\mathrm{t};\mathrm{r}) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{X}(k;r)}{M(k)}$$

$$\overline{x}(\mathrm{t};\mathrm{r}) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\overline{X}(k;r)}{M(k)}$$

(2.7)

When $x(t)$is (1)-differentiable and

$$\underline{x}(t;r) = \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\overline{X}(k;r)}{M(k)} + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{X}(k;r)}{M(k)},$$

$$\overline{x}(t;r) = \sum_{k=1,odd}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{X}(k;r)}{M(k)} + \sum_{k=0,even}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\overline{X}(k;r)}{M(k)} \quad (2.8)$$

if x(t) be (2)-differentiable.

$M(k)$is defined as the weighting factor and $M(k) > 0$. In this work, $M(k) = \frac{H^k}{k!}$ is implemented where H is the time horizon on interest.

Definition 2.1.5: If x(t) is (1)-differentiable, then:

$$\underline{X}(k;r) = \frac{H^k}{k!} \frac{d^k \underline{x}(t;r)}{dt^k}\bigg|_{t=t_0}, \quad \overline{X}(k;r) = \frac{H^k}{k!} \frac{d^k \overline{x}(t;r)}{dt^k}\bigg|_{t=t_0}, \quad k \leq K, 0 \leq r \leq 1 \quad (2.9)$$

and if x(t) is (2)-differentiable,

$$\begin{cases} \underline{X}(k;r) = \frac{H^k}{k!} \frac{d^k \overline{x}(t;r)}{dt^k}\Big|_{t=t_0} \\ \overline{X}(k;r) = \frac{H^k}{k!} \frac{d^k \underline{x}(t;r)}{dt^k}\Big|_{t=t_0} \end{cases} (k \text{ is odd})$$

$$\begin{cases} \underline{X}(k;r) = \frac{H^k}{k!} \frac{d^k \underline{x}(t;r)}{dt^k}\Big|_{t=t_0} \\ \overline{X}(k;r) = \frac{H^k}{k!} \frac{d^k \overline{x}(t;r)}{dt^k}\Big|_{t=t_0} \end{cases} (k \text{ is even}) \quad (2.10)$$

We can transform the fuzzy equation on the defined domain of the specific problem to a regular algebraic equation on the domain defined above as K using the fuzzy transform method. Moreover, $x(t)$ can be derived from the Tylor series finite term adding a reminder as shown by the following:

For $k \leq K, 0 \leq r \leq 1$

$$\underline{x}(t;r) = \sum_{k=0}^{N} \frac{(t-t_0)^k}{k!} \frac{\underline{X}(k;r)}{M(k)} + R_{(N+1)}(t)$$

$$= \sum_{k=0}^{N} \left(\frac{t-t_0}{H}\right)^k \underline{X}(k;r) + R_{(N+1)}(t),$$

$$\overline{x}(t;r) = \sum_{k=0}^{N} \frac{(t-t_0)^k}{k!} \frac{\overline{X}(k;r)}{M(k)} + R_{(N+1)}(t)$$

$$= \sum_{k=0}^{N} \left(\frac{t-t_0}{H}\right)^k \overline{X}(k;r) + R_{(N+1)}(t)$$

when $x(t)$ is (1)-differentiable and

$$\underline{x}(t;r) = \left(\sum_{k=1,odd}^{N} \frac{(t-t_0)^k}{k!} \frac{\overline{X}(k;r)}{M(k)} + \sum_{k=0,even}^{N} \frac{(t-t_0)^k}{k!} \frac{\underline{X}(k;r)}{M(k)}\right) + R_{(N+1)}$$

$$\overline{x}(t;r) = \left(\sum_{k=1,odd}^{N} \frac{(t-t_0)^k}{k!} \frac{\underline{X}(k;r)}{M(k)} + \sum_{k=0,even}^{N} \frac{(t-t_0)^k}{k!} \frac{\overline{X}(k;r)}{M(k)}\right) + R_{(N+1)}$$

when x(t) is (2)-differentiable.

Definition 2.1.6:

Let us define $t_0, \ldots, t_N$ as an equally spaced points on a grid of interest where $t_i = a + ih$ and while $i = 0, 1, \ldots, N and$ defining $h = \frac{b-a}{N}$. Using the previous assumption, we divided the domain to an N subdomains while the fuzzy function $X_i(t, r)$ approximates the subdomains where $i = 0, 1, \ldots, N-1$.

Using the presented initial conditions, we can obtain the following:

$$\underline{X}(0, r) = \underline{x}(0, r), \overline{X}(0, r) = \overline{x}(0, r), \ 0 \leq r \leq 1$$

Using the first subdomain, $\underline{x}(t, r) and \overline{x}(t, r)$can be represented by $\underline{X}(0, r) = \underline{x}_0(r), and \overline{X}(0, r) = \overline{x}_0(r)$; respectively. Moreover, we can represent them by the nth order Taylor series considering t_0 as follows

$$\underline{x}(t_0, r) = \underline{x}_0(0, r) + \underline{x}_0(1, r)(t - t_0) + \underline{x}_0(2, r)(t - t_0)^2 + \cdots + \underline{x}_0(N, r)(t - t_0)^N$$

$$\overline{x}(t_0, r) = \overline{x}_0(0, r) + \overline{x}_0(1, r)(t - t_0) + \overline{x}_0(2, r)(t - t_0)^2 + \cdots + \overline{x}_0(N, r)(t - t_0)^N$$

Additionally, using Taylor series for $x(t_i, r)$, thesolution on the grid pointst_{i+1}can be obtained as follows:

$$\underline{x}(t_{i+1}, r) \approx \underline{x}_i(t_{i+1}, r) = \underline{x}_i(0, r) + \underline{x}_i(1, r)(t_{i+1} - t_i) + \cdots + \underline{x}_i(N, r)(t_{i+1} - t_i)^N.$$

Then$\underline{x}(t_{i+1}, r) \approx \sum_{j=0}^{N} \underline{x}_i(j, r)\, h^j$.(2.11)

Also,

$$\overline{x}(t_{i+1}, r) \approx \overline{x}_i(t_{i+1}, r) = \overline{x}_i(0, r) + \overline{x}_i(1, r)(t_{i+1} - t_i) + \cdots + \overline{x}_i(N, r)(t_{i+1} - t_i)^N$$

And, $\overline{x}(t_{i+1}, r) \approx \sum_{j=0}^{N} \overline{x}_i(j, r)\, h^j$. (2.12)

2.2Properties Of One-Dimensional Fuzzy Differential transform Method:

Let us start this section with the following result:

Theorem:2.2.1:
Let $u(t)$ and $v(t)$ be two fuzzy functions while the corresponding fuzzy differential transformations are denoted by $U(k)$ and $V(k)$, respectively. Then

1. If$f(\mathrm{t}) = u(t) + v(t)$, then $\mathrm{F(k)} = \mathrm{U(k)} + \mathrm{V(k)}$, $k \in K$.
2. if $f(\mathrm{t}) = u(t) - v(t)$, then $\mathrm{F(k)} = \mathrm{U(k)} - \mathrm{V(k)}$, $k \in K$.
3. if$f(\mathrm{t}) = u(t) \ominus v(t)$, then $\mathrm{F(k)} = \mathrm{U(k)} \ominus \mathrm{V(k)}$, $k \in K$.

provided the Hukuhara difference as presented in Definition [1.1.6] exists.

Theorem:2.2.2:

Consider the fuzzy-valued function $g \in E$ and$f(x)$= $\int_{x_0}^{x} g(t)dt$,then

$F(k) = \frac{G(k-1)}{k}$, $k \geq 1$(2.13)

Where $F(k)$ and $G(k)$ are the differential transform of $f$and g , respectively.

Theorem:2.2.3:

Consider $f(x)$=$\int_{x_0}^{x_n} \ldots \int_{x_0}^{x_2} \int_{x_0}^{x_1} g(t)\, dt dx_1\, dx_2 \ldots dx_{n-1}$

Then $F(k) = \frac{(k-n)!}{k!} G(k-n; r) k \geq n$(2.14)

Where $F(k)$ and $G(k)$ are the fuzzy differential transformation of fuzzy valued functions of $f$and g, respectively.

Theorem:2.2.4:

Suppose that $U(k)$ and $G(k)$ are the differential transformations of the functions$u(x)$ and $g(x)$ such that they both are positivevalued functions, respectively.

If $f(x) = \int_{x_0}^{x} g(t).u(t)dt$ then

Under (1) differentiability of f we have, for $0 \leq r \leq 1$,

$$\underline{F}(k;r) = \sum_{l=0}^{k-1} \frac{G(l).\underline{U}(k-l-1;r)}{k}$$

$$\overline{F}(k;r) = \sum_{l=0}^{k-1} \frac{G(l).\overline{U}(k-l-1;r)}{k}$$

(2.15)

Under (2) differentiability of$f$ we have, for $0 \leq r \leq 1$,

$$\underline{F}(k;r) = \sum_{l=0,even}^{k-1} \frac{G(l).\underline{U}(k-l-1;r)}{k} + \sum_{l=1,odd}^{k-1} \frac{G(l).\overline{U}(k-l-1;r)}{k} F(0;r)$$

$= (0)$

$$\overline{F}(k;r) = \sum_{l=0,even}^{k-1} \frac{G(l).\overline{U}(k-l-1;r)}{k} + \sum_{l=1,odd}^{k-1} \frac{G(l).\underline{U}(k-l-1;r)}{k} F(0;r)$$

$= (0)$(2.16)

Theorem:2.2.5:

Suppose $U(k)$ and $G(k)$are the differential transformation for the functions, $u(x)$ and $g(x)$, respectively.

If $f(x)=[g(x)\int_{x_0}^{x} u(t;r)dt]$

then following the (1)-differentiability of f, we have, for $0 \leq r \leq 1$:

$$\underline{F}(k;r) = \sum_{l=0}^{k-1} \frac{G(l).\underline{U}(k-l-1;r)}{(k-l)}$$

$$\overline{F}(k;r) = \sum_{l=0}^{k-1} \frac{G(l).\overline{U}(k-l-1;r)}{(k-l)}$$

(2.17)

And under (2)-differentiability of f, we have, for $0 \le r \le 1$:

$$\underline{F}(k;r) = \sum_{l=0,even}^{k-1} \frac{G(l).\underline{U}(k-l-1;r)}{(k-l)} + \sum_{l=0,odd}^{k-1} \frac{G(l).\overline{U}(k-l-1;r)}{(k-l)}$$

$$\overline{F}(k;r) = \sum_{l=0,even}^{k-1} \frac{G(l).\overline{U}(k-l-1;r)}{(k-l)} + \sum_{l=0,odd}^{k-1} \frac{G(l).\underline{U}(k-l-1;r)}{(k-l)}$$

(2.18)

Theorem:2.2.6:

Suppose that $U(k)$ and $G(k)$are both the differential transformation of the functions,$u(x)$ and $g(x)$,respectively.

If$f(x) = g(x)\,u(x)$, then

$F(k;r) = \sum_{l=0}^{k} G(l) \odot U(k-l;r)$,$0 \le r \le 1$(2.19)

Chapter three:

Two-Dimensional Fuzzy Differential Transform Method

In this chapter, we present the main construction criteria for the two-dimensional FDTM.

3.1 Construction of Two Dimensional Fuzzy Differential Transform Method (2.D.FDTM):

Let$F(x,t)$ be a two-variable fuzzy function. The following can be defined:

Definition 3.1.1: Let us consider $x(s,t)$ is differentiable of order $k+h$ on the time domain T, then

$$\underline{x}_{ij}(k,h;r)=\frac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\Big]_{s=s_i,t=t_j},$$

$$\overline{x}_{ij}(k,h;r)=\frac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\Big]_{s=s_i,t=t_j}, \qquad \forall k,h\in K=0,1,\ldots(3.1)$$

when $x(s,t)$ is (1)-differentiable and

$$\left\{\begin{aligned}\underline{X}_{ij}(k,h;r) &= \frac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\Big]_{s=s_i,t=t_j}\\ \overline{X}_{ij}(k,h;r) &= \frac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\Big]_{s=s_i,t=t_j}\end{aligned}\right\}(k \text{ is odd, } h \text{ is even) or inverse}$$

and

$$\left\{\begin{aligned}\underline{X}_{ij}(k,h;r) &= \frac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\Big]_{s=s_i,t=t_j}\\ \overline{x}_{ij}(k,h;r) &= \frac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\Big]_{s=s_i,t=t_j}\end{aligned}\right\} \qquad \text{(both of} k,h \text{are even or odd) (3.2)}$$

when $x(s,t)$ is (2)-differentiable.

Note that $\underline{X}_{ij}(k,h;r) and \overline{X}_{ij}(k,h;r)$are called the lower and upper spectrums of$x(s,t) at s=s_i,\ t=t_j$, respectively.

Definition 3.1.2: If $x(s,t)$ is (1)-differentiable, then $x(s,t)$can be represented as follows, for $0\leq r \leq 1$:

$$\underline{x}(s,t;r) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!\,h!}\underline{X}(k,h;r)$$

$$\overline{x}(s,t;r) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\overline{X}(k,h;r) \quad (3.3)$$

and if $x(s,t)$ be (2)-differentiable, then $x(s,t)$can berepresented as follows:

$$\underline{x}(s,t;r) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!\,h!}\overline{X}(k,h;r)_{(\text{ is}k\text{ odd,}h\text{ is even })\text{or inverse}} + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!\,h!}\underline{X}(k,h;r)_{(\text{both of }k,h\text{ are even or odd})}$$

$$\overline{x}(s,t;r) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!\,h!}\underline{X}(k,h;r)_{(\ k\text{ is odd, }\ h\text{ is even })\text{ or inverse}} + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!\,h!}\overline{X}(k,h;r)_{(\text{both of }k,h\text{ are even or odd})}$$

(3.4)

Definition 3.1.3: The inverse transformation of $X(k,h,r)$. $X(k,h,r)$ is defined as·

$$\underline{X}(k,h;r) = M(k,h)\frac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\bigg]_{s=s_0,t=t_0},$$

$$\overline{X}(k,h;r) = M(k,h)\frac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\bigg]_{s=s_0,t=t_0} \quad (3.5)$$

when $x(s,t)$ is (1)-differentiable and

$$\left\{\begin{array}{l}\underline{X}(k,h;r) = \mathrm{M(k,h)}\dfrac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\bigg]_{s=s_0,t=t_0} \\ \overline{X}(k,h;r) = \mathrm{M(k,h)}\dfrac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\bigg]_{s=s_0,t=t_0}\end{array}\right\} (kisodd, hiseven)\text{or inverse}$$

$$\left\{\begin{array}{l}\underline{X}(k,h;r) = \mathrm{M(k,h)}\left.\dfrac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\right]_{s=s_0,t=t_0} \\ \overline{X}(k,h;r) = \mathrm{M(k,h)}\left.\dfrac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\right]_{s=s_0,t=t_0}\end{array}\right\} \text{(both of } k,h \text{ are even or odd)} \quad (3.6)$$

when $x(s,t)$ is (2)-differentiable.

Definition 3.1.4: The function $x(s,t)$ can be represented as follows, for $0 \le r \le 1$:

$$\underline{x}(s,t;r) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\frac{\underline{X}(k,h;r)}{M(k,h)},$$

$$\overline{x}(s,t;r) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\frac{\overline{X}(k,h;r)}{M(k,h)} \quad (3.7)$$

when $x(s,t)$ is (1)-differentiable and

$$\underline{x}(s,t;r) = \frac{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\frac{\overline{X}(k,h;r)}{M(k,h)}}{(k is odd, h is even) or inverse} + \frac{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\frac{\underline{X}(k,h;r)}{M(k,h)}}{((\text{both of } k,h \text{ are even or odd}))}$$

$$\overline{x}(s,t;r) = \frac{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\frac{\underline{X}(k,h;r)}{M(k,h)}}{(k \text{ is odd}, h \text{ is even}) \text{or inverse}} + \frac{\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{(s-s_i)^k(t-t_j)^h}{k!h!}\frac{\overline{X}(k,h;r)}{M(k,h)}}{((\text{both of } k,h \text{ are even or odd}))} \quad (3.8)$$

When $x(s,t)$ be (2)-differentiable .

where$M(k,h)$is the weighting factor and $M(k,h) > 0$. In this thesis we consider $M(k,h) = \frac{G^k H^h}{k!h!}$ where $G and H$ the time horizon on the domain of interest.

Definition 3.1.5: If $x(s,t)$ is (1)-differentiable, then, for $k \le K$, $0 \le r \le 1$:

$$\underline{X}(k,h;r) = \frac{G^k H^h}{k!h!}\left.\frac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\right]_{s=s_0,t=t_0}, \quad \overline{X}(k,h;r) = \frac{G^k H^h}{k!h!}\left.\frac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\right]_{s=s_0,t=t_0} \quad (3.9)$$

and if $x(s,t)$ is (2)-differentiable ,

$$\left\{\begin{array}{l}\underline{X}(k,h;r) = \dfrac{G^k H^h}{k!h!}\left.\dfrac{\partial^{k+h}\overline{x}(s,t;r)}{\partial s^k \partial t^h}\right]_{s=s_0,t=t_0} \\ \overline{X}(k,h;r) = \dfrac{G^k H^h}{k!h!}\left.\dfrac{\partial^{k+h}\underline{x}(s,t;r)}{\partial s^k \partial t^h}\right]_{s=s_0,t=t_0}\end{array}\right. (k \text{ is odd}, h \text{ is even }) \text{ or inverse}$$

$$\begin{cases} \underline{X}(k,h;r) = \frac{G^k H^h}{k!h!} \frac{\partial^{k+h} \underline{x}(s,t;r)}{\partial s^k \partial t^h} \Big]_{s=s_0, t=t_0} \\ \overline{X}(k,h;r) = \frac{G^k H^h}{k!h!} \frac{\partial^{k+h} \overline{x}(s,t;r)}{\partial s^k \partial t^h} \Big]_{s=s_0, t=t_0} \end{cases} \text{(both of } k,h \text{ are even or odd)} \quad (3.10)$$

We can transform the fuzzy equation on the defined domain of the specific problem to a regular algebraic equation on the domain defined above as K using the fuzzy transform method. Moreover, $x(s,t)$ can be derived from the Tylor series finite term adding a reminder as shown by the following, for $k \leq K, 0 \leq r \leq 1$:

$$\underline{x}(\mathrm{s},\mathrm{t};\mathrm{r}) = \sum_{k=0}^{N} \sum_{h=0}^{M} \frac{(s-s_0)^k (t-t_0)^h}{k!\,h!} \frac{\underline{X}(k,h;r)}{M(k,h)} + R_{(N+1,M+1)}(s,t)$$

$$= \sum_{k=0}^{N} \sum_{h=0}^{M} \left(\frac{s-s_0}{G}\right)^k \left(\frac{t-t_0}{H}\right)^h \underline{X}(k,h;r) + R_{(N+1,M+1)}(s,t)$$

,

$$\overline{x}(\mathrm{s},\mathrm{t};\mathrm{r}) = \sum_{k=0}^{N} \sum_{h=0}^{M} \frac{(s-s_0)^k (t-t_0)^h}{k!\,h!} \frac{\overline{X}(k,h;r)}{M(k,h)} + R_{(N+1,M+1)}(s,t)$$

$$= \sum_{k=0}^{N} \sum_{h=0}^{M} \left(\frac{s-s_0}{G}\right)^k \left(\frac{t-t_0}{H}\right)^h \overline{X}(k,h;r) + R_{(N+1,M+1)}(s,t)$$

when $x(s,t)$ is (1)-differentiable and

$$\underline{x}(\mathrm{s},\mathrm{t};\mathrm{r}) = \sum_{k=0}^{N} \sum_{h=0}^{M} \frac{(s-s_0)^k (t-t_0)^h}{k!\,h!} \frac{\overline{X}(k,h;r)}{M(k,h)}_{\text{(k is odd,h is even)or inverse}} + R_{(N+1,M+1)}(s,t)$$

$$+ \sum_{k=0}^{N} \sum_{h=0}^{M} \frac{(s-s_0)^k (t-t_0)^h}{k!\,h!} \frac{\underline{X}(k,h;r)}{M(k,h)}_{\text{(both of k,h are even or odd)}} + R_{(N+1,M+1)}(s,t)$$

$$= \sum_{k=0}^{N} \sum_{h=0}^{M} \left(\frac{s-s_0}{G}\right)^k \left(\frac{t-t_0}{H}\right)^h \overline{X}(k,h;r)_{\text{(k is odd,h is even)or inverse}}$$

$$+R_{(N+1,M+1)}(s,t)$$

$$+\sum_{k=0}^{N}\sum_{h=0}^{M}\left(\frac{s-s_0}{G}\right)^k\left(\frac{t-t_0}{H}\right)^h \underline{X}(k,h;r)_{\text{(both of k,h are even or odd)}}$$

$$+R_{(N+1,M+1)}(s,t)$$

$$\overline{x}(s,t;r)=\sum_{k=0}^{N}\sum_{h=0}^{M}\frac{(s-s_0)^k(t-t_0)^h}{k!\,h!}\frac{\underline{X}(k,h;r)}{M(k,h)}_{\text{(k is odd,h is even)or inverse}}$$

$$+R_{(N+1,M+1)}(s,t)$$

$$+\sum_{k=0}^{N}\sum_{h=0}^{M}\frac{(s-s_0)^k(t-t_0)^h}{k!\,h!}\frac{\overline{X}(k,h;r)}{M(k,h)}_{\text{(both of k,h are even or odd)}}$$

$$+\quad R_{(N+1,M+1)}(s,t),$$

$$\overline{x}(s,t;r)=\sum_{k=0}^{N}\sum_{h=0}^{M}\left(\frac{s-s_0}{G}\right)^k\left(\frac{t-t_0}{H}\right)^h \underline{X}(k,h;r)_{\text{(k is odd,h is even)or inverse}}$$

$$+R_{(N+1,M+1)}$$

$$+\sum_{k=0}^{N}\sum_{h=0}^{M}\left(\frac{s-s_0}{G}\right)^k\left(\frac{t-t_0}{H}\right)^h \overline{X}(k,h;r)_{\text{(both of k,h are even or odd)}}$$

$$+R_{(N+1,M+1)}(s,t)$$

when $x(s,t)$ is (2)-differentiable .

Definition 3.1.6: *Let* $t_0,\dots,t_N$ *and* $s_0,\dots,s_{N'}$ are the equally spaced grid points where

$s_i=a+ih$ and $t_j=a+jh'$ where $i = 0,1,\dots,N$, $j = 0,1,\dots,N'$ *and*

$h=\frac{b-a}{N}$, $h'=\frac{b'-a'}{N'}$. Here the domain $[a,b]\times[a',b']$is divided to

N× N'subdomain, and the fuzzy approximation functions in each subdomain are$X_i(s,t,r)$ for

$i = 0,1,\dots,N-1$ and $X_j(s,t,r)$ for $j = 0,1,\dots,N'-1$, respectively.

From the initial conditions, the following can obtained:

$\underline{X}(0,0,r) = \underline{x}(0,0,r)$, $\overline{X}(0,0,r) = \overline{x}(0,0,r)$, $0 \le r \le 1$

In the first subdomain, $\underline{x}(s,t,r) and \overline{x}(s,t,r)$can be described by

$\underline{X}(0,0,r) = \underline{x}_{0,0}(r)$, $\overline{X}(0,0,r) = \overline{x}_{0,0}(r)$; respectively. They can be represented in terms oftheir $(N,N') - t(h,h')$order bivariate Taylor series with respect tos_0, t_0. That is,

$$\begin{aligned}\underline{x}(s_0,t_0,r) = {} & \underline{x}_0(0,0,r) + \underline{x}_0(0,1,r)(t-t_0) + \underline{x}_0(0,2,r)(t-t_0)^2 + \cdots \\ & + \underline{x}_0(0,N',r)(t-t_0)^{N'} \\ & + \underline{x}_0(1,0,r)(s-s_0) + \underline{x}_0(1,1,r)(s-s_0)\ (t-t_0) \\ & + \underline{x}_0(1,2,r)(s-s_0)\ (t-t_0)^2 + \cdots \\ & + \underline{x}_0(1,N'-1,r)(s-s_0)\ (t-t_0)^{N'-1} \\ & + \underline{x}_0(N,N'-1,r)(s-s_0)^N(t-t_0)^{N'-1} \\ & + \underline{x}_0(N,N',r)(s-s_0)^N(t-t_0)^{N'}\end{aligned}$$

$$\begin{aligned}\overline{x}(s_0,t_0,r) = {} & \overline{x}_0(0,0,r) + \overline{x}_0(0,1,r)(t-t_0) + \overline{x}_0(0,2,r)(t-t_0)^2 + \cdots \\ & + \overline{x}_0(0,N',r)(t-t_0)^{N'} \\ & + \overline{x}_0(1,0,r)(s-s_0) + \overline{x}_0(1,1,r)(s-s_0)\ (t-t_0) \\ & + \overline{x}_0(1,2,r)(s-s_0)\ (t-t_0)^2 + \cdots \\ & + \overline{x}_0(1,N'-1,r)(s-s_0)\ (t-t_0)^{N'-1} \\ & + \overline{x}_0(N,N'-1,r)(s-s_0)^N(t-t_0)^{N'-1} \\ & + \overline{x}_0(N,N',r)(s-s_0)^N(t-t_0)^{N'}\end{aligned}$$

Additionally, using Taylor series for $x(s_i,t_j,r)$, thesolution on the grid pointss_{i+1}, t_{j+1}can be obtained as follows:

$$
\begin{aligned}
\underline{x}(s_{i+1}, t_{j+1}, r) = \underline{x}_{ij}(s_{i+1}, t_{j+1}, r) \\
&= \underline{x}_{ij}(0,0,r) + \underline{x}_{ij}(0,1,r)(t_{j+1} - t_j) \quad + \underline{x}_{ij}(0,2,r)(t_{j+1} - t_j)^2 + \cdots \\
&+ \underline{x}_{ij}(0,N',r)(t_{j+1} - t_j)^{N'} + \underline{x}_{ij}(1,0,r)(s_{i+1} - s_i) \\
&+ \underline{x}_{ij}(1,1,r)(s_{i+1} - s_i) \quad (t_{j+1} - t_j) \\
&+ \underline{x}_{ij}(1,2,r)(s_{i+1} - s_i) \quad (t_{j+1} - t_j)^2 + \cdots \\
&+ \underline{x}_{ij}(1,N'-1,r)(s_{i+1} - s_i) \quad (t_{j+1} - t_j)^{N'-1} \\
&+ \underline{x}_{ij}(N,N'-1,r)(s_{i+1} - s_i)^N (t_{j+1} - t_j)^{N'-1} \\
&+ \ \underline{x}_{ij}(N,N',r)(s_{i+1} - s_i)^N (t_{j+1} - t_j)^{N'} \\
&= \sum_{m=0}^{N} \sum_{n=0}^{N'} \underline{x}_{ij}(m,n,r)\, h^m h'^n
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
\overline{x}(s_{i+1}, t_{j+1}, r) = \overline{x}_{ij}(s_{i+1}, t_{j+1}, r) \\
&= \overline{x}_{ij}(0,0,r) + \overline{x}_{ij}(0,1,r)(t_{j+1} - t_j) \quad + \overline{x}_{ij}(0,2,r)(t_{j+1} - t_j)^2 + \cdots \\
&+ \overline{x}_{ij}(0,N',r)(t_{j+1} - t_j)^{N'} \quad + \overline{x}_{ij}(1,0,r)(s_{i+1} - s_i) \\
&+ \overline{x}_{ij}(1,1,r)(s_{i+1} - s_i) \quad (t_{j+1} - t_j) \\
&+ \overline{x}_{ij}(1,2,r)(s_{i+1} - s_i) \quad (t_{j+1} - t_j)^2 + \cdots \\
&+ \overline{x}_{ij}(1,N'-1,r)(s_{i+1} - s_i) \quad (t_{j+1} - t_j)^{N'-1} \\
&+ \overline{x}_{ij}(N,N'-1,r)(s_{i+1} - s_i)^N (t_{j+1} - t_j)^{N'-1} \\
&+ \ \overline{x}_{ij}(N,N',r)(s_{i+1} - s_i)^N (t_{j+1} - t_j)^{N'} \\
&= \sum_{m=0}^{N} \sum_{n=0}^{N'} \overline{x}_{ij}(m,n,r)\, h^m h'^n
\end{aligned} \tag{3.12}
$$

Chapter four:

Solving two dimensional fuzzy Volterra integral equation

In this Chapter, we apply FDTM for the solution of the linear two-dimensional volterra fuzzy integral equations of the second kind with separable kernels. We conclude this chapter by presenting some numerical examples.

4.1 Solution Using (2D.FDTM):

Let $u(x)$ be a fuzzy function which is the main solution, $f(x)$ is any given function, and let $k(x,t)$ be a separable function kernel. Then

We can define the general Volterra integral equation of fuzzy nature and a second kind as in the following form:

$$u(x,y) + \iint_{ba}^{yx} \big(k_1(x,y,s,t) \odot u(s,t)\big) dtds$$

$$= f(x,y) + \iint_{ba}^{yx} \big(k_2(x,y,s,t) \odot u(s,t)\big) dtds \quad (4.1)$$

Where $f(x,y)$:R $\longrightarrow$ $\mathbb{E}$and $x \in (a,c), y \in (b,d)\, , c < \infty, d < \infty$.

In addition, we use FDTM to solve Equation (1) with separable kernels,

i.e., $k_1(x,y,s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} Q_{j\,(x)} P_{j(t)} G_{i\,(y)} H_{i(s)}$

and similarly for k_2.

4.2 Some Useful properties of Fuzzy Differential Transform Method (two-Dimension)

Theorem:4.2.1:

Let us consider $u(t,s)$ and $v(t,s)$ to be fuzzy functions combined with their fuzzy differential transformations denoted by $U(k,h)$ and$V(k,h)$, respectively. Then, for $k \in K$, $h \in H$:

1. If$f(\mathrm{t},\mathrm{s}) = u(t,s) + v(t,s)$, then $F(\mathrm{k},\mathrm{h}) = U(k,h) + V(k,h)$.
2. If $f(t,s) = u(t,s) - v(t,s)$, then$F(k,h) = U(k,h) - V(k,h)$.
3. If $f(t,s) = u(t,s) \ominus v(t,s)$, then $F(k,h) = U(k,h) \ominus V(k,h)$.

This applies when ensuring that the Hukuhara difference defined in Definition[1.1.6] exists.

Proof. For more details see [3.1], the proof is obvious.

Theorem:4.2.2:

considering the fuzzy function $g \in E$ and $f(x,y) = \int_{y0}^{y} \int_{x0}^{x} g(t,s)dtds$, then

$$F(k,h) = \frac{G(k-1,h-1)}{kh},\ k,h \geq 1 \qquad (4.2)$$

Where $F(k,h)$and$G(k,h)$ are the differential transform of f and g , respectively.

Proof:

Using definition of FDTM

We get $for\ \ 0 \leq r \leq 1$

$$f(x,y;r) = \int_{y_0}^{y} \int_{x0}^{x} g(t,s;r)dtds$$

$$f(x,y;r) = [\int_{y_0}^{y} \int_{x0}^{x} \underline{g}(t,s;r)dtds\ ,\ \int_{y0}^{y} \int_{x0}^{x} \overline{g}(t,s;r)dtds\ ]$$

$$f(x,y;r)=\int_{y_0}^{y}\int_{x_0}^{x}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\underline{G}(k,h;r)(s-y_0)^h(t-x_0)^k\,dtds\,,\int_{y0}^{y}\int_{x0}^{x}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\overline{G}(k,h;r)(s-y_0)^h(t-x_0)^k\,dtds$$

$$=[\sum_{h,k=0}^{\infty}\underline{G}(k,h;r)\frac{(y-y_0)^{h+1}(x-x_0)^{k+1}}{(h+1)(k+1)},\sum_{h,k=0}^{\infty}\overline{G}(k,h;r)\frac{(y-y_0)^{h+1}(x-x_0)^{k+1}}{(h+1)(k+1)}]$$

Then, let us change the indexing from $k=0\ to k=1$,we deduce that

$$=\begin{bmatrix}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\underline{G}(k-1,h-1;r)\frac{(y-y_0)^h(x-x_0)^k}{(h)(k)},\\ \sum_{h=0}^{\infty}\sum_{k=0}^{\infty}\overline{G}(k-1,h-1;r)\frac{(y-y_0)^h(x-x_0)^k}{(h)(k)}\end{bmatrix}$$

Finally using definition of F.D.T.M .T.D,

We obtain

$F(k,h;r)=\frac{G(k-1,h-1;r)}{kh}$,$0\ \leq r\leq 1$, $k,h\geq 1$.

Theorem:4.2.3:

If f(x,y)=$\int_{y_0}^{y_n}\int_{x_0}^{x_n}\ldots\int_{y_0}^{y_2}\int_{x_0}^{x_2}\int_{y_0}^{y_1}\int_{x_0}^{x_1}g(t,s)\,dtdsdx_1\,dy_1\,dx_2\,dy_2\ldots dx_{n-1}\,dy_{n-1}$

Then

$F(k,k)=\frac{(k-n)!(h-n)!}{k!h!}G(k-n,h-n;r)k\geq n$ (4.3)

where $F(k,h)\ and G(k,h)$ are the fuzzy differential transformation of fuzzy functions of f and g, respectively.

Proof:

$$\underline{f}(x,y;r)=\int_{y_0}^{y_n}\int_{x_0}^{x_n}\ldots\int_{y_0}^{y_2}\int_{x_0}^{x_2}\int_{y_0}^{y_1}\int_{x_0}^{x_1}g(t,s;r)\,dtdsdx_1\,dy_1\,dx_2\,dy_2\ldots dx_{n-1}\,dy_{n-1}$$

$$=\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_1}\int_{x_0}^{x_1}\sum_{h,k=0}^{\infty}\underline{G}(k,h;r)(s-y_0)^h(t-x_0)^k\,dtdsdx_1\,dy_1\ldots dx_{n-1}\,dy_{n-1}$$

$$=\sum_{h,k=0}^{\infty}\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_1}\int_{x_0}^{x_1}\underline{G}(k,h;r)(s-y_0)^h(t-x_0)^k dtdsdx_1\,dy_1\ldots dx_{n-1}\,dy_{n-1}$$

$$=\sum_{h,k=0}^{\infty}\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_2}\int_{x_0}^{x_2}\frac{\underline{G}(k,h;r)(y_1-y_0)^{h+1}(x_1-x_0)^{k+1}}{(k+1)(h+1)}dx_1dy_1\ldots dx_{n-1}\,dy_{n-1}$$

⋮

$$=\sum_{h,k=0}^{\infty}\frac{\underline{G}(k,h;r)(y-y_0)^{h+n}(x-x_0)^{k+m}}{(k+1)(k+2)\ldots(k+m)(h+1)(h+2)\ldots(h+n)}$$

$$\overline{f}(x,y;r)=\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_1}\int_{x_0}^{x_1}g(t,s;r)\,dtdsdx_1\,dy_1\ \ldots\ dx_{n-1}\,dy_{n-1}$$

$$=\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_1}\int_{x_0}^{x_1}\sum_{h,k=0}^{\infty}\overline{G}(k,h;r)(s-y_0)^h(t-x_0)^k\,dtdsdx_1\,dy_1\ldots dx_{n-1}\,dy_{n-1}$$

$$=\sum_{h,k=0}^{\infty}\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_1}\int_{x_0}^{x_1}\overline{G}(k,h;r)(s-y_0)^h(t-x_0)^k dtdsdx_1\,dy_1\ldots dx_{n-1}\,dy_{n-1}$$

$$=\sum_{h,k=0}^{\infty}\int_{y_0}^{y}\int_{x_0}^{x}\ldots\int_{y_0}^{y_2}\int_{x_0}^{x_2}\frac{\overline{G}(k,h;r)(y_1-y_0)^{h+1}(x_1-x_0)^{k+1}}{(k+1)(h+1)}dx_1dy_1dx_2\,dy_2\ldots dx_{n-1}\,dy_{n-1}$$

$$\vdots$$

$$=\sum_{h,k=0}^{\infty}\frac{\overline{G}(k,h;r)(y-y_0)^{h+n}(x-x_0)^{k+m}}{(k+1)(k+2)\ldots(k+m)(h+1)(h+2)\ldots(h+n)}$$

Let us start the indexing of the series from $k=n\,,\ h=ninsteadofk=0$

$and\ h=0$

We have, for $0\le r\le 1$:

$$\underline{f}(x,y;r)$$

$$=\sum_{h=n}^{\infty}\sum_{k=n}^{\infty}\frac{\underline{G}(k-n,h-n;r)(y-y_0)^h(x-x_0)^k}{(k+1-n)(k+2-n)\ldots(k)(h+1-n)(h+2-n)\ldots(h)}$$

$$=\sum_{h=n}^{\infty}\sum_{k=n}^{\infty}\frac{(k-n)!\,(h-n)!\,\underline{G}(k-n,h-n;r)(y-y_0)^h(x-x_0)^k}{k!\,h!}$$

$$\overline{f}(x,y;r)$$

$$=\sum_{h=n}^{\infty}\sum_{k=n}^{\infty}\frac{\overline{G}(k-n,h-n;r)(y-y_0)^h(x-x_0)^k}{(k+1-n)(k+2-n)\ldots(k)(h+1-n)(h+2-n)\ldots(h)}$$

$$=\sum_{h=n}^{\infty}\sum_{k=n}^{\infty}\frac{(k-n)!\,(h-n)!\,\overline{G}(k-n,h-n;r)(y-y_0)^h(x-x_0)^k}{k!\,h!}$$

Now using (3.3) of the FDTM, we have, for $0\le r\le 1$:

$$\underline{F}(k,h;r)=\frac{(k-n)!\,(h-n)!\,\underline{G}(k-n,h-n;r)(y-y_0)^h(x-x_0)^k}{k!\,h!}$$

$$\overline{F}(k,h;r)=\frac{(k-n)!\,(h-n)!\,\overline{G}(k-n,h-n;r)(y-y_0)^h(x-x_0)^k}{k!\,h!}$$

Which completes this proof.

Theorem:4.2.4:

Let us suppose that$U(k,h) and G(k,h)$are the differential transformations of the function $u(x,y) and g(x,y)$, respectively.

If $f(x,y)=\int_{y_0}^{y}\int_{x_0}^{x} g(t,s).u(t,s)dtds$ then, for $0\le r\le 1$:

Under (1) differentiability of f we have

$$\underline{F}(k,h;r)=\sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{kh}$$

$$\overline{F}(k,h;r)=\sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{kh}$$

(4.4)

Using (2) differentiability in (1.1.9) of the function$f$, we have

$$\underline{F}(k,h;r)=\sum_{i=0,even}^{h-1}\sum_{l=0,even}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{kh}+\sum_{i=1,odd}^{h-1}\sum_{l=1,odd}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{kh}F(0,0;r)=(0,0)$$

$$\overline{F}(k,h;r)=\sum_{i=0,even}^{h-1}\sum_{l=0,even}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{kh}+\sum_{i=1,odd}^{h-1}\sum_{l=1,odd}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{kh}F(0,0;r)=(0,0)\ ,$$

(4.5)

Proof:

Using definition (3.1.2) of FDTM with (1) differentiability of f,we have

For $0\le r\le 1$

$$F(0,0;r) = \int_{y_0}^{y}\int_{x_0}^{x} g(t,s).u(t,s;r)dtds$$

$$= \begin{bmatrix} \int_{y_0}^{y}\int_{x_0}^{x} g(t,s).\underline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds\,, \\ \int_{y_0}^{y}\int_{x_0}^{x} g(t,s).\overline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds \end{bmatrix}$$

$$= \left[\frac{G(k,h)\underline{U}(k,h;r)(x-x_0)^{k+1}(y-y_0)^{h+1}}{(k+1)(h+1)}, \frac{G(k,h)\overline{U}(k,h;r)(x-x_0)^{k+1}(y-y_0)^{h+1}}{(k+1)(h+1)}\right]$$

$k = 0, h = 0, x = x_0, y = y_0$

$$F(0,0;r) = \begin{bmatrix} G(0,0)\underline{U}(0,0;r)(x_0-x_0)^1(y_0-y_0)^1, \\ G(0,0)\overline{U}(0,0;r)(x_0-x_0)^1(y_0-y_0)^1 \end{bmatrix}$$

$$= (0,0)$$

$$\mathrm{F}(1,1;r) = \frac{d}{dt}\frac{d}{ds}[\int_{y_0}^{y}\int_{x_0}^{x} g(t,s).u(t,s;r)dtds]$$

$$= [\frac{d}{dt}\frac{d}{ds}[\int_{y_0}^{y}\int_{x_0}^{x} g(t,s).\underline{u}(t,s;r)dtds]\,, \frac{d}{dt}\frac{d}{ds}[\int_{y_0}^{y}\int_{x_0}^{x} g(t,s).\overline{u}(t,s;r)dtds]\,]$$

$$= [g(x,y).\underline{u}(x,y;r), g(x,y).\overline{u}(x,y;r)]$$

$$= G(k,h)\,U(k,h;r)$$

$$= G(0,0)\,U(0,0;r)$$

$$F(2,2;r) = \frac{1}{2\times 2}\frac{d^2}{dt^2}\frac{d^2}{ds^2}[\int_{y_0}^{y}\int_{x_0}^{x} g(t,s).u(t,s;r)dtds]$$

$$= \frac{1}{2\times 2}\,[G(1,1)\,U(0,0;r), [G(0,0)\,U(1,1;r)]$$

In general we have, for $0 \leq r \leq 1$:

$$\underline{F}(k,h;r) = \sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{kh}$$

$$\overline{F}(k,h;r) = \sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{kh}$$

We can consider the function f which is (2) differentiable in the same way to obtain the needed result that fulfill this proof. This part of proof is omitted since it follows the same way above.

Theorem:4.2.5:

Suppose $U(k,h)$ $and G(k,h)$are the differential transformation of the functions,$u(x,y) and g(x,y)$(is positive real-valued function), respectively.

If $f(x,y)=[g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}u(t,s;r)dtds]$

Then using (1)-differentiability of f, we have, for $0\le r\le 1$:

$$\underline{F}(k,h;r)=\sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

$$\overline{F}(k,h;r)=\sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)} \qquad (4.6)$$

And when (2)-differentiability of $f$is considered, we have, for $0\le r\le 1$:

$\underline{F}(k,h;r)=$

$$\sum_{i=0,even}^{h-1}\sum_{l=0,even}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}+\sum_{i=0,odd}^{h-1}\sum_{l=0,odd}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

$\overline{F}(k,h;r)=$

$$\sum_{i=0,even}^{h-1}\sum_{l=0,even}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}+\sum_{i=0,odd}^{h-1}\sum_{l=0,odd}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

(4.7)

Proof:

$$\mathrm{F}(0,0;r)=[g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}u(t,s;r)dtds]$$

$$=\left[g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\underline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds,\ g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\overline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds\right]$$

$$= [\frac{G(k,h)\underline{U}(k,h;r)(x-x_0)^{k+1}(y-y_0)^{h+1}}{(k+1)(h+1)}, \frac{G(k,h)\overline{U}(k,h;r)(x-x_0)^{k+1}(y-y_0)^{h+1}}{(k+1)(h+1)}]$$

$$= [0,0]$$

$$F(1,1;r) = \frac{d}{dt}\frac{d}{ds}[g(x,y)\int_{y_0}^{y}\int_{x_0}^{x} u(t,s;r)dtds]$$

$$= \left[\frac{d}{dt}\frac{d}{ds}g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\underline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds,\ \frac{d}{dt}\frac{d}{ds}g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\overline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds]\right]$$

$$= [g(x,y)\underline{u}(x,y;r), g(x,y)\overline{u}(x,y;r)]$$

$$= G(0,0)U(0,0;r)$$

$$F(2,2;r) = \frac{d^2}{dt^2}\frac{d^2}{ds^2}[g(x,y)\int_{y_0}^{y}\int_{x_0}^{x} u(t,s;r)dtds]$$

$$= [\frac{d^2}{dt^2}\frac{d^2}{ds^2}g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\underline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds, \frac{d^2}{dt^2}\frac{d^2}{ds^2}g(x,y)\int_{y_0}^{y}\int_{x_0}^{x}\overline{u}(t,s;r)(t-x_0)^k(s-y_0)^h dtds]\,]$$

$$= [G(1,1)U(0,0;r), U(1,1;r)G(0,0)]$$

In general, we have, for $0 \le r \le 1$:

$\underline{F}(k,h;r)=$

$$\sum_{i=0,even}^{h-1}\sum_{l=0,even}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)} + \sum_{i=0,odd}^{h-1}\sum_{l=0,odd}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

$\overline{F}(k,h;r)=$

$$\sum_{i=0,even}^{h-1}\sum_{l=0,even}^{k-1}\frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)} + \sum_{i=0,odd}^{h-1}\sum_{l=0,odd}^{k-1}\frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

Which completes the proof

Theorem 4.2.6:

Suppose that $U(k,h) and G(k,h)$ are the differential transformation of the functions, $u(x,y) and g(x,y)$ (is positive real-valued function), respectively.

If $f(x,y) = g(x,y)\, u(x,y)$, then

$$F(k,h;r) = \sum_{i=0}^{h}\sum_{l=0}^{k} G(l,i) \odot U(k-l,h-i;r) \quad ,0 \le r \le 1 \tag{4.8}$$

Proof:

Using (3.11) and (3.12) of 2D. FDTM.

we have :

$$f(x,y;r) \approx \left(\sum_{i=0}^{h}\sum_{l=0}^{k} G(k,h)\,(x-x_0)^k(y-y_0)^h\right) \odot \left(\sum_{i=0}^{h}\sum_{l=0}^{k} U(k,h;r)\,(x-x_0)^k(y-y_0)^h\right)$$

$$\begin{aligned} = {} & [G(0,0) + (G(1,1)(x-x_0)(y-y_0)) + (G(2,2)(x-x_0)^2(y-y_0)^2) + \cdots \\ & + (G(n,m)(x-x_0)^n(y-y_0)^m)] \odot [U(0,0;r) + (U(1,1;r)(x \\ & - x_0)(y-y_0)) + (U(2,2;r)(x-x_0)^2(y-y_0)^2) + \cdots \\ & + (U(n,m;r)(x-x_0)^n(y-y_0)^m)] \end{aligned}$$

$$\begin{aligned} = {} & [G(0,0)U(0,0;r)] + [G(0,0)U(1,1;r) + G(1,1)U(0,0;r)][(x-x_0)(y-y_0)] \\ & + [(G(0,0)U(2,2;r)) + (G(1,1)U(1,1;r)) \\ & + (G(2,2)U(0,0;r)][(x-x_0)^2(y-y_0)^2] + \cdots + [(G(0,0)U(n,m;r)) \\ & + (G(1,1)U(n-1,m-1;r)) + \cdots \\ & + (G(n-1,m-1)U(1,1;r)) + (G(n,m)U(0,0;r))][(x-x_0)^n(y-y_0)^m] \end{aligned}$$

In general:

$$F(k,h;r) = \sum_{i=0}^{h}\sum_{l=0}^{k} G(l,i) \odot U(k-l,h-i;r), \quad 0 \le r \le 1$$

Which completes the proof.

Chapter five

Numerical Examples

5.1 Examples for the Solution of the One Dimensional Case:

In this section, we present some numerical examples using the fuzzy Volterra integral equations in the one dimensional case and using the fuzzy differential transform method (FDTM).

Example 5.1.1.[2]: Consider the following fuzzy Volterra integral equation:

$$u(x) = f(x) + \int_0^x (x - t)u(t)\,dt$$

where $f(x;r) = \left(1 - x - \frac{x^2}{2}\right) \odot [r, 2 - r]$.

Solution: On the basis of Theorem 2.2.6 and while using the equation's (2.17) and (2.15) we have the following :

For $0 \leq r \leq 1, k \geq 1$.

$$\underline{U}(k,r) = \left(\delta(k) - \delta(k-1) - \frac{\delta(k-2)}{2}\right)r + \sum_{i=0}^{k-1} \delta(l-1)\frac{\underline{U}(k-l-1;r)}{k-l} - \sum_{i=0}^{k-1} \delta(l-1)\frac{\underline{U}(k-l-1;r)}{k}$$

And

$$\overline{U}(k,r) = \left(\delta(k) - \delta(k-1) - \frac{\delta(k-2)}{2}\right)(2-r) + \sum_{i=0}^{k-1} \delta(l-1)\frac{\overline{U}(k-l-1;r)}{k-l} - \sum_{i=0}^{k-1} \delta(l-1)\frac{\overline{U}(k-l-1;r)}{k}$$

Where $\underline{U}(0;r) = rand\overline{U}(0;r) = 2-r$. Therefore,

$\underline{U}(1;r) = -r$	$\underline{U}(2;r) = 0$
$\underline{U}(3;r) = \frac{-r}{3!}$	$\underline{U}(4;r) = 0$
$\underline{U}(5;r) = \frac{-r}{5!}$	$\underline{U}(6;r) = 0$
$\underline{U}(7;r) = \frac{-r}{7!}$	$\underline{U}(8;r) = 0$
$\vdots$	$\vdots$

$\overline{U}(1;r) = -(2-r)$	$\overline{U}(2;r) = 0$
$\overline{U}(3;r) = \frac{-(2-r)}{3!}$	$\overline{U}(4;r) = 0$
$\overline{U}(5;r) = \frac{-(2-r)}{5!}$	$\overline{U}(6;r) = 0$
$\overline{U}(7;r) = \frac{-(2-r)}{7!}$	$\overline{U}(8;r) = 0$
$\vdots$	$\vdots$

The Taylor series of $sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$, the $r-cut$ representation of solutionis:

$$u(x;r) = [r, 2-r] \odot (1 - \sinh(x)), 0 \leq r \leq 1.$$

which is in practice the exact solution of this problem.

Example 5.1.2.[12]: Consider the following fuzzy Volterra integro-differential equation:

$$u'(x) = (1+x)(r+1, r-2) + \int_0^x u(t)\,dt$$
$$u(0) = (0,0), \;\; \acute{u}(0) = (r+1, r-2),$$

Solution:

Based on Theorem 2.2.6 and using properties of FDTM, we have the following for all $0 \le r \le 1, k \ge 1.$:

$$\underline{U}(k+1, r) = [(\delta(k) + \delta(k-1))(r+1) + \frac{\underline{U}(k-1; r)}{k}]\frac{k}{(k+1)!}$$

And

$$\overline{U}(k+1, r) = [(\delta(k) + \delta(k-1))(r-2) + \frac{\overline{U}(k-1; r)}{k}]\frac{k}{(k+1)!}$$

Where

$\underline{U}(0; r) = 0$	$\overline{U}(0; r) = 0$
$\underline{U}(1; r) = r+1$	$\overline{U}(1; r) = r-2$
$\underline{U}(2; r) = \frac{r+1}{2!}$	$\overline{U}(2; r) = \frac{r-2}{2!}$
$\underline{U}(3; r) = \frac{r+1}{3!}$	$\overline{U}(3; r) = \frac{r-2}{3!}$
$\underline{U}(4; r) = \frac{r+1}{4!}$	$\overline{U}(4; r) = \frac{r-2}{4!}$
$\underline{U}(5; r) = \frac{r+1}{5!}$	$\overline{U}(5; r) = \frac{r-2}{5!}$
$\underline{U}(6; r) = \frac{r+1}{6!}$	$\overline{U}(6; r) = \frac{r-2}{6!}$
$\underline{U}(7; r) = \frac{r+1}{7!}$	$\overline{U}(7; r) = \frac{r-2}{7!}$
$\underline{U}(8; r) = \frac{r+1}{8!}$	$\overline{U}(8; r) = \frac{r-2}{8!}$
$\underline{U}(9; r) = \frac{r+1}{9!}$	$\overline{U}(9; r) = \frac{r-2}{9!}$
$\vdots$	$\vdots$

Therefore, the solution of fuzzy Volterra equation will be

$$u(x) = (r+1, r-2)(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \cdots)$$

$$u(x) = (r+1, r-2)(e^x - 1)$$

which is the exact solution of fuzzy Volterra equation.

5.2 Examples for the Solution of the Two Dimensional Case:

In this section, we use numerical examples to find the numerical solution of fuzzy Volterra integral equations in two dimensions using 2D.FDTM in order to show the applicability of the proposed method.

Example 5.2.1: Define the fuzzy Volterra integral equation for this example as follows:

$$U(x,y)= f(x,y) + \int_0^y \int_0^x (1+xy)u(t,s)dtds$$

where $f(x,y;r) = [(3,3)+r,(8,8)-2r]$. By Theorem 4.1.6 and the equation's (4.2) , (4.6), we have for $0 \le r \le 1, k \ge 1, h \ge 1$:

$$U(x,y;r)= f(x,y) + \int_0^y \int_0^x u(t,s)dtds + xy\int_0^y \int_0^x u(t,s)dtds$$

$$\underline{u}(x,y;r) = \big((3,3)+r\big)[\delta(k,h)+\delta(k-1,h-1)] + \frac{\underline{U}(k-1,h-1)}{kh} + \sum_{i=0}^{h-1}\sum_{l=0}^{k-1} \frac{G(l,i).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

$$\overline{u}(x,y;r) = ((8,8)-2r)[\delta(k,h)+\delta(k-1,h-1)] + \frac{\overline{U}(k-1,h-1)}{kh} + \sum_{i=0}^{h-1}\sum_{l=0}^{k-1} \frac{G(l,i).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)}$$

Where
$\underline{u}(0,0;r) = \big((3,3)+r\big) and \overline{u}(0,0;r) = ((8,8)-2r)$. Consequently, we obtain:

$\underline{u}(1,1;r)= 2\big((3,3)+r\big)$	$\underline{u}(0,1;r)= 0$	$\underline{u}(1,2;r)= 0$
$\underline{u}(2,2;r)=\frac{3\,((3,3)+r)}{2}$	$\underline{u}(0,2;r)= 0$	$\underline{u}(1,3;r)= 0$
$\underline{u}(3,3;r)=\frac{4((3,3)+r)}{(2)(3)}$	$\underline{u}(0,3;r)= 0$	$\underline{u}(1,4;r)= 0$

$\underline{u}(4,4;r)=\frac{5\,((3,3)+r)}{(2)(3)(4)}$	$\underline{u}(1,0;r)=0$	$\underline{u}(2,1;r)=0$
$\underline{u}(5,5;r)=\frac{6\,((3,3)+r)}{(2)(3)(4)(5)}$	$\underline{u}(2,0;r)=0$	$\underline{u}(2,3;r)=0$
$\underline{u}(6,6;r)=\frac{7\,((3,3)+r)}{(2)(3)(4)(5)(6)}$	$\underline{u}(3,0;r)=0$	$\underline{u}(2,4;r)=0$

And

$\overline{u}\,(1,1;r)=2\,((8,8)-2r)$	$\overline{u}(0,1;r)=0$	$\overline{u}(1,2;r)=0$
$\overline{u}(2,2;r)=\frac{3\,((8,8)-2r)}{2}$	$(\overline{u}0,2;r)=0$	$\overline{u}(1,3;r)=0$
$\overline{u}\,(3,3;r)=\frac{4((8,8)-2r)}{(2)(3)}$	$\overline{u}(0,3;r)=0$	$\overline{u}(1,4;r)=0$
$\overline{u}\,(4,4;r)=\frac{5\,((8,8)-2r)}{(2)(3)(4)}$	$\overline{u}(1,0;r)=0$	$\overline{u}(2,1;r)=0$
$\overline{u}\,(5,5;r)=\frac{6\,((8,8)-2r)}{(2)(3)(4)(5)}$	$\overline{u}(2,0;r)=0$	$\overline{u}(2,3;r)=0$
$\overline{u}\,(6,6;r)=\frac{7\,((8,8)-2r)}{(2)(3)(4)(5)(6)}$	$\overline{u}(3,0;r)=0$	$\overline{u}(2,4;r)=0$

Example 5.2.2: In this example, we use the following fuzzy Volterra integral equation:

$$U(x,y)=f(x,y)+\int_0^y\int_0^x(x+y-t)u(t,s)dtds$$

Where$f(x,y;r)=[(3,3)+r,(8,8)-2r]$. Applying Theorem 4.1.6 and using equation's (4.6) and (4.4), we have for $0\le r\le 1, k\ge 1, h\ge 1$:

$$\mathrm{U}(x,y;r)= f(x,y)+(x+y)\int_0^y\int_0^x u(t,s)dtds + t\int_0^y\int_0^x u(t,s)dtds$$

$$\underline{u}(x,y;r) = \big((3,3)+r\big)[\delta(k)+\delta(h)] + \sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{\delta(l-1,i-1).\underline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)} - \sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{\delta(l-1,i-1).\underline{U}(k-l-1,h-i-1;r)}{kh}$$

$$\overline{u}(x,y;r) = (8,8)-2r)[\delta(\mathrm{k})+\delta(\mathrm{h})] + \sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{\delta(l-1,i-1).\overline{U}(k-l-1,h-i-1;r)}{(k-l)(h-i)} - \sum_{i=0}^{h-1}\sum_{l=0}^{k-1}\frac{\delta(l-1,i-1).\overline{U}(k-l-1,h-i-1;r)}{kh}$$

Where
$\underline{u}(0,0;r)=\big((3.3)+r\big)and\overline{u}(0.0;r)=((8,8)-2r)$. Consequently, we obtain:

$\underline{u}(1,1;r)=0$	$\underline{u}(0,1;r)=0$	$\underline{u}(1,2;r)=0$
$\underline{u}(2,2;r)=\frac{(3,3)+r}{2}$	$\underline{u}(0,2;r)=0$	$\underline{u}(1,3;r)=0$
$\underline{u}(3,3;r)=0$	$\underline{u}(0,3;r)=0$	$\underline{u}(1,4;r)=0$
$\underline{u}(4,4;r)=\frac{7((3,3)+r)}{(2)(3)(4)(3)(4)}$	$\underline{u}(1,0;r)=0$	$\underline{u}(2,1;r)=0$
$\underline{u}(5,5;r)=0$	$\underline{u}(2,0;r)=0$	$\underline{u}(2,3;r)=0$
$\underline{u}(6,6;r)=\frac{(11)(7)\big((3,3)+r\big)}{(2)(3)(4)(5)(6)(3)(4)(5)(6)}$	$\underline{u}(3,0;r)=0$	$\underline{u}(2,4;r)=0$

And

$\overline{u}\,(1,1;r) = 0$	$\overline{u}(0,1;r) = 0$	$\overline{u}(1,2;r) = 0$
$\overline{u}(2,2;r) = \frac{((8,8)-2r)}{2}$	$\overline{u}(0,2;r) = 0$	$\overline{u}\,(1,3;r) = 0$
$\overline{u}\,(3,3;r) = 0$	$\overline{u}(0,3;r) = 0$	$\overline{u}\,(1,4;r) = 0$
$\overline{u}\,(4,4;r) = \frac{7\,((8,8)-2r)}{(2)(3)(4)(3)(4)}$	$\overline{u}(1,0;r) = 0$	$\overline{u}(2,1;r) = 0$
$\overline{u}\,(5,5;r) = 0$	$\overline{u}(2,0;r) = 0$	$\overline{u}(2,3;r) = 0$
$\overline{u}(6,6;r) = \frac{(11)(7)\,((8,8)-2r)}{(2)(3)(4)(5)(6)(3)(4)(5)(6)}$	$\overline{u}(3,0;r) = 0$	$\overline{u}(2,4;r) = 0$

5.3 Conclusion and Future Work:

we present a fuzzy differential transform method to approximate the solution of Fuzzy Volterra Integral equations in 2 Dimensions with separable kernels. To this end, we develop a 2 dimensional framework for the method by expanding the theoretical work to entertain 2 dimensional problems, we apply our proposed method to solve some examples and present the results to confirm the theoretical work. And we will be expanding this method toentertain 3 dimensional problems in the future.

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ملخص

حل معادلة فولتيرا التكاملية الضبابية في بعدين بطريقة تحويل المشتقة الضبابي

غدير رواجبة

ركزت العديد من الطرق الحالية لحل المعادلات التكاملية الضبابية على حل مثل هذه المشكلات في بُعد واحد. في حين أن هذه الأساليب فعالة إلا أن هناك حاجة إلى حلول ذات أبعاد أعلى للتعامل مع الأنواع الخارجية أو التطبيقات العملية. إحدى هذه المعادلات هي معادلات فولتيرا التكاملية الضبابية بنواة قابلة للفصل.

في هذا العمل ، نركز على المعادلات التكاملية الضبابية في بُعد واحد ونقدم طريقة تحويل المشتقة الضبابية لتقريب الحل في بعدين. نقوم بتوسيع العمل في المعادلات التكاملية الضبابية في بُعد واحد وتطوير إطار ثنائي الأبعاد للطريقة من خلال توسيع العمل النظري لحل المشاكل ثنائية الأبعاد. بعد ذلك ، نطبق طريقتنا المقترحة لحل بعض الأمثلة المعيارية وتقديم النتائج لتأكيد العمل النظري.