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Solving Two-Dimensional Fredholm Integral Equations Using Radial Basis Functions

By
Ghadeer Abed-Elbaset Mahmoud Mosleh

Supervisor
Dr. Abdelhalim Ziqan

Co-Supervisor
Dr. Ammar Qarariyeh

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Thesis Approval

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This thesis was defended successfully on //2023 and approved by

Committee Members — Signature

1. Supervisor Name: Dr. Abdelhalim Ziqan
2. Co-Supervisor Name: Dr. Ammar Qarariyeh
3. Internal Examiner Name: Dr.Salah Afaneh
4. External Examiner Name: Dr.Hasan Idais

Declaration

My name is Ghadeer Abed-elbaset Mosleh, and I am an rab American University graduate student (number 201712842). I confirm that I have completed this Master's thesis myself. And I declare that I have complied with the Arab American University regulations, instructions, and standards of academic codes of conduct. I also adhere to the Deans' Council regulations in withdrawing their confirmation of this degree in case of any violations.

Name: Ghadeer Mosleh

Date: 11.6.2023

Signature:

غدير مصلح

Dedication

I dedicate this thesis

to my parents,

to my husband Kefah,

and to my children Salma, Alma , Aze Adeen and Mohammad.

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First, all gratitude is due to God Almighty, for guiding, and inspiring me and blessing me with patience and strength to complete my academic degreee satisfactorily.

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Abstract

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As its title indicates, this study was devoted to solving Fredholm integral equation in two dimensions, using Radial Basis Functions (RBF) and RBF thin plate splines in particular. Fredholm integral equation is an import equation with many applications in science and engineering. There are different RBF bases in scholarly research. Researchers have used Gaussian and the inverse multi-quadric RBFs given their attributes when dealing with engineering applications. However, new types of RBFs have been introduced with more flexible mathematical properties. In this study, we used the thin plate splines as basis functions to introduce a solution for Fredholm integral equation. After finding the analytical solution for Fredholm integral equation, we introduced a number of RBF bases to solve the integral equation using the introduced types, including the thin plate splines. Then we conducted a numerical comparison for the error between these different types, using numerical examples. we concluded with a testing of the efficiency of existing approaches compared to this approach.

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Chapter One

Introduction

A function of a numerical variable, for example, may appear under an integral in an equation known as an integral equation. The unknown function is thereby subsumed under one or more integrals in a functional equation. Equation of linear integrals in its generic form is:

$$h(x)y(x) = f(x) + \int_{a(x)}^{b(x)} k(x, y)y(t)dt \tag{1.1}$$

The unknown function is h(x), the function k(x,y) is referred to as the integral equation's kernel, and f(x) , also known as the perturbation function [25].
Integral equations naturally appear in a wide variety of scientific and engineering disciplines. In scientific study, a computational approach to solving integral equations is crucial. Fredholm and Volterra integral equations are two examples of integral equations. One of the most significant integral equations is the Fredholm equation [28].
The most standard form of Fredholm linear integral equations is given by equation(1.1), where the limits of integration a(x) and b(x) are constants and the unknown function h(x) appears linearly under the integral sign, and k(x,t) is the kernel of the equation [21].
In integral equations theory, when y(x)=1 and the kernel is degenerate of the

form:

$$K(x,y) = \alpha_1(x)\beta_1(x) + ... + \alpha_n(x)\beta_n(x) \tag{1.2}$$

where the functions $\alpha_1, \ldots \alpha_n, \beta_1, \ldots \beta_n$ are continuous and $\alpha_1, \ldots \alpha_n$ are linearly independent, the integral equation (1.1)can be transformed into a system of finite dimensional linear equations, and the precise solution of a Fredholm integral equation can be easily derived using straightforward linear algebra techniques [21].

There are four methods for resolving the second type of Fredholm integral equation: piecewise Bernoulli polynomials, mean value theorem, singular kernels, and basis functions.

A component of a specific basis for a function space is a basis function. A linear combination of basis functions can be used to represent any continuous function in the function space, much as a linear combination of basis vectors can be used to represent any vector in a vector space. Sorts of basis function are triangle function, wavelet, spline and radial basis function [10]. Solving integral equations is one of the most significant topics in applied mathematics when dealing with applied problems including those that arise naturally in signal processing theory. It appears in inverse issues and frontal linear modeling. In physics, problems in fluid mechanics involving hydrodynamic interactions close to flexible interfaces of finite size are also represented by the integration of Fredholm equations. The Fredholm equation can be solved in several ways but there is a more straightforward way.

This research focuses on Radial Basis Functions. (RBFs). These formulas are created using a distance variable that has a single dimension. When compared to the conventional coordinates-based functions, they seem to have a few advantages. The RBF collocation methods are mathematically straightforward and fully meshless, in contrast to the conventional meshed-based approaches. They stay away from problematic mesh formation for issues in high dimensions with irregular or moving boundaries. RBF types are polyharmony spline, thin plate splines (TPS), MQ, IMQ and Gaussian [12]. The thin plate splines are spline-based techniques for data interpolation and smoothing. Introduced to geometric design by Duchon, [15] they are an important special case of a polyharmony spline. Robust Point Matching (RPM) is a common extension and shortly known as the TPS-RPM algorithm. This type of radial basis is used in this research to solve the equation in hand by building up a solution space where the basis functions are the thin plate splines. Then the method is implemented numerically and finally, we caps with comparison of the results with other methods based on classical radial basis. This topic has been an active area of research for a number of years. The interest in this area increased with the advancement of different technologies in fields of science. In this respect, the researcher has made a review of related literature, analyzed the methods, followed one of them in this proposal. She concluded with the testing of the efficiency of such approaches. The unique equation has been solved by using several methods of radial basis functions, in the literature. The Galerkin and collocation methods are the two commonly used methods for the numerical solutions of the two-dimensional integral equations. The analysis for convergence of these methods is well doc- umented in the literature [24] [8] [7]. Han and Wang approximated the two-dimensional Fredholm integral equations by the Galerkin iterative method. In [18], Hadizadeh and Asgary by using the bivariate Chebyshev collocation method solved the linear Volterra–Fredholm integral equations of the second kind. However, in this thesis, it was solved by using the thin plate spline and comparing the results obtained through numerical examples. The present work will form a new approach that will be added to the literature already present in this field. The new approach and numerical comparison conducted will help further the knowledge in this field in the future. In this research, the researcher used RBFs collocation method to solve problems in integral equations. For this purpose, chapter one was devoted to Fredholm integral equations, while chapter two was devoted to the space radial basis function. The researcher studied and analyzed this problem. She explained the RBFs in terms of types, interpolation and properties. Chapter three dwelt on the Thin Plate Spline-RBF for Fredholm Integral Equation in One-Dimension. A numerical example of TPS radial basis functions (RBFs) was applied to resolve the one-dimensional Fredholm integral equation. After solving it, its results were compared with the results of the method. Chapter four discussed solving the Thin Plate Spline-RBF for Fredholm Integral Equation in Two-Dimensions. The two-dimensional Fredholm integral problem was resolved using a numerical example at the TPS radial basis functions (RBFs). After being solved, the outcomes were contrasted with those of the approach.

Chapter Two

Preliminaries

This chapter, presents some basic concepts and definitions.

2.1 Fredholm Integral Equation

A solution to the Fredholm integral equation leads to the development of the Fredholm theory.

Definition 2.1.1 *An integral equation is an equation in which the unknown funcion $h(x)y(x)$ to be determined appears under the integral sign. A typical form of an integral equation in is of the form*

$$h(x)y(x) = f(x) + \int_a^b k(x,y)y(t)dt \tag{2.1}$$

where f(x) and K(x, y) are given continuous functions defined, respectively, on D=[a, b] , k(x, y) is called the kernal of the integral equation and h, y are unknown functions on D.

The integral equations (2.1) is called a Fredholm Integral Equation of the first kind when $h(x) = 0$ as follows

$$0 = f(x) + \int_a^b k(x,y)y(t)dt \tag{2.2}$$

also, The integral equations (2.1) is called a Fredholm Integral Equation of the second kinds when $h(x) = 1$ as follows

$$y(x) = f(x) + \int_a^b k(x, y)y(t)dt \tag{2.3}$$

The theory of signal processing naturally leads to Fredholm equations. David Slepian, for instance, popularized the infamous spectral concentration problem. The same operators that are used in linear filters are used here. Additionally, they frequently appear in inverse issues and linear forward modeling. When such integral equations are solved, it is possible to relate experimental spectra to different underlying distributions in physics. One example is the distribution of polymers by mass in a polymeric melt [2]. The creation of photo-realistic images through computer graphics, in which the Fredholm equation is used to describe light transport from the virtual light sources to the image plane, is one specific application of Fredholm equation. The Liouville–Neumann series is an infinite series that cor-responds to the resovent formalism technique of solving the Fredholm integral equations in Fred- holm theory. The second kind of Fredholm integral equation can be resolved numerically. These analytical techniques include the degenerate kernel methods, the Adomain decomposition method, the modified decomposition method, the method of consecutive approximations, and transforming the Fredholm integral equation to an ODE [26]. The solution of the Fred- holm Integral Equation is analytic. To solve by the Series solution method, the Saberi-Nadja and Heidari applied the technique of Trapezoidal Rule and Simp- son's Rule, as two numerical approximation methods in order to solve the Fredholm integral equations [2].

2.2 Radial Basis Function

Due to the lack of grid data, there are modern ways to approximate multivariate functions. These are called Radial Basis Function methods. For several years, these methods have been known, tested and analyzed. Researchers have identified a number of positive properties in them. They are typically employed as a method of approximating functions that are only known at a finite set of centers or points. Additionally, they are employed to quickly perform evaluations by approximating function at other points or centers. Moreover, these methods have received increasing attention to solve PDE in irregular domains. In his study, According to Hardy [20] , multiquadric RBF are connected to an accurate solution of the biharmonic potential problem. This therefore has a physical basis. RBF are associated to prewavelets, according to research by Buhmann and Micchelli [11] and Chiuet al (wavelets that do not have orthogonality properties). RBF has also been employed by Alipanah and Dehghan [3] to resolve a nonlinear integral equation in the one-dimensional situation..

Definition 2.2.1 *A radial is function $\phi : R^N \longrightarrow R$. If there is a univariate function φ : $[0, \infty) \rightarrow R$, such that $\phi(x) = \varphi(r)$, where r = $\| x \|$; and $\| . \|$ is a norm on R^N a typically Euclidean norm [13].*

Definition 2.2.2 *Let $R^+ = X \in R, x \geq 0$ be the non-negative half-line and let $\psi : R^+ \longrightarrow R$ be a continuous function with $\psi(0) \geq 0$. Radial basis functions on R^d. can be defined as functions of the form $\psi(\| (x - x_j) \|)$, in which x, $x_j \in R^d$, and $\| . \|$denotes the Euclidean distance between x and x_js. If one selects N points $\{x_j\}_{j=1}^N inR^d$ then by custom f(x) = $\Sigma_{j=1}^N \lambda_j \psi(\| (x - x_j) \|)$, $\lambda_j \in R$ is called the Radial Basis Functions as well [5].*

2.2.1 Types of Radial Basis Functions

In the following table, we can see different types.

Tab. 2.1: Types of Raial Basis Functions.

Name of RBF	φ (r), r ≥ 0	Smoothness.
Gaussian(GA)	e^{-cr^2}	Infinite
Multiquadric(MQ)	$\sqrt{c^2 + r^2}$	Infinite.
Inverse Multiquadric(IMQ)	$\frac{1}{\sqrt{c^2+r^2}}$	Infinite
Cubic(CU)	r^3	Piecewise
Thin Plate Spline(TPS)	$r^2 \ln(r)$	Piecewise

The $r = \| x - x_i \|$ denotes the distance between x and the i^{th} nodal point x_i while the c is shape parameter. Controlling the shape of functions, which impacts the pace of convergence, is done using the parameter c. There are two main classes of the common Radial Basis Functions [17].

1- The first class is the infinitely smooth RBFs. The shape parameter c determines these infinitely differentiable basis functions and makes them ' heavily depend on it. Hardy multiquadric (MQ), Gaussian (GA), inverse multiquadric (IMQ), and in- verse quadric(IQ) are examples of these functions [17].

2- The second class is the infinitely smooth (except at points) RBFs or piecewise smooth RBFs. Unlike Class 1, the basis functions of Class 2 are not infinitely differentiable. They are both shape parameters and free. They are also relatively less accurate than the basis. Examples of these functions, are the Thin plate spline, cubic r^3 and linear r [17].

The shape parameter can play a significant part in the accuracy of approximation. The following figures are examples of the shape of Radial Basis Function:

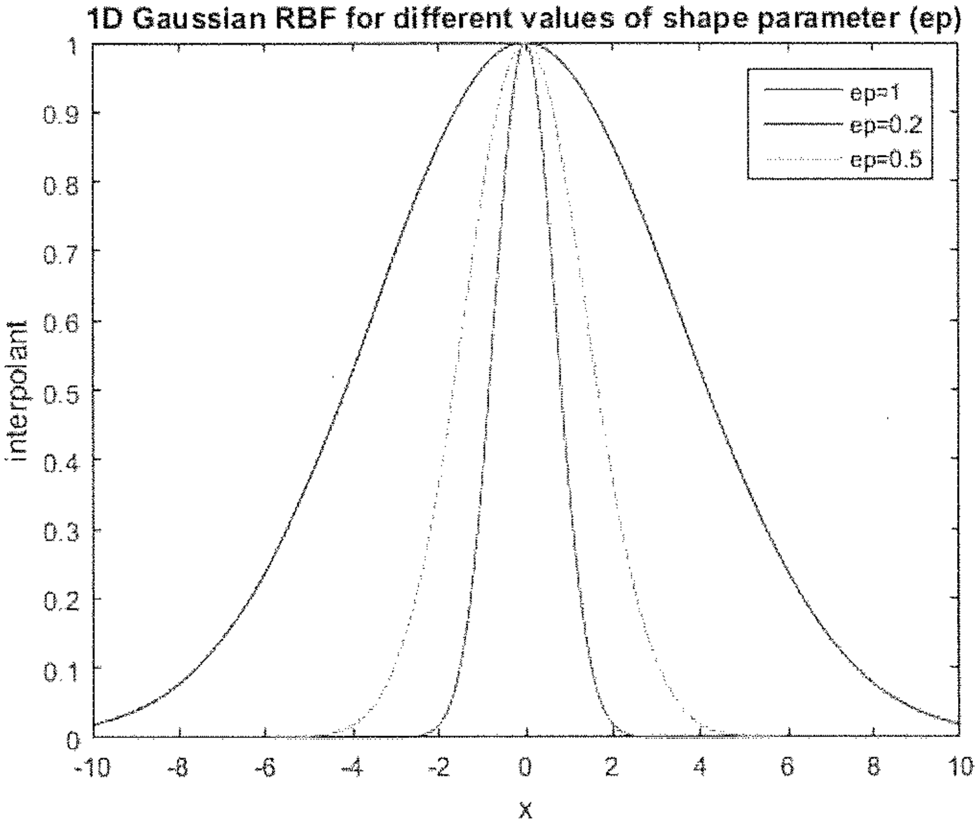
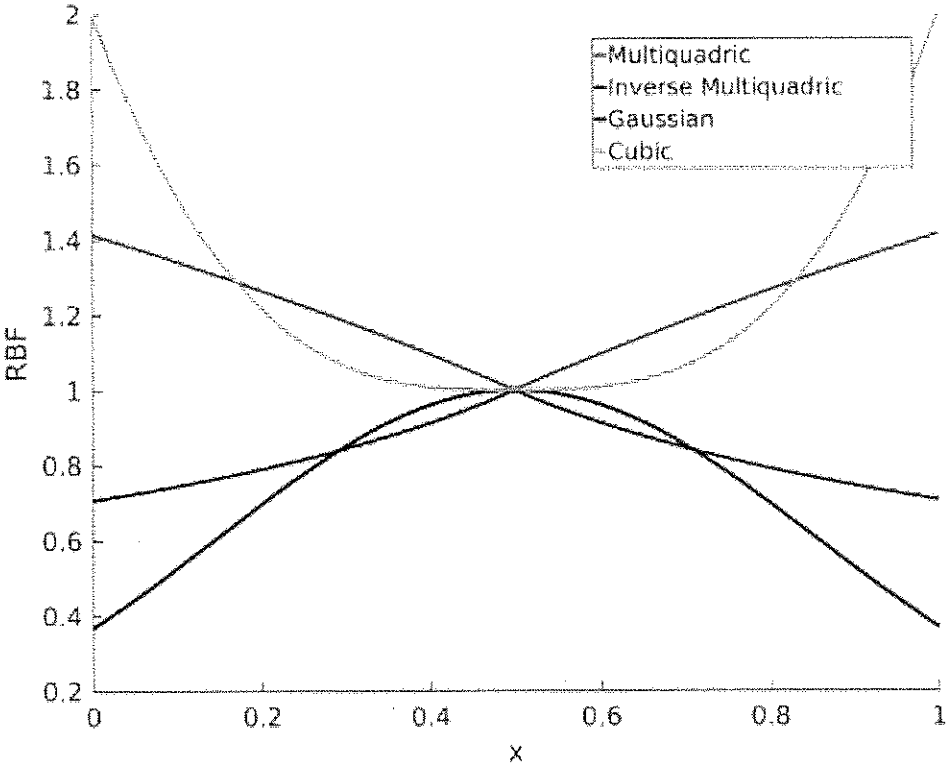


Fig. 2.1: One-Dimension Gaussian RBF

This thesis discusses the Thin Plate Spline, and the way of using it in solving Fredholm Integral Equation.



Radial Basis function shapes in 1D, centered in 0.5 with r = 0.5

Fig. 2.2: Radial Basis Function in One-Dimension

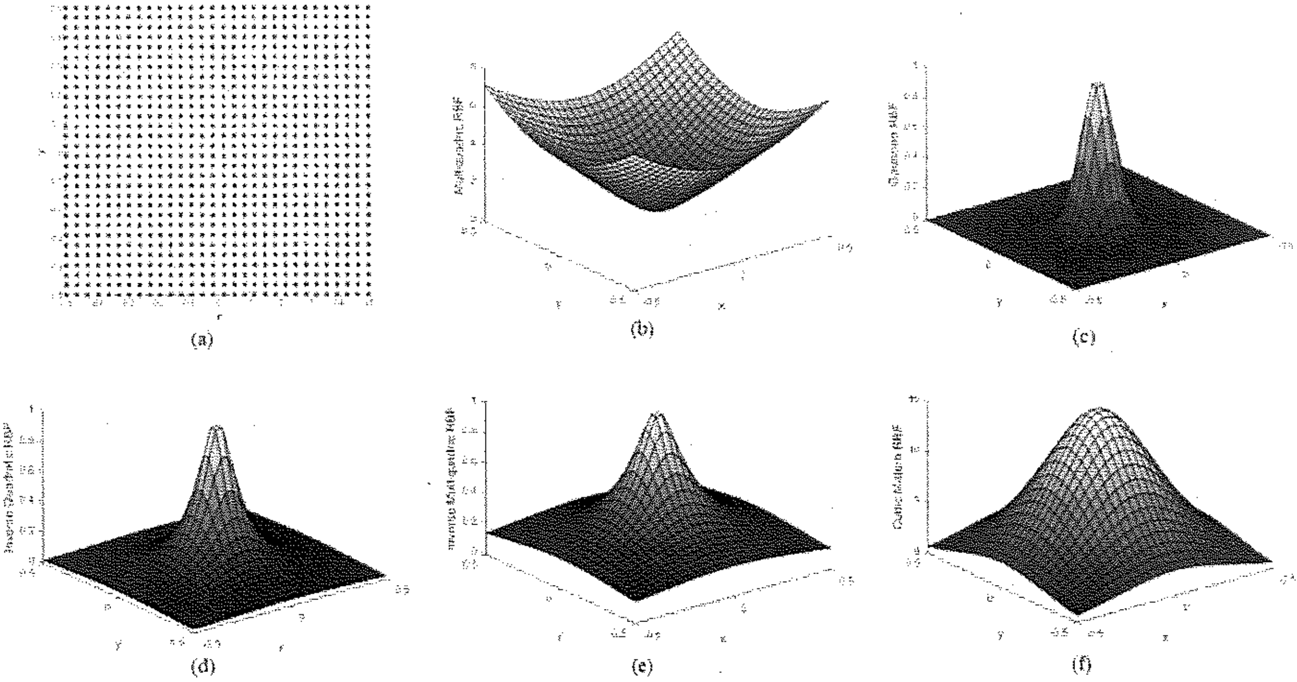


Fig. 2.3: Radial Basis Function in Two-Dimension Profiles at c =10.00; (a) Nodes, (b) Multiquadric , (c) Gaussian , (d) Inverse Quadratic , (e) Inverse Multiquadric , and (f) Cubic Matérn .

2.2.2 Radial Basis Function Interpolation

According to Mairhuber Curtis theorem, to have a well-posed multivariate scattered data interpolation problem, there is a need to prefix a set of basis functions planned to be used for interpolation of arbitrary scattered data. In lieu of fixing the set of basis functions, we make the basis depend on data location [16].

RBF can be defined as a a limited set of distinct points $X \subset R^N$. These are also called centers. The function value $\phi(x)$, $x \in X$ solely depends on $\| x \|$ rather than on the coordinates of x [1].

Given the centers set X = $x_i; i = 1, ..., N$, which is a set of distinct points in R^N, and the function values $f(x_i); i = 1, ..., N$.

The Radial Basis Function approximation to the function f is of the form

$$S(x) = \sum_{i=1}^{N} \lambda_i \phi(\|x - x_i\|_2).$$

The coefficients λ_i are determined by applying the interpolation condition $S(x_i) = f(x_i)$, at the center set X.

Implementing the interpolation condition at N centers re- sults in a $N \times N$ linear system

$$A\,\lambda = f,$$

where the entries of the matrix A are$A_{i,j} = \phi(\|x_i - x_j\|)_{1 \le i,j \le N}$ and

$f = [f_1, ..., f_N]^T$,

$\lambda = [\lambda_1, ..., \lambda_N]^T$.

The interpolant of f(x) is unique if and only if the matrix A is non-singular.It has been discussed about adequate conditions for $\phi(r)$ to guarantee non-singularity of the A matrix [1].

Definition 2.2.3 *A function ϕ on X is assumed to be positive definite on X, if for any set of points $x_1, x_2, ..., x_N$ in X the $N \times N$ matrix A_{ij} = $\phi(x_i$ - $x_j)$ is nonnegative definite, That is.*

$V^T AV = \sum_{i=1}^{N} \sum_{j=1}^{N} v_i v_j A_{ij} \geq 0$

for all nonzero $V \in R^N$. if $V^T AV \geq 0$. Whenever the points x_i are distinct and $V \neq 0$, then we say that $\phi(r)$ is strictly a positive definite function [29].

The eigenvalues and determinant of A are positive, if $\phi(r)$ is to be strictly positive definite function on a linear space.

Theorem 2.2.1 *(Bochner's theorem) On R. f is non-negative Borel function. If $0 < \int Rf < \infty$, then f* is a strictly positive definite, where f* is the Fourier transform of function f, That is. $f^* = \int_{-\infty}^{+\infty} f(y) e^{ixy} dy$. On an inner-product space, a real-value function F is said to be radial if F(x) = F(y) whenever $\|x\| = \|y\|$. On using the aforementioned theorem, One finds many strictly positive definite functions or RBFs. For instance if $f(x) = \frac{e^{-|x|}}{2}$ then $f^*(x) = \frac{1}{1+x^2}$ [27].*

2.2.3 Thin Plate Spline

Computational geometry, a branch of computer science, was where Thin Plate Splines were initially applied. The modeling approach is now widely employed in both the social and natural sciences. It is utilized, for instance, in ecology (e.g., population growth), anthropology (e.g., geometric morphometrics), engineering (e.g., structural design), and pattern recognition (e.g. fingerprint iden-

tification). The study of health sciences also makes use of it. It is specifically employed in the creation and evaluation of medical imaging methods [9].
The 2D generalization of the cubic spline is called the thin plate spline. The technique bases its discrete analysis on thin plate splines (TPSs) constructed on scattered points. collocation strategy TPSs are viewed as a particular category of free form parameter random basis functions. They provide reliable and efficient techniques for estimating unknown functions. In order to approximate integrals, this thesis has used a unique accurate quadrature formula based on the non-uniform Gauss-Legendre integration rule. A common issue in geosciences is the modeling of scattered data, which has led to the development of the Thin Plate Spline (TPS), an interpolation method [6] [14]. The Fredholm Integral Equation was solved in one and two dimensions using the Thin Plate Spline RBF.

2.2.4 Thin Plate Splines Approximation

Let v_i represent the target function values at locations (x_i, y_i) in the plane, with i = 1, ..., N. In particular, the researcher has set v_i equal to the target coordinates(x_i^*, y_i^*) in turn to obtain one continuous transformation for each coordinate.

We assumes that the locations (x_i, y_i) are all different and are not collinear. The TPS interpolant f (x , y) minimizes the bending energy
$I_f = \int\int_{R^2}(f_{xx}^2 + 2f_{xy}^2 + f_{yy}^2)dxdy$
and has the form
$f(x, y) = \sum_{i=1}^{N} w_i U(\|(x_i, y_i) - (x, y)\|)$
where U(r)=$r^2 \ln r$.

In order for f (x, y) to have square integral second derivatives, we require that
$\sum_{i=1}^{N} w_i = 0$, $\sum_{i=1}^{N} w_i x_i = 0$, $\sum_{i=1}^{N} w_i y_i = 0$.

Together with the interpolation condition, $f(x_i, y_i) = v_i$.

Finally, the TPS-value in any point can be calculated according to:

$f(x, y) = \sum_{i=1}^{N} w_i r_i^2 \ln r_i$ [14].

Chapter Three

Thin Plate Spline-RBF for Fredholm Integral Equation in One-Dimention

The one-dimensional Fredholm Integral Equation's approximate solution is presented in this chapter using the Thin Plate Spline RBF. One-Dimensional Fredholm Integral Equations are introduced in the first section. The theoretical basis for this strategy is presented in the second section, and in third section shows the example of solutin fredholm integral eqution using thin plate spline and Compare it with my brother's methods.

3.1 One- Dimensional Fredholm Integral Equations

The one-Dimensional Fredholm Integral Equations of the second kind of the form are considered,

$$y(x) = f(x) + \int_a^b k(x, y)y(t)dt \tag{3.1}$$

where f(x) and K(x, y) are given continuous functions defined, respectively, on D=[a, b] and y is unknown functions on D.

3.2 Solution Method

In RBF interpolation, the unknown function y(x) is expressed as a combination of RBF i.e $y(x) \approx \sum_{i=1}^{N} \lambda_i \phi_i(x)$, where $\phi_i(x) = \phi(|x - x_i|) = |x - x_i|^2 ln(|x - x_i|)$ then the equation(3.1) become:

$$\sum_{i=1}^{N} \lambda_i |x - x_i|^2 ln(|x - x_i|) = f(x) + \int_a^b k(x,y)(\sum_{i=1}^{N} \lambda_i |t - x_i|^2 ln(|t - x_i|))dt \qquad (3.2)$$

Additionally, using the linear property of an integral, the specific expression of the aforementioned equation is found for every collocation pointx_j in $[a, b]$ as

$$\sum_{i=1}^{N} \lambda_i |x - x_i|^2 ln(|x - x_i|) = f(x_j) + \int_a^b k(x_j,y)(\sum_{i=1}^{N} \lambda_i |t - x_i|^2 ln(|t - x_i|))dt \qquad (3.3)$$

x_j collection point.

when selecting M collocation point, thus forming the collocation equations in matrix for

$$[\Phi - K][\lambda] = [F], \qquad (3.4)$$

where $F = (f(x_1), f(x_2), ..., f(x_m))$,

and

$\lambda = (\lambda_1, \lambda_2, ..., \lambda_N)$

The elements in matrix Φ and K are defined

$$\Phi_{ij} = |x_j - x_i|^2 ln|x_j - x_i| \qquad (3.5)$$

$$K_{ij} = \int_a^b k(x_j, t)(|t - x_i|^2 ln|t - x_i|)dt \qquad (3.6)$$

The notation x_i and x_j represent correspondingly the collocation point and RBF center, wherease the component K_{ij} is the defininte integral of function $h(t)$ which can be calculated throughout the follwing Gauss qudratic formula:

$$\int_{-1}^{1} h(u)du \approx \sum_{d=1}^{D} w_d h(u_d) \tag{3.7}$$

The integral range can be transformed [23]

$-1 \leq u \leq 1 \longleftrightarrow a \leq t \leq b$

$if \frac{u-(-1)}{1-(-1)} = \frac{x-a}{b-a}$

so that t=$\frac{b-a}{2}$u + $\frac{b+a}{2}$ = p(u)

$K_{ij} = \int_a^b k(x_j,t)(|t-x_i|^2 ln|t-x_i|$

$=\frac{b-a}{2}\int_{-1}^{1} k(x_j,p(u))|p(u)-x_i|^2 ln|p(u)-x_i|)du$

$\approx \frac{b-a}{2}\sum_{d=1}^{D} w_d k(x_j,p(u_d))(|p(u_d)-x_i|^2 ln|p(u_d)-x_i|)$

3.3 Example of Solution Fredholm Integral Equation in One-Dimension by Thin Plate Spline

Example 3.3.1 *We consider the Fredholm Integral Equation given in,*

$$f(x) = \int_0^1 \sqrt{x+y}y(t)dt, \tag{3.8}$$

where f(x)= $\frac{-16}{105}x^{\frac{7}{2}}$ + $\frac{16}{105}x^2(x+1)^{\frac{3}{2}}$ - $\frac{8}{35}x(x+1)^{\frac{5}{2}}$+$\frac{2}{7}(x+1)^{\frac{7}{2}}$, *$0 \leq x \leq 1$.*

which the precise solution is y(x)=x^2, and 0$\leq$ x $\leq$ 1

setting Radial basis function(TPS-RBF (ϕ(r)=r^{2}ln(r)), GS, and MQ) centers with interval 0.2 the coolocation points are the same as the center points N=10, For MQ the saphe parameter β= 12 . Then, the 20-point Gauss quadrature formula in matrix element calculation, and the numerical result was shown in table 3.1. As Tabel 3.1 shows, comparison between RBF methods.

The maximum errors of types RBF is in the table 3.2.

Tab. 3.1: comparison between RBF methods.

x_i	Exact solutions	TPS RBF	GA RBF	MQ RBF
0.1	0.01	0.0136914898	0.010431787	0.010008990
0.2	0.04	-0.02391498	0.03758911	0.04000413
0.3	0.09	0.14584667	0.0895160	0.0900153654
0.4	0.16	0.28759641	0.1666516087	0.160022667
0.5	0.25	0.183308272	0.255764401	0.250009480
0.6	0.36	0.13743517	0.349716785	0.35998681
0.7	0.49	0.500982125	0.4723170664	0.4899874717
0.8	0.64	0.866245903	0.649983924	0.640029827
0.9	0.81	0.836212529	0.839325022	0.810057024
1.0	1.00	0.775988339	0.923076622	0.999862821

Tab. 3.2: Maximum errors of RBF methods.

Types of RBF	MAX errors	RMSE
TPS RBF	0.226328748386308	0.134065863186407
GS RBF	0.076923377343589	0.027173263649900
MQ RBF	1.371786452182278e-04	4.9222e-05

Comments these two table illustrates, numerical results show Tps rbf is simplicity and very good of the method

Another example

Example 3.3.2 *We consider the Fredholm Integral Equation given in,*

$$f(x) = \int_0^1 (x+y)^2 y(t)dt,$$

(3.9)

where $f(x)= \frac{7}{2}x^2 + \frac{13}{3}x + \frac{19}{12}$, $0 \leq x \leq 1$.

which the precise solution is y(x)=5 x +1, and $0\leq x \leq 1$ this paper used the method presented in it, thin plate spline radial basis function $\phi(r) = r^2 \ln r$ for solving the one dimension fredholm inteqral equation, since this method is simple and involve less computation use N= 10. In the table 3.3 show the exact solution , approximation solutions and maximum error we calculate them by Matlab as a follow

Tab. 3.3: Approximate solution of RBF methods.

x_i	Exact solution	TPS RBF	GA RBF
0.1000	1.50000	1.77950	1.7054
0.2000	2	1.383068	1.9031
0.3000	2.50000	1.67681	2.1372
0.4000	3	2.35860	2.5629
0.5000	3.50000	3.4522566	3.2632
0.6000	4	4.53645	4.1692
0.7000	4.50000	5.38139	5.0474
0.8000	5	5.716698	5.5723
0.9000	5.50000	5.315442	5.4700
1.0000	6	4.149618	4.6764

The maximum errors of types RBF is in the table 3.2.

In the table below show the approximate solution of types of RBF and shows the maximum error of TPS RBF is 1.850381572941764 and the maximum error of GA RBF is 1.3236 . show the maximum error of GA less than the TPS but

Tab. 3.4: Maximum errors of RBF methods.

Types of RBF	MAX errors	RMSE
TPS RBF	1.8504	0.81157
GS RBF	1.3236	0.5328

the difference is simple between them.

Chapter Four

Thin Plate Spline RBF for Solving Two-Dimensional Fredholm Integral Equation

In this chapter, the Thin Plate Spline RBF is used to approximate the solution of the Two-Dimensional Fredholm Integral Equation. Section one, introduces Two-Dimensional Fredholm Integral Equations while section two presents the theoretical path of this method and in third section shows the example of solutin fredholm integral eqution using thin plate spline and Compare it with my brother's methods.

4.1 Two- Dimensional Fredholm Integral Equations

The Two-Dimensional Fredholm Integral Equations of the second kind of the form

$$u(\tau,\mu) = f(\tau,\mu) + \int_a^b \int_c^d K(\tau,\mu,\lambda,\eta,u(\lambda,\eta))d\lambda d\eta, (\tau,\mu) \in D \qquad (4.1)$$

where $f(\tau,\mu)$ and $K(\tau,\mu,\lambda,\eta,u)$ are given continuous functions defined, respectively on D=[a, b]$\times$ [c, d], E =D $\times$ D and u is unknown functions on D. [4]

The second type of two-dimensional Fredholm integral equations can be used to solve a wide variety of engineering and mechanical issues. When calculating plasma physics, for instance, Fredholm integral equations must typically be solved. There are numerous initiatives to create and analyze The second type of Fredholm Integral Equations can be solved numerically.

The aim of this thesis is to apply the TPS-RBF method for solving the Two-Dimensional Fredholm Integral Equations.

4.2 Solution Method

In this section, the Radial Basis Functions are used to approximate the solution of the Fredholm Integral Equation,using the Thin Plate Spline.

The equation(3.1) the in previous section was solved, using TPS-RBF. Changing the variables τ = (b-a)x + a, μ = (d-c)y + c, λ = (b-a)t + a and η = (d-c)s + c, Eq. (3.1) can be written as

$$u(x,y) = f_1(x,y) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x,y,t,s,u(t,s))dtds, (x,y) \in D \qquad (4.2)$$

where f_1(x, y) = f ((b-a)x + a, (d-c)y + c),

K_1(x, y, t, s, u(t, s)) = K ((b-a)x + a, (d-c)y + c, (b-a)t + a, (d-c)s + c, u((b-a)t + a, (d-c)s + c)) and D = [0, 1] $\times$[0, 1]. [4]

Let ϕ(r)be a strictly positive definite function or Radial Basis Function, and we approximate u(x, y) with interpolation by function Φ(r) i.e.,

$$u(x,y) \approx \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x,y) = C^T\Phi(x,y) \qquad (4.3)$$

where, ϕ_{ij}(x, y)=$\phi\|(x_i,y_j)-(x,y)\|$.

$$\Phi(x,y)=\begin{pmatrix} \phi_{00} & \phi_{01} & \dots & \phi_{0M} \\ \phi_{10} & \phi_{11} & \dots & \phi_{1M} \\ . & . & . & \\ . & . & . & \\ . & . & . & \\ \phi_{N0} & \phi_{N1} & \dots & \phi_{NM} \end{pmatrix}$$

$$\text{and } C^T=\begin{pmatrix} C_{00} & C_{01} & \dots & C_{0M} \\ C_{10} & C_{11} & \dots & C_{1M} \\ . & . & . & \\ . & . & . & \\ . & . & . & \\ C_{N0} & C_{N1} & \dots & C_{NM} \end{pmatrix}$$

Also (x_i, y_j) are the node points. Now by substituting Eq. (4.3) in Eq. (4.2) we have:

$C^T\Phi(x,y) = f_1(x,y) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x,y,t,s,C^T\Phi(t,s))dtds,$

or

$\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x,y) \approx f_1(x,y) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x,y,t,s,\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(t,s)dtds,$

For obtaining c_{ij}, i = 0, 1, . . . , N and j=0, 1, ..., M in the Eq. (3.5), we consider the points (x, y) = (x_i, y_j) for i = 0, 1, . . . , N and j=0, 1, ..., M we have that

$C^T\Phi(x_i,y_j) = f_1(x_i,y_j) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x_i,y_j,t,s,C^T\Phi(t,s))dtds,$

or

$\sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(x_i, y_j) \approx f_1(x_i, y_j) + (b-a)(d-c)\int_0^1\int_0^1 K_1(x_i, y_j, t, s, \sum_{i=0}^{N}\sum_{j=0}^{M} c_{ij}\phi_{ij}(t, s)dtds,$
By applying numerical integration method approximate the integral of f on [0, 1] as $\int_0^1$f(x) dx= $\sum_{i=0}^{N} w_i f(x_i)$ where x_i nodes point , and w_i is the weights , we can approximate the integral in Eq. (3.2) and hence the above equation can be written as follow
$C^T\Phi(x_i, y_j) = f_1(x_i, y_j) + (b-a)(d-c)\sum_{r_1=0}^{N}\sum_{r_2=0}^{M} w_{r_1} w_{r_2} K_1(x_i, y_j, t_{r_1}, s_{r_2}, C^T\Phi(t, s))$
for i=0, 1, ..., N, j=0, 1, ..., M

This is a nonlinear system of equations, by solving this system, we obtain the unknown vector C^T.

4.3 Example of Solution Fredholm Integral Equation in Two-Dimension by Thin Plate Spline

Example 4.3.1 *We consider the Fredholm Integral Equation in two dimension given in,*

$$f(x, y) = \int_0^1\int_0^1 e^{x+y+s+t}u(s, t)dsdt, \tag{4.4}$$

where f(x,y)= $\frac{(-1+e^2)*e^{x+y}}{2}$ and the exact solution is u(x,y)=$2*\frac{e^{x+y}}{-1+e^2}$, and 0 $\leq$ x,y $\leq$ 1

When we applied TPS-RBF (ϕ(r)=r^2 ln(r)), the
collocation points were the same as the center points four points. (0,0) , (0,1),(1,0),(1,1) . For Gauss quadratiqur formula we choose Q=3. wa calculate the wights and nodes to approximate the two dimensional integral $\int_0^1\int_0^1 e^{x+y+s+t}u(s, t)dsdt$, then we apply the thr TPS-RBF that is described in section 4 and solve $C^T\Phi(x_i, y_j) = f_1(x_i, y_j) + (b-a)(d-c)\sum_{i=1}^{N}\sum_{j=1}^{M} w_i w_j K(x_i, y_j, t_i, s_j, C^T\Phi(t, s))$ to find the unkown matrix C^T. first approximate the two dimension integral where N=M=3, using nodes and weights

we Calculates the nodes and weights of the Gaussian quadrature polynomial,

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x)$ - $nP_{n-1}(x)$.

apply n=3 gets

$P_3(x) = \frac{1}{2}(5\ x^3$ - 3 x) Now,

nodes x_i the i-th zeros of $P_n(x)$,

weights $w_i = \frac{2(1-x_i^2)}{(nP_{n-1}(x_i))^2}$

Tab. 4.1: weights and nodes

i	weight(w_i)	Nodes(x_i)
1	0.555555555556	0.7745966692414834
2	0.888888888	0.0000000000000000
3	0.55555555556	-0.7745966692414834

K_{ij}= $(b-a)(d-c)\sum_{i=1}^{N}\sum_{j=1}^{M} w_i w_j K(x_i, y_j, t_i, s_j, \Phi(t_i, s_j))$.

$K_{(1,1)}(0,0)$=

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{3} w_j e^{0+0+t_1+s_j} \sqrt{(0-t_1)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_1)^2+(0-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+0+t_2+s_j} \sqrt{(0-t_2)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_2)^2+(0-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+0+t_3+s_j} \sqrt{(0-t_3)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_3)^2+(0-s_j)^2} \end{pmatrix}$$

$K_{(1,1)}(0,0)$=

$$\begin{pmatrix} 0.55555 & 0.888888 & 0.55555 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{3} w_j e^{0+0+t_1+s_j} \sqrt{(0-t_1)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_1)^2+(0-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+0+t_2+s_j} \sqrt{(0-t_2)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_2)^2+(0-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+0+t_3+s_j} \sqrt{(0-t_3)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_3)^2+(0-s_j)^2} \end{pmatrix}$$

where

$\sum_{j=1}^{3} w_j e^{0+0+t_1+s_j} \sqrt{(0-t_1)^2+(0-s_j)^2}^2 \ln\sqrt{(0-t_1)^2+(0-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.888888 & 0.55555 \end{pmatrix} \begin{pmatrix} 0.471313 \\ -0.33468 \\ 0.10104 \end{pmatrix} = 0.0209$$

$\sum_{j=1}^{3} w_j e^{0+0+t_2+s_j} \sqrt{(0-t_2)^2+(0-s_j)^2}^2 \ln \sqrt{(0-t_2)^2+(0-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} -0.33429 \\ 0 \\ -0.07164 \end{pmatrix} = -0.2255$$

and

$\sum_{j=1}^{3} w_j e^{0+0+t_3+s_j} \sqrt{(0-t_3)^2+(0-s_j)^2}^2 \ln \sqrt{(0-t_3)^2+(0-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} 0.10099 \\ -0.041287 \\ 0.02163 \end{pmatrix} = 0.00442.$$

then the

$K_{(1,1)}(0,0)=$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} 0.0209 \\ -0.02255 \\ 0.00442 \end{pmatrix} = -0.18622$$

$K_{(1,2)}(0,1)=$

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{3} w_j e^{0+1+t_1+s_j} \sqrt{(0-t_1)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_1)^2+(1-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+1+t_2+s_j} \sqrt{(0-t_2)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_2)^2+(1-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+1+t_3+s_j} \sqrt{(0-t_3)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_3)^2+(1-s_j)^2} \end{pmatrix}$$

$K_{(1,2)}(0,1)=$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{3} w_j e^{0+1+t_1+s_j} \sqrt{(0-t_1)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_1)^2+(1-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+1+t_2+s_j} \sqrt{(0-t_2)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_2)^2+(1-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{0+1+t_3+s_j} \sqrt{(0-t_3)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_3)^2+(1-s_j)^2} \end{pmatrix}$$

where

$\sum_{j=1}^{3} w_j e^{0+1+t_1+s_j} \sqrt{(0-t_1)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_1)^2+(1-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.888888 & 0.55555 \end{pmatrix} \begin{pmatrix} -1.79011 \\ 2.17655 \\ 6.66031 \end{pmatrix} = 4.63546$$

$\sum_{j=1}^{3} w_j e^{0+1+t_2+s_j} \sqrt{(0-t_2)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_2)^2+(1-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} -0.455445 \\ 0 \\ 2.251305 \end{pmatrix} = 0.99771$$

and

$\sum_{j=1}^{3} w_j e^{0+1+t_3+s_j} \sqrt{(0-t_3)^2+(1-s_j)^2}^2 \ln \sqrt{(0-t_3)^2+(1-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.5555 \end{pmatrix} \begin{pmatrix} -0.383403 \\ 0.466667 \\ 1.42651 \end{pmatrix} = 0.99325.$$

then the

$K_{(1,2)}(0,1)$=

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.5555 \end{pmatrix} \begin{pmatrix} 4.63546 \\ 0.99771 \\ 0.99325 \end{pmatrix} = 4.0152$$

$K_{(2,1)}(1,0)$=

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{3} w_j e^{1+0+t_1+s_j} \sqrt{(1-t_1)^2+(0-s_j)^2}^2 \ln \sqrt{(1-t_1)^2+(0-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{1+0+t_2+s_j} \sqrt{(1-t_2)^2+(0-s_j)^2}^2 \ln \sqrt{(1-t_2)^2+(0-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{1+0+t_3+s_j} \sqrt{(1-t_3)^2+(0-s_j)^2}^2 \ln \sqrt{(1-t_3)^2+(0-s_j)^2} \end{pmatrix}$$

where

$\sum_{j=1}^{3} w_j e^{1+0+t_1+s_j} \sqrt{(1-t_1)^2+(0-s_j)^2}^2 \ln \sqrt{(1-t_1)^2+(0-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} -1.78037 \\ -0.45630 \\ -0.38343 \end{pmatrix} = -1.60562$$

$\sum_{j=1}^{3} w_j e^{1+0+t_2+s_j} \sqrt{(1-t_2)^2+(0-s_j)^2}^2 \ln \sqrt{(1-t_2)^2+(0-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} 2.176778 \\ 0 \\ 0.46660 \end{pmatrix} = 1.46814$$

and

$\sum_{j=1}^{3} w_j e^{1+0+t_3+s_j} \sqrt{(1-t_3)^2+(0-s_j)^2}^2 \ln \sqrt{(1-t_3)^2+(0-s_j)^2} =$

$$\begin{pmatrix} 0.5555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} 6.659706 \\ 2.24999 \\ 1.42651 \end{pmatrix} = 6.48533.$$

then the

$K_{(2,1)}(1,0)$=

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} -1.60562 \\ 1.46814 \\ 6.48533 \end{pmatrix} = 4.0152$$

and

$K_{(2,2)}(1,1)$=

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} \sum_{j=1}^{3} w_j e^{1+1+t_1+s_j} \sqrt{(1-t_1)^2+(1-s_j)^2}^2 \ln \sqrt{(1-t_1)^2+(1-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{1+1+t_2+s_j} \sqrt{(1-t_2)^2+(1-s_j)^2}^2 \ln \sqrt{(1-t_2)^2+(1-s_j)^2} \\ \sum_{j=1}^{3} w_j e^{1+0+t_3+s_j} \sqrt{(1-t_3)^2+(1-s_j)^2}^2 \ln \sqrt{(1-t_3)^2+(1-s_j)^2} \end{pmatrix}$$

where

$\sum_{j=1}^{3} w_j e^{1+1+t_1+s_j} \sqrt{(1-t_1)^2+(1-s_j)^2}^2 \ln \sqrt{(1-t_1)^2+(1-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} -4.095049 \\ .432605 \\ 13.63780 \end{pmatrix} = 5.68515$$

$\sum_{j=1}^{3} w_j e^{1+1+t_2+s_j} \sqrt{(1-t_2)^2+(1-s_j)^2}^2 \ln \sqrt{(1-t_2)^2+(1-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} 0.433069 \\ 5.117011 \\ 10.03096 \end{pmatrix} = 12.52587$$

and

$\sum_{j=1}^{3} w_j e^{1+1+t_3+s_j} \sqrt{(1-t_3)^2+(1-s_j)^2}^2 \ln \sqrt{(1-t_3)^2+(1-s_j)^2} =$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.55555 \end{pmatrix} \begin{pmatrix} 13.63514 \\ 10.03150 \\ 9.1071107 \end{pmatrix} = 21.5482.$$

then the

$K_{(2,2)}(1,1)=$

$$\begin{pmatrix} 0.55555 & 0.88888 & 0.5555 \end{pmatrix} \begin{pmatrix} 5.68515 \\ 12.52587 \\ 21.5482 \end{pmatrix} = 26.262$$

The matrix of

$$K_{i,j}=\begin{pmatrix} -0.18622 & 4.0152 \\ 4.0152 & 26.262 \end{pmatrix}$$

Now, we find the matrix λ_{ij} for solve the system

[k] [λ] = [F]

where f(x,y)= $\frac{-1+e^2*e^{x+y}}{2}$

f(0,0)= $\frac{-1+e^2*e^{0+0}}{2} = 3.1945$

f(0,1)=$\frac{-1+e^2*e^{0+1}}{2} = 8.625$

f(1,0)= $\frac{-1+e^2*e^{1+0}}{2} = 8.625$

f(1,1)= $\frac{-1+e^2*e^{1+1}}{2} = 23.6045$

then

$$[F]=\begin{pmatrix} 3.1945 & 8.625 \\ 8.625 & 23.6045 \end{pmatrix}$$

here we solve the system

$$\begin{pmatrix} -0.18622 & 4.0152 \\ 4.0152 & 26.262 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{01} \\ \lambda_{10} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} 3.1945 & 8.625 \\ 8.625 & 23.6045 \end{pmatrix}$$

then the

$$\lambda_{i,j}=\begin{pmatrix} 2.344 & 6.267 \\ 0.6868 & 1.857 \end{pmatrix}$$

Now, we find to the approximate U(x,y)

$$u(x,y) \approx \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_{ij} \phi_{ij}(x,y) = \lambda^T \Phi(x,y) \tag{4.5}$$

$$\begin{pmatrix} U(0,0) & U(0,1) \\ U(1,0) & U(1,1) \end{pmatrix} = \begin{pmatrix} 2.344 & 0.6865 \\ 6.267 & 1.857 \end{pmatrix} \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$$

The table below shows the comparison between the thin plates and the Gaussian method in solving 2D fredholm inegral equation:

(x,y)	U(x,y)	TPS RBF	GA RBF
(0,0)	0.313	0.34318	0.33239
(0,1)	0.8508	0.8691	0.22123
(1,0)	0.8508	1.0195	1.086
(1,1)	2.3130	1.4572	0.72288

It is clear from the table that the values of approximate tps are closest to the exact solution more than gauss method.

The absolute errors are reported in the table below :

(x,y)	$err_max(TPS)$	$err_max(GA)$
(0,0)	0.03018	0.01939
(0,1)	0.0183	0.62957
(1,0)	0.1687	0.2352
(1,1)	0.8558	1.59012

comments This table illustrates, the max. erorr difference between thin plate spline and gauss radial basis function and shows that the amount of max erorr of TPS less than GA.

4.4 conclusion

In this thesis, we have to solve Fredholm integral equation in one dimension and in two dimensions using Thin plate spline Radial basis function. This method is very straightforward and requires little computation. Additionally, we may extend this method to issues with larger dimensions. In this thesis we have solved Fredholm's integral equation in one and two dimensions, using radial basis functions (RBF) and RBF thin plate spline in particular. Fredholm's integral equation is an equation that has many applications in science and engineering. After finding the analytical solution to the Fredholm integral equation, we presented in the numerical solution other ways to solve the Fredholm integral equation using Gauss and MQ Radial basis function. A numerical comparison of the error between these different types was then performed using numerical examples. It concluded by testing the efficiency of existing methods against this approach. In the future it is possible to expand the solution of the Fredholm integral equation in higher dimension using the thin-plate spline Radial basis function method and solution of the Fredholm integral equation in Two dimension using softeware like matlab.

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ملخص

حل معادلات فريدهولم التكاملية ثنائية الأبعاد باستخدام وظائف الأساس الشعاعي

غدير عبد الباسط مصلح

كما يشير عنوانها ، تم تخصيص هذه الدراسة لحل معادلة فريدهولم التكامليه في بعدين ، باستخدام وظائف الأساس الشعاعي (RBF) وشرائح الصفائح الرقيقة RBF على وجه الخصوص. معادلة فريدهولم التكامليه هي معادلة استيراد لها العديد من التطبيقات في العلوم والهندسة. هناك قواعد مختلفة لـ RBF في البحث العلمي. استخدم الباحثون غاوسي وعكس رباعي RBF نظرًا لسماتهم عند التعامل مع التطبيقات الهندسية. ومع ذلك ، تم إدخال أنواع جديدة من RBF مع خصائص رياضية أكثر مرونة. في هذه الدراسة ، استخدمنا شرائح الصفائح الرقيقة كوظائف أساسية لتقديم حل لمعادلة فريدهولم التكامليه. بعد إيجاد الحل التحليلي لمعادلة فريدهولم التكاملية ، قدمنا عددًا من أنواع RBF لحل المعادلة التكاملية باستخدام الأنواع المقدمة ، بما في ذلك شرائح الألواح الرقيقة. ثم أجرينا مقارنة عددية للخطأ بين هذه الأنواع المختلفة باستخدام أمثلة عددية. اختتمنا باختبار كفاءة الأساليب الحالية مقارنة بهذا النهج.